EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 1, 2021, 53-64
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Coefficient problems in a class of functions with bounded turning associated with Sine function

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Abstract. The Hankel determinant for a function having power series was first defined by Pommerenke. The growth of Hankel determinant has been evaluated for different subcollections of univalent functions. Many subclasses with bounded turning have several interesting geometric properties. In this paper, some classes of functions with bounded turning which connect to the sine functions, are studied in the region of the unit disc in order. Our purpose is to obtain some upper bounds for the third and fourth Hankel determinants related to such classes.

2020 Mathematics Subject Classifications: 30C45, 30C50

Key Words and Phrases: Holomorphic functions, Subordinations, Trigonometric function, Hankel determinant

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of all functions $f(z)$ which are holomorphic in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. Therefore, for $f(z) \in \mathcal{A}$, one has

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D}). \tag{1}$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i1.3902

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Let $\mathcal{S} \subset \mathcal{A}$ represent all functions that are univalent in $\mathbb{D}$. For a function $f \in \mathcal{S}$ of the form (1), Bieberbach conjectured in 1916 that $|a_n| \leq n$, $n = 2, 3, \ldots$. De Branges proved this in 1985, see [7]. During this period, a lot of coefficients results were established for some subfamilies of $\mathcal{S}$. For example, the class $\mathcal{S}^*$ of starlike functions, $\mathcal{K}$ of convex functions and $\mathcal{R}$ of bounded turning functions:

$$\begin{aligned}
\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D} \right\}, \qquad (2) \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D} \right\}, \\
\mathcal{R} &= \left\{ f \in \mathcal{S} : f'(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D} \right\},
\end{aligned}$$

where " $\prec$ " represents the subordination.

We write $g_1 \prec g_2$, if there is an analytic function v in $\mathbb{D}$, with limitations $v(0) = 0$ and $|v(z)| < 1$, such that $g_1(z) = g_2(v(z))$, $z \in \mathbb{D}$. In case of univalency of g_2 in $\mathbb{D}$, the following relation holds;

$$g_1(z) \prec g_2(z), \quad z \in \mathbb{D} \iff g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

By varying the function right hand side of subordinations in (2), we can define some subclasses of the set $\mathcal{S}$ which have several interesting geometric properties, see [9–12, 15–17, 23, 28, 29, 35]. From among these subfamilies we recall here the families that are associated with trigonometric function as follows;

$$\begin{aligned}
\mathcal{K}_{\sin} &= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \sin(z), \quad z \in \mathbb{D} \right\}, \qquad (3) \\
\mathcal{S}^*_{\sin} &= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin(z), \quad z \in \mathbb{D} \right\}, \qquad (4) \\
\mathcal{R}_{\sin} &= \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \sin(z), \quad z \in \mathbb{D} \right\}. \qquad (5)
\end{aligned}$$

The set defined in (4) was established by Cho et.al [10] and studied the radii problems. Here we investigated only the class (5).

For given parameters $q, n \in \mathbb{N} = \{1, 2, \ldots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [32, 33] for a function $f \in \mathcal{S}$ having power series expansion (1) as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \qquad (6)$$

The growth of $H_{q,n}(f)$ has been evaluated for different subcollections of univalent functions. Exceptionally, for each of the sets $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{R}$ the sharp bound of the determinant $H_{2,2}(f) = \left|a_2 a_4 - a_3^2\right|$ were found by Janteng et al. [13, 14] while for the family of close-to-convex functions the sharp estimation is still unknown (see, [38]). On the other hand,

for the set of Bazilevič functions, the best estimate of $|H_{2,2}(f)|$ was proved by Krishna and RamReddy [21]. For more work on $H_{2,2}(f)$, see [5, 26, 27, 30, 31].
The determinant

