Coercive Singular Perturbations

I. - A priori Estimates (*).

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Summary. – *We consider general boundary value problems with small parameter ε in the operator and boundary conditions. Both the perturbed and reduced operators are supposed to be elliptic. We point out* necessary *and* sufficient *conditions of Shapiro-Lopatinsky type for the singularly perturbed problem to be coercive, i.e. for a two-sided a priori estimate to hold for its solutions uniformly with respect to ε.*

I. – Singular perturbations of an elliptic boundary value problem.

The Dirichlet problem for singularly perturbed differential equations has been widely studied in the literature. There are also some attempts to investigate more general boundary value problems for operators containing small parameter (see [20], [22] for the survey of results and references).

One of the interesting aspects connected with singular perturbations is asymptotic formulas for their solutions so that the attention of many authors was mainly concentrated on this subject. A non-less important problem is related with convergence of the asymptotic expansions to the corresponding solutions, the latter being closely connected with the stability of singular perturbations uniformly with respect to the small parameter ε. Some stability results concerning singular perturbations were obtained in [6], [8], [13], [16], [23], [25], and others. In [8] the author indicates some sufficient conditions on the boundary operators and the operator in the domain for a non-adequate a priori estimate to hold uniformly with respect to ε in Schauder type norms. Some conditions for the Vishik-Lyusternik procedure (see [28], [29]) to be operative are also mentioned, the latter being necessary but not sufficient for the stability of singularly perturbed boundary value problem.

We point out necessary and sufficient conditions on the operator in the domain and boundary conditions for a two-sided a priori estimate to hold for the solution of elliptic singular perturbation uniformly with respect to ε. Those conditions are algebraic and have a great deal in common with well-known Shapiro-Lopatisky coerciveness condition (see [18], [24]), which is necessary and sufficient for an adequate a priori estimate for the solutions to an elliptic boundary value problem without

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small parameter to hold up to the boundary (see [1], [26] where these conditions are used in order to establish the a priori estimates).

Let us sketch briefly the stability phenomena connected with singular perturbations of elliptic boundary value problems.

An elliptic boundary value problem might happen to be perturbed by another one containing small parameter ε in its coefficients in such a way that the stability of the original problem is destroyed. In other words, the a priori estimate which is the basic property of the original reduced problem (with $\varepsilon = 0$) does not hold uniformly with respect to ε for the solutions to inconveniently perturbed problems. Being more specific, we illustrate the situation by the following examples.

EXAMPLE 1. – Consider the Dirichlet problem:

$$-\Delta u^0(x) = f^0(x)\,, \qquad x \in U\,, \tag{0.1}$$

$$u^0(x') = 0\,, \qquad x' \in \partial U \tag{0.2}$$

in a compact domain U with smooth boundary ∂U.

For the solutions of (0.1), (0.2), the a priori estimation holds:

$$\|u^0\|_1 \leq C\|f^0\|_{-1}\,, \tag{0.3}$$

where $\|\cdot\|_K$ are norms in the Sobolev spaces $H_K(U)$.

Denoting the problem (0.1), (0.2) by

$$\mathfrak{A}^0 = (-\Delta; \{1\})$$

consider two singular perturbations for the operator $-\Delta$ in U:

$$\mathfrak{A}_1^\varepsilon = \left(\varepsilon^2\Delta^2 - \Delta; \left\{1, \frac{\partial}{\partial N}\right\}\right) \tag{0.4}$$

and

$$\mathfrak{A}_2^\varepsilon = \left(-\varepsilon^2\Delta - \Delta; \left\{1, \frac{\partial}{\partial N}\right\}\right) \tag{0.5}$$

with homogeneous boundary conditions:

$$u^\varepsilon(x') = \frac{\partial u^\varepsilon}{\partial N}(x') = 0\,, \qquad x' \in \partial U\,, \tag{0.6}$$

N being the inward normal to ∂U.

The singular perturbation $\mathfrak{A}_1^\varepsilon$ occurs when considering linearized model of thin elastic plates.

It is immediate that the a priori estimation (0.3) holds for the solution $u^\varepsilon(x)$ of the problem $\mathfrak{A}_1^\varepsilon$ with constant C which does not depend on ε, $u^\varepsilon(x)$.

On the other hand (0.3) can not hold any longer for the solutions $v^\varepsilon(x)$ of $\mathfrak{A}_2^\varepsilon$. Moreover, even some weaker estimation of the type:

$$\|v^\varepsilon\|_1 \leq C(\|-(\varepsilon^2\Delta^2 + \Delta)v^\varepsilon\|_{-1} + \|v^\varepsilon\|_0) \tag{0.7}$$

does not hold with C which does not depend on ε.

Indeed, assuming that (0.7) is true and substituting there for $v^\varepsilon(x)$ the functions:

$$v^\varepsilon(x) = \exp(ix\cdot\eta/\varepsilon)\,\psi(x) \tag{0.8}$$

with $\eta \in \boldsymbol{R}^n$, $|\eta| = 1$, $\psi(x) \in C_0^\infty(U)$, $\|\psi\|_0 = 1$, one gets a contradiction, given that for the functions (0.8) the left hand side in (0.7) grows like ε^{-1} and the right hand side is bounded as $\varepsilon \to 0$.

The instability of singular perturbation might occur because of an inconvenient perturbation of boundary operator.

EXAMPLE 2. – Consider two singular perturbations of the Dirichlet problem $\mathfrak{A}^0 = (-\Delta; 1)$ by Neumann's ones:

$$\mathfrak{A}_-^\varepsilon = \left(-\Delta; \left\{-\varepsilon\frac{\partial}{\partial N} + 1\right\}\right) \tag{0.9}$$

and

$$\mathfrak{A}_+^\varepsilon = \left(-\Delta; \left\{\varepsilon\frac{\partial}{\partial N} + 1\right\}\right) \tag{0.10}$$

where again N is the inward normal to ∂U.

For $\mathfrak{A}_-^\varepsilon$ the a priori estimation

$$\|u^\varepsilon\|_1 \leq C\left(\|-\Delta u^\varepsilon\|_{-1} + \left[\left(-\varepsilon\frac{\partial}{\partial N} + 1\right)u^\varepsilon\right]_{\frac{1}{2}} + \|u^\varepsilon\|_0\right) \tag{0.11}$$

with $[\cdot]_s$ the Sobolev's norm in $H_s(\partial U)$, holds with constant C which does not depend on ε, u^ε. When $\varepsilon \to 0$ one gets from (0.11) the usual a priori estimation for a second order elliptic partial differential equation.

For the solutions of $\mathfrak{A}_+^\varepsilon$ the estimation (0.11) is impossible with C which does not depend on ε. Indeed, assuming (0.11) and subistituting $u^\varepsilon(x)$ of the form

$$u^\varepsilon(x) = \exp(-x_n/\varepsilon + ix'\cdot\eta'/\varepsilon)\psi(x') \tag{0.12}$$

where (x', x_n) are local coordinates related with a point $x_0' \in \partial U$, $\eta' \in R^{n-1}$, $|\eta'| = 1$, $\psi(x')$ has its support in a small neighborhood of $x_0' \in \partial U$ and $[\psi]_0 = 1$, one gets a contradiction, given that the left-hand side in (0.11) with (0.12) substituted for $u^\varepsilon(x)$ grows as $\varepsilon^{-\frac{1}{2}}$ while the right-hand-side is bounded when $\varepsilon \to +0$.

Therefore, there is a need to classify the singular perturbations of elliptic boundary value problems.

One introduces algebraic conditions of ellipticity and coerciveness for singularly perturbed boundary value problems, which are *necessary* and *sufficient* for the stability of the singular perturbation uniformly with respect to the small parameter ε.

Introducing Sobolev spaces $H_{(s)}$ with vectorial subscripts $s = (s_1, s_2, s_3)$ which describe the behaviour of function $u^\varepsilon \in H_s$ when $\varepsilon \to 0$ and its smoothness with respect to D and εD, where D is the differentiation, one establishes a two-sided a priori estimates which are the characteristic property of coercive singular perturbation.

Let us briefly sketch the contents of the paper.

In Section 1 we introduce the spaces $H_{(s)}$ with subscripts $s \in \mathcal{R}^3$ which are used later to establish the a priori estimates. Spaces H_{s_1,s_2} with two subscripts were introduced previously in [4] to investigate Dirichlet problem for singular pseudo-differential operators. It turns out that spaces with three subscripts are better suited for studying general boundary value problems for singular perturbations.

In Section 2 we establish two-sided a priori estimates for elliptic singular perturbations in $\mathcal{R}^n$ and show the necessity of the ellipticity condition for such an estimate to hold uniformly with respect to ε. The motivation to include this section was mainly to make the paper self-contained.

In Section 3, which is the central part of the paper, we consider first singularly perturbed boundary value problem for ordinary differential equations with small parameter $\varepsilon > 0$ and large parameter $\xi' \in \mathcal{R}^{n-1}$ and establish a priori estimates uniformly with respect to ε and ξ'. Afterwards, the results are extended to singular perturbations with constant coefficients in $\mathcal{R}^n_+$. Further we state the main results for singular perturbations with variable coefficients in a bounded domain with smooth boundary. It is interesting to point out that for singular perturbations there are situations similar to the ones for pseudodifferential operators with negative factorization index (see [5]) when solutions to an elliptic pseudodifferential equation are not in general sufficiently smooth up to the boundary and there is a need to introduce Poisson operators with unknown densities for the solution to be in corresponding Sobolev space. We end the Section 3 by discussing some examples of elliptic and non-elliptic and also of coercive and non-coercive singular perturbations.

The results in this paper are a slight modification of the ones summarized in [10] (see also [11], [12] for singularly perturbed ordinary differential and difference operators).

1. – Spaces $\mathcal{H}_{(s)}$, $H_{(s)}$.

1.1. *Spaces* $\mathcal{H}_{(s)}(\mathcal{R}^n)$, $H_{(s)}(\mathcal{R}^n)$.

Throughout this paper ε is a small positive or nonnegative parameter. We consider functions $u(\varepsilon, x)$, $\varepsilon \in [0, \varepsilon_0]$, $x \in \mathcal{R}^n$ with complex values. Sometimes $u(\varepsilon, x)$ will be interpreted as a family of functions of $\varepsilon \in [0, \varepsilon_0]$ with values in some topological vector space.

Let $u\colon [0, \varepsilon_0] \to C_0^\infty(\mathcal{R}_x^n)$ be L^∞mapping with respect to the topology induced by $C_0^\infty(\mathcal{R}^n)$. We write it in a short way: $u \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n))$. For such u and any vector $s = (s_1, s_2, s_3) \in \mathcal{R}^3$ we define the norms with subscript s:

$$\|u\|_{(s)} = \varepsilon^{-s_1}\|\langle D\rangle^{s_2}\langle \varepsilon D\rangle^{s_3} u\|_{L^2(\mathcal{R}_x^n)} \tag{1.1.1}$$

where $D = (D_1, \ldots, D_n)$, $D_K = -i\partial/\partial x_K$, $\langle D\rangle^2 = 1 + |D|^2$, $|D|^2 = -\Delta$, Δ being the Laplacian in $\mathcal{R}_x^n$, and the norms (1.1.1) being well defined a.e. in $[0, \varepsilon_0]$.

We denote by $\mathcal{H}_{(s)}(\mathcal{R}^n)$ the closure of $L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n))$ with respect to the norm:

$$\|u\|_{(s)} = \operatorname*{vrai\,max}_{\varepsilon \in [0, \varepsilon_0]} \|u\|_{(s)} . \tag{1.1.2}$$

It is quite obvious that for non-negative integer s_2, s_3 the norms (1.1.1) are equivalent uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$ to the following ones:

$$\Bigg(\sum_{|\alpha| \leqslant s_2, |\beta| \leqslant s_3} \varepsilon^{2(|\beta| - s_1)} \int\limits_{\mathcal{R}^n} |D^{\alpha+\beta} u(\varepsilon, x)|^2 dx\Bigg)^{\frac{1}{2}} . \tag{1.1.3}$$

PROPOSITION 1.1.1. – *$\mathcal{H}_{(s)}(\mathcal{R}^n)$ is a Banach space.*

PROOF. – According to the definition of $\mathcal{H}_{(s)}(\mathcal{R}^n)$ a function $u \in \mathcal{H}_{(s)}(\mathcal{R}^n)$ iff the function $v = \varepsilon^{-s_1}\langle D\rangle^{s_2}\langle \varepsilon D\rangle^{s_3} u \in L^\infty([0, \varepsilon_0]; L^2(\mathcal{R}_x^n))$. Therefore, the mapping $\varepsilon^{-s_1}\langle D\rangle^{s_2} \cdot \langle D\rangle^{s_3}$ maps isometrically $\mathcal{H}_{(s)}(\mathcal{R}^n)$ onto $L^\infty([0, \varepsilon_0]; L^2(\mathcal{R}_x^n))$. ■

For $u \in \mathcal{H}_{(s)}(\mathcal{R}^n)$ we introduce the seminorm

$$|u|_{(s)} = \overline{\lim_{\varepsilon \to 0}} \|u\|_{(s)} \tag{1.1.4}$$

and we denote by $\mathring{\mathcal{H}}_{(s)}(\mathcal{R}^n)$ the set of all functions $u \in \mathcal{H}_{(s)}(\mathcal{R}^n)$ such that $|u|_{(s)} = 0$:

PROPOSITION 1.1.2. – *$\mathring{\mathcal{H}}_{(s)}(\mathcal{R}^n)$ is a closed subspace in $\mathcal{H}_{(s)}(\mathcal{R}^n)$.*

PROOF. – It is quite obvious that $\mathring{\mathcal{H}}_{(s)}(\mathcal{R}^n)$ is a vector space, given that for the seminorm $|\cdot|_{(s)}$ the triangle inequality holds. For the same reason it is a closed set. Indeed, if $\{u_n\}_{n \geqslant 1}$ is a Cauchy sequence in $\mathcal{H}_{(s)}(\mathcal{R}^n)$, and $|u_n|_{(s)} = 0$, then, u being the limit of $\{u_n\}_{n \geqslant 1}$ in $\mathcal{H}_{(s)}(\mathcal{R}^n)$, one can write

$$|u|_{(s)} \leqslant |u - u_n|_{(s)} + |u_n|_{(s)} = |u - u_n|_{(s)} . \tag{1.1.5}$$

The left hand side in (1.1.5) does not depend on n and the right hand side vanishes as $n \to +\infty$. ■

We define the spaces $H_{(s)}(\mathcal{R}^n)$ to be:

$$H_{(s)}(\mathcal{R}^n) = \mathcal{K}_{(s)}(\mathcal{R}^n)/\mathring{\mathcal{K}}_{(s)}(\mathcal{R}^n) . \tag{1.1.6}$$

They are provided with the corresponding factor-norm, denoted again by $|\cdot|_{(s)}$,

$$|U|_{(s)} = \overline{\lim_{\varepsilon\to 0}}\, \|u\|_{(s)} \tag{1.1.7}$$

with u any element belonging to the same equivalency class U.

Given an L^∞-mapping $u\colon [0, \varepsilon_0] \to S'(\mathcal{R}^n_x)$ we use the notation $F_{x\to\xi}\, u$ or $\hat{u}(\varepsilon, \xi)$ for its Fourier transform in the sense of tempered distributions. For $u\colon [0, \varepsilon_0] \to L^1(\mathcal{R}^n_x)$ it is defined by the usual formula:

$$(F_{x\to\xi}\, u)(\varepsilon, \xi) = \int_{\mathcal{R}^n} \exp(-\, ix\cdot\xi)\, u(\varepsilon, x)\, dx . \tag{1.1.8}$$

The inverse Fourier transform is denoted by $F^{-1}_{\xi\to x}$.

The norm $|\cdot|_{(s)}$ is said to be stronger than $|\cdot|_{(s')}$ if there exists a constant $C_{s,s'}$, which depends only on its subscripts such that

$$|U|_{s'} \leqslant C_{s,s'} |U|_{s'} , \qquad \forall U \in H_{(s)}(\mathcal{R}^n) \tag{1.1.9}$$

or, equivalently,

$$\|u\|_{s'} \leqslant C_{s,s'} \|u\|_{s'} , \qquad \forall \varepsilon \in [0, \varepsilon_0], \quad u \in \mathcal{K}_{(s)}(\mathcal{R}^n) . \tag{1.1.10}$$

It is immediate that the necessary and sufficient condition for (1.1.10) to hold is the inequality

$$\varepsilon^{-s'_1} \langle\xi\rangle^{s'_2} \langle\varepsilon\xi\rangle^{s'_3} \leqslant C_{s,s'}\, \varepsilon^{-s_1} \langle\xi\rangle^{s_2} \langle\varepsilon\xi\rangle^{s_3}, \qquad \forall \varepsilon \in (0, \varepsilon_0],\ \forall \xi \in \mathcal{R}^n. \tag{1.1.11}$$

PROPOSITION 1.1.3. – *The inequality* (1.1.11) *holds iff*

$$s_1 \geqslant s'_1 , \qquad s_1 + s_2 \geqslant s'_1 + s'_2 , \qquad s_2 + s_3 \geqslant s'_2 + s'_3 . \tag{1.1.12}$$

PROOF. – It is quite obvious that the conditions (1.1.12) are necessary for (1.1.11) to hold (by taking in (1.1.11) alternately $\varepsilon \to 0$, $|\xi| = 1$, afterwards $\varepsilon = \langle\xi\rangle^{-1}$, $\xi \to \infty$ and, finally, $\varepsilon = 1$, $\xi \to \infty$).

To show that (1.1.12) is sufficient, it is enough to consider $s' = 0$. Now, (1.1.12) with $s' = 0$ yields for $\varepsilon_0 \leqslant 1$:

$$\varepsilon^{-s_1} \langle\xi\rangle^{s_2} \langle\varepsilon\xi\rangle^{s_3} = \big(\varepsilon\langle\xi\rangle\big)^{-s_1} \langle\xi\rangle^{s_1+s_2} \langle\varepsilon\xi\rangle^{s_3} \geqslant \langle\xi\rangle^{s_1+s_2} \langle\varepsilon\xi\rangle^{s_3-s_1} \geqslant \langle\varepsilon\xi\rangle^{s_2+s_3} \geqslant 1 . \quad \blacksquare \tag{1.1.13}$$

For a given $s \in \mathfrak{R}^3$ the set of points $s' \in \mathfrak{R}^3$ satisfying (1.1.12) is a pyramid with its vertex located at the point s. We use the notation $s' \leqslant s$ for writing (1.1.12) in a short way and the notation $s' < s$ if there is at least one strict inequality in (1.1.12); the sums $s_1 + s_2$ and $s_2 + s_3$ are denoted respectively by $|s|$ and $[s]$.

With the relations $\leqslant$, $<$, the norms $|\cdot|_{(s')}$, $\|\cdot\|_{(s)}$ and the spaces $H_{(s)}(\mathfrak{R}^n)$, $\mathcal{K}_{(s)}(\mathfrak{R}^n)$ become partially ordered, i.e.

$$H_{(s)}(\mathfrak{R}^n) \subseteq H_{(s')}(\mathfrak{R}^n)\,, \qquad \mathcal{K}_{(s)} \subseteq \mathcal{K}_{(s')}(\mathfrak{R}^n) \qquad \text{iff } s' \leqslant s\,. \tag{1.1.14}$$

We will need later a substitute for the interpolation inequality in the usual Sobolev spaces. It is convenient to be able to estimate a given norm of order $s \in \mathfrak{R}^3$ by some stronger one of order $\sigma \in \mathfrak{R}^3$ with small coefficient and by a weaker one of order $\varrho \in \mathfrak{R}^3$ with a large coefficient. It is easy to check that such an inequality cannot hold for any σ, s, ϱ such that $\varrho < s < \sigma$. However, there is a useful substitute for the interpolation inequality.

PROPOSITION 1.1.14. – *Let* $\sigma, s \in \mathfrak{R}^3$, $s < \sigma$ *and*

$$\varrho = s - t(\sigma - s)\,, \qquad t > 0\,. \tag{1.1.15}$$

Then

$$\|u\|^2_{(s)} \leqslant \delta^2 \|u\|^2_{(\sigma)} + \delta^{-2t}\|u\|^2_{(\varrho)}\,, \qquad \forall u \in \mathcal{K}_{(\sigma)}(\mathfrak{R}^n), \quad \forall \delta > 0\,. \tag{1.1.16}$$

PROOF. – It is quite obvious that it suffices to show (1.1.16) for $s = 0$. In that case the inequality (1.1.16) is equivalent to the following one:

$$1 \leqslant \delta^2 \varepsilon^{-2\sigma_1} \langle \xi \rangle^{2\sigma_2} \langle \varepsilon\xi \rangle^{2\sigma_3} + \delta^{-2t} \varepsilon^{2t\sigma_1} \langle \xi \rangle^{-2t\sigma_2} \langle \varepsilon\xi \rangle^{-2t\sigma_3} \tag{1.1.17}$$

or, whith $\lambda = \delta^2 \varepsilon^{-2\sigma_1} \langle \xi \rangle^{2\sigma_2} \langle \varepsilon\xi \rangle^{2\sigma_3}$, to the inequality

$$1 \leqslant \lambda + \lambda^{-t}, \qquad \forall \lambda > 0\,. \quad \blacksquare \tag{1.1.18}$$

For $s \in \mathfrak{R}^3$ we denote

$$\Gamma_s = \{s' \in \mathfrak{R}^3 | s' \leqslant s\}\,, \qquad \Gamma_s^* = \{s' \in \mathfrak{R}^3 | s' \geqslant s\}\,. \tag{1.1.19}$$

The pyramid Γ_s^* is called reciprocal to Γ_s.

COROLLARY 1.1.5. – Let $s \in \mathfrak{R}^3$ and $\sigma \in \operatorname{Int} \Gamma_s^*$. Then for $\forall \varrho \in \Gamma_s$ there exists a $t > 0$ such that (1.1.16) holds.

Indeed, the condition $\sigma \in \operatorname{Int} \Gamma_s^*$ means that $\sigma_1 - s_1 > 0$, $\sigma_1 + \sigma_2 - (s_1 + s_2) > 0$, $\sigma_2 + \sigma_3 - (s_2 + s_3) > 0$. Therefore there exists $t > 0$ such that $\varrho > s - t(\sigma - s)$.

For $s \in \mathcal{R}^3$ we denote by $\partial\Gamma_s$ the boundary of the pyramid Γ_s which has three plane components:

$$\begin{aligned} \partial_1\Gamma_s &= \{s' \leqslant s \,|\, s_1' = s_1\}\,, \\ \partial_2\Gamma_s &= \{s' \leqslant s \,|\, s_1' + s_2' = s_1 + s_2\}\,, \\ \partial_3\Gamma_s &= \{s' \leqslant s \,|\, s_2' + s_3' = s_2 + s_3\}\,. \end{aligned} \tag{1.1.20}$$

We define $\partial_j\Gamma_s^*$ to be the component of $\partial\Gamma_s^*$ such that $\partial_j\Gamma_s$ and $\partial_j\Gamma_s^*$ are contained in the same plane passing through s.

COROLLARY 1.1.6. – Let $s \in \partial_j\Gamma_s$ and $\sigma \in \operatorname{Int}\partial_j\Gamma_s^*$. Then for $\forall\varrho \in \partial_j\Gamma_s$ there exists $t > 0$ such that (1.1.16) holds.

Indeed, one uses the same argument as in the Corollary 1.1.5.

1.2. *Restriction to a hyperplane.*

For $u \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n))$ we denote by $\pi_0 u$ its restriction to the hyperplane $\{x_n = 0\}$ and consider the mapping

$$L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n_x)) \ni u \to \pi_0\{D_n^{j-1}u\}_{1\leqslant j\leqslant K} \in L^\infty\big([0, \varepsilon_0]; \prod_{1\leqslant j\leqslant K} C_0^\infty(\mathcal{R}^{n-1}_{x'})\big)\,, \tag{1.2.1}$$

where $x = (x', x_n)$; $[\pi_0 u]_{(\sigma)}$ denotes the corresponding norm of order $\sigma \in \mathcal{R}^3$.

The mapping (1.2.1) is well defined a.e. on $[0, \varepsilon_0]$.

LEMMA 1.2.1. – *Let $s \in \mathcal{R}^3$ be such that $s_2 = \frac{1}{2}$, $s_3 > 0$, and let $\sigma = s - \frac{1}{2}e_2 = (s_1, 0, s_3)$. Then for $u \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n_x))$ the following inequality holds*:

$$[\pi_0 u]_{(\sigma)} \leqslant C(1 + |\ln \varepsilon|)\, \|u\|_{(s)}\,. \tag{1.2.2}$$

PROOF. – Let $\tilde{u}(\varepsilon, \xi', x_n) = F_{x'\to\xi'} u$ be the partial Fourier transform of u with respect to $x' = (x_1, \dots x_{n-1}) \in \mathcal{R}^{n-1}_{x'}$. Now, $\hat{u}(\varepsilon, \xi) = F_{x\to\xi} u$ being the Fourier transform of u with respect to $x \in \mathcal{R}^n$, one can write down:

$$|\tilde{u}(\varepsilon, \xi', 0)|^2 \leqslant (2\pi)^{-2}\int_{\mathcal{R}} \langle\xi\rangle^{-1}\langle\varepsilon\xi\rangle^{2-s_3}\, d\xi_n \int_{\mathcal{R}} \langle\xi\rangle\langle\varepsilon\xi\rangle^{2s_3}|\hat{u}(\varepsilon, \xi)|^2\, d\xi_n. \tag{1.2.3}$$

After the change of variable $\xi_n = \varepsilon^{-1}\langle\varepsilon\xi'\rangle t$, the first integral in the right hand side of the last inequality becomes:

$$\int_{\mathcal{R}} \langle\xi\rangle^{-1}\langle\varepsilon\xi\rangle^{-2s_3}\, d\xi_n = \langle\varepsilon\xi'\rangle^{-2s_3}\int_{\mathcal{R}} \big(\varepsilon^2\langle\xi'\rangle^2\langle\varepsilon\xi'\rangle^{-2} + t^2\big)^{-\frac{1}{2}}\langle t\rangle^{-2s_3}\, dt\,.$$

Given that for $0<q<1$ and $s_3>0$ holds:

$$\int_{\mathcal{R}} (q^2+t^2)^{-\frac{1}{2}} \langle t\rangle^{-2s_3} dt \leqslant C \langle \ln q \rangle$$

one can rewrite (1.2.3) in the following way:

$$\varepsilon^{-2s_1} \langle \varepsilon\xi' \rangle^{2s_3} |\tilde{u}(\varepsilon, \xi', 0)|^2 \leqslant C \langle \ln (\varepsilon \langle \xi' \rangle \langle \varepsilon\xi' \rangle^{-1}) \rangle \int_{\mathcal{R}} \varepsilon^{-2s_1} \langle \xi \rangle \langle \varepsilon\xi \rangle^{2s_3} |\hat{u}(\varepsilon, \xi)|^2 d\xi \,. \tag{1.2.4}$$

Given that the logarithm in the right hand side of (1.2.4) does not exceed $|\ln \varepsilon|$, $\forall \varepsilon \in (0, \frac{1}{2}]$, one gets (1.2.2) by integrating (1.2.4) with respect to $\xi' \in \mathcal{R}^{n-1}$ and applying the Parceval's identity. ■

LEMMA 1.2.2. – *Let* $u \in \mathcal{H}_{(s)}(\mathcal{R}^n)$ *with* $s_1 > \frac{1}{2}$, $s_2 + s_3 > \frac{1}{2}$. *Then for* $K < \min\{s_2 + \frac{1}{2}, s_2 + s_3 + \frac{1}{2}\}$ *the mapping*:

$$\mathcal{H}_{(s)}(\mathcal{R}^n) \ni u \to \pi_0 \{D_n^{p-1} u\}_{1\leqslant p \leqslant K} \in \prod_{1\leqslant p\leqslant K} \mathcal{H}_{(\varrho_p)}(\mathcal{R}^{n-1}) \tag{1.2.5}$$

with $\varrho_p = (s_1, s_2 - p + \frac{1}{2}, s_3)$ *is continuous linear one.*

PROOF. – Using the notation introduced above in the proof of the Lemma 1.2.1, one can write down

$$\begin{aligned} |D_n^{p-1}\tilde{u}(\varepsilon, \xi', 0)|^2 &= (2\pi)^{-2} \left| \int_{\mathcal{R}} \xi_n^{p-1} \hat{u}(\varepsilon, \xi) d\xi_n \right|^2 \leqslant \\ &\leqslant (2\pi)^{-2} \int_{\mathcal{R}} \langle \xi \rangle^{-2s_2} \langle \varepsilon\xi \rangle^{-2s_3} |\xi_n|^{2(p-1)} d\xi_n \int_{\mathcal{R}} \langle \xi \rangle^{2s_2} \langle \varepsilon\xi \rangle^{2s_3} |\hat{u}(\varepsilon, \xi)| d\xi_n \,. \end{aligned} \tag{1.2.6}$$

After the change of variable $\xi_n = \langle \xi' \rangle t$, the first integral in the right hand side of (1.2.6) becomes:

$$\begin{aligned} \int_{\mathcal{R}} \langle \xi \rangle^{-2s_2} \langle \varepsilon\xi \rangle^{-2s_3} |\xi_n|^{2(p-1)} d\xi_n &= \langle \xi' \rangle^{-2(s_2 - p + \frac{1}{2})} \langle \varepsilon\xi \rangle^{-2s_3} \cdot \\ &\int_{\mathcal{R}} \langle t \rangle^{-2s_2} |t|^{2(p-1)} (1 + t^2 \varepsilon^2 \langle \xi' \rangle^2 \langle \varepsilon\xi' \rangle^{-2})^{-s_3} dt \,. \end{aligned} \tag{1.2.7}$$

If $s_3 \geqslant 0$, then the integral in the right hand side of (1.2.7) is bounded by the following one:

$$\int_{\mathcal{R}} \langle t \rangle^{-2s_2} |t|^{2(p-1)} dt < \infty \qquad \text{for } 1 \leqslant p < s_2 + \tfrac{1}{2} \,. \tag{1.2.8}$$

On the other hand, if $s_3 < 0$, then the same integral in the right hand side of (1.2.7) is bounded by

$$\int_{\mathcal{R}} \langle t\rangle^{-2(s_2+s_3)} |t|^{2(p-1)} dt < \infty, \qquad 1 \leqslant p < s_2 + s_3 + \tfrac{1}{2}. \tag{1.2.9}$$

Multiplying (1.2.6) by $\varepsilon^{-2s_1}\langle\xi'\rangle^{2(s_2-p+\frac{1}{2})}\langle\varepsilon\xi'\rangle^{2s_3}$, one gets:

$$\varepsilon^{-2s_1}\langle\xi'\rangle^{2(s_2-p+\frac{1}{2})}\langle\varepsilon\xi'\rangle^{2s_3} |D_n^{p-1}\tilde{u}(\varepsilon, \xi', 0)|^2 \leqslant C\int_{\mathcal{R}} \varepsilon^{-2s_1}\langle\xi\rangle^{2s_2}\langle\varepsilon\xi\rangle^{2s_3} |\hat{u}(\varepsilon, \xi)|^2 d\xi_n \tag{1.2.10}$$

and that along with Parceval's identity ends the proof. ■

LEMMA 1.2.3. – *Let* $u \in \mathcal{H}_{(s)}(\mathcal{R}^n)$ *with* $s_2 < \frac{1}{2}$, $s_2 + s_3 > \frac{1}{2}$. *Then the mapping*

$$\mathcal{H}_{(s)}(\mathcal{R}^n) \subset u \to \pi_0 u \in \mathcal{H}_{(\sigma)}(\mathcal{R}^{n-1})$$

with $\sigma = (s_1 + s_2 - \frac{1}{2}, 0, s_2 + s_3 - \frac{1}{2})$ *is continuous linear one.*

PROOF. – With the same notations ad above, one can write down:

$$|\tilde{u}(\varepsilon, \xi', 0)|^2 \leqslant (2\pi)^{-2}\int_{\mathcal{R}} \langle\xi\rangle^{-2s_2}\langle\varepsilon\xi\rangle^{-2s_3} d\xi_n \int_{\mathcal{R}} \langle\xi\rangle^{2s_2}\langle\varepsilon\xi\rangle^{2s_3} |\hat{u}(\varepsilon, \xi)|^2 d\xi_n. \tag{1.2.11}$$

Making the change of variables $\xi_n = \varepsilon^{-1}\langle\varepsilon\xi'\rangle t$ in the first integral in the right hand side of (1.2.11), this integral becomes:

$$\int_{\mathcal{R}} \langle\xi\rangle^{-2s_2}\langle\varepsilon\xi\rangle^{-2s_3} d\xi_n = \varepsilon^{-1+2s_2}\langle\varepsilon\xi'\rangle^{1-2s_2-2s_3} \int_{\mathcal{R}} (\varepsilon^2\langle\xi'\rangle^2\langle\varepsilon\xi'\rangle^{-2} + t^2)^{-s_2}\langle t\rangle^{-2s_3} dt. \tag{1.2.12}$$

The integral in the right hand side of (1.2.12) is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, given that $s_2 < \frac{1}{2}$ and $s_2 + s_3 > \frac{1}{2}$.

