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COEFFICIENTS OF MEROMORPHIC SCHLICHT FUNCTIONS

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ABSTRACT. This paper presents an elementary proof of a known theorem on the coefficients of meromorphic schlicht functions: if $f\in\Sigma$ and $b_k=0$ for $1\leqq k<n/2$, then $|b_n|\leqq 2/(n+1)$.

Let Σ denote the class of functions

$$f(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

analytic and schlicht in $|z|>1$ except for a simple pole at ∞ with residue 1. Let Σ_0 be the subclass of Σ for which $b_0=0$. It follows from the area theorem that $|b_1|\leqq 1$, and Schiffer [7] obtained the sharp estimate $|b_2|\leqq 2/3$. The form of the extremal functions suggested that $|b_n|\leqq 2/(n+1)$ for all n. This has proved to be quite false, but it is true for certain subclasses of Σ. For example, the following theorem goes back to Goluzin [3].

THEOREM. *Let $f\in\Sigma$ and suppose $b_1=b_2=\cdots=b_{m-1}=0$ for some $m\geqq 1$. Then $|b_n|\leqq 2/(n+1)$, $n=m, m+1, \cdots, 2m$.*

The inequality $|b_2|\leqq 2/3$ is a special case of this theorem. Jenkins [5] proved the theorem by the method of quadratic differentials. I claimed [1] to give an elementary proof based on the Grunsky inequalities. However, Professor Jenkins has pointed out to me that my proof contains an error, since the square-root transformation $\sqrt{(f(z^2))}$ will introduce a branch point wherever $f(z^2)=0$. Furthermore, Goluzin made essentially the same mistake (see the footnote in [4, p. 279]).

The purpose of this paper is to correct the error in [1] and thus to deduce the theorem in an elementary way from the Grunsky inequalities. The main idea was suggested by Pommerenke's recent proof [6] that $|b_2|\leqq 2/3$.

The nth Faber polynomial $F_n(w)$ of a function $f\in\Sigma$ is defined by

$$F_n[f(z)] = z^n + \sum_{k=1}^{\infty} \beta_{nk} z^{-k}.$$

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The inequality

$$(1) \qquad |\beta_{nn}| \leqq 1, \qquad n = 1, 2, \cdots,$$

is a very special case of the Grunsky inequalities.

To prove the theorem, it is enough to show $|b_{2m-1}| \leqq 1/m$ and $|b_{2m}| \leqq 2/(2m+1)$. For the first of these inequalities, assume without loss of generality that $f \in \Sigma_0$ and observe (as in [1]) that

$$[f(z)]^m = F_m[f(z)] = z^m + \sum_{j=1}^{\infty} \beta_{mj} z^{-j},$$

with $\beta_{mm} = m b_{2m-1}$. Hence $|b_{2m-1}| \leqq 1/m$, by (1).

To obtain the estimate for $|b_{2m}|$, assume b_0 is chosen so that $f(z) \neq 0$ in $|z| > 1$, and let

$$g(z) = \sqrt{(f(z^2))} = z\{1 + b_0 z^{-2} + b_m z^{-2m-2} + b_{m+1} z^{-2m-4} + \cdots\}^{1/2}.$$

Let $F_n^*(w)$ be the nth Faber polynomial of g, so that

$$F_n^*[g(z)] = z^n + \sum_{j=1}^{\infty} \beta_{nj}^* z^{-j}.$$

The proof depends on the following lemma.

LEMMA. *For each fixed m, there exist real numbers r_k such that*

$$\beta_{2k+1,2m+1}^* = ((2k+1)/2) b_{m+k} + r_k b_0^{m+k+1}, \qquad k = 0, 1, \cdots, m.$$

PROOF. The lemma will be proved by induction on k. For $k=0$, it is clear that $F_1[g(z)] = g(z)$, and one computes

$$\beta_{1,2m+1}^* = \tfrac{1}{2} b_m + \binom{\frac{1}{2}}{m+1} b_0^{m+1},$$

where $\binom{x}{n} = x(x-1) \cdots (x-n+1)/n!$ is a binomial coefficient. Assuming now that the lemma has been proved for $0, 1, \cdots, k-1$, consider

$$\begin{aligned}
[g(z)]^{2k+1} &= z^{2k+1}\{1 + b_0 z^{-2} + b_m z^{-2m-2} + \cdots\}^{k+1/2} \\
&= z^{2k+1} \sum_{j=0}^{\infty} \binom{k+\frac{1}{2}}{j} (1 + b_0 z^{-2})^{k-j+1/2} (b_m z^{-2m-2} + \cdots)^j \\
&= (1 + b_0 z^{-2})^{k+1/2} z^{2k+1} + (k + \tfrac{1}{2})(1 + b_0 z^{-2})^{k-1/2} \\
&\quad \cdot (b_m + b_{m+1} z^{-2} + \cdots) z^{2(k-m)-1} + O(z^{-2m-2}) \\
&= z^{2k+1} + \sum_{j=1}^{k} \binom{k+\frac{1}{2}}{j} b_0^j z^{2(k-j)+1} + \sum_{j=1}^{\infty} \gamma_{kj} z^{-j},
\end{aligned}$$