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \tag{7}$$

is known as third order Hankel determinant and the estimation of this determinant $|H_{3,1}(f)|$ is a challenging task. In 2010, the first article on $H_{3,1}(f)$ by Babalola [4], in which he obtained the upper bound of $|H_{3,1}(f)|$ for the groups of $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$. Later on, a few creators distributed their work regarding $|H_{3,1}(f)|$ for various subcollections of holomorphic and univalent functions, see [1, 2, 6, 8, 19, 22, 37, 39]. In 2017, the consequences of Babalola [4] improved by Zaprawa [40], by proving

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{49}{540}, & \text{for} \quad f \in \mathcal{K}, \\ \frac{41}{60}, & \text{for} \quad f \in \mathcal{R}. \end{cases}$$

and asserted that these inequalities are as yet not sharp. Additionally for the sharpness, he thought about the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ comprising of functions with m-fold symmetry and acquired the sharp bounds. Recently in 2018, Kowalczyk et.al [20] and Lecko et.al [25] evaluated the sharp inequalities

$$|H_{3,1}(f)| \leq 4/135, \quad \text{and} \quad |H_{3,1}(f)| \leq 1/9,$$

for the recognizable sets $\mathcal{K}$ and $\mathcal{S}^*(1/2)$ respectively, where the symbol $\mathcal{S}^*(1/2)$ indicates the family of starlike functions of order 1/2. Additionally in 2018, the authors [24] got an improved bound $|H_{3,1}(f)| \leq 8/9$ for $f \in \mathcal{S}^*$, yet not best possible. Now in this paper, our main purpose is to study third and fourth order Hankel determinants family defined in (5).

2. A Set of Lemmas

Let $\mathcal{P}$ be the family of functions p that are holomorphic in $\mathbb{D}$ with $\mathfrak{Re}\,p(z) > 0$ and the power series form as follow;

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}). \tag{8}$$

Lemma 1. *If $p \in \mathcal{P}$ be expressed in series expansion* (8)*, then*

$$|c_n| \leq 2 \quad \text{for} \quad n \geq 1, \tag{9}$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}, \tag{10}$$

$$|c_{i+j} - \mu c_i c_j| \leq 2, \quad \text{for } 0 \leq \mu \leq 1. \tag{11}$$

and for complex number ρ, *we have*

$$\left|c_2 - \rho c_1^2\right| \leq 2\max\{1, |2\rho - 1|\}. \tag{12}$$

where the inequalities (9), (10), (11) *are taken from [34] and* (12) *is obtained in [18].*

Lemma 2. *If* $p(z) \in \mathcal{P}$ *be expressed in series expansion* (8), *then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

for some x, $|x| \leq 1$ *and*

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - \left(4 - c_1^2\right)c_1 x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z$$

for some z, $|z| \leq 1$.

Lemma 3. ([3]) *Let* $p \in \mathcal{P}$ *has power series* (8), *then*

$$\left|Jc_1^3 - Kc_1c_2 + Lc_3\right| \leq 2|J| + 2|K - 2J| + 2|J - K + L| \tag{13}$$

Corollary 1. ([34]) *Let* $p \in \mathcal{P}$ *has power series* (8), *then*

$$\left|c_1^3 - 2c_1c_2 + c_3\right| \leq 2,$$

Lemma 4. ([36]) *Let* m, n, l *and* a *satisfy the inequalities* $0 < m < 1, 0 < r < 1$, *and*

$$8r(1-r)\left[(mn - 2l)^2 + (m(r+m) - n)^2\right] + m(1-m)(n - 2rm)^2 \leq 4m^2(1-m)^2 r(1-r).$$

If $p(z) \in \mathcal{P}$ *and has power series* (8) *then*

$$\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4\right| \leq 2.$$

3. Improved bound of $|H_{3,1}(f)|$ for the Set $\mathcal{R}_{\sin}$

Theorem 1. *If* $f(z)$ *of the form* (1) *belongs to* $\mathcal{R}_{\sin}$, *then*

$$|a_k| \leq \frac{1}{k}, \quad k = 2, 3, 4, 5. \tag{14}$$

The results are sharp.