After multiplication by $\varepsilon^{1-2s_1-2s_2}\langle\varepsilon\xi'\rangle^{2s_2+2s_3-1}$ the inequality (1.2.11) becomes:

$$\varepsilon^{1-2s_1-2s_2}\langle\varepsilon\xi'\rangle^{2s_2+2s_3-1} |\tilde{u}(\varepsilon, \xi', 0)|^2 \leqslant C\int_{\mathcal{R}} \varepsilon^{-2s_1}\langle\xi\rangle^{2s_2}\langle\varepsilon\xi\rangle^{2s_3} |\hat{u}(\xi)|^2 d\xi_n. \tag{1.2.13}$$

Finally one has to use the Parceval's identity. ■

REMARK 1.2.4. – The statements of the Lemmas 1.2.1-1.2.3 are valid for the spaces $H_{(s)}(\mathcal{R}^n)$ and corresponding $H_{(\sigma_j)}(\mathcal{R}^{n-1})$.

1.3. *Spaces* $\mathcal{H}_{(s)}(\mathcal{R}^n_+)$, $H_{(s)}(\mathcal{R}^n_+)$.

We denote by $\mathcal{R}^n_+$ the halfspace $\{x_n > 0\}$ and for s_2, s_3 non-negative integer define the norms $\|\cdot\|^+_{(s)}$, $s = (s_1, s_2, s_3)$, by the formula:

$$\|u\|^+_{(s)} = \Big(\sum_{|\alpha|\leqslant s_2, |\beta|\leqslant s_3} \varepsilon^{2|\beta|-2s_1} \int_{\mathcal{R}^n_+} |D^{\alpha+\beta} u(\varepsilon, x)|^2\, dx\Big)^{\frac{1}{2}}. \tag{1.3.1}$$

We denote by $\mathcal{H}_{(s)}(\mathcal{R}^n_+)$ the space of all $u \in L^\infty([0, \varepsilon_0]; S^1(\mathcal{R}^n))$ such that

$$\|u\|^+_{(s)} = \operatorname*{vrai\,max}_{\varepsilon\in[0,\varepsilon_0]} \|u\|^+_{(s)} < \infty\,. \tag{1.3.2}$$

For functions $u \in L^\infty([0, \varepsilon_0]; \overline{C_0^\infty(\mathcal{R}^n_+)})$ we use the usual extention operator l, defined by the formula:

$$lu(\varepsilon, x) = \chi(x_n)\, u(\varepsilon, x) + \chi(-x_n) \sum_{1\leqslant p\leqslant s_2+s_3} C_p\, u(\varepsilon, x', -p x_n) \tag{1.3.3}$$

where $\chi(x_n)$ is the Heviside's function and c_p, $1 \leqslant p \leqslant s_2 + s_3$, satisfy the system of linear equations:

$$\sum_{1\leqslant p\leqslant s_2+s_3} (-p)^{j-1} C_p = 1\,, \qquad 1 \leqslant j \leqslant s_2 + s_3\,. \tag{1.3.4}$$

LEMMA 1.3.1. – *For s_2, s_3 non-negative integers, the operator l defined by* (1.3.3), (1.3.4) *can be extended as linear continuous mapping from $\mathcal{H}_{(s)}(\mathcal{R}^n_+)$ into $\mathcal{H}_{(s)}(\mathcal{R}^n)$.*

PROOF. – The Lemma 1.3.1 follows immediately from the inequalities

$$\|D_n^j(lu)\|_{L^2(\mathcal{R}^n)} \leqslant C_j \|D_n^j\|_{L^2(\mathcal{R}^n_+)}\,, \qquad \text{for any } u \in \overline{C_0^\infty(\mathcal{R}^n_+)},\ (1 \leqslant j < s_2 + s_3)\,. \qquad \blacksquare$$

For $s \in \mathcal{R}^3$ and $u \in L^\infty([0, \varepsilon_0]; S'(\mathcal{R}^n))$ the norms $\|\cdot\|^+_{(s)}$ are defined as follows:

$$\|u\|^+_{(s)} = \inf_l \|lu\|_{(s)}\,. \tag{1.3.5}$$

We denote by $\mathcal{H}_{(s)}(\mathcal{R}^n_+)$ the space of all tempered distributions $u \in L^\infty([0, \varepsilon_0]; S'(\mathcal{R}^n))$ such that

$$\|u\|^+_{(s)} = \operatorname*{vrai\,max}_{\varepsilon\in[0,\varepsilon_0]} \|u\|^+_{(s)} < \infty\,. \tag{1.3.6}$$

Introducing the subspace $\mathcal{H}^-_{(s)}(\mathcal{R}^n) \subset \mathcal{H}_{(s)}(\mathcal{R}^n)$ of distribution $u(\varepsilon, x)$ with their support contained in $\overline{\mathcal{R}^n_-} = \{x_n \leqslant 0\}$, one has the isomorphism:

$$\mathcal{H}_{(s)}(\mathcal{R}^n_+) \cong \mathcal{H}_{(s)}(\mathcal{R}^n)/\mathcal{H}^-_{(s)}(\mathcal{R}^n)\,. \tag{1.3.7}$$

Now, the spaces $H_{(s)}(\mathcal{R}^n_+)$ and corresponding norms $|\cdot|^+_{(s)}$ are defined in the same way as in the Section 1.1.

1.4. *Spaces* $\mathcal{H}_{(s)}(G)$, $H_{(s)}(G)$.

Let G be a compact domain, $G \subset \mathcal{R}^n_x$, with boundary ∂G which is supposed to be C^∞ oriented manifold. Denoting by $\mathcal{H}^-_{(s)}$, the subspace of $\mathcal{H}_{(s)}(\mathcal{R}^n)$ of all tempered distributions $u \in L^\infty([0, \varepsilon_0]; S'(\mathcal{R}^n_x))$, such that their support is contained in the complementary set to G in $\mathcal{R}^n_x$, one defines the spaces $\mathcal{H}_{(s)}(G)$ by the formula:

$$\mathcal{H}_{(s)}(G) = \mathcal{H}_{(s)}(\mathcal{R}^n)/\mathcal{H}^-_{(s)} . \tag{1.4.1}$$

the spaces $\mathcal{H}_{(s)}(G)$ being provided with the factor-norms induced by the topology of factor-space.

Again, the spaces $H_{(s)}(G)$ are defined in the same way as it was done in the Section 1.1.

1.5. *Spaces* $\mathcal{H}_{(\sigma)}(\partial G)$, $H_{(\sigma)}(\partial G)$.

The boundary ∂G being an oriented C^∞ compact manifold, we denote by $\{O_j\}_{1\leqslant j\leqslant N}$ a finite covering of ∂G by open balls $\{O_j\}$ and by $\{\psi_j\}$ the partition of unity subordinated to the covering $\{O_j\}$.

Using a local diffeomorphism $x = x(y)$ in the neighborhood of a point $P_j \in \partial U \cap O_j$ with Jacobian $J(y)$, $J(P_j) = I$ (I is identity in $\mathrm{Hom}(\mathcal{R}^n; \mathcal{R}^n)$), such that $x \to y$ maps the set $O_j \cap G$ into the half-space $\mathcal{R}^n_+ = \{y_n > 0\}$, one can define for a function $\varphi(\varepsilon, x)$: $[0, \varepsilon_0] \times \partial G \to C$ the norm with subscript $\sigma \in \mathcal{R}^3$ by the formula:

$$[\varphi]_{(\sigma)} = \sum_{1\leqslant j\leqslant N} [\psi_j \varphi]_{(\sigma)} \tag{1.5.1}$$

where the norms $[\psi_j\varphi]_{(\sigma)}$ are computed over $\mathcal{R}^{n-1}_{y'}$, $y' = (y_1, \dots, y_{n-1})$ after application of the diffeomorphism $x = x(y)$ above.

We denote by $\mathcal{H}_{(\sigma)}(\partial G)$ the space of all functions (distributions) $\varphi \in L^\infty([0, \varepsilon_0]; S'(\partial G))$ such that

$$|[\varphi]|_{(\sigma)} = \operatorname*{vrai\,max}_{\varepsilon\in[0,\varepsilon_0]} [\varphi]_{(\sigma)} < \infty . \tag{1.5.2}$$

Of course, the norms (1.5.1), (1.5.2) depend upon the choice of the covering $\{O_j\}$ and the subordinated partition of unity $\{\psi_j\}$, but the topology in $\mathcal{H}_{(\sigma)}(\partial G)$ does not and all the norms thus defined are equivalent each to other uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$.

The definition of the spaces $H_{(\sigma)}(\partial G)$ can be given by the procedure, used above in the Section 1.1.

We restricted ourselves to the minimum of definitions and facts about the spaces $\mathcal{H}_{(s)}$, $H_{(s)}$ that will be used later to investigate the boundary value problems for singularly perturbed Differential Operators.

Of course, it is possible to develop the theory of spaces $\mathcal{H}^{\mu}$, H^{μ} with $\mu = \mu(\varepsilon, \xi)$ family of weights depending on parameter $\varepsilon \in [0, \varepsilon_0]$, this theory being in many points similar to this one developed in [30] for $\mu = \mu(\xi)$.

2. – Singular perturbations in $\mathcal{R}^n$.

The aim of this section is to establish a two-sided a priori estimate for elliptic perturbations (defined below) in spaces $\mathcal{H}_{(s)}(\mathcal{R}^n)$ and also to show that the ellipticity condition is necessary for the validity of such an estimate. To do this one uses the classical technique of partition of unity. It goes without saying that one can consider a more general class of pseudodifferential singular perturbations and develop the calculus analogue to the algebras of pseudodifferential and difference operators investigated formerly in [9], [15], [17]. The main purpose in adding this section is to make the paper self-contained.

2.1. *Classes* $\mathscr{P}_\nu$.

Let δ be a small non-negative parameter, $\delta \in [0, \delta_0]$, and $Q^\delta(x, \xi)$ a family of polynomials in $\xi \in \mathcal{R}^n$ with coefficients C^∞ functions of $x \in \mathcal{R}^n$, $\delta \in (0, \delta_0]$, which are supposed to be stabilizing to functions of δ when x is large enough. We assume also that

$$\deg_\xi Q^\delta = m\,, \qquad \forall \delta \in (0, \delta_0],\ \deg_\xi Q^0 = m_1 < m\,. \tag{2.1.1}$$

Expanding Q^δ in a finite sum of homogeneous polunomials Q^δ_K,

$$Q^\delta(x, \xi) = \sum_{0 \leqslant K \leqslant m} Q^\delta_K(x, \xi)\,, \qquad \deg_\xi Q^\delta_K = K\,, \tag{2.1.2}$$

we assume that $Q^\delta_K(x, \xi)$ can be represented in the form:

$$Q^\delta_K(x, \xi) = \delta^{t_K} q^\delta_K(x, \xi)\,, \qquad t_K \geqslant 0,\ 0 \leqslant K \leqslant m \tag{2.1.3}$$

where the homogeneous polynomials $q^\delta_K(x, \xi)$ are supposed to have Hölder continuous coefficients at $\delta = 0$ and $q^0_K(x, \xi)$, $0 \leqslant K \leqslant m$, does not vanish identically.

The assumption (2.1.1) upon the degree of Q^δ yields:

$$t_{m_1} = 0\,, \quad t_K > 0\,, \qquad \text{for } m_1 < K \leqslant m\,. \tag{2.1.4}$$

Denoting $m_2 = m - m_1$, we restrict ourselves to the polynomials Q^δ such that

$$m_2 t_K \geqslant t_m (K - m_1)\,, \qquad \text{for } m_1 < K < m\,. \tag{2.1.5}$$

The condition (2.1.5) being fulfilled we introduce a new small parameter ε,

$$\varepsilon^{m_2} = \delta^{t_m}. \tag{2.1.6}$$

Denoting by $Q(x, \varepsilon, \xi)$ and $q_K(x, \varepsilon, \xi)$ the polynomials which are thus obtained from Q^δ and q_K^δ respectively, one can write down

$$Q(x, \varepsilon, \xi) = \sum_{m_1 \leqslant K \leqslant m} \varepsilon^{m_2 t_K / t_m} q_K(x, \varepsilon, \xi) + r(x, \varepsilon, \xi)\,, \tag{2.1.7}$$

where $\deg_\xi r(x, \varepsilon, \xi) < m_1$.

Introducing the numbers

$$\gamma_K = m_1 - K + m_2 t_K / t_m\,, \qquad m_1 \leqslant K \leqslant m\,, \tag{2.1.8}$$

which are non-negative, given the condition (2.1.5), and $\gamma_{m_1} = \gamma_m = 0$, we can rewrite (2.1.7) in the fashion

$$Q(x, \varepsilon, \xi) = \sum_{m_1 \leqslant K \leqslant m,\, \gamma_K = 0} \varepsilon^{K - m_1} q_K(x, 0, \xi) + R(x, \varepsilon, \xi) \tag{2.1.9}$$

where the remainder $R(x, \varepsilon, \xi)$ of degree at most m satisfies the inequality:

$$|R(x, \varepsilon, \xi)| \leqslant C\big(\langle \xi \rangle^{m_1 - 1} + \varepsilon^{\gamma} \langle \varepsilon \rangle^{m_1} \langle \varepsilon \xi \rangle^{m_2}\big) \tag{2.1.10}$$

with some positive constants γ and C.

The polynomial

$$Q_0(x, \varepsilon, \xi) = \sum_{m_1 \leqslant K \leqslant m,\, \gamma_K = 0} \varepsilon^{K - m_1} q_K(x, 0, \xi) \tag{2.1.11}$$

is called principal symbol of $Q(x, \varepsilon, \xi)$.

Obviously, $Q_0(x, \varepsilon, \xi)$ is homogeneous function in (ε^{-1}, ξ) of order m_1,

$$Q_0(x, \varepsilon, \xi) = \varepsilon^{-m_1} Q_0(x, 1, \varepsilon\xi)\,, \tag{2.1.12}$$

which satisfies the inequality

$$|Q_0(x, \varepsilon, \xi)| \leqslant C |\xi|^{m_1} \langle \varepsilon \xi \rangle^{m_2}, \qquad \forall x \in \mathfrak{R}^n,\ \forall \xi \in \mathfrak{R}^n,\ \forall \varepsilon \geqslant 0\,. \tag{2.1.13}$$

The representation (2.1.9) with R and Q_0 satisfying (2.1.10), (2.1.12), (2.1.13), is called canonical for the family Q^δ.

We will consider a slightly larger class of polynomials $Q(x, \varepsilon, \xi)$ further called symbols whose canonical representation satisfies the conditions:

$$(2.1.14)\qquad \begin{aligned} &Q_0(x, \varepsilon, \xi) = \varepsilon^{-(\nu_1+\nu_3)} Q_0(x, 1, \varepsilon\xi)\,,\\ &|Q_0(x, \varepsilon, \xi)| \leqslant C\varepsilon^{-\nu_1} |\xi|^{\nu_2} \langle \varepsilon\xi \rangle^{\nu_3},\\ &|R(x, \varepsilon, \xi)| \leqslant C\varepsilon^{-\nu_1}\big(\langle \xi \rangle^{\nu_2-1} + \varepsilon^{\gamma} \langle \xi \rangle^{\nu_2} \langle \varepsilon\xi \rangle^{\nu_3}\big)\end{aligned}$$

with some positive constants γ and C.

$Q(x, \varepsilon, \xi)$ is supposed also to stabilize to a symbol $Q(\varepsilon, \xi)$ as $x \to \infty$ in the following sense:

$$(2.1.15)\qquad |Q(x, \varepsilon, \xi) - Q(\varepsilon, \xi)| \leqslant \varphi_0(x)\, \varepsilon^{-\nu_1} \langle \xi \rangle^{\nu_2} \langle \varepsilon\xi \rangle^{\nu_3}$$

where $\varphi_0(x) \in S(\mathcal{R}^n)$.

The least vector $\nu \in \mathcal{R}^3$ such that (2.1.14) still holds, is called order of the symbol Q.

The class of the symbols $Q(x, \varepsilon, \xi)$ of order ν is denoted by $\mathscr{P}_\nu$ and the class of corresponding singularly perturbed Partial Differential Operators $Q(x, \varepsilon, D)$ is denoted by P_ν.

2.2. *Singular perturbations in the spaces $\mathcal{H}_{(s)}(\mathcal{R}^n)$, $H_{(s)}(\mathcal{R}^n)$.*

For $s \in \mathcal{R}^3$ with integer $s_j \geqslant \nu_j$, $j = 2, 3$, it is immediate that

$$(2.2.1)\qquad Q \in \operatorname{Hom}\big(\mathcal{H}_{(s)}(\mathcal{R}^n); \mathcal{H}_{(s-\nu)}(\mathcal{R}^n)\big)\,, \qquad \forall Q \in P_\nu\,.$$

PROPOSITION 2.2.1. – *The inclusion* (2.2.1) *holds for* $\forall s \in \mathcal{R}^3$ *and* $\forall Q \in P_\nu$.

PROOF. – If Q does not depend on x then (2.2.1) follows immediately. Therefore, it suffices to show (2.2.1) for Q whose coefficients, as functions of $x \in \mathcal{R}^n$, belong to $S(\mathcal{R}^n_x)$.

The inclusion (2.2.1) is equivalent to the inequality

$$(2.2.2)\qquad \|K\hat{v}\|_{L^2(\mathcal{R}^n_\xi)} \leqslant C \|\hat{v}\|_{L^2(\mathcal{R}^n_\xi)}$$

where $\hat{v} \to K\hat{v}$ is integral operator with the kernel:

$$(2.2.3)\qquad K(\xi, \varepsilon, \eta) = \langle \xi \rangle^{s_2-\nu_2} \langle \varepsilon\xi \rangle^{s_3-\nu_3} \varepsilon^{\nu_1} \hat{Q}(\xi - \eta, \varepsilon, \eta) \langle \eta \rangle^{-s_2} \langle \varepsilon\eta \rangle^{-s_3}$$

and the constant C does not depend on ε and $\hat{v}$.

Given that $Q \in S(\mathcal{R}^n_x)$ as function of x, one can write down for $\hat{Q} = F_{x\to\xi-\eta} Q$:

$$(2.2.4)\qquad |\varepsilon^{\nu_1} \hat{Q}(\xi - \eta, \varepsilon, \eta)| \leqslant C_N \langle \xi - \eta \rangle^{-N} \langle \eta \rangle^{\nu_2} \langle \varepsilon\eta \rangle^{\nu_3}$$

with $\forall N > 0$.

Now using the Peetre's inequality (see, for instance, [3]):

$$\langle\xi\rangle^{\varrho}\langle\eta\rangle^{-\varrho}\leqslant 2^{|\varrho|}\langle\xi-\eta\rangle^{|\varrho|}, \qquad \forall\varrho\in\mathfrak{R}\,, \tag{2.2.5}$$

(2.2.3)-(2.2.5) yield

$$\begin{aligned}\int_{\mathfrak{R}^n}|K(\xi,\varepsilon,\eta)|\,d\xi\leqslant C_N\int_{\mathfrak{R}^n}\langle\xi-\eta\rangle^{-N_1}d\xi\leqslant C\\ \int_{\mathfrak{R}^n}|K(\xi,\varepsilon,\eta)|\,d\eta\leqslant C_N\int_{\mathfrak{R}^n}\langle\varepsilon-\eta\rangle^{-N_1}d\eta\leqslant C\end{aligned} \tag{2.2.6}$$

where $N_1=N-|s_2-\nu_2|-|s_3-\nu_3|>n$, and C does not depend on ε, ξ, η.

The inequalities (2.2.6) show that the family of operators $\hat{v}\to K\hat{v}$ is uniformly bounded in $L^2(\mathfrak{R}^n_\xi)$ with respect to $\varepsilon\in[0,\varepsilon_0]$. ■

2.3. *Singular perturbation and diffeomorphisms on* $\mathfrak{R}^n$.

Let $x=\alpha(y)$ be a diffeomorphism of $\mathfrak{R}^n_x$ onto $\mathfrak{R}^n_y$: We denote by $J(y)$ the Jacobian of $\alpha(y)$ and by ${}^tJ(y)$ the transpose of the matrix $J(y)$.

PROPOSITION 2.3.1. – *Let $Q\in P_\nu$, and $Q_0(x,\varepsilon,\xi)$ be the principal symbol of Q. Then after the diffeomorphism $x=\alpha(y)$ the principal symbol of the corresponding singular perturbation in variables y becomes $Q_0\big(\alpha(y),\varepsilon,({}^tJ)^{-1}(y)\eta\big)$.*

PROOF. – Breaking down the principal symbol $Q_0(x,\varepsilon,\xi)$ into the sum of homgeneous components:

$$Q_0(x,\varepsilon,\xi)=\sum_{\nu_2\leqslant K\leqslant\nu_2+\nu_3}\varepsilon^{K-\nu_1-\nu_2}q_K(x,\xi) \tag{2.3.1}$$

the statement of the Proposition 2.3.1 follows immediately from the corresponding formula for the principal symbol of differential operator $q_K(x,D)$ in variables y. ■

2.4. *Elliptic singular perturbations.*

We start with the definition:

DEFINITION 2.4.1. – *Singular perturbation $Q\in P_\nu$ is called elliptic of order ν if its principal symbol Q_0 satisfies the condition*

$$|Q_0(x,\varrho,\xi)|\geqslant C\varrho^{-\nu_1}\langle\varrho\rangle^{\nu_3}, \qquad \forall\varrho>0,\ \forall\xi\in\Omega_n,\ \forall x\in\mathfrak{R}^n \tag{2.4.1}$$

with some constant $C>0$ which does not depend on ϱ, ξ, x, where by Ω_n we denote the unit sphere in $\mathfrak{R}^n_\xi$: $\Omega_n=\{\xi\in\mathfrak{R}^n\,|\,|\xi|=1\}$.

REMARK 2.4.2. – Introducing reduced principal symbol $Q_0^0(x, \xi)$,

$$Q_0^0(x, \xi) = \lim_{\varepsilon \to +0} \varepsilon^{\nu_1} Q_0(x, \varepsilon, \xi)\,, \tag{2.4.2}$$

the condition (2.4.1) is equivalent to the following ones:

(i) $Q_0(x, 1, \xi)$ is an elliptic symbol in the usual sense, that is Q_{00} being the principal homogeneous symbol of order $\nu_2+\nu_3$ to the symbol Q_0, the inequality holds:

$$Q_{00}(x, 1, \xi) \neq 0\,, \qquad \forall x \in \mathfrak{R}^n,\ \forall \xi \in \Omega_n\,.$$

(ii) $Q_0^0(x, \xi)$ is elliptic of order ν_2, that is

$$Q_0^0(x, \xi) \neq 0\,, \qquad \forall x \in \mathfrak{R}^n,\ \forall \xi \in \Omega_n\,.$$

(iii) $Q_0(x, 1, \xi)$ does not vanish:

$$Q_0(x, 1, \xi) \neq 0\,, \qquad \forall x \in \mathfrak{R}^n,\ \forall \xi \in \mathfrak{R}^n.$$

It is quite obvious that (2.4.1) involves (i)-(iii) (by taking in (2.4.1) alternately $\varrho \to 0$ after multiplication by ϱ^{ν_1}, afterwards, multiplying (2.4.1) by $\varrho^{\nu_1-\nu_3}$ and letting $\varrho \to +\infty$ and finally, using the homogeneity of Q_0).

On the other hand, using again the homogeneity and (iii) one gets

$$Q_0(x, \varrho, \xi) \neq 0\,, \qquad \forall \varrho > 0,\ \forall \xi \in \mathfrak{R}^n,\ |\xi| = 1,\ \forall x \in \mathfrak{R}^n.$$

Finally, using (ii) one gets for ϱ small enough:

$$|Q_0(x, \varrho, \xi)| \geq C\varrho^{-\nu_1}, \qquad \forall \xi \in \Omega_n,\ \forall x \in \mathfrak{R}^n,$$

and from (i) one obtains for ϱ large enough:

$$|Q_0(x, \varrho, \xi)| \geq C\varrho^{\nu_3-\nu_1}, \qquad \forall \xi \in \Omega_n,\ \forall x \in \mathfrak{R}^n.$$

Sometimes it will be convenient to use another equivalent definition of ellipticity:

$$|Q_0(x, \varrho, \xi)| \geq C\varrho^{-\nu_1}|\xi|^{\nu_2}\langle \varrho\xi \rangle^{\nu_3}, \qquad \forall \xi \in \mathfrak{R}^n,\ \forall x \in \mathfrak{R}^n,\ \forall \varrho > 0\,. \tag{2.4.3}$$

Its equivalency to (2.4.1) is quite obvious given the homogeneity of Q_0 in (ϱ^{-1}, ξ).

The class of elliptic singular perturbations of order ν is denoted by E_ν.