say. A calculation gives

$$(2)\quad \gamma_{k,2m+1} = \binom{k+\frac{1}{2}}{k+m+1} b_0^{k+m+1} + (k+\tfrac{1}{2}) \sum_{n=0}^{k} \binom{k-\frac{1}{2}}{n} b_0^n b_{m+k-n}.$$

From the form of $[g(z)]^{2k+1}$, it is clear that

$$F_{2k+1}^*[g(z)] = [g(z)]^{2k+1} - \sum_{j=0}^{k-1} \binom{k+\frac{1}{2}}{k-j} b_0^{k-j} F_{2j+1}^*[g(z)].$$

Thus by (2) and our inductive hypothesis, we have

$$\begin{aligned}\beta_{2k+1,2m+1}^* &= \gamma_{k,2m+1} - \sum_{j=0}^{k-1} \binom{k+\frac{1}{2}}{k-j} b_0^{k-j} [(j+\tfrac{1}{2}) b_{m+j} + r_j b_0^{m+j+1}] \\ &= (k+\tfrac{1}{2}) b_{m+k} + r_k b_0^{m+k+1},\end{aligned}$$

since

$$(k+\tfrac{1}{2}) \binom{k-\frac{1}{2}}{k-j} = (j+\tfrac{1}{2}) \binom{k+\frac{1}{2}}{k-j}, \qquad j = 0, 1, \cdots, k-1.$$

This proves the lemma.

In view of (1), the lemma gives the inequality

$$(3)\qquad | ((2m+1)/2) b_{2m} + r_m b_0^{2m+1} | \leqq 1$$

for the case in which $f(z) \neq 0$ in $|z| > 1$. Now let

$$f(z) = z + b_m z^{-m} + b_{m+1} z^{-m-1} + \cdots \in \Sigma_0,$$

and let E be the complement of the range of f. If $0 \in E$, then (3) gives at once the inequality $|b_{2m}| \leqq 2/(2m+1)$. The proof is complete also if $r_m = 0$.

Suppose next that $0 \notin E$ and $r_m \neq 0$. Assume without loss of generality that $b_{2m} > 0$. If $\alpha \in E$, then $[f(z) - \alpha]$ has no zeros in $|z| > 1$, so by (3),

$$b_{2m} \leqq (2/(2m+1))[1 + r_m \operatorname{Re}\{\alpha^{2m+1}\}], \qquad \alpha \in E.$$

The proof will be complete, then, if we show that $\alpha \in E$ can be chosen so that

$$(4)\qquad r_m \operatorname{Re}\{\alpha^{2m+1}\} \leqq 0.$$

But if this expression is positive for all $\alpha \in E$, then since E is connected and $0 \notin E$, it follows that E lies entirely in an open sector with vertex 0 and angle $\pi/(2m+1)$. On the other hand, 0 is in the convex hull of E, since $\int_0^{2\pi} f(re^{i\theta})d\theta = 0$, $r > 1$. This contradiction shows that $\alpha \in E$ can be chosen to satisfy (4), proving that $b_{2m} \leqq 2/(2m+1)$.

Garabedian and Schiffer [2] found that Re $\{b_3\}$ is maximized by a function $f \in \Sigma$ with

$$f(z) = z + 4ie^{-3}z^{-1} + (\tfrac{1}{2} + e^{-6})z^{-3} + \cdots.$$

However, the above theorem shows that $|b_3| \leqq \frac{1}{2}$ if $b_1 = 0$. This can be generalized as follows.

THEOREM. *If* $f \in \Sigma$ *and* $|\arg\{b_3\} - \arg\{b_1^2\}| \leqq \pi/2$, *then* $|b_3| \leqq \frac{1}{2}$.

PROOF. Assume that $b_0 = 0$ and, after a suitable rotation, that $b_3 > 0$ and Re$\{b_1^2\} \geqq 0$. Observe that

$$[f(z)]^2 = z^2 + 2b_1 + 2b_2 z^{-1} + (2b_3 + b_1^2)z^{-2} + \cdots.$$

Hence by (1), Re$\{2b_3 + b_1^2\} \leqq 1$, which shows $b_3 \leqq \frac{1}{2}$.

Similar generalizations can be made to higher coefficients.

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