Proof. Since $f(z) \in \mathcal{R}_{\sin}$, form subordination definition there exists a Schwarz function $v(z)$ with $v(0) = 0$ and $|v(z)| < 1$, in such a way that

$$f'(z) = 1 + \sin(v(z)), \quad (z \in \mathbb{D}).$$

Since,

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 + \cdots. \tag{15}$$

Define a function

$$h(z) = \frac{1+v(z)}{1-v(z)} = 1 + c_1 z + c_2 z^2 + \cdots . \tag{16}$$

Clearly, we have $h(z) \in \mathcal{P}$ and

$$v(z) = \frac{h(z)-1}{h(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}.$$

This gives

$$\begin{aligned} 1 + \sin(v(z)) &= 1 + \frac{1}{2}c_1 z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right) z^2 + \left(\frac{5c_1^3}{48} - \frac{c_1 c_2}{2} + \frac{c_3}{2}\right) z^3 \\ &\quad + \left(-\frac{1}{32}c_1^4 + \frac{5}{16}c_1^2 c_2 - \frac{1}{2}c_3 c_1 - \frac{1}{4}c_2^2 + \frac{1}{2}c_4\right) z^4 + \cdots . \end{aligned} \tag{17}$$

By comparing (15) and (17), we may get

$$a_2 = \frac{c_1}{4}, \tag{18}$$

$$a_3 = \frac{1}{3}\left(\frac{c_2}{2} - \frac{c_1^2}{4}\right), \tag{19}$$

$$a_4 = \frac{1}{4}\left(\frac{5}{48}c_1^3 + \frac{c_3}{2} - \frac{c_1 c_2}{2}\right), \tag{20}$$

$$a_5 = \frac{1}{5}\left(\frac{c_4}{2} + \frac{5}{16}c_1^2 c_2 - \frac{c_1^4}{32} - \frac{c_1 c_3}{2} - \frac{c_2^2}{4}\right). \tag{21}$$

Now implementing (9), in (18), we obtain

$$|a_2| \leq \frac{1}{2}.$$

Now using (10), in (19), we get

$$|a_3| \leq \frac{1}{6}\left(2 - \frac{|c_1|^2}{2}\right)$$

The maximum value of above function at $c_1 = 0$.

$$|a_3| \leq \frac{1}{3}.$$

Implementation of triangle inequality and Lemma 3, in (20), leads us to

$$|a_4| \leq \frac{1}{4}.$$

By applying Lemma 4 in (21), it provides

$$|a_5| \leq \frac{1}{5}.$$

If for $k = 2, 3, 4, 5$, we take the functions $f_k(z) = z + \cdots$ such that

$$f_k'(z) = 1 + \sin(z^{k-1}), \quad (z \in \mathbb{D}),$$

then $f_k'(z) \prec 1 + \sin z$ and so $f_k \in \mathcal{R}_{\sin}$ and

$$f_k(z) = z + \frac{1}{k}z^k - \frac{1}{3!(3k-2)}z^{3k-2} + \cdots, \quad (z \in \mathbb{D}) \tag{22}$$

which shows that the bounds are sharp.

Conjecture If $f(z)$ of the form (1) belongs to $\mathcal{R}_{\sin}$, then

$$|a_n| \leq \frac{1}{n}, \quad n \geq 6. \tag{23}$$

Theorem 2. *If $f(z)$ of the form* (1) *belongs to $\mathcal{R}_{\sin}$, then for any complex number ρ*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{3}\max\left\{1, \frac{3|\rho|}{4}\right\}. \tag{24}$$

The result is sharp.