PROPOSITION 2.4.3. – *Ellipticity is an invariant property with respect to any diffeomorphism of $\mathfrak{R}^n$ onto itself.*

PROOF. – The statement is an immediate consequence of the Proposition 2.3.1.

PROPOSITION 2.4.4. – *Let $Q \in E_\nu$ and $Q \equiv Q(\varepsilon, \xi)$ does not depend on x. Then for ε_0 small enough the following a priori estimate holds:*

$$\|u\|_{(s)} \leqslant C_{s,s'}(\|Qu\|_{(s-\nu)} + \|u\|_{(s')}) \tag{2.4.4}$$

with $\forall s' \in \partial_1 \Gamma_s$ and a constant $C_{s,s'}$ which depends only on its subscripts.

PROOF. – Introducing

$$Q_0^{\mathscr{C}}(\varepsilon, \xi) = Q_0(\varepsilon, \hat{\xi}), \quad \text{with} \quad \hat{\xi} = \langle\xi\rangle\omega, \quad \omega' = \xi/|\xi|, \tag{2.4.5}$$

one can write down:

$$Q(\varepsilon, \xi) = Q_0^{\mathscr{C}}(\varepsilon\hat{\xi}) + R^{\mathscr{C}}(\varepsilon, \xi) \tag{2.4.6}$$

where $R^{\mathscr{C}}(\varepsilon, \xi)$ satisfies the inequality:

$$|R^{\mathscr{C}}(\varepsilon, \xi)| \leqslant C\varepsilon^{-\nu_1}(\langle\xi\rangle^{\nu_2-1} + \varepsilon^{\gamma}\langle\xi\rangle^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}). \tag{2.4.7}$$

The ellipticity condition (2.4.3) and (2.4.5) yield:

$$\|Q_0^{\mathscr{C}} u\|_{(s-\nu)} \geqslant C\|u\|_{(s)}. \tag{2.4.8}$$

Now,

$$\|R^{\mathscr{C}} u\|_{(s-\nu)} \leqslant C(\|u\|_{(s'')} + \varepsilon_0^{\gamma}\|u\|_{(s)}) \tag{2.4.9}$$

with $s'' = (s_1, s_2 - 1, s_3 - \nu_3)$.

Using the interpolation inequality for $\|u\|_{(s'')}$ with convenient t, i.e. such that $s'' - t(s - s'') \leqslant s'$,

$$\|u\|_{(s'')}^2 \leqslant \delta^2\|u\|_{(s)}^2 + \delta^{-2t}\|u\|_{(s')}^2, \tag{2.4.10}$$

(2.4.8)-(2.4.10) yield

$$\|Qu\|_{(s-\nu)} \geqslant \|Q_0^{\mathscr{C}} u\|_{(s-\nu)} - \|R^{\mathscr{C}} u\|_{(s-\nu)} \geqslant (C - \varepsilon_0^{\gamma} - \delta)\|u\|_{(s)} - C_\delta\|u\|_{(s')},$$

which leads to (2.4.4) if ε_0 and δ are small enough. ■

REMARK 2.4.5. – If (2.4.4) holds with some $s' < s$ such that $s_1' < s_1$ then as a matter of fact, $Q(\varepsilon, D)$ for ε_0 small enough establishes the isomorphism:

$$Q \in \operatorname{Iso}(\mathcal{H}_{(s)}(\mathcal{R}^n); \mathcal{H}_{(s-\nu)}(\mathcal{R}^n)). \tag{2.4.11}$$

Indeed, in that case, $\|u\|_{(s')} \leqslant C\varepsilon^{s_1 - s_1'}\|u\|_{(s)}$.

Our aim now is to prove the a priori estimate (2.4.4) for elliptic singular perturbation of order ν with variable coefficients. To do this we need several lemmas.

LEMMA 2.4.6. – *For any $s_2, s_3 \in \mathcal{R}$ and $0 < \varepsilon_0 \leqslant 1$ the following inequality holds:*

$$|\langle\xi\rangle^{s_2}\langle\xi\varepsilon\rangle^{s_3} - \langle\eta\rangle^{s_2}\langle\varepsilon\eta\rangle^{s_3}|\langle\eta\rangle^{1-s_2}\langle\varepsilon\eta\rangle^{-s_3} \leqslant 2C\langle\xi-\eta\rangle^{N_0+1}, \tag{2.4.12}$$

where

$$C = \max\{|s_2 2^{|s_2-1|+|s_3|}, |s_3|2^{|s_2|+|s_3-1|}\}, \tag{2.4.13}$$
$$N_0 = \max\{|s_2-1| + |s_3|, |s_2| + |s_3-1|\}$$

PROOF. – It is immediate that

$$|\text{grad}\langle\xi\rangle^{\varrho}| \leqslant |\varrho|\langle\xi\rangle^{\varrho-1}. \tag{2.4.14}$$

Denoting the left hand side in (2.4.12) by $g(\varepsilon, \xi, \eta)$, the inequality (2.4.14) and Lagrange formula for the difference in the left hand side of (2.4.12) yield:

$$g(\varepsilon, \xi, \eta) \leqslant \langle\xi-\eta\rangle(|s_2|\langle\xi_\theta\rangle^{s_2-1}\langle\varepsilon\xi_\theta\rangle^{s_3} + \varepsilon|s_3|\langle\xi_\theta\rangle^{s_2}\langle\varepsilon\xi_\theta\rangle^{s_3-1}\langle\eta\rangle^{1-s_2}\langle\varepsilon\eta\rangle^{s_3}) \tag{2.4.15}$$

where $\xi_\theta = \xi + \theta(\eta - \xi)$ with some θ, $0 < \theta < 1$.

Now using Peetre's inequality (2.2.5) to estimate $\langle\xi_\theta\rangle^{\varrho_1}\langle\eta\rangle^{-\varrho_1}$ and $\langle\varepsilon\xi_\theta\rangle^{\varrho_2}\langle\varepsilon\eta\rangle^{-\varrho_2}$ with $\varrho_1 = s_2$, $\varrho_1 = s_2 - 1$ and $\varrho_2 = s_3$, $\varrho_2 = s_3 - 1$, one obtains for $g(\varepsilon, \xi, \eta)$ the inequality (2.4.12) with constants (2.4.13), noticing also that $\varepsilon\langle\eta\rangle \leqslant \langle\varepsilon\eta\rangle$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0 \leqslant 1$. ■

LEMMA 2.4.7. – *Let $R_0(x, \varepsilon, \xi) \in \mathcal{P}_\mu$ and assume that the coefficients of $\varepsilon^{\mu_1}R_0$ belong to $L^\infty([0, \varepsilon_0]; S(\mathcal{R}^n_x))$. Then for $\forall s \in \mathcal{R}^3$, $\forall u \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n))$ holds:*

$$\varepsilon^{-s_1}\langle D\rangle^{s_2}\langle\varepsilon D\rangle^{s_3}R_0(x, \varepsilon, D)u(x) = R_0(x, \varepsilon, D)\varepsilon^{-s_1}\langle D\rangle^{s_2}\langle\varepsilon D\rangle^{s_3}u(x) + Ku(x) \tag{2.4.16}$$

where the operator $u \to Ku$ satisfies the inequality

$$\|Ku\|_{(0)} \leqslant C\|u\|_{(s+\mu-e_2)} \tag{2.4.17}$$

with $e_2 = (0, 1, 0)$ and a constant C which does not depend on ε and u.

PROOF. – One finds immediately that $F_{x\to\xi}(Ku)$ is an integral operator: $\hat{u}(\xi) \to (\hat{K}\hat{u})(\xi)$ with the kernel:

$$\hat{K}(\xi, \varepsilon, \eta) = \varepsilon^{-s_1}[\langle\xi\rangle^{s_2}\langle\varepsilon\xi\rangle^{s_3} - \langle\eta\rangle^{s_2}\langle\varepsilon\eta\rangle^{s_3}]\hat{R}_0(\xi-\eta, \varepsilon, \eta) \tag{2.4.18}$$

where

$$\hat{R}_0(\xi-\eta, \varepsilon, \eta) = F_{x\to\xi-\eta} R_0(x, \varepsilon, \eta) .$$

Introducing $\hat{v}(\xi) = \varepsilon^{-(s_1+\mu_1)}\langle\xi\rangle^{s_2+\mu_2-1}\langle\varepsilon\xi\rangle^{s_3+\mu_3-1}\hat{u}(\xi)$ the inequality (2.4.17) is equivalent to the following one:

$$\|\hat{S}\hat{v}\|_{L^2(\mathcal{R}^n_\xi)} \leqslant C\|\hat{v}\|_{L^2(\mathcal{R}^n_\xi)} \tag{2.4.19}$$

where we denoted by $\hat{S}$ integral operator $\hat{v}\to\hat{S}\hat{v}$ with the kernel:

$$\hat{s}(\xi, \varepsilon, \eta) = [\langle\xi\rangle^{s_2}\langle\varepsilon\xi\rangle^{s_3} - \langle\eta\rangle^{s_2}\langle\varepsilon\eta\rangle^{s_3}]\,\hat{R}_0(\xi-\eta, \varepsilon, \eta)\,\varepsilon^{\mu_1}\langle\eta\rangle^{1-s_2-\mu_2}\langle\varepsilon\eta\rangle^{-s_3-\mu} . \tag{2.4.20}$$

Given that $R_0(x, \varepsilon, \eta)\in\mathcal{P}_\mu$ and has coefficients which are in $S(\mathcal{R}^n_x)$ as functions of x, one can write down for $\hat{R}(\xi-\eta, \varepsilon, \eta)$:

$$\langle\xi-\eta\rangle^N|\hat{R}_0(\xi-\eta, \varepsilon, \eta)|\,\varepsilon^{\mu_1}\langle\eta\rangle^{-\mu_2}\langle\varepsilon\eta\rangle^{-\mu_3} \leqslant C_N , \quad \forall N>0 \tag{2.4.21}$$

where the constant C_N depends only on N.

Now, the formula (2.4.20), the Lemma 2.4.6 and (2.4.21) with $N = N_0 + n + 2$ yield

$$\int_{\mathcal{R}^n}|\hat{s}(\xi, \varepsilon, \eta)|\,d\xi \leqslant C , \quad \int_{\mathcal{R}^n}|\hat{s}(\xi, \varepsilon, \eta)|\,d\eta \leqslant C \tag{2.4.22}$$

where the constant C does not depend on ε, ξ, η.

The inequalities (2.4.22) show that (2.4.19) is valid, and, therefore (2.4.17) holds for K. ■

LEMMA 2.4.8. – *Let $Q(x, \varepsilon, D)\in E_\nu$ and $u\in L^\infty([0, \varepsilon_0]; C_0^\infty(G_\delta))$ where* $\operatorname{diam} G_\delta < \delta$. *Then given $s\in\mathcal{R}^3$ and $\forall s'\in\partial_1\Gamma_s$, there exists a constant $C_{s,s'}$, such that for δ and ε_0 small enough the a priori estimate holds:*

$$\|u\|_{(s)} \leqslant C_{s,s'}(\|Qu\|_{(s-\nu)} + \|u\|_{(s')}) . \tag{2.4.23}$$

PROOF. – With x_0 any point in G_δ and introducing the notations:

$$Q_0(x) = Q_0(x, \varepsilon, D) , \quad Q(x) = Q(x, \varepsilon, D) , \quad R_0(x) = Q_0(x) - Q_0(x_0) , \quad R(x) = Q(x) - Q_0(x)$$

one can write down:

$$Q(x) = Q_0(x_0) + R_0(x) + R(x) . \tag{2.4.24}$$

By (2.1.14) and the Proposition 2.2.1 the following inequality holds for the operator $R(x)$:

$$\|R(x)u\|_{(s-\nu)} \leqslant C(\|u\|_{(s-e_2-\nu_3 e_3)} + \varepsilon^{\gamma}\|u\|_{(s)}) \tag{2.4.25}$$

with $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.

Now let, $O_\delta \subset \bar{G}_\delta$, $\operatorname{diam} O_\delta < 2\delta$ and $\varphi_\delta \in C_0^\infty(O_\delta)$ such that $\varphi_\delta(x) \equiv 1$ for $x \in \bar{G}_\delta$. Denote $R_0^\delta(x) = \varphi_\delta(x) R_0(x)$. Using the Lemma 2.4.7 one can write down:

$$\varepsilon^{\nu_1-s_1}\langle D\rangle^{s_2-\nu_2}\langle\varepsilon D\rangle^{s_3-\nu_3}R_0(x)u \equiv \varepsilon^{\nu_1-s_1}\langle D\rangle^{s_2-\nu_2}\langle\varepsilon D\rangle^{s_3-\nu_3}R_0^\delta(x)u = \\ = R_0^\delta(x)\varepsilon^{\nu_1-s_1}\langle D\rangle^{s_2-\nu_2}\langle\varepsilon D\rangle^{s_3-\nu_3}u + K^\delta u \tag{2.4.26}$$

where the operator $u \to K^\delta u$ satisfies the inequality

$$\|K^\delta u\|_{(0)} \leqslant C_\delta \|u\|_{(s-e_2)} \tag{2.4.27}$$

with some constant C_δ that depends only on δ.

The coefficients of $R_0^\delta(x)$ being smooth with their support in O_δ, $\operatorname{diam} O_\delta < 2\delta$ and $R_0^\delta(x_0) \equiv 0$, one gets immediately:

$$\|R_0^\delta(x)\varepsilon^{\nu_1-s_1}\langle D\rangle^{\nu_2-s_2}\langle\varepsilon D\rangle^{\nu_3-s_3}u\|_0 \leqslant C\delta\|u\|_{(s)}, \tag{2.4.28}$$

where C does not depend on δ, ε and u.

By the Proposition 2.4.4 one has

$$\|u\|_{(s)} \leqslant C(\|Q_0(x_0)u\|_{(s-\nu)} + \|u\|_{(s')}) \tag{2.4.29}$$

with $\forall s' \in \partial_1\Gamma_{(s)}$ and C which depends only on s, s'.

Finally (2.4.24)-(2.4.29) and the interpolation inequality (1.1.16) with δ_1 instead of δ applied to the term $\|u\|_{(s-e_2)}$ yield:

$$\|u\|_{(s)} \leqslant C'_\delta(\|Q(x)u\|_{(s-\nu)} + (\varepsilon_0^\gamma + \delta_1)\|u\|_{(s)} + \|u\|_{(s')}) \tag{2.4.30}$$

the last inequality leading to (2.4.23) for ε_0 and δ_1 sufficiently small. ■

REMARK 2.4.9. – If $Q(x, \varepsilon, D) \in E_\nu$ and $u \in L^\infty([0, \varepsilon_0]; C_0^\infty(G_r))$ where $G_r = \{x||x| > r\}$, then for r large enough and ε_0 sufficiently small (2.4.23) still holds.

Indeed, the fact that the supp u is smaller than δ was used in the proof of the Lemma 2.4.8 only to estimate the term $R_0(x)$. Now denoting by $R_0(x)$ the difference

$$R_0(x) = Q_0(x) - Q_0(\infty), \tag{2.4.31}$$

$\varepsilon^{\nu_1}R_0(x)$ has its coefficients in $L^\infty([0, \varepsilon_0]; S(\mathcal{R}^n))$. Therefore for $r = \delta^{-1}$ and δ small enough the inequality (2.4.28) still holds.

THEOREM 2.4.10. – *Let $Q \in E_\nu$. Then for ε_0 sufficiently small the a priori estimation holds:*

$$\|u\|_{(s)} \leqslant C(\|Qu\|_{(s-\nu)} + \|u\|_{(s')}) \,, \qquad \forall u \in \mathcal{H}_{(s)}(\mathcal{R}^n) \,, \tag{2.4.32}$$

for $\forall s \in \mathcal{R}^n$, $\forall s' \in \partial_1 \Gamma_s$ where the constant C does not depend on ε and u.

PROOF. – Let B_r be a ball with radius $r = \delta^{-1}$ and let $\{G_j\}_{1 \leqslant j \leqslant N}$ be a finite covering of the ball B_r such that $\operatorname{diam} G_j < \delta$, so that, denoting $CB_r = G_{N+1}$, the collection of open sets $\{G_j\}_{1 \leqslant j \leqslant N+1}$ is a finite covering of $\mathcal{R}^n_x$. Let $\{\psi_j\}_{1 \leqslant j \leqslant N+1}$ be the partition of identity subordinated to the covering $\{G_j\}_{1 \leqslant j \leqslant N+1}$, so that for $\forall u \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n_x))$ one has

$$u = \sum_{1 \leqslant j \leqslant N+1} \psi_j u \,. \tag{2.4.33}$$

Now by Lemma 2.4.8 and the Remark 2.4.9 one obtains

$$\|u\|_{(s)} \leqslant \sum_{1 \leqslant j \leqslant N+1} \|\psi_j u\|_{(s)} \leqslant C \sum_{1 \leqslant j \leqslant N+1} (\|Q(\psi_j u)\|_{(s-\nu)} + \|\psi_j u\|_{(s')}) \,. \tag{2.4.34}$$

Applying again the Lemma 2.4.7, and the interpolation inequality (1.1.16), one can write down

$$\begin{aligned} &\|Q(\psi_j u)\|_{(s-\nu)} \leqslant \|Qu\|_{(s-\nu)} + C\|u\|_{(s')} \\ &\|\psi_j u\|_{(s')} \leqslant C\|u\|_{(s')} \,, \end{aligned} \tag{2.4.35}$$

where the constant C depends only on s, s', ν, δ and ε_0.

This ends the proof of the Theorem 2.4.10. ∎

THEOREM 2.4.11. – *If for $Q \in P_\nu$ the a priori estimation* (2.4.32) *holds for some $s \in \mathcal{R}^3$ and $s' \in \partial_1 \Gamma_s$, $s' < s$, then $Q \in E_\nu$.*

PROOF. – With $\eta \in \mathcal{R}^n$ and $\varphi(x) \in C_0^\infty(\mathcal{R}^n_x)$, we will substitute into (2.4.32) the functions $u(\varepsilon, x)$

$$u(\varepsilon, x) = \exp(i\varepsilon^{-1} x \cdot \eta)\, \varphi(x) \,. \tag{2.4.36}$$

It is immediate by the Leibnitz's formula that for such functions $u(\varepsilon, x)$ holds

$$Q(x, \varepsilon, D)\, u = \varepsilon^{-(\nu_1 + \nu_2)} \exp(i\varepsilon^{-1} x \cdot \eta)[Q_0(x, 1, \eta)\, \varphi(x) + \varepsilon^{\gamma_0} \psi_\eta(\varepsilon, x)] \tag{2.4.37}$$

where $\psi_\eta(\varepsilon, x) \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}^n_x))$, $\operatorname{supp} \psi_\eta \subseteq \operatorname{supp} \varphi$ and $\gamma_0 = \min\{1, \gamma\}$, γ being the same in (2.1.14).

It is easy to check that for $\forall \varrho \in \mathcal{R}^3$, $\forall \psi \in L^\infty([0, \varepsilon_0]; C_0^\infty(\mathcal{R}_x^n))$ holds

$$\|\exp(i\varepsilon^{-1}x\cdot\eta)\psi\|_{(\varrho)} = \varepsilon^{-(\varrho_1+\varrho_2)}|\eta|^{\varrho_2}\langle\eta\rangle^{\varrho_3}[\|\psi\|_{(0)} + C_\psi \varepsilon] \tag{2.4.38}$$

where C_ψ depends on ψ but is bounded uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$ and η with $|\eta| \geqslant r_0 > 0$.

Indeed,

$$\begin{aligned} \|\exp(i\varepsilon^{-1}x\cdot\eta)\psi\|_{(\varrho)} &= \|\varepsilon^{-\varrho_1}\langle\xi + \varepsilon^{-1}\eta\rangle^{\varrho_2}\langle\varepsilon\xi + \eta\rangle^{\varrho_3}\hat{\psi}(\xi)\|_{(0)} = \\ &= \varepsilon^{-(\varrho_1+\varrho_2)}\{\|\varepsilon^{\varrho_2}\langle\varepsilon^{-1}\eta\rangle^{\varrho_2}\langle\eta\rangle^{\varrho_3}\hat{\psi}(\xi)\|_{(0)} + [\|\varepsilon^{\varrho_2}\langle\xi + \varepsilon^{-1}\eta\rangle^{\varrho_2}\langle\varepsilon\xi + \eta\rangle^{\varrho_3}\hat{\psi}(\xi)\|_{(0)} - \\ &\quad - \|\varepsilon^{\varrho_2}\langle\varepsilon^{-1}\eta\rangle^{\varrho_2}\langle\eta\rangle^{\varrho_3}\hat{\psi}(\xi)\|]_{(0)}\}\,. \end{aligned} \tag{2.4.39}$$

Denoting the difference within the brackets in the right hand side of (2.4.39), by $g(\varepsilon, \eta)$ using the triangle inequality for the norms and the Lemma 2.4.6, one gets:

$$|g(\varepsilon, \eta)| \leqslant C\varepsilon\|\varepsilon^{\varrho_2-1}\langle\varepsilon^{-1}\eta\rangle^{\varrho_2-1}\langle\eta\rangle^{\varrho_3}\langle\xi\rangle^{N_0+1}\hat{\psi}(\xi)\|_{(0)}\,. \tag{2.4.40}$$

Now, noticing that for $|\eta| \geqslant r_0 > 0$ holds:

$$|\varepsilon^{\varrho}\langle\varepsilon^{-1}\eta\rangle^{\varrho} - |\eta|^{\varrho}| \leqslant C\varepsilon|\eta|^{\varrho-1}, \quad \forall \varrho \in \mathcal{R}\,, \tag{2.4.41}$$

one obtains (2.4.38) with

$$C_\psi \leqslant C\langle\eta\rangle^{-1}\|\langle\xi\rangle^{N_0+1}\hat{\psi}(\xi)\|_{(0)} \tag{4.4.42}$$

where the constant C does not depend on ε, ξ, η and ψ.

Applying (2.4.38) to both the left and right hand sides of (2.4.32) and taking into the consideration (2.4.37), one gets after the multiplication by $\varepsilon^{s_1+s_2}|\eta|^{\nu_2-s_2}\langle\eta\rangle^{\nu_3-s_3}$

$$|\eta|^{\nu_2}\langle\eta\rangle^{\nu_3}\|\varphi\|_{(0)} \leqslant C\|Q_0(x, 1, \eta)\varphi\|_{(0)} + C_\varphi \varepsilon \tag{2.4.43}$$

where C is the same as in the right hand side of (2.4.32) and C_φ does not depend on ε and is uniformly bounded when $|\eta| \geqslant r_0 > 0$.

Letting $\varepsilon \to 0$ in (2.4.43), one gets:

$$|\eta|^{\nu_2}\langle\eta\rangle^{\nu_3}\|\varphi\|_{(0)} \leqslant C\|Q_0(x, 1, \eta)\varphi\|_{(0)} \tag{2.4.44}$$

and that shows the ellipticity of Q, given that (2.4.44) holds for $\forall \varphi \in C_0^\infty(\mathcal{R}^n)$, and C does not depend on $\eta \in \mathcal{R}^n$. ∎

3. – Coercive boundary value problems for elliptic singular perturbations.

The aim of this section is to investigate boundary value problems in a compact domain for elliptic singular perturbations. We will point out necessary and sufficient algebraic condition, called coerciveness condition, on principal symbols of the operators for a two-sided a priori estimate for the solutions to the singularly perturbed boundary value problem to hold in corresponding spaces $\mathcal{H}_{(s)}$. Some applications and examples will also be discussed.

3.1. *Algebraic preliminaries.*

We prove in this section some auxiliary statements that will be further used for establishing a priori estimates for singular boundary value problems.

LEMMA 3.1.1. – *Let $Q_0(\varepsilon, \xi)$ be elliptic principal symbol of order ν, $Q_{00}(\xi)$ the principal homogeneous part of $Q_0(1, \xi)$ of order $\nu_2 + \nu_3$ and $\lambda(\varrho, \omega')$ zero of the equation:*

$$Q_0(\varrho, \omega', \lambda) = 0 \tag{3.1.1}$$

where $\varrho > 0$, $\omega' \in \Omega_{n-1}$.

Then $\operatorname{Im} \lambda(\varrho, \omega') \neq 0$, $\varrho > 0$, $\forall \omega' \in \Omega_{n-1}$ *and there exist a rational number $q < 0$ and $C(\omega') \neq 0$ such that*

$$\lambda(\varrho, \omega') = \lambda_{00}(\omega') + \varrho^q[C(\omega') + o(1)]\,, \qquad \varrho \to +\infty\,, \tag{3.1.2}$$

where $\lambda_{00}(\omega')$ is zero of the equation

$$Q_{00}(\omega', \lambda) = 0\,, \tag{3.1.3}$$

and $o(1) \to 0$ uniformly with respect to $\omega' \in \Omega_{n-1}$ as $\varrho \to +\infty$.

PROOF. – The coefficients of the family of polynomials

$$\lambda \to \varrho^{\nu_1 - \nu_3} Q_0(\varrho, \omega', \lambda)$$

are continuous functions of $\varrho \in (0, \infty]$ and $\omega' \in \Omega_{\nu-1}$ and approach the ones of the polynomial

$$\lambda \to Q_{00}(\omega', \lambda)$$

uniformly with respect to $\omega' \in \Omega_{n-1}$, as $\varrho \to +\infty$.

Besides, given the ellipticity of $Q_{00}(\xi)$ in the usual Petrovsky's sense, the leading coefficients of both polynomials $\lambda \to \varrho^{\nu_1 - \nu_3} Q_0$, $\lambda \to Q_{00}$ do not depend on $\omega' \in \Omega_n$ and are non-vanishing constants.

Therefore, according to the theorem on continuous dependence of roots of polynomials on parameters, one obtains:

$$\lim_{\varrho\to+\infty} \lambda(\varrho, \omega') = \lambda_{00}(\omega') .$$

Now, $\lambda(\varrho, \omega')$ being an algebraic function of ϱ, it follows from the theory of Puiseux's series (see, for instance, [3]) that there exist a rational number $q<0$ and $C(\omega')\neq 0$ such that (3.1.2) holds, the number q being rational and continuous function of $\omega'\in\Omega_{n-1}$, it follows that q does not depend on ω'. ■

LEMMA 3.1.2. – *Let $Q_0(\varepsilon, \xi)$ be elliptic principal symbol of order ν, $Q_0^0(\xi)$ the homogeneous part of $Q_0(1, \xi)$ of the lowest degree,* $\deg Q_0^0 = \nu_2$, *and $\lambda(\varrho, \omega')$ zero of the equation* (3.1.1).

Then there exist rational numbers $q_1 > 0$, $q_2 > -1$ and $C_j(\omega')\neq 0$, $j=1, 2$, such that either

$$\lambda(\varrho, \omega') = \lambda_0^0(\omega') + \varrho^{q_1}[C_1(\omega') + o(1)] , \qquad \varrho\to 0 \tag{3.1.4}$$

with $\lambda_0^0(\omega')$ zero of the equation:

$$Q_0^0(\omega', \lambda) = 0 \tag{3.1.5}$$

or

$$\lambda(\varrho, \omega') = \varrho^{-1}\mu + \varrho^{q_2}[C_2(\omega') + o(1)] , \qquad \varrho\to 0 \tag{3.1.6}$$

with μ a non-vanishing zero of the equation

$$Q_0(1, 0, \mu) = 0 . \tag{3.1.7}$$

PROOF. – Again, since $\lambda(\varrho, \omega')$ is an algebraic function of ϱ, it can be expanded into Puiseux's series.

First assume that $\lambda(\varrho, \omega')$ is bounded when $\varrho\to 0$, $\omega'\in\Omega_n$. Then, as it follows from the theory of Puiseux's series, $\lambda(\varrho, \omega')$ has the following behaviour:

$$\lambda(\varrho, \omega') \sim \lambda_0^0(\omega')\,\varrho^{\alpha}, \qquad \varrho\to 0 , \tag{3.1.8}$$

with some $\alpha\geqslant 0$.

The number α being rational and depending continuously on $\omega'\in\Omega_n$, one gets the conclusion that α does not depend on ω'. Substituting (3.1.8) into (3.1.1) one finds out that the following relations have to hold:

$$\alpha = 0 , \quad Q_0^0(\omega', \lambda_0^0(\omega')) = 0 . \tag{3.1.9}$$

Again, the theory of Puiseux's series asserts that there exist a rational number q_1 and $C_1(\omega') \neq 0$ such that (3.1.4) holds.

Now assuming that $\lambda(\varrho, \omega') \to \infty$ when $\varrho \to 0$, writing down

$$\lambda(\varrho, \omega') \sim \varrho^{-\beta}\mu(\omega')\,, \qquad \mu(\omega') \neq 0\,, \qquad \omega' \in \Omega_n\,, \tag{3.1.10}$$

and substituting (3.1.10) into (3.1.1) one gets the conclusion that necessarily

$$\beta = 1\,, \qquad Q_0(1, 0, \mu) = 0\,. \tag{3.1.11}$$

Finally, one more use of the Puiseux series theory leads to the formula (3.1.6). ■

REMARK 3.1.3. – The ellipticity of $Q_0(\varepsilon, \xi)$ guarantees that $\operatorname{Im}\lambda_0^0(\omega') \neq 0$, $\operatorname{Im}\mu \neq 0$, $\forall\omega' \in \Omega_n$.