Proof. Utilizing (18) and (19), we may get

$$\left|a_3 - \rho a_2^2\right| = \left|\frac{c_2}{6} - \frac{c_1^2}{12} - \frac{\rho}{16}c_1^2\right|.$$

This gives

$$\left|a_3 - \rho a_2^2\right| = \frac{1}{6}\left|\left\{c_2 - \left(\frac{4+3\rho}{8}\right)c_1^2\right\}\right|.$$

Application of (12), leads us to

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{3}\max\left\{1, \frac{|3\rho|}{4}\right\}.$$

For the sharpness of (24) consider (22), with $k = 2$:

$$f_2(z) = z + \frac{1}{2}z^2 - \frac{1}{4!}z^4 + \cdots, \quad (z \in \mathbb{D}),$$

which gives equality in (24) when $|\rho| \geq 4/3$, namely

$$\left|a_3 - \rho a_2^2\right| = |\rho a_2^2| = \frac{|\rho|}{4}.$$

For the case $|\rho| \leq 4/3$ consider

$$f_3(z) = z + \frac{1}{3}z^3 - \frac{1}{42}z^7 + \cdots, \quad (z \in \mathbb{D}),$$

which gives

$$\left|a_3 - \rho a_2^2\right| = |a_3| = \frac{1}{3} = \frac{1}{3}\max\left\{1, \frac{3|\rho|}{4}\right\}.$$

Corollary 2. *If* $f \in \mathcal{R}_{\sin}$ *and* $|\rho| \leq 4/3$*, then*

$$\left|a_3 - \rho a_2^2\right| \leq \frac{1}{3}. \tag{25}$$

Theorem 3. *If* f *of the form* (1) *belongs to* $\mathcal{R}_{\sin}$*, then*

$$|a_2 a_3 - a_4| \leq \frac{1}{4}. \tag{26}$$

The result is sharp.

Proof. From (18),(19) and (20), we have

$$|a_2 a_3 - a_4| = \left|-\frac{3}{64}c_1^3 + \frac{1}{6}c_2 c_1 - \frac{1}{8}c_3\right| = \left|\frac{3}{64}c_1^3 - \frac{c_1 c_2}{6} + \frac{c_3}{8}\right|.$$

Implementation of triangle inequality and Lemma 3, in (20), leads us to

$$|a_2 a_3 - a_4| \leq \frac{1}{4}.$$

The sharpness of (26) shows $f_4(z) = z + z^4/4 - z^{10}/60 + \cdots$ which was defined in (22).

Theorem 4. *If* $f(z)$ *of the form* (1) *belongs to* $\mathcal{R}_{\sin}$*, then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{9}. \tag{27}$$

The result is sharp.

Proof. Now since from (18), (19), and (20), we have

$$\left|a_2 a_4 - a_3^2\right| = \left|\frac{c_1 c_3}{32} - \frac{c_1^2 c_2}{288} - \frac{c_1^4}{2304} - \frac{c_2^2}{36}\right|.$$

Now in terms of Lemma 2, we obtain

$$\begin{aligned} \left|a_2 a_4 - a_3^2\right| &= \left|\frac{c_1 c_3}{32} - \frac{c_1^2 c_2}{288} - \frac{c_1^4}{2304} - \frac{c_2^2}{36}\right| \\ &= \left|-\frac{c_1^4}{768} - \frac{c_1^2 x^2\left(4-c_1^2\right)}{128} - \frac{x^2\left(4-c_1^2\right)^2}{144} + \frac{c_1\left(4-c_1^2\right)\left(1-|x|^2\right)z}{64}\right|. \end{aligned}$$

Let $|z| = 1$, $|x| = t$, $t \in [0,1]$, $|c_1| = c \in [0,2]$. Then, using the triangle inequality, we get

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{c^4}{768} + \frac{t^2 c^2\left(4-c^2\right)}{128} + \frac{t^2\left(4-c^2\right)^2}{144} + \frac{\left(1-t^2\right)c\left(4-c^2\right)}{64}.$$

Putting

$$H(c,t)=\frac{c^4}{768}+\frac{t^2c^2\left(4-c^2\right)}{128}+\frac{t^2\left(4-c^2\right)^2}{144}+\frac{\left(1-t^2\right)c\left(4-c^2\right)}{64},$$

then,

$$\frac{\partial H(c,t)}{\partial t}=\frac{t\left(c^2-18c+32\right)\left(4-c^2\right)}{576}>0,$$

which shows that $H(c,t)$ increases on $[0,1]$ with respect t. That is $H(c,t)$ have maximum value at $t=1$, which is

$$\max H(c,t)=H(c,1)=\frac{c^4}{768}+\frac{c^2\left(4-c^2\right)}{128}+\frac{\left(4-c^2\right)^2}{144}.$$