REMARK 3.1.4. – It follows from the Lemma 3.1.2 that there are precisely ν_2 zeros (with their multiplicity) of (3 1.1) satisfying (3.1.4), (3.1.5) and precisely ν_3 zeros (with their multiplicity) of (3.1.1) satisfying (3.1.6), (3.1.7). Those zeros are further called respectively of the first and second type.

We assume further that $Q_{00}(\xi)$ is properly elliptic polynomial of order $\nu_2 + \nu_3$ in the usual sense (see for instance, [19]).

DEFINITION 3.1.4. – *An elliptic principal symbol $Q_0(\varepsilon, \xi)$ of order ν is called properly elliptic principal symbol, if $Q_{00}(\xi)$ is properly elliptic polynomial of order $\nu_2 + \nu_3$.*

LEMMA 3.1.5. – *Let $Q_0(\varepsilon, \xi)$ be properly elliptic principal symbol of order ν. Then the reduced symbol $Q_0^0(\xi)$ is properly elliptic too.*

PROOF. – It follows from the definition of the ellipticity (i)-(iii) of singular perturbation that $Q_0^0(\xi)$ is elliptic homogeneous polynomial of order ν_2. For $n \geqslant 3$ the class of Petrovsky's elliptic symbols coincides with this one of properly elliptic polynomials. Therefore, there is a need to prove the lemma only for $n = 2$.

The principal symbol $Q_0(\varepsilon, \xi)$ being elliptic, the inequality (2.4.3) with ε instead of ϱ holds, and it can be rewritten as follows:

$$|\varepsilon^{\nu_1}Q_0(\varepsilon, \xi)| \geqslant C|\xi|^{\nu_2}\langle\varepsilon\xi\rangle^{\nu_3}, \qquad \forall\xi, \varepsilon \geqslant 0\,. \tag{3.1.12}$$

In fact, the last inequality has to be valid also for $\varepsilon < 0$, given that $\varepsilon^{\nu_1}Q_0(\varepsilon, \xi)$ is homogeneous in (ε^{-1}, ξ) of order $\nu_2 + \nu_3$.

Without restricting the generality one can assume that $\nu_1 = 0$. Introducing a new parameter $\xi_0 = \varepsilon^{-1}$ and denoting

$$q_0(\xi_0, \xi) = \xi_0^{\nu_3} Q_0(\xi_0^{-1}, \xi)\,, \tag{3.1.13}$$

the polynomial $q_0(\xi_0, \xi)$ in (ξ_0, ξ) satisfies the inequality

$$|q_0(\xi_0, \xi)| \geq C|\xi|^{\nu_2}(|\xi_0| + |\xi|)^{\nu_3}. \tag{3.1.14}$$

Besides, $q_0(\xi_0, \xi)$ is homogeneous of order $\nu_2 + \nu_3$.

Consider now the polynomial $\xi_0 \to |\xi|^{-\nu_2} q_0(\xi_0, \xi)$ that is a homogeneous function in (ξ_0, ξ) of order ν_3 which does not vanish when $(\xi_0, \xi) \in \mathcal{R}^3 \setminus \{0\}$.

If $\lambda(\xi)$ is zero of the equation:

$$|\xi|^{-\nu_2} q_0(\lambda, \xi) = 0 \tag{3.1.15}$$

such that $\operatorname{Im}\lambda > 0$, then the complex number $\mu = -\lambda(-\xi)$ is a zero of the same equation such that $\operatorname{Im}\mu < 0$. When $\xi \in \mathcal{R}^2 \setminus \{0\}$, both $\lambda(\xi)$ and $-\lambda(-\xi)$ cannot meet the real line. Given that the circle $|\xi| = 1$ is a connex set in $\mathcal{R}^2 \setminus \{0\}$, one can deform $-\xi$ to ξ along this circle, so that ξ does not vanish. Therefore, the root $\mu(\xi) = -\lambda(-\xi)$ will be transformed into some other zero of the equation (3.1.15), say $\tilde{\mu}(\xi)$. Consequently, to any root of (3.1.15), say $\lambda(\xi)$, contained in the upper half plane corresponds a root $\tilde{\mu}(\xi)$ contained in the lower one, which shows that ν_3 has to be an even non-negative integer. $Q_{00}(\xi)$ being properly elliptic in the usual sense, the number $\nu_2 + \nu_3$ is also even, and, consequently, so it is for ν_2.

Denote: $\nu_j = 2r_j$. $Q_0(\varepsilon, \xi_1, \lambda)$ being properly elliptic in the usual sense for any fixed $\varepsilon \in (0, \varepsilon_0]$, it has precisely $r_2 + r_3$ roots in both upper and lower half planes. None of these roots can meet the real line if $\xi_1 \neq 0$, so that the polynomial $Q_0^0(\xi_1, \lambda) = \lim_{\varepsilon \to 0} Q_0(\varepsilon, \xi_1, \lambda)$ of order $\nu_2 = 2r_2$ must have the same number of roots in both upper and lower half-planes, given that precisely r_3 roots of $Q_0(\varepsilon, \xi_1, \lambda)$ in both upper and lower half planes go to infinity as $\varepsilon \to 0$. ∎

COROLLARY 3.1.6. – If $Q_0(\varepsilon, \xi)$ is properly elliptic principal symbol of order ν, then ν_2, ν_3 are even non-negative integers:

$$\nu_j = 2r_j. \tag{3.1.16}$$

Let $\lambda \to Q_0^+(\varepsilon, \xi', \lambda)$ be the factor of the polynomial $\lambda \to Q_0(\varepsilon, \xi', \lambda)$ that corresponds to the zeros of Q_0 contained within the upper half of the complex λ-plane, when $\xi' \in \mathcal{R}^{n-1} \setminus \{0\}$, $\varepsilon > 0$. Besides, $\lambda \to Q_0^+$ being well defined up to a coefficient depending on ε, ξ', the latter is supposed to be chosen in such a way that

$$\lim_{\lambda \to \infty} \lambda^{-(r_2+r_3)} Q_0^+(\varepsilon, \xi', \lambda) = \varepsilon^{r_3}. \tag{3.1.17}$$

LEMMA 3.1.7. – *Let $Q_0(\varepsilon, \xi)$ be properly elliptic of order ν and $\lambda \to Q_0^+(\varepsilon, \xi', \lambda)$ the factor of $\lambda \to Q_0$ as described above. Then $Q_0^+(\varepsilon, \xi', \lambda)$ is homogeneous function in*

$(\varepsilon^{-1}, \xi', \lambda)$ *of order* r_2 *which satisfies the inequalities*

$$(3.1.18) \qquad C^{-1}(|\xi'| + |\lambda|)^{r_2}(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{r_3} \leqslant \leqslant |Q_0^+(\varepsilon, \xi', \lambda)| \leqslant C(|\xi'| + |\lambda|)^{r_2}(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{r_3}$$

for any $\xi' \in \mathfrak{R}^{n-1}$, *any* $\lambda \in \boldsymbol{C}$ *such that* $\operatorname{Im}\lambda \leqslant 0$; *here the constant* C *does not depend on* $\varepsilon \in [0, \infty)$, $\xi' \in \mathfrak{R}^{n-1}$, *and* $\lambda \in \boldsymbol{C}$, $\operatorname{Im}\lambda \leqslant 0$.

Besides, for $\varepsilon \in [0, \varepsilon_0]$ *(with* ε_0 *sufficiently small) and* $|\xi'| \leqslant 2$ *the polynomial* $\lambda \to Q_0^+(\varepsilon, \xi', \lambda)$ *can be split into the product*

$$(3.1.19) \qquad Q_0^+(\varepsilon, \xi', \lambda) = Q_{01}^+(\varepsilon, \xi', \lambda)\, Q_{02}^+(\varepsilon, \xi', \lambda)\,, \qquad \varepsilon \in [0, \varepsilon_0],\ |\xi'| \leqslant 2\,,$$

where the zeros of the polynomial $\lambda \to Q_{01}^+$ *are contained within a compact domain in the upper half plane uniformly with respect to* $\varepsilon \in [0, \varepsilon_0]$, $|\xi'| \leqslant 2$ *and the zeros of* $\lambda \to Q_{02}^+$ *multiplied by* ε *also are contained in a compact domain in the upper half plane uniformly with respect to* $\varepsilon \in [0, \varepsilon_0]$, $|\xi'| \leqslant 2$; *the functions* $(\varepsilon^{-1}, \xi', \lambda) \to Q_{01}^+$, $(\varepsilon^{-1}, \xi', \lambda) \to Q_{02}^+$ *can be extended as homogeneous functions of* $(\varepsilon^{-1}, \xi', \lambda) \in (0, \infty) \times \mathfrak{R}_{\xi'}^{n-1} \times \boldsymbol{C}$ *respectively of orders* r_2 *and* 0.

The leading coefficients of $\lambda \to Q_{01}^+$, $\lambda \to Q_{02}^+$ *being respectively* 1 *and* ε^{r_3}, *the following inequalities hold*:

$$(3.1.20) \qquad \begin{aligned} &|Q_{01}^+(\varepsilon, \xi', \lambda)| \geqslant C(|\xi'| + |\lambda|)^{r_2}, \\ &|Q_{02}^+(\varepsilon, \xi', \lambda)| \geqslant C(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{r_3}, \end{aligned}$$

for any $\xi' \in \mathfrak{R}^{n-1}$, *any* $\lambda \in \boldsymbol{C}$, $\operatorname{Im}\lambda \leqslant 0$.

PROOF. – It is immediate that the roots $\lambda_j(\varepsilon, \xi')$ of the equation

$$Q_0(\varepsilon, \xi', \lambda) = 0$$

are homogeneous functions of order 1 in variable (ε^{-1}, ξ'), given that the polynomial $\lambda \to Q_0$ is homogeneous function in $(\varepsilon^{-1}, \xi', \lambda)$.

Therefore one can write down with $\varrho = \varepsilon|\xi'|$, $\omega' = \xi'|\xi'|^{-1}$:

$$(3.1.21) \qquad \lambda_j(\varepsilon, \xi') = |\xi'|\, \lambda_j(\varrho, \omega')$$

where $\lambda_j(\varrho, \omega')$ satisfies the equation (3.1.1).

According to the Lemma 3.1.2 either (3.1.4), (3.1.5) or (3.1.6), (3.1.7) holds for $\lambda_j(\varrho, \omega')$ as $\varrho \to +0$, and there are precisely r_2 roots λ_j, $1 \leqslant j \leqslant r_2$ of (3.1.1) satisfying (3.1.4), (3.1.5) with $\operatorname{Im}\lambda_0^0(\omega') \geqslant C_0 > 0$, $\forall \omega' \in \mathfrak{R}^{n-1}$, $|\omega'| = 1$, and precisely r_3 roots λ_j, $r_2 < j \leqslant r_2 + r_3$, of (3.1.1) satisfying (3.1.6), (3.1.7) with $\operatorname{Im}\mu_j > 0$.

Therefore, one has with some constant $C_0 > 0$:

(3.1.22) $\operatorname{Im} \lambda_j(\varepsilon, \xi') \geqslant C_0|\xi'|, \quad \forall \varepsilon \in [0, \infty), \ \forall \xi' \in \mathcal{R}^{n-1}$, when $1 \leqslant j \leqslant r_2$,

and

(3.1.23) $\operatorname{Im} \lambda_j(\varepsilon, \xi') \geqslant C_0 \varepsilon^{-1}, \quad \forall \xi' \in \mathcal{R}^{n-1}, \ \forall \varepsilon \in [0, \infty)$, when $r_2 < j \leqslant r_2 + r_3$.

Now if $\langle \varepsilon\xi' \rangle = \varrho \to \infty$, one gets by Lemma 3.1.1 and (3.1.21)

(3.1.24) $\operatorname{Im} \lambda_j(\varepsilon, \xi') \geqslant C_0 \varepsilon^{-1} \langle \varepsilon\xi' \rangle, \quad \forall \xi' \in \mathcal{R}^{n-1}, \ \forall \varepsilon \in (0, \infty], \ 1 \leqslant j \leqslant r_2 + r_3$.

The formulas (3.1.22)-(3.1.24) yield

(3.1.25)
$$\begin{aligned} &\operatorname{Im} \lambda_j(\varepsilon, \xi') \geqslant C_0|\xi'|, && \text{for } 1 \leqslant j \leqslant r_2, \\ &\operatorname{Im} \lambda_j(\varepsilon, \xi') \geqslant C_0 \varepsilon^{-1} \langle \varepsilon\xi' \rangle, && \text{for } r_2 < j \leqslant r_2 + r_3, \end{aligned}$$

for any $\xi' \in \mathcal{R}^{n-1}$, any $\varepsilon \in (0, \infty)$

Now, obviously, one has

(3.1.26)
$$Q_0^+(\varepsilon, \xi', \lambda) = \prod_{1 \leqslant j \leqslant r_2} (\lambda - \lambda_j(\varepsilon, \xi')) \prod_{r_2 \leqslant j \leqslant r_2 + r_3} (\varepsilon\lambda - \varepsilon\lambda_j(\varepsilon, \xi')),$$

where the factors in the right hand side are repeated according to the multiplicity of the corresponding zero λ_j.

Therefore, (3.1.25), (3.1.26) yield:

(3.1.27)
$$C(|\xi'| + |\lambda|)^{r_2}(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{r_3} \geqslant |Q_0^+(\varepsilon, \xi', \lambda)| \geqslant \geqslant C^{-1}(|\xi'| + |\lambda|)^{r_2}(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{r_3},$$

for any $\xi' \in \mathcal{R}^{n-1}$, any $\varepsilon \in (0, \infty)$ with some constant $C > 0$.

The homogeneity of Q_0^+ in $(\varepsilon^{-1}, \xi', \lambda)$ of order r_2 follows immediately from the fact that $\lambda_j(\varepsilon, \xi')$ are homogeneous in (ε^{-1}, ξ') of order 1.

Moreover, for $\varepsilon \in (0, \varepsilon_0]$, with ε_0 small enough and $|\xi'| \leqslant 2$ the zeros $\lambda_j(\varepsilon, \xi')$, $1 \leqslant j \leqslant r_2$, of the first type and the zeros $\lambda_j(\varepsilon, \xi')$, $r_2 < j \leqslant r_2 + r_3$ of the second one can be separated each from others by two Jordanian curves contained in the upper half of the complex λ-planes so that the breaking down (3.1.26) into the product of two polynomials:

(3.1.28)
$$Q_0^+(\varepsilon, \xi', \lambda) = Q_{01}^+(\varepsilon, \xi', \lambda) Q_{02}^+(\varepsilon, \xi', \lambda)$$

with Q_{01}^+ and Q_{02}^+ having respectively the zeros λ_j, $1 \leqslant j \leqslant r_2$, and $r_2 < j \leqslant r_2 + r_3$, is a smooth one with respect to the parameters $\xi' \in \mathcal{R}^{n-1}$, $|\xi'| \leqslant 2$, and $\varepsilon \in (0, \varepsilon_0]$.

It goes without saying that the splitting (3.1.28) is smooth in ξ' for ε_0 sufficiently small whenever $|\xi'| \leqslant C$. Of course, ε_0 depends on C in that case.

It is quite obvious that Q^+_{0j}, $j = 1, 2$, satisfy (3.1.20). ■

REMARK 3.1.8. – In fact, given the inequalities (3.1.25) and also the obvious inequalities:

$$|\lambda_j(\varepsilon, \xi')| \leqslant C|\xi'|\,, \qquad 1 \leqslant j_1 \leqslant r_2\,, \tag{3.1.29}$$
$$|\lambda_j(\varepsilon, \xi')| \leqslant C\varepsilon^{-1}\langle\varepsilon\xi'\rangle\,, \qquad r_2 < j \leqslant r_2 + r_3\,,$$

with C which does not depend on (ε, ξ'), there exists a contour Γ,

$$\Gamma = \{\lambda | \operatorname{Im}\lambda \geqslant 0,\ |\operatorname{Re}\lambda| = \gamma \operatorname{Im}\lambda\} \tag{3.1.30}$$

such that inequalities (3.1.18), (3.1.20) hold for $\lambda \in \Gamma$.

REMARK 3.1.9. – The function $Q^+_{02}(\varepsilon, \xi', \lambda)$ in (3.1.19) being homogeneous in $(\varepsilon^{-1}, \xi', \lambda)$ of order 0, one concludes immediately, taking into account (3.1.5), (3.1.6), that the zeros of the polynomial

$$\lambda \to Q^+_{02}(1, \varrho\omega', \lambda)$$

are contained in a compact domain within the upper half of complex λ-plane, when $\omega' \in \Omega_{n-1}$ and $\varrho \in [0, \varrho_0]$ with ϱ_0 sufficiently small.

3.2. *Singular perturbations in* $\mathfrak{R}_+$.

We will consider here a boundary value problem for singularly perturbed ordinary differential equation containing two parameters: $\varepsilon \in (0, \varepsilon_0]$ and $\xi' \in \mathfrak{R}^{n-1}$.

Let $Q_0(\varepsilon, \xi', \xi_n)$ be properly elliptic principal symbol with constant coefficients of order $\nu \in \mathfrak{R}^3$ and let $\nu_j = 2r_j$, $j = 1, 2$, r_j being non-negative integers.

Let $b_{j0}(\varepsilon, \xi', \xi_n)$ be principal symbols of orders $\mu_j \in \mathfrak{R}^3$, $\mu_j = (\gamma_j, m_j, p_j)$, $1 \leqslant j \leqslant r_2 + r_3$, where, of course, m_j, p_j are non-negative integers. Besides, we assume that b_{j0} are ordered in such a way that the sequence $\{m_j\}$, $1 \leqslant j \leqslant r_2 + r_3$, is non-decreasing and, moreover, we assume that

$$m_1 \leqslant m_2 \leqslant \ldots \leqslant m_{r_2} < m_{r_2+1} \leqslant \ldots \leqslant m_{r_2+r_3}\,. \tag{3.2.1}$$

We denote

$$\hat{\xi}' = \langle\xi'\rangle\,\omega' \qquad \text{with} \qquad \omega' = \xi'/|\xi'|\,, \tag{3.2.2}$$

so that $|\hat{\xi}'| = \langle\xi'\rangle$.

The following boundary value problem on a half line $\mathcal{R}_+ = \{t > 0\}$ is basic in investigating boundary value problems for singularly perturbed Partial Differential Equations:

$$Q_0(\varepsilon\ \xi', D_t)\,u(t) = f(t)\,, \qquad t \in \mathcal{R}_+ \tag{3.2.3}$$

$$\pi_0\big(b_{j0}(\varepsilon, \xi', D_t)\,u(t)\big) = \varphi_j\,, \qquad 1 \leqslant j \leqslant r_2 + r_3 \tag{3.2.4}$$

where by π_0 we denote the restriction of function $v(t)$ to the point $t = 0$.

Our first step is to study the homogeneous problem (3.2.3), (3.2.4), i.e. when $f(t) \equiv 0$, which we rewrite in the following fashion:

$$\varepsilon^{\nu_1} Q_0(\varepsilon, \xi', D_t)\,u(t) = 0\,, \qquad t \in \mathcal{R}_+ \tag{3.2.5}$$

$$\pi_0\big(\varepsilon^{\gamma_j} b_{j0}(\varepsilon, \xi', D_t)\,u(t)\big) = \varepsilon^{\gamma_j}\varphi_j\,, \qquad 1 \leqslant j \leqslant r_2 + r_3\,. \tag{3.2.6}$$

The solution $u(t)$ of (3.2.5), (3.2.6) is sought in the class of functions on $\mathcal{R}_+$ decreasing as $t \to +\infty$.

With $\lambda \to Q_0^+(\varepsilon, \xi', \lambda)$ the polynomial in the factorization of $\lambda \to Q_0(\varepsilon, \xi', \lambda)$ defined above in the previous section, the general solution of (3.2.5) in the class of functions decreasing as $t \to +\infty$, can be written in the following fashion:

$$u(t) = \frac{1}{2\pi i}\int_\Gamma \frac{M(\varepsilon, \xi', \lambda)}{Q_0^+(\varepsilon, \xi', \lambda)} \exp(it\lambda)\,d\lambda$$

where Γ is any Jordanian contour, which encloses all the roots of $Q_0^+(\varepsilon, \xi', \lambda)$ and $\lambda \to M(\varepsilon, \xi', \lambda)$ is a polynomial in λ.

Introducing a new variable $\lambda \to \langle \xi' \rangle \lambda$ and denoting $\varrho = \varepsilon \langle \xi' \rangle$,the last formula becomes

$$u(t) = \frac{1}{2\pi i}\int_\Gamma \frac{M(\varepsilon, \varrho, \omega', \lambda)}{Q_0^+(\varrho, \omega', \lambda)} \exp\big(it\langle \xi' \rangle \lambda\big)\,d\lambda \tag{3.2.7}$$

where $\lambda \to M(\varepsilon, \varrho, \omega', \lambda)$ is again some polynomial.

The advantage of using the formula (3.2.7) is that the contour Γ can now be chosen independently on $\varrho \geqslant \varrho_0 > 0$ and $\omega' \in \Omega_n$ if ϱ_0 is any positive constant. Indeed, according to the Lemma 3.1.1 the zeros of $\lambda \to Q_0^+(\varrho, \omega', \lambda)$ are contained within a compact domain in the upper half plane and they have as limits the corresponding zeros of the polynomial $\lambda \to Q_{00}^+(\omega', \lambda)$, where

$$Q_{00}^+(\omega', \lambda) = \lim_{\varrho \to +\infty} \varrho^{-r_3} Q_0^+(\varrho, \omega', \lambda)\,.$$

We will consider first the case when $\varrho \leqslant \varrho_0$ where ϱ_0 is sufficiently small.

In this case, as asserts the Lemma 3.1.7, the polynomial $\lambda \to Q_0^+(\varrho, \omega', \lambda)$ can be broken down into the product of two polynomials $Q_{0j}^+(\varrho, \omega', \lambda)$, $j = 1, 2$, the factors in the product being smooth in ϱ, ω'. Besides, the zeros of $Q_{01}^+(\varrho, \omega', \lambda)$ can be enclosed within compact Jordanian curve contained in the upper half plane say Γ_1, which does not depend on $\varrho \in [0, \varrho_0]$ and $\omega' \in \mathfrak{R}^{n-1}$, $|\omega'| = 1$, while the zeros of $Q_{02}^+(\varrho, \omega', \lambda)$ grow as $1/\varrho$ when $\varrho \to 0$ and have imaginary parts growing to $+\infty$ in the same fashion.

Using the splitting (3.1.19) one can rewrite the formula for general decreasing at infinity solution of (3.2.5) in the following way:

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{M_1(\varepsilon, \varrho, \omega', \lambda)}{Q_{01}^+(\varrho, \omega', \lambda)} \exp\left(i\langle\xi'\rangle t\lambda\right) d\lambda + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{M_2(\varepsilon, \varrho, \omega', \gamma)}{Q_{02}^+(1, \varrho\omega', \lambda)} \exp\left(i\varepsilon^{-1} t\lambda\right) d\lambda \tag{3.2.8}$$

where the contour Γ_2 is also a compact Jordanian curve in the upper half plane, which contains all the roots of $\lambda \to Q_{02}^+(1, \varrho\omega', \lambda)$ and does not depend on $\varrho, \omega' \in \in [0, \varrho_0] \times \Omega_{n-1}$.

Now, the polynomials $\lambda \to M_j(\varepsilon, \varrho, \omega', \lambda)$ of orders $r_j - 1$ at most, $j = 2, 3$ in the right hand side of (3.2.8) have to be found in such a way for the boundary conditions (3.2.6) to be satisfied.

Besides, it is quite obvious that there exist limits

$$\lim_{\varrho \to 0} Q_{01}^+(\varrho, \omega', \lambda) = Q_0^{0+}(\omega', \lambda) \tag{3.2.9}$$

$$\lim_{\varrho \to 0} Q_{02}^+(1, \varrho\omega', \lambda) = Q_{02}^+(1, 0, \lambda) \tag{3.2.10}$$

where $\lambda \to Q_0^{0+}(\omega', \lambda)$ is the polynomial in the factorization to the reduced symbol $\lambda \to Q_0^0(\omega', \lambda)$, which corresponds to the roots of $Q_0^0(\omega', \lambda)$ contained in the upper half plane when $\omega' \in \Omega_{n-1}$.

We will also denote:

$$Q^+(\lambda) = Q_{02}^+(1, 0, \lambda)\,. \tag{3.2.11}$$

One finds immediately that $Q^+(\lambda)$ is the factor in the factorization for the polynomial $\lambda \to \lambda^{-\nu_2} Q_0(1, 0, \lambda)$ which corresponds to the roots of $\lambda^{-\nu_2} Q_0(1, 0, \lambda)$ contained in the upper half plane. Of course, the ellipticity (2.4.1) of $Q_0(\varepsilon, \xi)$ guarantees that all the zeros of $\lambda^{-\nu''} Q_0(1, 0, \lambda)$ are non-real numbers and the proper ellipticity says that the number of zeros contained in the upper and lower half planes is r_3.

Now we will restrict the class of boundary operators in (3.2.6); the restrictions below will enable us to establish a two-sided a priori estimate for the solutions of (3.2.5)-(3.2.6).

Denoting by $b^0_{j0}(\xi', \lambda)$ the reduced symbols for $b_{j0}(\varepsilon, \xi)$:

(3.2.12) $$b^0_{j0}(\xi', \lambda) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma_j} b_{j0}(\varepsilon, \xi', \lambda),$$

and by $b_j(\lambda)$ the symbols:

(3.2.13) $$b_j(\lambda) = b_{j0}(1, 0, \lambda),$$

we put the conditions:

CONDITION 3.2.1. – *The polynomials*

(3.2.14) $$\lambda \to b^0_{j0}(\omega', \lambda), \quad 1 \leqslant j \leqslant r_2$$

are supposed to be linearly independent modulo $Q^{0+}_0(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$.

CONDITION 3.2.2. – *The polynomials*

(3.2.15) $$\lambda \to b_j(\lambda), \quad r_2 < j \leqslant r_2 + r_3$$

are supposed to be linearly independent modulo $Q^+(\lambda)$.

Let

(3.2.16) $$B_{j0}(\varepsilon, \xi', \lambda) = \varepsilon^{\gamma_j} b_{j0}(\varepsilon, \xi', \lambda)$$

With the notation (3.2.16) and the Conditions 3.2.1, 3.2.2 being fulfilled, the polynomials

(3.2.17) $$\lambda \to B_{j0}(\varrho, \omega', \lambda), \quad 1 \leqslant j \leqslant r_2$$

are linearly independent modulo $Q^+_{01}(\varrho, \omega', \lambda)$, as well as the polynomials:

(3.2.18) $$\lambda \to b_{j0}(1, \varrho\omega', \lambda), \quad r_2 < j \leqslant r_2 + r_3$$

are linearly independent modulo $Q^+_{02}(1, \varrho\omega' \lambda)$, for $\forall \omega' \in \mathfrak{R}^{n-1}$, $|\omega'| = 1$, $\forall \varrho \in [0, \varrho_0]$, if ϱ_0 is sufficiently small, given the continuous dependence of all these polynomials on the parameters $\varrho \in [0, \varrho_0]$ and $\omega' \in \mathfrak{R}^{n-1}$, $|\omega'| = 1$ and the compactness of the set $\Omega_{n-1} \times [0, \varrho_0]$ in $\mathfrak{R}^n$.

Consequently, there exist polynomials

(3.2.19) $$\lambda \to M_j(\varrho, \omega', \lambda), \quad 1 \leqslant j \leqslant r_2$$

of order $< r_2$, and the polynomials

(3.2.20) $$\lambda \to M_j(\varrho\omega', \lambda), \quad r_2 < j \leqslant r_2 + r_3,$$

of order $< r_3$ which depend continuously on the parameters $(\varrho, \omega') \in [0, \varrho_0] \times \Omega_{n-1}$ and satisfy the orthogonality relations (see, for instance [19]):

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{B_{k0}(\varrho, \omega', \lambda)\, M_j(\varrho, \omega', \lambda)}{Q_{01}^{+}(\varrho, \omega', \lambda)}\, d\lambda = \delta_{kj}\,, \qquad 1 \leqslant k,\ j \leqslant r_2 \tag{3.2.21}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{b_{k0}(1, \varrho\omega', \lambda)\, M_j(\varrho\omega', \lambda)}{Q_{02}^{+}(1, \varrho\omega', \lambda)}\, d\lambda = \delta_{kj}\,, \qquad r_2 < k,\ j \leqslant r_2 + r_3 \tag{3.2.22}$$

where $\delta_{kk} = 1$ and $\delta_{kj} = 0$ otherwise.

We seek the solution to the problem (3.2.5), (3.2.6) in the form:

$$\begin{aligned} u(t) = \sum_{1 \leqslant j \leqslant r_2} \psi_j \frac{1}{2\pi i}\int_{\Gamma_1} \frac{M_j(\varrho, \omega', \lambda)}{Q_{01}^{+}(\varrho, \omega', \lambda)} \exp\,(i\langle\xi'\rangle t\lambda)\, d\lambda\, + \\ + \sum_{r_2 < j \leqslant r_2 + r_3} \psi_j \frac{1}{2\pi i}\int_{\Gamma_2} \frac{M_j(\varrho\omega', \lambda)}{Q_{02}^{+}(1, \varrho\omega', \lambda)} \exp\,(i\varepsilon^{-1} t\lambda)\, d\lambda \end{aligned} \tag{3.2.23}$$

where $\psi_j = \psi_j(\varrho, \omega')$ have to be found.

Substituting (3.2.23) into the boundary conditions (3.2.6), using the homogeneity and the orthogonality conditions (3.2.21), (3.2.22), one gets:

$$\psi_k + \varrho^{-m_k} \sum_{r_2 < j \leqslant r_2 + r_3} \psi_j \frac{1}{2\pi i}\int_{\Gamma_2} \frac{b_{k0}(1, \varrho\omega', \lambda)\, M_j(\varrho\omega', \lambda)}{Q_{02}^{+}(1, \varrho\omega', \lambda)}\, d\lambda = \varepsilon^{\gamma_k} \langle\xi'\rangle^{-m_k} \varphi_k \tag{3.2.24}$$

for $1 \leqslant k \leqslant r_2$, and

$$\varrho^{m_k} \sum_{1 \leqslant j \leqslant r_2} \psi_j \frac{1}{2\pi i}\int_{\Gamma_2} \frac{B_{k0}(\varrho, \omega', \lambda)\, M_j(\varrho, \omega', \lambda)}{Q_{01}^{+}(\varrho, \omega', \lambda)}\, d\lambda + \psi_k = \varepsilon^{\gamma_k + m_k} \varphi_k \tag{3.2.25}$$

for $r_2 < k \leqslant r_2 + r_3$.

Introducing the notations:

$$a_{kj}(\varrho, \omega') = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{b_{k0}(1, \varrho\omega', \lambda)\, M_j(\varrho\omega', \lambda)}{Q_{02}^{+}(1, \varrho\omega', \lambda)}\, d\lambda\,, \qquad 1 \leqslant k \leqslant r_2 < j \leqslant r_2 + r_3 \tag{3.2.26}$$

and

$$a_{kj}(\varrho, \omega') = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{B_{k0}(\varrho, \omega', \lambda)\, M_j(\varrho, \omega', \lambda)}{Q_{01}^{+}(\varrho, \omega', \lambda)}\, d\lambda\,, \qquad 1 \leqslant j \leqslant r_2 < k \leqslant r_2 + r_3\,. \tag{3.2.27}$$

the system of linear equations (3.2.24), (3.2.25) for ψ_k can be rewritten in the following fashion:

$$\begin{pmatrix} I_{r_2}, & \operatorname{diag}(\varrho^{-m_k})_1^{r_2} A_{12}(\varrho, \omega') \\ \operatorname{diag}(\varrho^{m_k})_{r_2+1}^{r_2+r_3} A_{21}(\varrho, \omega'), & I_{r_3} \end{pmatrix} \psi = \begin{pmatrix} \operatorname{diag}(\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k})_1^{r_2} \\ \operatorname{diag}(\varepsilon^{\gamma_k + m_k})_{r_2+1}^{r_2+r_3} \end{pmatrix} \varphi \tag{3.2.28}$$

where I_{r_j} is identity in $\mathrm{Hom}(\boldsymbol{C}^{r_j}; \boldsymbol{C}^{r_j})$, $\mathrm{diag}(c_k)_l^m$ is diagonal matrix having the complex numbers c_j, $l \leqslant j \leqslant m$, on its principal diagonal, and the matrices $A_{12}(\varrho\ \omega')$ with the elements (3.2.26) and $A_{21}(\varrho\ \omega')$ with the elements (3.2.27) depend continuously on $(\varrho, \omega') \in [0, \varrho_0] \times \Omega_{n-1}$ with their values in $\mathrm{Hom}(\boldsymbol{C}^{r_3}; \boldsymbol{C}^{r_2})$ and $\mathrm{Hom}(\boldsymbol{C}^{r_2}; \boldsymbol{C}^{r_3})$ respectively.

Let us denote by $\mathscr{T}_{\varrho,\omega'}$, $(\varrho, \omega') \in [0, \varrho_0] \times \Omega_n$, the class of matrices of the form:

$$\mathscr{T}_{\varrho,\omega'} = I + \begin{pmatrix} 0, & \varrho^{-\alpha} A_{12}(\varrho, \omega') \\ \varrho^{\beta} A_{21}(\varrho, \omega'), & 0 \end{pmatrix}, \tag{3.2.29}$$

where $\mathscr{T}_{\varrho,\omega'} \in \mathrm{Hom}(\boldsymbol{C}^{r_2+r_3}; \boldsymbol{C}^{r_2+r_3})$, $A_{12}(\varrho, \omega')$, $A_{21}(\varrho, \omega')$ are continuous functions of $\varrho, \omega' \in [0, \varrho_0] \times \Omega_{n-1}$ with values respectively in $\mathrm{Hom}(\boldsymbol{C}^{r_3}; \boldsymbol{C}^{r_2})$ and $\mathrm{Hom}(\boldsymbol{C}^{r_2}; \boldsymbol{C}^{r_3})$.

LEMMA 3.2.3. – *If* $0 \leqslant \alpha < \beta$ *then for* ϱ_0 *sufficiently small there exists the inverse* $\mathscr{T}^{-1}_{\varrho,\omega'}$ *to the matrix* (3.2.29) *which has the form:*

$$\mathscr{T}^{-1}_{\varrho,\omega'} = I + \varrho^{\beta-\alpha} \begin{pmatrix} B_{11}(\varrho, \omega'), & 0 \\ 0, & B_{22}(\varrho, \omega') \end{pmatrix} + \begin{pmatrix} 0, & \varrho^{-\alpha} B_{12}(\varrho, \omega') \\ \varrho^{\beta} B_{21}(\varrho, \omega'), & 0 \end{pmatrix} \tag{3.2.30}$$

where $B_{jj}(\varrho, \omega')$ *and* $B_{jk}(\varrho, \omega')$ *are continuous functions of* $\varrho, \omega' \in [0, \varrho_0] \times \Omega_{n-1}$ *with values respectively in* $\mathrm{Hom}(\boldsymbol{C}^{r_{j+1}}; \boldsymbol{C}^{r_{j+1}})$ *and* $\mathrm{Hom}(\boldsymbol{C}^{r_{k+1}}; \boldsymbol{C}^{r_{j+1}})$, $\mathrm{Hom}(\boldsymbol{C}^{r_{j+1}}; \boldsymbol{C}^{r_{k+1}})$.

PROOF. – First of all, it is immediate that

$$\det \mathscr{T}_{\varrho'\omega,} = \det\left(I_{r_2} - \varrho^{\beta-\alpha} A_{12}(\varrho, \omega') A_{21}(\varrho, \omega')\right) = 1 + O(\varrho^{\beta-\alpha}), \qquad \varrho \to 0$$

and, consequently, the inverse matrix $\mathscr{T}^{-1}_{\varrho,\omega'}$ exists for ϱ_0 sufficiently small. Let us denote by $K(\varrho, \omega')$ the matrix with the block structure

$$K(\varrho, \omega') = \begin{pmatrix} 0, & \varrho^{-\alpha} A_{12}(\varrho, \omega') \\ \varrho^{\beta} A_{21}(\varrho, \omega'), & 0 \end{pmatrix}. \tag{3.2.31}$$

Given that for matrices with block structure the multiplication is performed as though the corresponding block-matrices were numbers (see, for instance, [7]), one finds easily, for any integer $p > 0$:

$$K^{2p}(\varrho, \omega') = \varrho^{p(\beta-\alpha)} \begin{pmatrix} (A_{12}A_{21})^p, & 0 \\ 0, & (A_{21}A_{12})^p \end{pmatrix}. \tag{3.2.32}$$

Therefore the Neumann's series for $\mathscr{T}^{-1}_{\varrho,\omega'}$.

$$\mathscr{T}^{-1}_{\varrho,\omega'} = (I + K(\varrho, \omega'))^{-1} = I + \sum_{j \geqslant 1} (-K(\varrho, \omega'))^j$$

converges uniformly with respect to $(\varrho, \omega') \in [0, \varrho_0] \times \Omega_{n-1}$ if ϱ_0 is sufficiently small, which leads us immediantely to the form (3.2.30) for the inverse matrix $\mathscr{T}^{-1}_{\varrho,\omega'}$ (*). ∎

(*) Another argument which can be used in order to prove the Lemma 3.2.3 is the multiplication of $\mathscr{T}_{\varrho,\omega'}$ by $I - K(\varrho, \omega')$ and the use of (3.2.32) with $p = 1$, so that $\mathscr{T}^{-1}_{\varrho,\omega'} = I - K(\varrho, \omega') + O(\varrho^{\beta-\alpha})$, $\varrho \to 0$, with $K(\varrho, \omega)$ defined in (3.2.31).

With $\varphi = (\varphi', \varphi'')$, $\varphi' \in C^{r_2}$, $\varphi'' \in C^{r_3}$ we introduce the notations:

$$\chi' = \operatorname{diag}\left(\varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\right)_1^{r_2}\varphi', \qquad M' = \{M_K\}_1^{r_2} \in C^{r_2} \tag{3.2.33}$$
$$\chi'' = \operatorname{diag}\left(\varepsilon^{\gamma_k + m_k}\right)_{r_2+1}^{r_2+r_3}\varphi'', \qquad M'' = \{M_K\}_{r_2+1}^{r_2+r_3} \in C^{r_3}.$$

Applying the Lemma 3.2.3 (with $\alpha = m_{r_2}$, $\beta = m_{r_2+1}$) and using the notations (3.2.33) one can write down the following formula for the solution $u(t)$ to the problem (3.2.5), (3.2.6) valid for $\varrho, \omega' \in [0, \varrho_0] \times \Omega_{n-1}$ with ϱ_0 sufficiently small:

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma_1}[(\chi' + \varrho^{\beta-\alpha}B_{11}\chi')\cdot M' + \varrho^{-\alpha}B_{12}\chi''\cdot M'](Q_{01}^+)^{-1}\exp\left(i\langle\xi'\rangle t\lambda\right)d\lambda + \tag{3.2.34}$$
$$+ \frac{1}{2\pi i}\int_{\Gamma_2}[(\chi'' + \varrho^{\beta-\alpha}B_{22}\chi'')\cdot M'' + \varrho^{\beta}B_{21}\chi'\cdot M''](Q_{0\varepsilon}^+)^{-1}\exp\left(i\varepsilon^{-1}t\lambda\right)d\lambda$$

where $\alpha = m_{r_2}$, $\beta = m_{r_2+1}$ and $a\cdot b = \sum a_k b_k$, with $a, b \in C^{r_j}$, $t \in \mathcal{R}_+$.

Now our purpose is to estimate the following norms with parameters of the solution $u(t)$ given by (3.2.34):

$$\|u\|_{(s),\xi'}^{+} = \inf_{l}\|\varepsilon^{s_1}\langle\xi\rangle^{s_2}\langle\varepsilon\xi\rangle^{s_3}l\hat{u}\|_{L^2(\mathcal{R}_{\xi_n})} \tag{3.2.35}$$

where $(lu)(t)$ is an extention of $u(t)$ on the whole line $t \in \mathcal{R}$, $(l\hat{u})(\xi_n) = F_{t\to\xi_n}lu$ and the infinum is taken over all possible extention operators l.

It is well known that the norm in the right hand side of (3.2.25) is equivalent to the following one (see, for instance, [5] or [21])

$$\|u\|_{(s),\xi'}^{+} = \varepsilon^{s_1}\|\Pi^+(\xi_n + i\langle\xi'\rangle)^{s_2}(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle^{s_3})l_0\hat{u}\|_{L^2(\mathcal{R}_{\xi_n})}, \tag{3.2.36}$$

where $l_0u(t) = 0$ for $t < 0$, $l_0u(t) = u(t)$ for $t > 0$, and Π^+ is Fourier transform of the multiplication operator by the characteristic function of $\mathcal{R}_+$.

For $v \in S(\mathcal{R}_{\xi_n})$, Π^+ is Cauchy integral type operator which can be written in this case in the following fashion:

$$\Pi^+v(\xi_n) = \lim_{\delta\to+0}\frac{1}{2\pi i}\int_{\mathcal{R}}\frac{v(\eta_n)}{\xi_n - \eta_n - i\delta}d\eta_n = \frac{1}{2\pi i}\int_{\mathcal{R}}\frac{v(\eta_n)\,d\eta_n}{\xi_n - \eta_n - io}. \tag{3.2.37}$$

It projects functions from $L^2(\mathcal{R}^n_{\xi_n})$ onto the subspace of those functions in this space that can be extended ad analytic function onto the lower complex half plane $\operatorname{Im}\xi_n < 0$.

Taking the Fourier transform of $l_0u(t)$ with $u(t)$ given by (3.2.34) for $t \in \mathcal{R}_+$, we will have to compute (or to estimate) the following projections (*):

$$w_1(\xi_n) = \Pi^+(\xi_n + i\langle\xi'\rangle)^{s_2}(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3}(\xi_n - \lambda\langle\xi'\rangle)^{-1} \tag{3.2.38}$$

(*) We recall that l_0 is extension by zero for $t < 0$.

and

$$(3.2.39)\qquad w_2(\xi_n) = \Pi^+(\xi_n + i\langle\xi'\rangle)^{s_2}(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3}(\xi_n - \lambda\varepsilon^{-1})^{-1}$$

where λ is contained within Γ_1 or Γ_2 and, therefore, $\operatorname{Im}\lambda \geqslant C_0 > 0$, $|\lambda| \leqslant C$ with some positive constant C_0, C. We start with the projection (3.2.38).

For $s_2 < \frac{1}{2}$, $s_2 + s_3 < \frac{1}{2}$ the function in the right hand side of (3.2.38) under the Π^+ sign belongs to $L_2(\mathcal{R}_{\xi_n})$. Therefore,

$$(3.2.40)\qquad \|w_1\|_{L^2(\mathcal{R}_{\xi_n})} \leqslant \|(\xi_n + i\langle\xi'\rangle^{s_2})(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle^{s_3})(\xi_n - \lambda\langle\xi'\rangle)^{-1}\|_{L^2(\mathcal{R}^{\xi}_{n_n})} \leqslant$$
$$\leqslant C\langle\xi'\rangle^{s_2-\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_3} \leqslant C_1\langle\xi'\rangle^{s_2-\frac{1}{2}},$$

where C_1 does not depend on $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, $\varrho \in [0, \varrho_0]$.

Now assume that $s_2 = n_2 + \alpha_2$, $s_2 + s_3 = n_3 + \alpha_3$, where n_2, n_3 are integer $m = \max\{n_2, n_3\} > 0$, and $\alpha_j < \frac{1}{2}$.

Using the formula

$$(3.2.41)\qquad \frac{1}{\xi_n - \lambda\langle\xi'\rangle} = \sum_{0\leqslant k\leqslant m-1} \frac{(\lambda+i)^k\langle\xi'\rangle^k}{(\xi_n + i\langle\xi'\rangle)^{k+1}} + \frac{(\lambda+i)^m\langle\xi'\rangle^m}{(\xi_n + i\langle\xi'\rangle)^m}\,\frac{1}{\xi_n - \lambda\langle\xi'\rangle}$$

and noticing that

$$(3.2.42)\qquad \Pi^+(\xi_n + i\langle\xi'\rangle)^{s_2-k-1}(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3} = 0$$

for $\forall k$, s_2, s_3 (because the function in (3.2.42) is analytic in the upper half plane and, consequently, its inverse Fourier transform is a distribution whose support is contained in $\overline{\mathcal{R}}_- = \{t \leqslant 0\}$), one can write down for $w_1(\xi_n)$ the following expression:

$$(3.2.43)\qquad w_1(\xi_n) = \Pi^+(\xi_n + i\langle\xi'\rangle)^{s_1-m}(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3}(\lambda+i)^m\langle\xi'\rangle^m(\xi_n - \lambda\langle\xi'\rangle)^{-1}.$$

Given that $s_2 - m < \frac{1}{2}$, $s_2 + s_3 - m < \frac{1}{2}$, one gets again the estimate (3.2.40) for w_1, in this case, too.

Now we will estimate $w_2(\xi_n)$.

Again for $s_2 > -\frac{1}{2}$, $s_2 + s_3 < \frac{1}{2}$ the function in the right hand side of (3.2.39) (under the Π^+ sign) being in $L^2(\mathcal{R}_{\xi_n})$, one can write down:

$$(3.2.44)\qquad \|w_2\|_{L^2(\mathcal{R}_{\xi_n})} \leqslant \|(\xi_n + i\langle\xi'\rangle)^{s_2}(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3}(\xi_n - \lambda\varepsilon^{-1})^{-1}\|_{L^2(\mathcal{R}_{\xi_n})} \leqslant C\varepsilon^{\frac{1}{2}-s_2},$$

where C does not depend on $\varepsilon \in [0, \varrho_0]$, $\xi' \in \mathcal{R}^{n-1}$, $\varrho \in [0, \varrho_0]$.

If $s_2 + s_3 = m + \alpha$ with integer $m > 0$ and $\alpha < \frac{1}{2}$, then using again the formula (3.2.41) with ε^{-1} instead of $\langle\xi'\rangle$, one gets for $w_2(\xi_n)$ the same estimate (3.2.44) (*).

(*) The same argument can be used when $s_2 < -\frac{1}{2}$.

The inequalities (3.2.40), (3.2.44) and (3.2.34) yield:

$$\|u\|^{+}_{(s),\xi'} \leqslant C[\varepsilon^{-s_1}\langle\xi'\rangle^{s_2-\frac{1}{2}}(|\chi'| + \varrho^{-\alpha}|\chi''|) + \varepsilon^{\frac{1}{2}-s_1-s_2}(\varrho^{\beta}|\chi'| + |\chi''|)] \leqslant \\ \leqslant C\Big(\sum_{1\leqslant j\leqslant r_2} \varepsilon^{\gamma_j-s_1}\langle\xi'\rangle^{s_2-m_j-\frac{1}{2}}|\varphi_j| + \sum_{r_2<j\leqslant r_2+r_3} \varepsilon^{\gamma_j-\alpha+m_j}\langle\xi'\rangle^{s_2-\alpha-\frac{1}{2}}|\varphi_j| + \\ + \sum_{1\leqslant j\leqslant r_2} \varepsilon^{\gamma_j-s_1+\beta-s_2+\frac{1}{2}}\langle\xi'\rangle^{\beta-m_j}|\varphi_j| + \sum_{r_2<j\leqslant r_2+r_3} \varepsilon^{\gamma_j-s_1+m_j-s_2+\frac{1}{2}}|\varphi_j|\Big) \tag{3.2.45}$$

where the constant C does not depend on $\varepsilon\in[0, \varepsilon_0]$, $\xi'\in\mathfrak{R}^{n-1}$, $\varrho\in[0, \varrho_0]$ and u, φ_j.

For s_2 satisfying the inequalities:

$$\alpha + \tfrac{1}{2} < s_2 < \beta + \tfrac{1}{2} \tag{3.2.46}$$

the last estimate can be rewritten in a simpler equivalent form, given that the second and the third sums in the right hand side of (3.2.45) can be estimated respectively by the fourth and the first ones. In this case, when (3.2.46) holds, the estimate (3.2.45) becomes:

$$\|u\|^{+}_{(s),\xi'} \leqslant C\Big[\sum_{1\leqslant j\leqslant r_1} \varepsilon^{\gamma_j-s_1}\langle\xi'\rangle^{s_2-m_j-\frac{1}{2}}|\varphi_j| + \sum_{r_2<j\leqslant r_2+r_3} \varepsilon^{\gamma_j+m_j-s_1-s_2+\frac{1}{2}}|\varphi_j|\Big], \tag{3.2.47}$$

where the constant C does not depend on $\varepsilon\in[0, \varepsilon_0]$, $\xi'\in\mathfrak{R}^{n-1}$, $\varrho\in[0, \varrho_0]$, u and φ_j.

We have proved the following statement:

LEMMA 3.2.4. – *If $Q_0(\varepsilon, \xi)$ is elliptic principal symbol of order ν and principal symbols $b_{j0}(\varepsilon, \xi)$ of orders $\mu_j = (\gamma_j, m_j, p_j)$ satisfy the Conditions* 3.2.1, 3.2.2, *then for $\varrho = \varepsilon\langle\xi'\rangle \leqslant \varrho_0$ with ϱ_0 sufficiently small there exists a unique solution to the problem* (3.2.5), (3.2.6) *which satisfies the a priori estimate* (3.2.45). *If in addition s_2 satisfies* (3.2.46), *then for the solution $u(t)$ to the problem* (3.2.5), (3.2.6) *the a priori estimation* (3.2.47) *holds.*

REMARK 3.2.5. – Given that $1 \leqslant \langle\varepsilon\xi'\rangle^2 \leqslant 1+\varrho_0^2$, one can rewrite the estimate (3.2.47) in the following equivalent form:

$$\|u\|^{+}_{(s),\xi'} \leqslant C\varepsilon^{-s_1}\Big[\sum_{1\leqslant j\leqslant r_2} \varepsilon^{\gamma_j}\langle\xi'\rangle^{s_2-m_j-\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_3-p_j}|\varphi_j| + \\ + \sum_{r_2<j\leqslant r_2+r_3} \varepsilon^{\gamma_j+m_j-s_2+\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_2+s_3-m_j-p_j-\frac{1}{2}}|\varphi_j|\Big]. \tag{3.2.48}$$

Now we will investigate the case when $\varrho \geqslant \varrho_0$ where ϱ_0 is the positive number fixed above in the Lemma 3.2.4.

In this case the roots of the first and second type to the polynomial $\lambda \to Q_0^+(\varrho, \omega', \lambda)$ cannot be separated from each others and the solution to the problem (3.2.5), (3.2.6) has to be sought in the form (3.2.7) with $\varrho \geqslant \varrho_0 > 0$.

More precisely, according to (3.1.18) $Q_0^+(\varrho, \omega', \lambda)$ grows like ϱ^{r_3} as $\varrho \to \infty$. Therefore, we seek the solution $u(t)$ to the problem (3.2.5), (3.2.6) in the form

$$u(t) = \frac{1}{2\pi i} \int_\Gamma \frac{M(\varepsilon, \varrho, \omega', \lambda)}{\varrho^{-r_3} Q_0^+(\varrho, \omega', \lambda)} \exp\left(it\langle\xi'\rangle\lambda\right) d\lambda$$

where the polynomial $\lambda \to M(\varepsilon, \varrho, \omega', \lambda)$ has to be found, and the contour Γ does not depend on $\varrho \in [\varrho_0, +\infty]$, $\omega' \in \Omega_{n-1}$.

The roots of the polynomial $Q_0^+(\varrho, \omega', \lambda)$ being of the form (Lemma 3.1.1)

$$\lambda_j(\varrho, \omega') = \varrho^{-1}\lambda_j(\varrho\omega') = \lambda_{j00}(\omega') + O(\varrho^{-\gamma}), \quad \varrho \to +\infty \tag{3.2.49}$$

with some $\gamma > 0$, the contour Γ can be chosen independently on $\varrho \in [\varrho_0, \infty]$, $\omega' \in \Omega_{n-1}$.

It turns out from (3.1.18) that there exists the limit

$$\lim_{\varrho \to +\infty} \varrho^{-r_3} Q_0^+(\varrho, \omega', \lambda) = Q_{00}^+(\omega', \lambda), \tag{3.2.50}$$

where $\lambda \to Q_{00}^+(\omega', \lambda)$ is the polynomial in the factorization for $\lambda \to Q_{00}(\omega', \lambda)$, which corresponds to the roots of $Q_{00}(\omega', \lambda)$, contained in the upper half of the complex λ-plane, when $\omega' \in \Omega_{n-1}$.

Denoting by $b_{j00}(\xi', \lambda)$ the homogeneous symbol of $b_{j0}(1, \xi', \lambda)$ of the highest order $m_j + p_j$ one gets immediately

$$\lim_{\varrho \to +\infty} \varrho^{\gamma_j - p_j} b_{j0}(\varrho, \omega', \lambda) = b_{j00}(\omega', \lambda), \quad 1 \leqslant j \leqslant r_2 + r_3.$$

We restrict the class of boundary operators in (3.2.6) by putting the following conditions:

CONDITION 3.2.6. – *The polynomials*

$$\lambda \to b_{j0}(\varrho, \omega', \lambda), \quad 1 \leqslant j \leqslant r_2 + r_3 \tag{3.2.51}$$

are supposed to be linearly independent modulo $Q_0^+(\varrho, \omega', \lambda)$, $\varrho \in (0, +\infty)$, $\forall \omega' \in \Omega_{n-1}$.

CONDITION 3.2.7. – *The polynomials*

$$\lambda \to b_{j00}(\omega', \lambda), \quad 1 \leqslant j \leqslant r_2 + r_3 \tag{3.2.52}$$

are supposed to be linearly independent modulo $Q_{00}^+(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$.

As a consequence of the conditions 3.2.6, 3.2.7, there exist poltnomials

$$\lambda \to M_j(\varrho, \omega', \lambda), \quad 1 \leqslant j \leqslant r_2 + r_3 \tag{3.2.53}$$

of order $< r_2 + r_3$, which depend continuously on $\varrho \in [\varrho_0, +\infty)$, $\omega' \in \Omega_{n-1}$ and have a finite limit when $\varrho \to +\infty$, the latter being continuous functions of $\omega' \in \Omega_{n-1}$ such that the orthogonality conditions are fulfilled uniformly with respect to $\varrho \in [\varrho_0, +\infty]$, $\omega' \in \Omega_n$:

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{\varrho^{\gamma_j - \nu_j} b_{j0}(\varrho, \omega', \lambda)\, M_k(\varrho, \omega', \lambda)}{\varrho^{-r_3} Q_0^+(\varrho, \omega', \lambda)}\, d\lambda = \delta_{kj}\,, \quad 1 \leqslant k,\ j \leqslant r_2 + r_3\,, \tag{3.2.54}$$

$\forall \omega' \in \Omega_{n-1}$, $\forall \varrho \in [0, +\infty]$.

REMARK 3.2.8. – As it was mentioned above, one has

$$\lim_{\varrho \to +\infty} \varrho^{-\nu_k + \gamma_k} b_{k0}(\varrho, \omega', \lambda) = b_{k00}(\omega', \lambda)\,. \tag{3.2.55}$$

Therefore, for $\varrho = +\infty$ the orthogonality condition is this one for the polynomials (3.2.52).

We seek the solution $u(t)$ to the problem (3.2.5), (3.2.6) in the form:

$$u(t) = \sum_{1 \leqslant j \leqslant r_2 + r_3} \psi_j \frac{1}{2\pi i}\int_{\Gamma} \frac{M_j(\varrho, \omega', \lambda)}{\varrho^{-r_3} Q_0^+(\varrho, \omega', \lambda)} \exp\left(it\langle\xi'\rangle\lambda\right) d\lambda \tag{3.2.56}$$

where ψ_j have to be found from the boundary conditions.

Substituting (3.2.56) into (3.2.6) one gets using the orthogonality conditions (3.2.54) and the homogeneity of $\varepsilon^{\gamma_k} b_{k0}(\varepsilon, \xi', \lambda)$ in $(\varepsilon^{-1}, \xi', \lambda)$ of order m_k:

$$\psi_k = \varepsilon^{\gamma_k}\langle\xi'\rangle^{-m_k}\varrho^{-\nu_k}\varphi_k\,, \quad 1 \leqslant k \leqslant r_2 + r_3\,. \tag{3.2.57}$$

Hence, the formula for $u(t)$ takes the form:

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\psi \cdot M}{\varrho^{-r_3} Q_0^+} \exp\left(it\langle\xi'\rangle\lambda\right) d\lambda \tag{3.2.58}$$

with vectors $\psi = \{\psi_j\}$ and $M = \{M_j\}$ defined by (3.2.57), (3.2.54). Besides, $M(\varrho, \omega')$ depends continuously on $\varrho, \omega' \in [\varrho_0, +\infty) \times \Omega_{n-1}$ and is uniformly bounded for $\varrho, \omega' \in [\varrho_0, +\infty] \times \Omega_{n-1}$.

We stress again that the contour Γ does not depend on $\varrho \in [\varrho_0, +\infty]$ and $\omega' \in \Omega_{n-1}$.