Setting

$$G(c)=\frac{c^4}{768}+\frac{c^2\left(4-c^2\right)}{128}+\frac{\left(4-c^2\right)^2}{144},$$

then we have

$$G'(c)=\frac{c^3}{192}+\frac{c\left(4-c^2\right)}{64}-\frac{c^3}{64}-\frac{c\left(4-c^2\right)}{36}.$$

If $G'(c)=0$, then the root is $c=0$. Further, since $G''(c)=-\frac{7}{144}<0$, so the function $G(c)$ can attain the maximum value at $c=0$, which is

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{9}.$$

The sharpness of (27) shows $f_3(z)=z+z^3/3-z^7/42+\cdots$ which was defined in (22).

Theorem 5. *If $f(z)=z+a_2z^2+a_3z^3+\cdots$ belongs to $\mathcal{R}_{\sin}$, then*

$$|H_{3,1}(f)|\le\frac{359}{2160}=0.16620\ldots. \tag{28}$$

Proof. Third order Hankel determinant form equation (7) one may written as;

$$H_{3,1}(f)=a_3\left(a_2a_4-a_3^2\right)-a_4\left(a_4-a_2a_3\right)+a_5\left(a_3-a_2^2\right).$$

where $a_1=1$. This provides that

$$|H_{3,1}(f)|\le|a_3|\left|a_2a_4-a_3^2\right|+|a_4|\left|a_4-a_2a_3\right|+|a_5|\left|a_3-a_2^2\right|.$$

By implementing (14), (25), (26) and (27), we obtain our desired result.

4. Bound of $|H_{4,1}(f)|$ for the Set $\mathcal{R}_{\sin}$

First we can write $H_{4,1}(f)$ in the form

$$\begin{aligned} H_{4,1}(f) &= a_7 H_{3,1}(f) - 2a_4 a_6 \left(a_2 a_4 - a_3^2\right) - 2a_5 a_6 \left(a_2 a_3 - a_4\right) - a_6^2 \left(a_3 - a_2^2\right) \\ &\quad + a_5^2 \left(a_2 a_4 - a_3^2\right) + a_5^2 \left(a_2 a_4 + 2a_3^2\right) - a_5^3 + a_4^4 - 3a_3 a_4^2 a_5. \end{aligned} \tag{29}$$

Also

$$\left|a_2 a_4 + 2a_3^2\right| \leq \left|a_2 a_4 - a_3^2\right| + 3\left|a_3\right|^2,$$

using (14) and (27), we get

$$\begin{aligned} \left|a_2 a_4 + 2a_3^2\right| &\leq \frac{1}{9} + \frac{1}{3} \\ &= \frac{4}{9}. \end{aligned} \tag{30}$$

Theorem 6. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belongs to $\mathcal{R}_{\sin}$, then*

$$|H_{4,1}(f)| \leq 0.10556.$$

Proof. Using triangle inequality in (29), we obtain

$$\begin{aligned} |H_{4,1}(f)| &\leq |a_7| \, |H_{3,1}(f)| + 2|a_4| \, |a_6| \left|a_2 a_4 - a_3^2\right| + 2|a_5| \, |a_6| \, |a_2 a_3 - a_4| + |a_6|^2 \left|a_3 - a_2^2\right| \\ &\quad + |a_5|^2 \left|a_2 a_4 - a_3^2\right| + |a_5|^2 \left|a_2 a_4 + 2a_3^2\right| + |a_5|^3 + |a_4|^4 + 3|a_3| \, |a_4|^2 \, |a_5|. \end{aligned}$$

By using(14), (23), (25), (26), (27), (28) and (30), we get the required result.

Acknowledgements

The authors would like to express our appreciation to the anonymous referees for the comprehensive reading of this paper and their valuable comments and suggestions.

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