Estimating the norm $\|u\|^+_{(s),\xi'}$ as we did it before for $w_1(\xi_n)$, when we had $\varrho \in [0, \varrho_0]$, one finds out

$$\|u\|^+_{(s),\xi'} \leqslant C\varepsilon^{-s_1}\langle\xi'\rangle^{s_2 - \frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_3}|\psi| \leqslant C \sum_{1 \leqslant j \leqslant r_2 + r_3} \varepsilon^{\gamma_j - s_1}\langle\xi'\rangle^{s_2 - m_j - \frac{1}{2}}\langle\varepsilon\xi\rangle^{s_3 - \nu_j}|\varphi_j| \tag{3.2.59}$$

with a constant C which does not depend on $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, $\varrho \geqslant \varrho_0 > 0$ with ϱ_0 fixed above.

Given that for $\varrho = \varepsilon\langle\xi'\rangle \geqslant \varrho_0 > 0$ and $\varepsilon \in (0, \varepsilon_0]$, $(\varepsilon_0 \leqslant 1)$, there is the equivalency

$$\langle\xi'\rangle^2 \leqslant \varepsilon^{-2}\langle\varepsilon\xi'\rangle^2 \leqslant (1 + \varrho_0^{-2})\langle\xi'\rangle^2 \tag{3.2.60}$$

one can rewrite the estimate (3.2.59) in the following fashion:

$$\|u\|^+_{(s),\xi'} \leqslant C\Big(\sum_{1\leqslant j\leqslant r_2} [\varphi]_{(\tau_j),\xi'} + \sum_{r_2<j\leqslant r_2+r_3} [\varphi_j]_{(\sigma_j),\xi'}\Big) \tag{3.2.61}$$

with τ_j, $\sigma_j \in \mathcal{R}^3$,

$$\tau_j = s - \mu_j - \tfrac{1}{2}e_2\,, \qquad e_2 = (0, 1, 0)$$

$$\sigma_j = \tau_j + (s_2 - m_j - \tfrac{1}{2})\,e\,, \qquad e = (1, -1, 1)$$

and the following notation:

$$[\varphi]_{(\sigma),\xi'} = \varepsilon^{-\sigma_1}\langle\xi'\rangle^{\sigma_2}\langle\varepsilon\xi'\rangle^{\sigma_3}|\varphi|\,, \qquad \sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathcal{R}^3. \tag{3.2.62}$$

Therefore we have proved the following statement:

LEMMA 3.2.9. – *$Q_0(\varepsilon, \xi)$ being elliptic symbol of order ν and principal symbols $b_{j0}(\varepsilon, \xi)$ of orders $\mu_j = (\gamma_j, m_j, p_j)$ satisfying the conditions* (3.2.6), (3.2.7), *there exists for $\varrho \geqslant \varrho_0 > 0$ unique solution $u(t)$ to the problem* (3.2.5), (3.2.6) *which satisfies the a priori estimate* (3.2.61) *with a constant C which might depend on ϱ_0 (fixed above in the* Lemma 3.2.4) *but not upon $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, φ_j and u.*

LEMMA 3.2.10. – *Both solutions to the problem* (3.2.5), (3.2.6), *constructed for $0 < \varrho \leqslant \varrho_0$ and $\varrho \geqslant \varrho_0$ coincide for $\varrho = \varrho_0$ and are continuous functions of the parameters $\varrho > 0$, $\xi' \in \mathcal{R}^{n-1}$, $\varepsilon \in (0, \varepsilon_0]$.*

PROOF. – Immediately follows from the uniquess of the solution to the problem (3.2.6), (3.2.7) for $\varrho \leqslant \varrho_0$ and $\varrho \geqslant \varrho_0$, and from the theorem on continuous dependence of solutions to ordinary differential equations on parameters. ■

Therefore, when $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$ with $\alpha = m_{r_2}$, $\beta = m_{r_2+1}$, the a priori estimate (3.2.61) holds for the solutions to the ptoblem (3.2.5), (3.2.6) uniformly with respect to the parameters $\varrho \in [0, +\infty]$, $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$ given that for $\varrho \leqslant \varrho_0$ the estimate (3.2.61) coincides with (3.2.48).

Now let $s_2 \geqslant \beta + \frac{1}{2}$, $\beta = m_{r_2+1}$. Then the estimate (3.2.45) can be rewritten in the following equivalent form:

$$\|u\|^+_{(s),\xi'} \leqslant C\Big(\sum_{1\leqslant j\leqslant r_2} \varepsilon^{\gamma_j - s_1 + \beta - s_2 + \frac{1}{2}}\langle\xi'\rangle^{\beta - m_j}|\varphi_j| + \\ + \sum_{r_2<j\leqslant r_2+r_3} \varepsilon^{\gamma_j - s_1 + m_j - s_2 + \frac{1}{2}}|\varphi_j|\Big)\,, \qquad 0 \leqslant \varrho \leqslant \varrho_0\,, \tag{3.2.63}$$

given that in this case holds:

$$\varepsilon^{\gamma_j - s_1}\langle\xi'\rangle^{s_2 - m_j - \frac{1}{2}} \leqslant \varrho_0^{s_2-\beta-\frac{1}{2}}\,\varepsilon^{\gamma_j - s_1+\beta - s_2 + \frac{1}{2}}\langle\xi'\rangle^{\beta - m_j}. \tag{3.2.64}$$

On the other hand the estimate (3.2.59) for $\varrho \geqslant \varrho_0$ can be rewritten in the following way:

$$\|u\|^{+}_{(s),\xi'} \leqslant C\Big(\sum_{1\leqslant j\leqslant r_2}[\varphi_j]_{(\varrho_j),\xi'} + \sum_{r_2<j\leqslant r_2+r_3}[\varphi_j]_{(\sigma_j),\xi'}\Big) \tag{3.2.65}$$

with

$$\varrho_j = \tau_j + (s_2 - \beta - \tfrac{1}{2})e\,, \qquad e = (1, -1, 1)\,,$$

the vector subscript τ_j, σ_j, e being the same as in (3.2.61) and the norms $[\varphi]_{(\sigma),\xi'}$ defined by (3.2.62).

Given that for $\varrho \in [0, \varrho_0]$ the factor $\langle\varepsilon\xi'\rangle^s$ for any $s \in \mathcal{R}$ is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, one gets the conclusion that (3.2.65) holds for any $\varrho \in [0, +\infty]$ with a constant C which does not depend on $\varepsilon \in [0, \varepsilon_0]$, $\varrho \in [0, +\infty]$, $\xi' \in \mathcal{R}^{n-1}$, u and φ_j, the constant ϱ_0 being fixed in the Lemma 3.2.4.

Let $s_2 \leqslant \alpha + \frac{1}{2}$, $\alpha = m_{r_2}$. Then the estimate (3.2.45) can be rewritten in the following equivalent form:

$$\|u\|^{+}_{(s),\xi'} \leqslant C\Big[\sum_{1\leqslant j\leqslant r_2}\varepsilon^{\gamma_j - s_1}\langle\xi'\rangle^{s_2 - m_j - \frac{1}{2}}|\varphi_j| + \sum_{r_2<j\leqslant r_2+r_3}\varepsilon^{\gamma_j - s_1 + m_j - \alpha}\langle\xi'\rangle^{s_2-\alpha-\frac{1}{2}}|\varphi_j|\Big]\,, \qquad 0 \leqslant \varrho \leqslant \varrho_0\,, \tag{3.2.66}$$

given that in this case holds:

$$\varepsilon^{\gamma_j - s_1 + m_j - s_2 + \frac{1}{2}} \leqslant \varrho_0^{\alpha - s_2 + \frac{1}{2}}\,\varepsilon^{\gamma_j - s_1 + m_j - \alpha}\langle\xi'\rangle^{s_2 - \alpha - \frac{1}{2}}. \tag{3.2.67}$$

Rewriting the estimate (3.2.59) in the convenient form one gets:

$$\|u\|^{+}_{(s),\xi'} \leqslant C\Big(\sum_{1\leqslant j\leqslant r_2}[\varphi_j]_{(\tau_j),\xi'} + \sum_{r_2<j\leqslant r_2+r_3}[\varphi_j]_{(\delta_j),\xi'}\Big) \tag{3.2.68}$$

with

$$\delta_j = \sigma_j + (\alpha - s_2 + \tfrac{1}{2})\,e\,,$$

the vector subscripts τ_j, σ_j, e being the same as in (3.2.61) and the norm $[\varphi]_{(\sigma),\xi'}$ defined by (3.2.62).

Again the constant C in the right hand side of (3.2.68) does not depend on $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, u, φ_j, the constant $\varrho_0 \geqslant 0$ being fixed in the Lemma 3.2.4.

We will combine the conditions 3.2.1, 3.2.2, 3.2.6, 3.2.7 and later call them the coerciveness condition.

COERCIVENESS CONDITION:

(i) *The polynomials*

$$\lambda \to b_{j0}^0(\omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_2$$

which are homogeneous symbols of (the highest) order m_j *to the reduced boundary conditions, are linearly independent modulo* $Q_0^{0+}(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$, *where* Q_0^{0+} *is the factor in the factorization of the reduced symbol* Q_0^0 *which corresponds to the roots of* Q_0^0 *contained in the upper half of the complex* λ*-plane, when* $\omega' \in \Omega_{n-1}$.

(ii) *The polynomials*

$$\lambda \to b_j(\lambda)\,, \qquad r_2 < j \leqslant r_2 + r_3\,,$$

$b_j(\lambda) = b_{j0}(1, 0, \lambda)$, *are linearly independent modulo* $Q^+(\lambda)$, *where* $Q^+(\lambda)$ *is the factor of the polynomial* $\lambda^{-\nu_2}Q_0(1, 0, \lambda)$ *which corresponds to the roots of* $\lambda^{-\nu_2}Q_0(1, 0, \lambda)$ *contained in the upper half of the complex* λ*-plane.*

(iii) *The polynomials*

$$\lambda \to b_{j00}(\omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_2 + r_3$$

which are homogeneous parts of the highest order $m_j + p_j$ *of the polynomials* $b_{j0}(1, \omega', \lambda)$ *are linearly independent modulo* $Q_{00}^+(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$, *where* $Q_{00}^+(\omega', \lambda)$ *is the factor of the homogeneous polynomial* $Q_{00}(\omega', \lambda)$ *of the highest order* $\nu_2 + \nu_3$ *to the polynomial* $Q_0(\omega', \lambda)$, *which corresponds to the roots of* $Q_{00}(\omega', \lambda)$ *contained in the upper half of the complex* λ*-plane.*

(iv) *The polynomials*

$$\lambda \to b_{j0}(\varrho, \omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_2 + r_3\,,$$

are linearly independent modulo $Q_0^+(\varrho, \omega', \lambda)$, $\forall \varrho \in (0, +\infty)$, $\forall \omega' \in \Omega_{n-1}$, *where* $Q_0^+(\varrho, \omega', \lambda)$ *is the factor of* $Q_0(\varrho, \omega', \lambda)$ *which corresponds to the roots of* Q_0 *contained in the upper half of the complex* λ*-plane.*

Finally, we have proved the following main statement:

THEOREM 3.2.11. – *The polynomial* $Q_0(\varepsilon, \xi)$ *being elliptic of order* ν *and the polynomials* $b_{j0}(\varepsilon, \xi')$, *of orders* μ_j, $1 \leqslant j \leqslant r_2 + r_3$, *satisfying the coerciveness condition* (i)-(iv), *there exists unique solution to the singularly perturbed boundary value problem* (3.2.5), (3.2.6), *which satisfies alternately the a priori estimate* (3.2.61) *when* $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$, *the a priori estimate* (3.2.65) *when* $s_2 \geqslant \beta + \frac{1}{2}$, *and the a priori estimate* (3.2.68) *when* $s_2 \leqslant \alpha + \frac{1}{2}$, *with a constant* C *which does not depend on* $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathfrak{R}^{n-1}$ *and* φ_j, u.

Now our aim is to estimate the norm $\|\cdot\|^+_{(s),\xi'}$ of the solution $u(t)$ to the problem (3.2.3), (3.2.4) with second member $f(t)$ which will be supposed to have a finite norm $\|\cdot\|^+_{(s-\nu),\xi'}$, the norms $\|\cdot\|^+_{(s),\xi'}$ being defined by (3.2.35) (or, equivalently, by (3.2.36)).

We have to put some restrictions on s_2, s_3 for the traces of $u(t)$ and its derivatives up to the highest order in the boundary conditions (3.2.4) at the point $t=0$ to exist.

From now on we assume that the following condition is fulfilled:

$$s_2+s_3>\max_{1\leq j\leq r_2+r_3}\{m_j+p_j\}+\tfrac{1}{2}\,. \tag{3.2.69}$$

According to the definition (3.2.35), there exists an extention $lf(t)$ for $f(t)$ with the finite norm $\|\cdot\|_{(s),\xi'}$,

$$\|lf\|_{(s),\xi'}=\|\varepsilon^{-s_1}\langle\xi\rangle^{s_2}\langle\varepsilon\xi'\rangle\hat{lf}\|_{L^2(\mathcal{R}_{\xi_n})} \tag{3.2.70}$$

where $\hat{lf}=F_{t\to\xi_n}lf$.

We define $v(t)$ to be

$$v(t)=F^{-1}_{\xi_n\to t}Q_0^{-1}(\varepsilon,\xi',\xi_n)F_{t\to\xi_n}lf \tag{3.2.71}$$

so that $v(t)$ is the solution of the equation (3.2.3), such that

$$\|v\|^+_{(s),\xi'}\leq\|v\|_{(s),\xi'}\leq C\|lf\|_{s-\nu,\xi'}\leq C_1\|f\|^+_{s-\nu,\xi'} \tag{3.2.72}$$

with some constant C_1 which does not depend on $\varepsilon\in[0,\varepsilon_0]$, $\xi'\in\mathcal{R}^{n-1}$.

Of course, (3.2.72) holds iff $Q_0(\varepsilon,\xi)$ is elliptic of order ν.

Looking for the solution to the problem (3.2.3), (3.2.4) in the form:

$$u(t)=v(t)+w(t)\,, \tag{3.2.73}$$

and denoting

$$h_j(t)=b_{j0}(\varepsilon,\xi',D_t)v(t)\,, \tag{3.2.74}$$

we will have to estimate $\pi_0(h_j(t))$ multiplied by suitable function of ε, ξ'.

First let $\alpha+\frac{1}{2}<s_2<\beta+\frac{1}{2}$. Using (1.2.10) (with $p=1$ and $s-\mu_j$ instead of s one finds out for $1\leq j\leq r_2$:

$$[\pi_0(h_j)]_{(\tau_j),\xi'}=\varepsilon^{\gamma_j-s_1}\langle\xi'\rangle^{s_2-m_j-\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_3-p_j}|\pi_0(h_j)|\leq C\|h_j\|_{(s-\mu_j),\xi'}\leq C\|v\|_{(s),\xi'}\,. \tag{3.2.75}$$

Now using (1.2.13) (with $s-\mu_j$ instead of s), one gets for $r_2<j\leq r_2+r_3$:

$$[\pi_0(h_j)]_{(\sigma_j),\xi'}=\varepsilon^{\gamma_j+m_j-s_1-s_2+\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_2+s_3-m_j-p_j-\frac{1}{2}}|\pi_0(h_j)|\leq C\|h_j\|_{(s-\mu_j),\xi'}\leq C\|v\|_{(s),\xi'}. \tag{3.2.76}$$

Consequently, assuming (3.2.46), we obtained for the solution $u(t)$ to the problem (3.2.3), (3.2.4) the a priori estimate:

$$\|u\|^{+}_{(s),\xi'} \leqslant C\Big(\|f\|^{+}_{(s-\nu),\xi'} + \sum_{1\leqslant j\leqslant r_2} [\varphi_j]_{(\tau_j),\xi'} + \sum_{r_2<j\leqslant r_2+r_3} [\varphi_j]_{(\sigma_j),\xi'}\Big)\,, \tag{3.2.77}$$

with $\tau_j, \sigma_j \in \mathfrak{R}^3$ the same as in (3.2.61) and a constant C which does not depend on $\varepsilon\in[0,\varepsilon_0]$, $\xi'\in\mathfrak{R}^{n-1}$, u and φ_j.

Now, let $s_2\geqslant\beta+\frac{1}{2}$. Given that $\beta-m_j>0$ for $1\leqslant j\leqslant r_2$, one gets using (1.2.10) (with $p=1$):

$$[\pi_0(h_j)]_{(\varrho_j),\xi} = \varepsilon'^{-s_1+\gamma_j-s_2+\beta+\frac{1}{2}}\langle\xi'\rangle^{\beta-m_j}\langle\varepsilon\xi'\rangle^{s_2+s_3-\nu_j-\beta-\frac{1}{2}}|\pi_0(h_j)| \leqslant$$
$$\leqslant C\|h_j\|_{(\varrho_j+\frac{1}{2}e_2),\xi'} \leqslant C\|v\|_{(s+\Delta_\beta e),\xi'}\,. \tag{3.2.78}$$

where we denoted:

$$\Delta_\beta = (s_2-\beta-\tfrac{1}{2})\,. \tag{3.2.79}$$

We use here the fact that the order of b_{j0} is μ_j and that $\varrho_j+\mu_j+\frac{1}{2}e_2 = s+\Delta_\beta e$.
Using again (1.2.13) (with $s_2=0$) one gets for $r_2+1<j\leqslant r_2+r_3$:

$$[\pi_0(h_j)]_{(\sigma_j),\xi'} \leqslant C\|\varepsilon^{-s_1-s_2+\gamma_j+m_j}\langle\varepsilon\xi\rangle^{s_2+s_3-\nu_j-m_j}\hat{h}_j\|_{L^2(\mathfrak{R}_{\xi_n})} \leqslant$$
$$\leqslant C\|\varepsilon^{-s_1-s_2+m_j}\langle\xi\rangle^{m_j}\langle\varepsilon\xi\rangle^{s_2+s_3-m_j}\hat{v}\|_{L^2(\mathfrak{R}_{\xi_n})} =$$
$$= C\|(\varepsilon\langle\xi\rangle)^{m_j-\beta-\frac{1}{2}}\varepsilon^{-s_1-s_2+\beta+\frac{1}{2}}\langle\xi\rangle^{\beta+\frac{1}{2}}\langle\varepsilon\xi\rangle^{s_2+s_3-m_j}\hat{v}\|_{L^2(\mathfrak{R}_{\xi_n})} \leqslant$$
$$\leqslant C\|\varepsilon^{-s_1-s_2+\beta+\frac{1}{2}}\langle\xi\rangle^{\beta+\frac{1}{2}}\langle\varepsilon\xi\rangle^{s_2+s_3-\beta-\frac{1}{2}}\hat{v}\|_{L^2(\mathfrak{R}^{\xi}_{\varrho_n})} = C\|v\|_{(s+\Delta e),\xi'}\,. \tag{3.2.80}$$

Consider now the term $j=r_2+1$, $m_j=\beta$.
Using (1.2.4), one can write in this case: (with $q=\varepsilon\langle\xi'\rangle\langle\varepsilon\xi'\rangle^{-1}$)

$$[\pi_0(h_j)]_{(\sigma_j),\xi'} \leqslant C(1+|\ln q|)\,\|\varepsilon^{-s_1-s_2+\beta+\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}}\langle\varepsilon\xi\rangle^{s_2+s_3-\beta-\nu_j-\frac{1}{2}}\hat{h}_j\|_{L^2(\mathfrak{R}_{\xi_n})} \leqslant$$
$$\leqslant C(1+|\ln q|)\,\|\varepsilon^{-s_1-s_2+\beta+\frac{1}{2}}\langle\xi\rangle^{\beta+\frac{1}{2}}\langle\varepsilon\xi\rangle^{s_2+s_3-\beta-\frac{1}{2}}\hat{v}\|_{L^2(\mathfrak{R}_{\xi_n})} = C(1+|\ln q|)\,\|v\|_{(s+\Delta e),\xi'}\,. \tag{3.2.81}$$

Therefore, for $s_2\geqslant\beta+\frac{1}{2}$, the following a priori estimate holds for the solutions to the problem (3.2.3), (3.2.4):

$$\|u\|^{+}_{(s)} \leqslant C\Big[(1+|\ln q|)^{\frac{1}{2}}\|f\|^{+}_{(s-\nu+\Delta_\beta e),\xi'} + \sum_{1\leqslant j\leqslant r_2}[\varphi_j]_{(\tau_j+\Delta_\beta e),\xi,} + \sum_{r_2<j\leqslant r_2+r_3}[\varphi_j]_{(\sigma_j),\xi'}\Big] \tag{3.2.82}$$

with $\Delta_\beta=(s_2-\beta-\frac{1}{2})$, $e=(1,-1,1)$, $q=\varepsilon\langle\xi'\rangle\langle\varepsilon\xi'\rangle^{-1}$.
We point out that $s+\Delta_\beta l > s$ if $\Delta_\beta\neq 0$.
Finally, consider the case $s_2<\alpha+\frac{1}{2}$.

Assume first that $s_2 - m_j - \frac{1}{2} > 0$ and estimate the corresponding term. We will have, using (1.2.10) (with $(p = 1)$:

$$[\pi_0(h_j)]_{(\tau_j),\xi'} \leqslant C\|h_j\|_{(s-\mu_j),\xi'} \leqslant C\|v\|_{(s),\xi'}. \tag{3.2.83}$$

Now, let $s_2 - m_j - \frac{1}{2} < 0$ and $1 \leqslant j \leqslant r_2$. We can write in this case:

$$[\pi_0(h_j)]_{(\tau_j),\xi'} = \varepsilon^{-s_1+\gamma_j}\langle\xi'\rangle^{s_2-m_j-\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_3-p_j}|\pi_0(h_j)| \leqslant \\ \leqslant \varepsilon^{-s_1+\gamma}\langle\varepsilon\xi'\rangle^{s_2+s_3-m-p-\frac{1}{2}}|\pi_0(h_j)|. \tag{3.2.84}$$

Using (1.2.13) with $s - \mu_j$ instead of s, the last inequality becomes:

$$[\pi_0(h_j)]_{(\tau_j),\xi'} \leqslant C\varepsilon^{s_2-m_j-\frac{1}{2}}\|h_j\|_{(s-\mu_j),\xi'} \leqslant C\varepsilon^{s_2-m_j-\frac{1}{2}}\|v\|_{(s),\xi'} \leqslant C\|v\|_{(s+\Delta_\alpha e_1),\xi'}, \tag{3.2.85}$$

where we denoted:

$$\Delta_\alpha = (\alpha + \tfrac{1}{2} - s_2), \qquad e_1 = (1, 0, 0). \tag{3.2.86}$$

If $s_2 - m_j - \frac{1}{2} < 0$ and $r_2 < j \leqslant r_2 + r_3$, we can write:

$$[\pi_0(h_j)]_{(\delta_j),\xi'} = \varepsilon^{-s_1+\gamma_j+m_j-\alpha}\langle\xi'\rangle^{s_2-\alpha-\frac{1}{2}}\langle\varepsilon\xi'\rangle^{s_3-p_j-m_j+\alpha}|\pi_0(h_j)| \leqslant \\ \leqslant \varepsilon^{-s_1+\gamma_j+m_j-\alpha}\langle\varepsilon\xi'\rangle^{s_2+s_3-m_j-p_j-\frac{1}{2}}|\pi_0(h_j)|. \tag{3.2.87}$$

Using again (1.2.13) with $s - \mu_j$ instead of s, one gets:

$$[\pi_0(h_j)]_{(\delta_j),\xi'} \leqslant C\varepsilon^{s_2-\alpha-\frac{1}{2}}\|h_j\|_{(s-\mu_j),\xi'} \leqslant C\varepsilon^{s_2-\alpha-\frac{1}{2}}\|v\|_{(s),\xi'} = C\|v\|_{(s+\Delta_\alpha e_1),\xi'}, \tag{3.2.88}$$

with Δ_α defined in (3.2.86).

It is easy to check that $\delta_j = \sigma_j + \Delta_\alpha e$.

Therefore for $s_2 < \alpha + \frac{1}{2}$, the following a priori estimate holds for the solution to the problem (3.2.3), (3.2.4):

$$\|u\|^+_{(s),\xi'} \leqslant C\Big(\|f\|^+_{(s-\nu+\Delta_\alpha e_1),\xi'} + \sum_{1\leqslant j\leqslant r_2}[\varphi_j]_{(\tau_j),\xi'} + \sum_{r_2<j\leqslant r_2+r_3}[\varphi_j]_{(\sigma_j+\Delta_\alpha),\xi'}\Big) \tag{3.2.89}$$

with τ_j, σ_j the same as in (3.2.61) and Δ_α defined in (3.2.86). Of course, the constant C does not depend on $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathfrak{R}^{n-1}$, u, f and φ_j.

We have proved the following main result:

THEOREM 3.2.12. – *The polynomial $Q_0(\varepsilon, \xi)$ being elliptic of order ν and the polynomials $b_{j0}(\varepsilon, \xi')$ of orders μ_j, $1 \leqslant j \leqslant r_2+r_3$, satisfying the coerciveness condition* (i)-(iv), *there exists unique solution to the problem* (3.2.3), (3.2.4), *which satisfies alternately* *the a priori estimate* (3.2.77) *when* $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$, *the a priori estimate* (3.2.82) *when* $s_2 \geq \beta + \frac{1}{2}$, *and the a priori estimate* (3.2.89) *when* $s_2 < \alpha + \frac{1}{2}$, *with a constant* C *which does not depend on* $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$, f, φ_j *and* u.

REMARK 3.2.13. – When $s_2 = \alpha + \frac{1}{2}$, there is a logarithmic coefficient which multiplies $\|f\|^+_{(s-\nu+\Delta e_1),\xi'}$ alike to the corresponding term in (3.2.82). By the way, the assumption that $s_2 > \beta + \frac{1}{2}$ does not help to remove the logarithmic coefficient in (3.2.82).

REMARK 3.2.14. – Taking $\varepsilon = 1$ in (3.2.77), (3.2.82), (3.2.89) one gets, after the integration with respect to $\xi' \in \mathcal{R}^{n-1}$ the usual a priori estimates for coercive elliptic boundary value problems without small parameter in usual Sobolev spaces. Indeed, for the vectoral subscripts τ_j, σ_j, ϱ_j, δ_j, the sum of the two last coordinates equals:

$$[\tau_j] = [\sigma_j] = [\varrho_j] = [\delta_j] = s_2 + s_3 - m_j - p_j - \tfrac{1}{2}\,, \qquad 1 \leq j \leq r_2 + r_3 \tag{3.2.90}$$

while the sum of the two last coordinate for $s - \nu$, $s - \nu + \Delta_\beta$, $s - \nu + \Delta_\alpha$ equals:

$$[s - \nu] = [s - \nu + \Delta_\beta e] = [s - \nu + \Delta_\alpha e\,] = s_2 + s_3 - \nu_2 - \nu_3\,. \tag{3.2.91}$$

REMARK 3.2.15. – Assume again that the functions of $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathcal{R}^{n-1}$ in the right hand side of (3.2.77), (3.2.82), (3.2.89) are contained in $L^\infty([0, \varepsilon_0]; L^2(\mathcal{R}^{n-1}_{\xi'}))$.
Let $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$. Denote

$$\begin{aligned} \psi_j &= \varepsilon^{\nu_j - s_1}\langle\varepsilon\xi'\rangle^{s_2-\nu_j}\varphi_j\,, \qquad 1 \leq j \leq r_2\,,\\ \hat{g} &= \varepsilon^{\nu^1 - s^1}\Pi^+(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3-\nu_3} l\hat{f}\,,\\ \hat{v} &= \varepsilon^{-s_1}\Pi^+(\varepsilon\xi_n + i\langle\varepsilon\xi'\rangle)^{s_3} lu\,, \end{aligned} \tag{3.2.92}$$

and assume that there exist the limits

$$\begin{aligned} &\lim_{\varepsilon\to 0}\,[\psi_j - \psi_j^0]_{(0,s_2-m_j-\frac{1}{2},0),\xi'} = 0\,, \qquad 1 \leq j \leq r_2\\ &\lim_{\varepsilon\to 0}\,[\varphi_j]_{(\sigma_j),\xi'} = 0\,, \qquad r_2 < j \leq r_2 + r_3\\ &\lim_{\varepsilon\to 0}\,\|g - g^0\|^+_{(0,s_2-\nu_2,0),\xi'} = 0\,,\\ &\lim_{\varepsilon\to 0}\,\|v - v^0\|^+_{(0,s_2,0)} = 0\,, \end{aligned} \tag{3.2.93}$$

where $\psi_j^0 \in \tilde{H}_{s_2-m_j-\frac{1}{2}}(\mathcal{R}^{n-1}_{\xi'})$, $1 \leq j \leq r_2$, $\hat{g}^0 \in \hat{H}_{s_2-\nu_2}(\mathcal{R}^n_\xi)$, $\hat{v}^0 \in \hat{H}_{s_2}(\mathcal{R}^n_\xi)$, $\tilde{H}_l(\mathcal{R}^{n-1}_{\xi'})$, $\hat{H}_p(\mathcal{R}^n_\xi)$ being the spaces of Fourier transform of functions respectively from the usual Sobolev spaces $H_l(\mathcal{R}^{n-1}_{x'})$, $H_p(\mathcal{R}^n_x)$.

Then for $\varepsilon \to 0$ the estimate (3.2.77) becomes (after integrating with respect to $\xi' \in \mathcal{R}^{n-1}$ and applying the Parceval's identity) the usual a priori estimate for v^0, solution to the reduced coercive elliptic boundary value problem, in usual Sobolev spaces.

If $s_2 \geqslant \beta + \frac{1}{2}$, then introducing

$$\begin{aligned}
(3.2.94)\qquad \psi_j &= \varepsilon^{-s_1+\gamma_j-s_2+\beta+\frac{1}{2}} \langle \xi' \rangle^{\beta-s_2+\frac{1}{2}} \langle \varepsilon\xi' \rangle^{s_2+s_3-\nu_j-\beta-\frac{1}{2}} \varphi_j\,, \qquad 1 \leqslant j \leqslant r_2 \\
\hat{g} &= \varepsilon^{j\nu_1-s_1-s_2+\beta+\frac{1}{2}} (1+|\ln q|)^{\frac{1}{2}} \Pi^+(\xi_n + i\langle \xi' \rangle^{\beta-s_2+\frac{1}{2}} (\varepsilon\xi_n + i\langle \varepsilon\xi' \rangle)^{s_2+s_3-\nu_3-\beta-\frac{1}{2}} lf \\
\hat{v} &= \varepsilon^{-s_1} \Pi^+(\vartheta\xi_n + i\langle \varepsilon\xi' \rangle)^{s_3} lu
\end{aligned}$$

assuming that there exist a.e. the limits:

$$\begin{aligned}
(3.2.95)\qquad &\lim_{\varepsilon\to 0} [\psi_j - \psi_j^0]_{(0,s_2-m_j-\frac{1}{2},0),\xi'} = 0\,, \quad 1 \leqslant j \leqslant r_2 \\
&\lim_{\varepsilon\to 0} [\varphi_j]_{(\sigma_j),\xi'} = 0\,, \quad r_2 < j \leqslant r_2 + r_3 \\
&\lim_{\varepsilon\to 0} \|g - g^0\|^{+}_{(0,s_2-\ _2,0),\xi'} = 0\,, \\
&\lim_{\varepsilon\to 0} \|v - v^0\|^{+}_{(0,s_2,0)} = 0\,,
\end{aligned}$$

with $\psi_j^0 \in \tilde{H}_{s_2-m_j-\frac{1}{2}}(\mathcal{R}^{n-1}_{\xi'})$, $1 \leqslant j \leqslant r_2$, $\hat{g}^0 \in \hat{H}_{s_2-\nu_2}(\mathcal{R}^n_{\xi})$, $\hat{v}^0 \in \hat{H}_{s_2}(\mathcal{R}^n_{\xi})$, as above, and letting $\varepsilon \to 0$ one obtains from (3.2.82) the usual a priori estimate for the coercive elliptic reduced problem.

Finally, if $s_2 < \alpha + \frac{1}{2}$, then introducing ψ_j, $1 \leqslant j \leqslant r_2$, like above in (3.2.92), putting

$$\begin{aligned}
(3.2.96)\qquad \hat{g} &= \varepsilon^{\nu_1-s_1+s_2-\alpha-\frac{1}{2}} \langle \varepsilon\xi \rangle^{s_3-\nu_3} \hat{f}\,, \\
\hat{g} &= \varepsilon^{\nu_1-s_1+s_2-\alpha-\frac{1}{2}} \Pi^+(\varepsilon\xi_n + i\langle \varepsilon\xi' \rangle)^{s_3-\nu_3} \hat{l}f\,, \\
\hat{v} &= \varepsilon^{-s_1} \Pi^+(\varepsilon\xi_n + i\langle \varepsilon\xi' \rangle)^{s_3} \hat{l}u
\end{aligned}$$

assuming that there exist the limits

$$\begin{aligned}
(3.2.97)\qquad &\lim_{\varepsilon\to 0} [\psi_j - \psi_j^0]_{(0,s_2-m_j-\frac{1}{2},0),\xi'} = 0\,, \quad 1 \leqslant j \leqslant r_2 \\
&\lim_{\varepsilon\to 0} [\varphi_j]_{(\delta_j),\xi'} = 0\,, \quad r_2 < j \leqslant r_2 + r_3 \\
&\lim_{\varepsilon\to 0} \|g - g^0\|^{+}_{(0,s_2-\nu_2,0),\xi'} = 0\,, \\
&\lim_{\varepsilon\to 0} \|v - v^0\|^{+}_{(0,s_2,0)} = 0\,,
\end{aligned}$$

and letting $\varepsilon \to 0$, one obtains from (3.2.89) the usual a priori estimate for the reduced elliptic problem (see, for instance, [19]).

REMARK 3.2.16. – It is convenient to point out the connection between the vectorial subscripts in the estimates (3.2.77), (3.2.82), (3.2.89). When $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$, then the subscripts are respectively:

$$s - \nu\,, \quad \{\tau_j\}_{1\leq j\leq r_2}\,, \quad \{\sigma_j\}_{r_2<j\leq r_2+r_3}\,. \tag{3.2.98}$$

One should expect the subscripts (3.2.98) to show up given that for $\varepsilon = 1$ the singularly perturbaed problem becomes just usual elliptic boundary value problem. Still, σ_j might be not that easy to guess. There is a connection between σ_j and τ_j given by the formula:

$$\sigma_j = \tau_j + (s_2 - m_j - \tfrac{1}{2})\,e\,, \qquad e = (1, -1, 1)\,. \tag{3.2.99}$$

When $s_2 \geq \beta + \frac{1}{2}$, then the subscripts become respectively

$$s - \nu + \Delta_\beta e\,, \quad \{\tau_j + \Delta_\beta e\}_{1\leq j\leq r_2}\,, \quad \{\sigma_j\}_{r_2<j\leq r_2+r_3} \tag{3.2.100}$$

with $\Delta_\beta = (s_2 - \beta - \frac{1}{2})$, $e = (1, -1, 1)$.

Finally, when $s_2 < \alpha + \frac{1}{2}$, then the subscripts become respectively

$$s - \nu + \Delta_\alpha e_1\,, \quad \{\tau_j\}_{1\leq j\leq r_2}\,, \quad \{\sigma_j + \Delta_\alpha e\}_{r_2<j\leq r_2+r_3} \tag{3.2.101}$$

with $\Delta_\alpha = \alpha + \frac{1}{2} - s_2$, $e_1 = (1, 0, 0)$, since $\delta_j = \sigma_j + \Delta_\alpha e$, $r_2 < j \leq r_2 + r_3$.

REMARK 3.2.17. – The condition (3.2.69) can be replaced by the following one: $s_2 + s_3 - \frac{1}{2}$ is strictly greater than the maximum of order of boundary operators with respect to the normal derivative D_t.

REMARK 3.2.18. – It is immediate that for $m_{r_2} + \frac{1}{2} < s_2 < m_{r_2+1} + \frac{1}{2}$ the estimate (3.2.77) is two-sided. In other words, the right hand side in (3.2.77) can be estimated by $C_1\|u\|^+_{(s),\xi'}$ with some constant C_1 which does not depend on ε, ξ', f, φ_j, u.

On the other hand, the estimates (3.2.82), (3.2.89) are not two-sided and the right hand side in these inequalities cannot be estimated by $C_1\|u\|^+_{(s),\xi'}$ with C_1 which does not depend on ε, ξ'. To discover the reason of such a lack of adequateness, consider the following boundary value problem:

$$\big(i\varepsilon D_t + \langle\varepsilon\xi'\rangle\big)\,u(t) = f(t)\,, \qquad t > 0 \tag{3.2.102}$$

$$\pi_0 u = \varphi\,. \tag{3.2.103}$$

One seeks solution $u(t)$ to the problem (3.2.102), (3.2.103) such that $\|u\|^+_{(s),\xi'}$ is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathfrak{R}^{n-1}$.

Assume that $s \in \mathfrak{R}^3$ is such that $s_2 > \frac{1}{2}$, $s_2 + s_3 > \frac{1}{2}$. Given that the order of the operator in (3.2.102) is $\nu = (0, 0, 1)$ and that the trace $\pi_0 u$ has the norm $[\pi_0 u]_{(s-\frac{1}{2}e_2),\xi'}$

uniformly bounded with respect to ε and ξ', it is natural to take $f(t)$ with uniformly bounded norm $\|f\|^+_{(s-\nu),\xi'}$ and φ with uniformly bounded norm $[\varphi]_{(s-\frac{1}{2}e),\xi'}$. It turns out that in the case considered solution $u(t)$ with uniformly bounded norm $\|u\|^+_{(s),\xi'}$ does not exist and there is a need of somewhat compatibility conditions between φ and the behaviour of $f(t)$ when $t \to +0$. For instance, if $f(t) \equiv 0$, a solution $u(t)$ with property above exists iff $[\varphi]_{(s+(s_2-\frac{1}{2})e),\xi'}$ is uniformly bounded with respect to ε, ξ', which is a stronger condition than just the uniform boundness of $[\varphi]_{(s-\frac{1}{2}e_2),\xi'}$ when $s_2 > \frac{1}{2}$ (Recall that $e = (1, -1, 1)$, $e_2 = (0, 1, 0)$).

Therefore to get an adequate two-sided a priori estimate for the solution to the problem (3.2.3), (3.2.4) in the case, when $s_2 > m_{r_2+1} + \frac{1}{2}$ and $s_2 < m_{r_2} + \frac{1}{2}$ respectively there is a need to modify in a convenient way the problem itself.

From now on we drop the assumption (3.2.1), namely, that $m_{r_2} < m_{r_2+1}$ and we assume only that the orders m_j of the boundary operators b_{j0} in (3.2.4) are non-decreasing:

$$m_1 \leqslant m_2 \leqslant \,..\leqslant m_{r_2} \leqslant m_{r_2+1} \leqslant \ldots \leqslant m_{r_2+r_3}\,. \tag{3.2.104}$$

To have a proper modification to the problem (3.2.3), (3.2.4) there is a need to introduce Poisson operators (see [2], [27]).

For our purposes it is enough to consider Poisson operators with symbols rational functions in ξ_n.

Let $L_0(\varepsilon\xi', \lambda)$ be elliptic principal symbol of order $\sigma \in \mathfrak{R}^3$ and $a_0(\varepsilon, \xi', \lambda)$ be principal symbol of order $\alpha \in \mathfrak{R}^3$. For $\psi = \psi(\varepsilon, \xi')$ we define the following Poisson operator:

$$\frac{a_0}{L_0}(\psi \times \delta(t)) = \frac{1}{2\pi}\int\limits_{\Gamma} \frac{a_0(\varepsilon, \xi', \lambda)\,\psi(\varepsilon, \xi')}{L_0(\varepsilon, \xi', \lambda)} \exp(it\lambda)\, d\lambda\,, \qquad t > 0\,, \tag{3.2.105}$$

where Γ is contour in the upper half plane which encloses all the zeros of the polynomial $\lambda \to L_0^+(\varepsilon, \xi', \lambda)$, L_0^+ being the factor of $\lambda \to L_0(\varepsilon, \xi', \lambda)$ which corresponds to its zeros contained in the upper half of the complex λ-plane; here $\delta(t)$ is the δ-function of Dirac. If

$$a_0'(\varepsilon, \xi', \lambda) \equiv a_0(\varepsilon, \xi', \lambda)(\operatorname{mod} L_0^+(\varepsilon, \xi', \lambda))\,, \qquad \deg_\lambda a_0' < \deg_\lambda L_0^+$$

then the definition (3.2.105) can be given another equivalent form

$$\frac{a_0}{L_0}(\psi \times \delta(t)) = \frac{1}{2\pi}\int\limits_{\mathfrak{R}} \frac{a_0'(\varepsilon, \xi', \xi_n)\psi(\varepsilon, \xi')}{L_0(\varepsilon, \xi', \xi_n)} \exp(it\xi_n)\, d\xi_n\,. \tag{3.2.106}$$

LEMMA 3.2.19. – *Let the order σ of L_0 be either $(0, m, 0)$ or $(0, 0, m)$ with $m \geqslant 1$.*

Then the following estimates hold:

$$\|(L_0^{-1})\, a_0(\psi\times\delta(t))\|^{+}_{(s),\xi'}\leqslant C[\psi]_{(\Delta)}\,, \qquad \textit{for } \sigma=(0,m,0)\,, \tag{3.2.107}$$
$$\|(L_0^{-1})\, a_0^{-}(\psi\times\delta(t))\|^{+}_{(s),\xi'}\leqslant C[\psi]_{(\beta)}\,, \qquad \textit{for } \sigma=(0,0,m)\,.$$

where $\Delta = s+\alpha-\sigma+\frac{1}{2}e_2$ *and* $\beta=\Delta+(s_2+\alpha_2-m+\frac{1}{2})e$, *with* $e=(1,-1,1)$.

PROOF. – Using the representation (3.2.106), the Parceval's identity and making the change of variables $\xi_n\to\langle\xi'\rangle\xi_n$ for $\sigma=(0,m,0)$ and the change of variables $\xi_n\to\varepsilon^{-1}\langle\varepsilon\xi'\rangle\xi_n$ for $\sigma=(0,0,m)$, one gets immediately the estimates (3.2.107).

Now, coming back to the problem (3.2.3), (3.2.4) with m_j satisfying (3.2.104), consider first the case when

$$s_2>m_{r_2+1}+\tfrac{1}{2}\,. \tag{3.2.108}$$

Let $j_0>r_2$ be such that

$$m_{j_0+1}+\tfrac{1}{2}>s_2>m_{j_0}+\tfrac{1}{2}\,. \tag{3.2.109}$$

If there is no such a subscript j_0, we set

$$j_0=r_2+r_3\,.$$

Let $L_0(\xi',\xi_n)$ be elliptic polynomial symbol of order

$$\sigma=\big(0,2(j_0-r_2),0\big) \tag{3.2.110}$$

and consider the following modification of the problem (3.2.3), (3.2.4):

$$Q_0(\varepsilon,\hat{\xi}',D_t)\,u(t)=f(t)+\frac{1}{L_0(\hat{\xi}',D_t)}\sum_{1\leqslant j\leqslant j_0-r_2}a_{j0}(\varepsilon,\hat{\xi}',D_t)\big(\psi_j\times\delta(t)\big)\,,\qquad t>0 \tag{3.2.111}$$

$$\pi_0\, b_{j0}(\varepsilon,\hat{\xi}',D_t)\,u=\varphi_j\,,\qquad 1\leqslant j\leqslant r_3+j_0 \tag{3.2.112}$$

where the orders of a_{j0} and b_{j0} are respectively $\alpha_j\in\mathfrak{R}^3$ and $\mu_j=(\gamma_j,m_j,p_j)\in\mathfrak{R}^3$, the orders of additional boundary operators being supposed to satisfy the condition:

$$m_j+\tfrac{1}{2}>s_2\,,\qquad j_0<j\leqslant r_3+j_0\,. \tag{3.2.113}$$

Besides, as previously, we assume that the condition:

$$s_2+s_3>\max_{1\leqslant j\leqslant r_3+j_0}\{m_j+p_j\}+\tfrac{1}{2} \tag{3.2.114}$$

is fulfilled.

We seek $u(t)$ and ψ_j such that $\|u\|^+_{(s),\xi'}$ and $[\psi_j]_{(s-\nu+\alpha_j-\sigma+\frac{1}{2}e_2),\xi'}$ are uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$ and $\xi' \in \mathcal{R}^{n-1}$, under the condition that so it is for $\|f\|^+_{(s-\nu),\xi'}$, $[\varphi_j]_{(\tau_j),\xi'}$, $1 \leq j \leq j_0$ and $[\varphi_j]_{(\sigma_j),\xi'}$, $j_0 < j \leq r_3 + j_0$.

Any decreasing at $+\infty$ solution of (3.2.111) is also solution of the equation

$$(3.2.115) \qquad L_0^+(\hat{\xi}', D_t)Q_0(\varepsilon, \hat{\xi}', D_t)u(t) = L_0^+(\varepsilon, \hat{\xi}', D_t)f(t), \quad t > 0,$$

where $\lambda \to L_0^+(\hat{\xi}', \lambda)$ is the factor of $L_0(\hat{\xi}', \lambda)$ which corresponds to its zeros contained in the upper half of the complex λ-plane.

For the reciprocal statement to be true, obviously, it is necessary and sufficient that any solution of the homogeneous equation

$$(3.2.116) \qquad L_0^+(\hat{\xi}', D_t)v(t) = 0, \quad t > 0$$

to be represented as

$$(3.2.117) \quad v(t) = \frac{1}{L_0^+(\hat{\xi}', D_t)} \sum_{1 \leq j \leq j_0 - r_2} a_{j0}(\varepsilon, \hat{\xi}', D_t)(\psi_j \times \delta(t)) = \\ = \frac{1}{2\pi i} \int_\gamma \sum_{1 \leq j \leq j_0 - r_2} \psi_j \frac{a_{j0}(\varepsilon, \hat{\xi}', \lambda)}{L_0^+(\hat{\xi}', \lambda)} \exp(it\lambda)\, d\lambda, \quad t > 0$$

with some $\psi_j = \psi_j(\varepsilon, \xi')$.

The last formula holds for the general solution $v(t)$ of (3.2.116) iff the polynomials

$$\lambda \to a_{j0}(\varepsilon, \hat{\xi}', \lambda), \quad 1 \leq j \leq j_0 - r_2,$$

are linearly independent modulo $L_0^+(\hat{\xi}', D_t)$, for any ε and ξ'.

More precisely, we assume that the polynomials $\lambda \to a_{j0}$, $1 \leq j \leq j_0 - r_2$ satisfy the following condition:

$(a)_1$ *The polynomials*

$$\lambda \to a_{j0}(\varrho, \omega', \lambda), \quad 1 \leq j \leq j_0 - r_2$$

are linearly independent modulo $L_0^+(\omega', \lambda)$, *for any* $\varrho > 0$, *any* $\omega' \in \Omega_{n-1}$.

$(b)_1$ *The polynomials*

$$\lambda \to a^0_{j0}(\omega', \lambda), \quad 1 \leq j \leq j_0 - r_2$$

are linearly independent modulo $L_0^+(\omega', \lambda)$, *for any* $\omega' \in \Omega_{n-1}$.

$(c)_1$ *The polynomials*

$$\lambda \to a_{j00}(\omega', \lambda), \quad 1 \leq j \leq j_0 - r_2$$

are linearly independent modulo $L_0^+(\omega', \lambda)$, *for any* $\omega' \in \Omega_{n-1}$.

Under the condition $(a)_1$-$(c)_1$, the problem (3.2.111), (3.2.112) is equivalent to (3.2.115), (3.2.112).

Now, considering the problem (3.2.115), (3.2.112) we find ourselves precisely in the same situation as previously while investigating the problem (3.2.3), (3.2.4) with the condition $m_{r_2} + \frac{1}{2} < s_2 < m_{r_2+1} + \frac{1}{2}$ fulfilled.

Therefore, assume the following *coerciveness* condition:

(i)$_1$ *The polynomials*

$$\lambda \to b^0_{j0}(\omega', \lambda)\,, \qquad 1 \leqslant j \leqslant j_0$$

are linearly independent modulo $Q^{0+}_0(\omega', \lambda)\, L^+_0(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$.

(ii)$_1$ *The polynomials*

$$\lambda \to b_j(\lambda)\,, \qquad j_0 < j \leqslant r_j + j_0\,,$$

with $b_j(\lambda) = b_{j0}(1, 0, \lambda)$, *are linearly independent modulo* $Q^+(\lambda)$, *where* $Q^+(\lambda) = \lambda^{-r_2} Q^+_0(1, 0, \lambda)$.

(iii)$_1$ *The polynomials*

$$\lambda \to b_{j00}(\omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_3 + j_0\,,$$

are linearly independent modulo $Q^+_{00}(\omega', \lambda)\, L^+_0(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$.

(iv)$_1$ *The polynomials*

$$\lambda \to b_{j0}(\varrho, \omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_3 + j_0\,,$$

are linearly independent modulo $Q^+_0(\varrho, \omega', \lambda)\, L^+_3(\omega', \lambda)$, $\forall \varrho \in (0, +\infty)$ $\forall \omega' \in \Omega_{n-1}$.

The Theorem 3.2.12 (with $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$) and the estimate (3.2.77) being applicable in the considered case, one gets the following result:

THEOREM 3.2.20. – *The condition* (1.2.114) *being fulfilled, the polynomial* Q_0 *being elliptic of order* ν, *the Poisson operators* $(L^+_0)^{-1} a_{j0}$, $1 \leqslant j \leqslant j_0 - r_2$ *and the boundary operators* b_{j0}, $1 \leqslant j \leqslant r_3 + j_0$, *satisfying the coerciveness condition* $(a)_1$-$(c)_1$, (i)$_1$-(iv)$_1$, *the problem* (3.2.111), (3.2.112) *has unique solution* $u(t)$, $\{\psi_j\}$, $1 \leqslant j \leqslant j_0 - r_2$ *which satisfies the a priori estimate*:

$$(3.2.118) \qquad \|u\|^+_{(s),\xi'} + \sum_{1 \leqslant j \leqslant j_0 - r_2} [\psi_j]_{(\Delta_j),\xi'} \leqslant$$
$$\leqslant C\Big(\|f\|^+_{(s-\nu),\xi'} + \sum_{m_j < s - \frac{1}{2}} [\varphi_j]_{(\tau_j),\xi'} + \sum_{m_j > s_2 - \frac{1}{2}} [\varphi_j]_{(\sigma_l),\xi'}\Big) \leqslant C_1\Big(\|u\|^+_{(s),\xi'} + \sum_{1 \leqslant j \leqslant j_0 - r_2} [\psi_j]_{(\Delta_j,\xi')}\Big)\,,$$

where $\Delta_j = (s - \nu + \alpha_j - \sigma + \frac{1}{2} e_2) \in \mathfrak{R}^3$ *with* σ *defined in* (3.2.110), $\tau_j \in \mathfrak{R}^3$ *and* $\sigma_j \in \mathfrak{R}^3$ *are as defined above in* (3.2.61) *and the constants* C, C_1 *do not depend on* $\varepsilon \in [0, \varepsilon_0]$, $\xi' \in \mathfrak{R}^{n-1}$, u, ψ_j, f, φ_j.

REMARK 3.2.21. – One has to use the first of the inequalities (3.2.107) to estimate $[\varphi_j]_{(\Delta_j),\xi'}$.

Consider now the case when

$$s_2 < m_{r_2} + \tfrac{1}{2}\,.$$

Again, let $j_0 < r_2$ be such that

$$m_{j_0} + \tfrac{1}{2} < s_2 < m_{j_0+1} + \tfrac{1}{2}\,. \tag{3.2.119}$$

If there is no such subscript j_0, we put

$$j_0 = 0\,.$$

Let $L_0(\varepsilon\xi', \varepsilon\xi_n)$ be elliptic polynomial of order:

$$\sigma = \big(0, 0, 2(r_2 - j_0)\big) \in \mathfrak{R}^3 \tag{3.2.120}$$

and consider the following modification on the problem (3.2.3), (3.2.4)

$$Q_0^+(\varepsilon, \xi', D_t)\, u(t) = f(t) + \frac{1}{L_0(\varepsilon\xi', \varepsilon D_t)} \sum_{1 \leqslant j \leqslant r_2 - j_0} a_{j0}(\varepsilon, \xi', D_t)\big(\varphi_j \times \delta(t)\big)\,, \quad t > 0\,, \tag{3.2.121}$$

$$\pi_0 b_{j0}(\varepsilon, \xi', D_t)\, u = \varphi_j\,, \qquad j_0 - r_2 < j \leqslant r_2 + r_3\,, \tag{3.2.122}$$

where the orders of a_{j0} and b_{j0} are respectively $\alpha_j = (\alpha_{j1}, \alpha_{j2}, \alpha_{j3}) \in \mathfrak{R}^3$, $\mu_j = (\gamma_j, m_j, p_j) \in \mathfrak{R}^3$ and the orders of additional boundary operators b_{j0} are supposed to satisfy the condition

$$m_j + \tfrac{1}{2} < s_2\,, \qquad j \leqslant j_0\,. \tag{3.2.123}$$

To be able to satisfy (3.2.123) and the (given further) coerciveness condition one has to admit also boundary conditions b_{j0} with m_j negative integers, in other words, some b_{j0} for $j \leqslant 0$ might be pseudodifferential operators with symbols rational functions of ξ_n.

Assume that it is possible to complete the boundary conditions (3.2.4) by b_{j0}, $j_0 - r_2 < j \leqslant 0$, with polynomial symbols in such a way as to satisfy the following *coereciveness* condition:

(i)$_2$ *The polynomials*

$$\lambda \to b_{j0}^0(\omega', \lambda)\,, \qquad j_0 - r_2 < j \leqslant j_0$$

are linearly independent modulo $Q_0^+(\omega', \lambda)$, *for any* $\omega' \in \Omega_{n-1}$.

(ii)$_2$ *The polynomials*

$$\lambda \to b_j(\lambda)\,, \qquad j_0 < j \leqslant r_2 + r_3$$

with $b_j(\lambda) = b_{j0}(1, 0, \lambda)$, *are linearly independent modulo* $Q^+(\lambda)$, *where* $Q^+(\lambda) = \lambda^{-r_2} Q_0^+(1, 0, \lambda) L_0^+(0, \lambda)$.

(iii)$_2$ *The polynomials*

$$\lambda \to b_{j00}(\omega', \lambda)\,, \qquad j_0 - r_2 < j \leqslant r_2 + r_3\,,$$

are linearly independent modulo $Q_{00}^+(\omega', \lambda) L_{00}^+(\omega', \lambda)$ *for any* $\omega' \in \Omega_{n-1}$.

(iv)$_2$ *The polynomials*

$$\lambda \to b_{j0}(\varrho, \omega', \lambda)\,, \qquad j_0 - r_2 < j \leqslant r_2 + r_3\,,$$

are linearly independent modulo $Q_0^+(\varrho, \omega', \lambda) L_0^+(\varrho\omega', \lambda)$, *for any* $\varrho \in (0, +\infty)$, *any* $\omega' \in \Omega_{n-1}$.

Besides, assume that a_{j0}, $j_0 - r_2 < j \leqslant j_0$ satisfy the condition:

$(a)_2$ *The polynomials*

$$\lambda \to a_{j0}(\varrho, \omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_2 - j_0$$

are linearly independent modulo $L_0^+(\varrho\omega', \lambda)$, $\forall \varrho > 0$, $\forall \omega' \in \Omega_{n-1}$.

$(b)_2$ *The polynomials*

$$\lambda \to a_{j00}(\omega', \lambda)\,, \qquad 1 \leqslant j \leqslant r_2 - j_0$$

are linearly independent modulo $L_{00}^+(\omega', \lambda)$, $\forall \omega' \in \Omega_{n-1}$.

$(c)_2$ *The polynomials*

$$\lambda \to a_j(\lambda)\,, \qquad 1 \leqslant j \leqslant r_2 - j_0$$

with $a_j(\lambda) = a_{j0}(1, 0, \lambda)$ *are linearly independent modulo* $L_0^+(0, \lambda)$.

The condition $(a)_2$-$(c)_2$ guarantee that any decreasing at $+\infty$ solution of the equation

$$(3.2.124) \qquad L_0^+(\varepsilon\xi', \varepsilon D_t) Q_0^+(\varepsilon, \xi', D_t) u(t) = L_0^+(\varepsilon\xi', \varepsilon D_t) f(t)\,, \qquad t > 0$$

to be also a solution of (3.2.121), so that the problem (3.2.121), (3.2.122) is equivalent to the problem (3.2.124), (3.2.122).

Again, for the problem (3.2.124), (3.2.122) the Theorem 3.2.12 (with $\alpha + \frac{1}{2} < s_2 < \beta + \frac{1}{2}$) and the estimate (3.2.77) are applicable, and we get the following result:

THEOREM 3.2.22. – *The condition* (3.2.114) *being fulfilled, the polynomial* Q_0 *being elliptic of order* ν, *the Poisson operators* $(L_0^+)^{-1}a_{j0}$, $1 \leq j \leq r_2 - j_0$, *and the boundary operators* b_{j0}, $j_0 - r_2 \leq j \leq r_2 + r_3$ *satisfying the coerciveness condition* $(a)_2$-$(c)_2$, $(\mathrm{i})_2$-$(\mathrm{iv})_2$, *there exists unique solution* $u(t)$, $\{\psi_j\}$, $1 \leq j \leq r_2 - j_0$, *to the problem* (3.2.121), (3.2.122), *which satisfies the a priori estimate:*

$$\|u\|^+_{(s),\xi'} + \sum_{1 \leq j \leq r_2 - j_0} [\psi_j]_{(\beta_j),\xi'} \leq C \left(\|f\|^+_{(s-\nu),\xi'} + \sum_{m_j < s_2 - \frac{1}{2}} [\varphi_j]_{(\tau_j),\xi'} + \sum_{m_j > s_2 - \frac{1}{2}} [\varphi_j]_{(\sigma_j),\xi'} \right) \leq \leq C_1 \left(\|u\|^+_{(s,)\xi'} + \sum_{1 \leq j \leq r_2 - j_0} [\psi_j]_{(\beta_j),\xi'} \right), \tag{3.2.125}$$

where $\beta_j = s - \nu + \alpha_j - \sigma + \frac{1}{2}e_2 + (s_2 - \nu_2 + \alpha_{j2} + \frac{1}{2})e \in \mathfrak{R}^3$, *with* σ *defined in* (3.2.120), τ_j, σ_j *are as defined above in* (3.2.61), $e = (1, -1, 1)$, *and the constants* C, C_1 *do not depend on* ε, ξ', u, ψ_j, f, φ_j.

REMARK 3.2.23. – One has to use the second of the inequalities (3.2.107) to estimate $[\psi_j]_{(\beta_j),\xi'}$.

If the additional boundary operators b_{j0}, $j_0 - r_2 < j \leq 0$ cannot be chosen as polynomials and such a requirement conflicts with the coerciveness condition $(\mathrm{i})_2$-$(\mathrm{iv})_2$, then these boundary conditions can be given as pseudodifferential operators with rational symbols. In that case the coerciveness condition is formulated in terms of some determinants which are supposed not to vanish uniformly with respect to the parameters $\varrho > 0$ and $\omega' \in \Omega_{n-1}$. Of course, one can consider more general Poisson and boundary operators a_{j0} and b_{j0} than the ones introduceed aobve. Again, the coerciveness condition is formulated in terms of some determinants which are supposed not to vanish uniformly with respect to the parameters $\varrho > 0$ and $\omega' \in \Omega_{n-1}$.

3.3. *Singular Wiener-Hopf equations in* $\mathfrak{R}_+$.

We will sketch briefly the extension of the results given in the previous section to pseudodifferential equations in $\mathfrak{R}_+$ with symbols rational functions of $\xi \in \mathfrak{R}^n$. The statements below with slight obvious modifications are also true for pseudodifferential operators with transmission property (see [2], [27]).

Let $Q_0(\varepsilon, \xi)$ and $\tilde{Q}_0(\varepsilon, \xi)$ be two properly elliptic principal symbols of orders respectively ν and $\tilde{\nu}$. Let $\nu_j = 2r_j$ and $\tilde{\nu}_j = 2\tilde{r}_j$, $j = 2, 3$, with r_j, $\tilde{r}_j$ non-negative integers.

We introduce the vector

$$\mathscr{R} = (0, r_2 - \tilde{r}_3, r_3 - \tilde{r}_3) \in \mathfrak{R} \times \mathbf{Z} \times \mathbf{Z} \tag{3.3.1}$$

called index of the rational function

$$L_0(\varepsilon, \xi) = Q_0(\varepsilon, \xi)/\tilde{Q}_0(\varepsilon, \xi) .$$

The polynomials Q_0, $\tilde{Q}_0$ being elliptic principal symbols the function $L_0(\varepsilon, \xi)$ is smooth in $\varepsilon \in (0, \infty)$, $\forall \xi \in \mathfrak{R}^n \setminus \{0\}$, homogeneous in (ε^{-1}, ξ) of order

$$|\nu| - |\tilde{\nu}| = \nu_1 + \nu_2 - \tilde{\nu}_1 - \tilde{\nu}_2 , \tag{3.3.2}$$

and does not vanish for $\forall \varepsilon \in (0, \infty)$, $\forall \xi \in \mathfrak{R}^n \setminus \{0\}$:

$$L_0(\varepsilon, \xi) \neq 0 , \quad \forall \varepsilon \in (0, \infty),\ \forall \xi \in \mathfrak{R}^n \setminus \{0\} . \tag{3.3.3}$$

It is natural to call the vector

$$\mu = \nu - \tilde{\nu} \in \mathfrak{R}^3 \tag{3.3.4}$$

order of the rational function $L_0(\varepsilon, \xi)$.

Besides, it is quite obvious that $L_0(\varepsilon, \xi)$ satisfies the following two-sided inequalities:

$$C^{-1} \varepsilon^{-\mu_1} |\xi|^{\mu_2} \langle \varepsilon \xi \rangle^{\mu_3} \leqslant |L_0(\varepsilon, \xi)| \leqslant C \varepsilon^{-\mu_1} |\xi|^{\mu_2} \langle \varepsilon \xi \rangle^{\mu_3}. \tag{3.3.5}$$

A symbol satisfying (3.3.5) is called *elliptic singular perturbation of order* μ.

First assume that vector $(\mathscr{R}_2, \mathscr{R}_3)$ has non-negative components: $\mathscr{R}_j = r_j - \tilde{r}_j$,

$$\hat{R}_j \geqslant 0 , \quad j = 2, 3 . \tag{3.3.6}$$

Let $L_{j0}(\varepsilon, \xi)$, $1 \leqslant j \leqslant \mathscr{R}_2 + \mathscr{R}_3$, be rational functions in (ε, ξ), with vectorial orders

$$\mu_j = (\gamma_j, m_j, p_j) \in \mathfrak{R} \times \mathbf{Z} \times \mathbf{Z} , \quad 1 \leqslant j \leqslant \mathscr{R}_2 + \mathscr{R}_3 , \tag{3.3.7}$$

which are supposed to be homogeneous in (ε^{-1}, ξ) of order

$$|\mu_j| = \gamma_j + m_j \tag{3.3.8}$$

and smooth for $\varepsilon \in (0, \infty)$, $\xi \in \mathfrak{R}^{n-1} \setminus \{0\}$.

Using again the notation $\hat{\xi}'$ for

$$\hat{\xi}' = \langle \xi' \rangle \omega', \quad \omega' = \xi' |\xi'|^{-1} \in \Omega_{\nu-1} \tag{3.3.9}$$

we consider the following Wiener-Hopf singularly perturbed problem in $\mathfrak{R}_+$:

$$\pi_+ L_0(\varepsilon, \hat{\xi}', D_t)\, u_+(t) = f(t) , \tag{3.3.10}$$

$$\pi_0 L_{j0}(\varepsilon, \hat{\xi}', D_t)\, u_+(t) = \varphi_j , \quad 1 \leqslant j \leqslant \mathscr{R}_2 + \mathscr{R}_3 \tag{3.3.11}$$

where $u_+(t)$ is defined for $t \in \mathfrak{R}$ and has its support in $\overline{\mathfrak{R}_+}$, the operator $L_0 u_+$ is defined in the usual way:

$$L_0(\varepsilon, \hat{\xi}', D_t)\, u_+(t) = F^{-1}_{\xi_n \to t} L_0(\varepsilon, \hat{\xi}', \xi_n)\, F_{t \to \xi_n} u_+ ,$$

π_+ is restriction to the half-line $\mathfrak{R}_+$, and π_0 is restriction to the point $\{t = 0\}$, as defined above in the Section 3.2.

The boundary operators L_{j0} are supposed to be ordered in such a way that the sequence of their orders $\{m_j\}$, $1 \leqslant j \leqslant \mathscr{R}_2 + \mathscr{R}_3$, is a non-decreasing one, and, moreover, we assume at this stage that

$$m_1 \leqslant m_2 \leqslant \ldots \leqslant m_{\mathscr{R}_2} < m_{\mathscr{R}_2+1} \leqslant \ldots \leqslant m_{\mathscr{R}_2+\mathscr{R}_3} . \tag{3.3.12}$$

We denote further $m_{\mathscr{R}_2}$ and $m_{\mathscr{R}_2+1}$ respectively by α and β.

We describe the coerciveness condition on the boundary operators $L_{j0}(\varepsilon, \xi)$.

It is immediate that the rational function $\lambda \to L_0(\varepsilon, \xi', \lambda)$ admits the factorization:

$$L_0(\varepsilon, \xi', \lambda) = L_0^+(\varepsilon, \xi', \lambda)\, L_0^-(\varepsilon, \xi', \lambda) , \tag{3.3.13}$$

where

$$L_0^+(\varepsilon, \xi', \lambda) = Q_0^+(\varepsilon, \xi', \lambda)/\tilde{Q}_0^+(\varepsilon, \xi', \lambda) \tag{3.3.14}$$

has all its zeros and singularities within the upper λ-half plane and $L_0^-(\varepsilon, \xi', \lambda)$ has all its zeros and singularities within the lower λ-plane, when $\varepsilon \in (0, \infty)$, $\xi' \in \mathfrak{R}^{n-1} \setminus \{0\}$.

Besides, it is quite obvious that the vectorial order of $L_0^+(\varepsilon, \xi', \lambda)$ is $\mathscr{R} = (0, \mathscr{R}_2, \mathscr{R}_3)$, and it is homogeneous function in $(\varepsilon^{-1}, \xi', \lambda)$ of order $\mathscr{R}_2$ which satisfies the two-sided inequalities:

$$C^{-1}(|\xi'| + |\lambda|)^{\mathscr{R}_2}(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{\mathscr{R}_3} \leqslant |L_0^+(\varepsilon, \xi', \lambda)| \leqslant \leqslant C(|\xi'| + |\lambda|)^{\mathscr{R}_2}(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{\mathscr{R}_3} \tag{3.3.15}$$

when $\xi' \in \mathfrak{R}^{n-1} \setminus \{0\}$, $\lambda \in \boldsymbol{C}$, $\operatorname{Im} \lambda \leqslant 0$, $\varepsilon > 0$.

One checks easily, using the representation

$$L_0^+ = Q_0^+/\tilde{Q}_0^+,$$

that $L_0^+(\varrho, \omega', \lambda)$ for $\forall \varrho \geqslant \varrho_0 > 0$, $\forall \omega' \in \Omega_{n-1}$ has its zeros and singularities enclosed within a compact domain in the upper half-plane. We denote the boundary of this domain by Γ; Γ does not depend on ϱ, ω' for $\varrho \geqslant \varrho_0 > 0$, $\omega' \in \Omega_n$.

Besides, by the Lemma 3.1.2 the rational function $\lambda \to L_0^+(\varepsilon, \xi', \lambda)$ admits for $|\xi'| \leqslant C$ and ε_0 sufficiently small the factorization

$$L_0^+(\varepsilon, \xi', \lambda) = \mathscr{L}_{01}^+(\varepsilon, \xi', \lambda)\, \mathscr{L}_{02}^+(\varepsilon, \xi', \lambda) \tag{3.3.16}$$

with $\mathscr{L}^+_{01}(\varepsilon, \xi', \lambda)$ homogeneous in $(\varepsilon^{-1}, \xi', \lambda)$ of order $\mathscr{R}_2$ and satisfying the inequalities:

(3.3.17) $$C^{-1}(|\xi'| + |\lambda|)^{\mathscr{R}_2} \leq |\mathscr{L}^+_{01}(\varepsilon, \xi', \lambda)| \leq C(|\xi'| + |\lambda|)^{\mathscr{R}_2},$$

$\forall\varepsilon \in (0, \infty)$, $\forall\xi' \in \mathcal{R}^{n-1}\setminus\{0\}$, $\forall\lambda \in \boldsymbol{C}$, $\operatorname{Im}\lambda < 0$, and $\mathscr{L}^+_{02}(\varepsilon, \xi', \lambda)$ homogeneous in $(\varepsilon^{-1}, \xi', \lambda)$ of order 0 and satisfying the inequalities:

(3.3.18) $$C(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{\mathscr{R}_3} \leq |\mathscr{L}^+_{02}(\varepsilon, \xi', \lambda)| \leq C(1 + \varepsilon|\xi'| + \varepsilon|\lambda|)^{\mathscr{R}_3},$$

$\forall\varepsilon \in (0, \infty)$, $\forall\xi' \in \mathcal{R}^{n-1}\setminus\{0\}$, $\forall\lambda \in \boldsymbol{C}$, $\operatorname{Im}\lambda < 0$.

Using the representation of L^+_0 by polynomials $\lambda \to Q^+_0$, $\lambda \to \tilde{Q}^+_0$, it is immediate that there exist the limits:

(3.3.19) $$\lim_{\varrho \to +0} \mathscr{L}^+_{01}(\varrho, \xi', \lambda) = L^{0+}_0(\xi', \lambda),$$

where $L^{0+}_0(\xi', \lambda)$ is homogeneous function in (ξ', λ) of order $\mathscr{R}_2$, given by the formula:

$$L^{0+}_0(\xi', \lambda) = Q^{0+}_0(\xi', \lambda)/\tilde{Q}^{0+}_0(\xi', \lambda).$$

One checks easily that all the singularities and zeros of $L^{0+}_0(\omega', \lambda)$ for $\forall\omega' \in \Omega_{n-1}$, are enclosed within a compact domain in the upper half plane. We denote the boundary of this domain by Γ_1, Γ_1 does not depend on $\omega' \in \Omega_n$.

We denote by $L^+(\lambda)$ the rational function

(3.3.20) $$L^+(\lambda) = \mathscr{L}^+_{02}(1, 0, \lambda),$$

and by Γ_2 the Jordanian curve in the upper half-plane which encloses all the zeros and singularities of $L^+(\lambda)$.

Finally, $L^+_0(\varrho, \xi', \lambda)$ having the growth like $\varrho^{\mathscr{R}_3}$ as $\varrho \to +\infty$, we introduce

(3.3.21) $$L^+_{00}(\xi', \lambda) = \lim_{\varrho \to +\infty} \varrho^{-\mathscr{R}_3} Q^+_0(\varrho, \xi', \lambda).$$

It is quite obvious that

$$L^+_{00}(\xi', \lambda) = Q^+_{00}(\xi', \lambda)/\tilde{Q}^+_{00}(\xi', \lambda)$$

and it is homogeneous function in (ξ', λ) of order $\mathscr{R}_2 + \mathscr{R}_3$.

Again, all the zeros and singularities of $L^+_{00}(\omega', \lambda)$ for $\omega' \in \Omega_{n-1}$ are enclosed within a compact domain in the upper half plane, whose boundary is denoted by Γ_3. Γ_3 does not depend on $\omega' \in \Omega_{n-1}$.

The boundary operators L_{j0} are supposed to satisfy the following *coerciveness condition*:

(i) *Let $L^0_{j0}(\xi)$ be the reduced symbol for $L_{j0}(\varepsilon, \xi)$,*

(3.3.22) $$L^0_{j0}(\xi) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma_j} L_{j0}(\varepsilon, \xi).$$

Since $L^0_{j0}(\xi)$ does not vanish for $\xi \in \mathcal{R}^n \setminus \{0\}$, the zero and singularities of the rational function $\lambda \to L^0_{j0}(\omega', \lambda)$ for $\omega' \in \Omega_{n-1}$ are contained within two disjoint compact domains respectively in the upper and lower halves of the complex λ-plane. Therefore, we can assume that the curve Γ_1, defined above, additionally to the zeros and singularities of $\lambda \to L^{0+}_0(\omega', \lambda)$, encloses also those zeros and singularities of $L^0_{j0}(\omega', \lambda)$, $1 \leqslant j \leqslant \mathscr{R}_2$, which are located in the upper half of the complex λ plane.

Introduce

$$q^0_{kj}(\omega') = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{L^0_{j0}(\omega', \lambda)}{L^{0+}_0(\omega' k\ \lambda)} \lambda^{k-1} d\lambda\,, \quad 1 \leqslant k,\ j \leqslant \mathscr{R}_2\,. \tag{3.3.23}$$

The matrix $\|q^0_{kj}(\omega')\|$ is supposed to be non-singular for any $\omega' \in \Omega_{n-1}$:

$$\det \|q^0_{kj}(\omega')\|_{1 \leqslant k,j \leqslant \mathscr{R}_2} \neq 0\,, \quad \forall \omega' \in \Omega_{n-1}\,. \tag{3.3.24}$$

(ii) *Let $L_j(\lambda) = L_{j0}(1, 0, \lambda)$. Again we can assume that the contour Γ_2 defined above, additionally to the zeros and singularities of $L^+(\lambda)$, encloses also those zeros and singularities of $L^+_j(\lambda)$, $\mathscr{R}_2 < j \leqslant \mathscr{R}_2 + \mathscr{R}_3$, which are located in the upper half of the complex λ plane. Introduce*

$$q_{kj} = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{L^+(\lambda)}{L_j(\lambda)} \lambda^{k-\mathscr{R}_2-1} d\lambda\,, \quad \mathscr{R}_2 < k,\ j \leqslant \mathscr{R}_2 + \mathscr{R}_3\,. \tag{3.3.25}$$

The matrix $\|q_{kj}\|$ is supposed to be non-singular:

$$\det \|q_{kj}\|_{\mathscr{R}_2 < k,j \leqslant \mathscr{R}_2+\mathscr{R}_3} \neq 0\,. \tag{3.3.26}$$

(iii) *Let $L_{j00}(\xi)$ be the principal homogeneous symbol of order $m_j + p_j$ for $L_{j0}(1, \xi)$, $1 \leqslant j \leqslant \mathscr{R}_2 + \mathscr{R}_3$ so that:*

$$L_{j00}(\omega', \lambda) = \lim_{\varrho \to +\infty} \varrho^{\gamma_j - p_j} L_{j0}(\varrho, \omega' k\ \lambda)\,. \tag{3.3.27}$$

The zeros and singularities of $\lambda \to L_{j00}(\omega', \lambda)$, for $\forall \omega' \in \Omega_{n-1}$, $1 \leqslant j \leqslant \mathscr{R}_2 + \mathscr{R}_3$, being contained within two disjoint compact domains located respectively in the upper and lower halves of the complex λ-plane, we can assume, that those in the upper half plane are enclosed within the curve Γ_3 defined above.

We introduce the matrix:

$$q^{00}_{kj}(\omega') = \frac{1}{2\pi i}\int_{\Gamma_3} \frac{L_{j00}(\omega', \lambda)}{L^+_{00}(\omega', \lambda)} \lambda^{k-1} d\lambda\,, \quad 1 \leqslant k,\ j \leqslant \mathscr{R}_2 + \mathscr{R}_3\,. \tag{3.3.28}$$

The matrix $\|q^{00}_{kj}(\omega')\|$ is supposed to be non-singular:

$$\det \|q^{00}_{kj}(\omega')\|_{1 \leqslant k,j \leqslant \mathscr{R}_2+\mathscr{R}_3} \neq 0\,, \quad \forall \omega' \in \Omega_{n-1}\,. \tag{3.3.29}$$

(iv) *The zeros and singularities of* $\lambda \to b_{j0}(\varrho, \omega', \lambda)$. $1 \leqslant j \leqslant \mathcal{R}_2 + \mathcal{R}_3$, *for* $\forall \varrho \geqslant \varrho_0 > 0$, $\forall \omega' \in \Omega_{n-1}$, *being contained within two disjoint compact domains located respectively in the upper and lower halves of the complex* λ*-plane, we can assume that those in the upper half plane are enclosed within the contour* Γ *defined above. Of course the contour* Γ *depends on* ϱ_0.

Introduce the matrix:

$$(3.3.30) \qquad Q_{kj}(\varrho, \omega') = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varrho^{\gamma_j - p_j + \mathcal{R}_3} b_{j0}(\varrho, \omega', \lambda)}{Q_0^+(\varrho, \omega', \lambda)} \lambda^{k-1} d\lambda , \qquad 1 \leqslant k,\ j \leqslant \mathcal{R}_2 + \mathcal{R}_3 .$$

The matrix $\|Q_{kj}(\varrho, \omega')\|$ *is supposed to be non-singular*:

$$(3.3.31) \qquad \det \|Q_{kj}(\varrho, \omega')\|_{1 \leqslant k,j \leqslant \mathcal{R}_2 + \mathcal{R}_3} \neq 0 , \qquad \forall (\varrho, \omega') \in (0, +\infty) \times \Omega_{n-1} .$$

THEOREM 3.3.1. – *The conditions* (3.2.114) *and* $m_{\mathcal{R}_2} + \frac{1}{2} < s_2 < m_{\mathcal{R}_2+1} + \frac{1}{2}$ *being fulfilled*, $L_0(\varepsilon, \xi)$ *being elliptic singularly perturbed rational symbol of order* μ, *the boundary symbols* $L_{j0}(\varepsilon, \xi)$, $1 \leqslant j \leqslant \mathcal{R}_2 + \mathcal{R}_3$ *satisfying the coerciveness condition* (i)-(iv), *the problem* (3.3.10), (3.3.11) *has unique solution* $u(t)$ *which satisfies the a priori estimate* (3.2.77) *where* r_j *should be replaced by* $\mathcal{R}_j$, $j = 2, 3$, *and* v *by* μ, *the vectorial order of* $L_0(\varepsilon, \xi)$.

The proof of this theorem is absolutely similar to this one of the Theorem 3.2.12. Indeed, using the Wiener-Hopf method one represents the general solution to (3.3.10) decreasing at $+\infty$ in the form (3.2.8) with $\mathcal{L}_{0j}^+$ instead of Q_{0j}^+ when $\varrho = \varepsilon\langle\xi'\rangle \leqslant \varrho_0$ with ϱ_0 sufficiently small, and in the form (3.2.7) with L_0^+ instead of Q_0^+ when $\varrho \geqslant \varrho_0$. Substituting these expressions into (3.3.11) one finds out the coerciveness conditions (i)-(iv) and gets the a priori estimates in the same way as it was done in the previous section.

Now, if the condition $m_{\mathcal{R}_2} + \frac{1}{2} < s_2 < m_{\mathcal{R}_2+1} + \frac{1}{2}$ is not fulfilled or one of the numbers $\mathcal{R}_2, \mathcal{R}_3$ is negative integer, then introducing into the equation (3.3.10) Poisson operators with unknown densities, one gets a well posed boundary value problem in this case too and establishes a two-sided a priori estimate of the same kind as in the previous section.

3.4. *Singular perturbation in* $\mathfrak{R}_+^n$.

We consider here coercive singular perturbations with constant coefficients in $\mathfrak{R}_+^n$ and establish the basic a priori estimate for its solutions.

Let $Q(\varepsilon, \xi)$ be elliptic singular symbol of order $v = (v_1, v_2, v_3)$ with principal symbol $Q_0(\varepsilon, \xi)$ and $b_j(\varepsilon, \xi)$, $1 \leqslant j \leqslant r_2 + r_3$, singular symbols of order $\mu_j = (\gamma_j, m_j, p_j)$ with principal symbols $b_{j0}(\varepsilon, \xi)$, satisfying the coerciveness condition (i)-(iv).

Consider the singularly perturbed boundary value problem in $\mathfrak{R}_+^n$:

$$(3.4.1) \qquad Q(\varepsilon, D)\, u(x) = f(x) , \qquad x \in \mathfrak{R}_+^n$$

$$(3.4.2) \qquad \pi_0 b_j(\varepsilon, D)\, u(x') = \varphi_j(x') , \qquad x' \in \mathfrak{R}^{n-1},\ 1 \leqslant j \leqslant r_2 + r_3 .$$

The orders m_j of the operators b_j are supposed to satisfy the condition (3.2.1). The solution to the problem (3.4.1), (3.4.2) is sought in $H_{(s)}(\mathcal{R}^n_+)$ under the assumption (3.2.69).

THEOREM 3.4.1. – *The problem* (3.4.1), (3.4.2) *being elliptic coercive singular perturbation and the conditions* (3.2.1), (3.2.46), (3.2.69) *being fulfilled, the following a priori estimate holds for the solution* u: (*for* ε_0 *small enough*)

$$\|u\|^+_{(s)} \leqslant C\Big(\|f\|^+_{(s-\nu)} + \sum_{1\leqslant j\leqslant r_2} [\varphi_j]_{(\tau_j)} + \sum_{r_2<j\leqslant r_2+r_3} [\varphi_j]_{(\sigma_j)} + \|u\|^+_{(s')}\Big) \leqslant C_1 \|u\|^+_{(s)} \tag{3.4.3}$$

with any $s'\in\partial_1\Gamma_s$ *and the constants* C, C_1 *which depend only on* s *and* s'.

PROOF. – Noticing that the polynomials

$$\xi_n \to Q(\varepsilon, \xi', \xi_n)\,, \qquad \xi_n \to Q_0(\varepsilon, \hat{\xi}', \xi_n)$$

and

$$\xi_n \to b_j(\varepsilon, \xi', \xi_n)\,, \qquad \xi_n \to b_{j0}(\varepsilon, \hat{\xi}', \xi_n)$$

have the same principal symbols, applying the Fourier transform with respect to x' in (3.4.1), (3.4.2), one gets (3.4.3) using the a priori estimate (3.2.77) obtained in the previous section for the solutions to singularity perturbed boundary value problem for ordinary differential operators with parameters ε and ξ'. ∎

Consider now the case when $s_2 > m_{r_2+1} + \frac{1}{2}$. We now no longer assume that $m_{r_2} < m_{r_2+1}$. Let j_0 be such a subscript that

$$m_{j_0} + \tfrac{1}{2} < s_2 < m_{j_0+1} + \tfrac{1}{2}\,. \tag{3.4.4}$$

If there is no such a subscript j_0, we put

$$j_0 = r_2 + r_3\,.$$

Consider the following modification to the problem (3.4.1), (3.4.2):

$$Q(\varepsilon, D)\,u(x) = f(x) + L^{-1}(\varepsilon, D) \sum_{1\leqslant j\leqslant j_0} a_j(\varepsilon, D)\big(\psi_j(x')\times\delta(x_n)\big)\,, \qquad x\in\mathcal{R}^n_+\,, \tag{3.4.5}$$

$$\pi_0 b_j(\varepsilon, D)\,u(x') = \varphi_j(x')\,, \qquad 1\leqslant j\leqslant r_3 + j_0\,, \tag{3.4.6}$$

where $Q(\varepsilon, D)$ is elliptic singular perturbation of order $\nu\in\mathcal{R}^3$, $L(\varepsilon, D)$ is elliptic singular perturbation of order $\sigma = \big(0, 2(j_0 - r\,), 0\big)$, $a_j(\varepsilon, D)$ and $b_j(\varepsilon, D)$ are singular perturbations of orders $\alpha_j = (\alpha_{j1}, \alpha_{j2}, \alpha_{j3})$ and $\mu_j = (\gamma_j, m_j, p_j)$ respectively, satisfying the coerciveness condition $(a)_1$-$(c)_1$, (i)$_1$-(iv)$_1$ in the Section 3.2, and the Poisson operators $L^{-1}(D)\,a_j(\varepsilon, D)$ are defined by the formula (3.2.105) with $\tilde{\psi}_j(\xi') = F'_{x'\to\xi'}\psi$ instead of $\psi(\varepsilon, \xi')$.

THEOREM 3.4.2. – *The problem* (3.4.5), (3.4.6) *being elliptic coercive singular perturbation and the conditions* (3.2.114), (3.4.4) *being fulfilled, the following a priori estimate holds for its solution* u, $\{\psi_j\}$, $1 \leqslant j \leqslant j_0 - r_2$: *(for* ε_0 *small enough)*

$$\|u\|^+_{(s)} + \sum_{1\leqslant j\leqslant j_0-r_2} [\psi_j]_{(\Delta_j)} \leqslant C\Big(\|f\|^+_{(s-\nu)} + \sum_{m_j<s_2-\frac{1}{2}} [\varphi_j]_{(\tau_j)} + \sum_{m_j>s_2-\frac{1}{2}} [\varphi_j]_{(\sigma_j)} + \|u\|^+_{(s')}\Big) \leqslant$$
$$\leqslant C_1\Big(\|u\|^+_{(s)} + \sum_{1\leqslant j\leqslant j_0-r_2} [\psi_j]_{(\Delta_j)}\Big)\,, \tag{3.4.7}$$

where Δ_j, τ_j, $\sigma_j \in \mathfrak{R}^3$ *are the same vectorial subscripts as in* (3.2.61), (3.2.118), s' *is any vector in* $\partial_1\Gamma_s$ *and the constants* C, C_1 *depend only on* s, s'.

PROOF. – The inequality (3.4.7) is an immediate consequence of the Theorem 3.2.20 given that for any singular perturbation $S(\varepsilon, D)$ the polynomials

$$\xi_n \to S(\varepsilon, \xi', \xi_n)\,, \qquad \xi_n \to S_0(\varepsilon, \hat{\xi}', \xi_n)$$

have the same principal symbol $S_0(\varepsilon, \xi', \xi_n)$.

Besides, the estimate (3.2.118) being two-sided, one gets first (3.4.7) with some $s'' < s$, $s'' \in \partial_1\Gamma_s$. Using afterwards the interpolation inequality (1.1.16), one obtains the estimate (3.4.7) for any $s' \in \partial_1\Gamma_s$. ■

Assuming now that $s_2 < m_{r_2} + \frac{1}{2}$ and choosing $j_0 < r_2$ such that

$$m_{j_0} + \tfrac{1}{2} < s_2 < m_{j_0+1} + \tfrac{1}{2} \tag{3.4.8}$$

(if there is no such a subscript j_0 we put $j_0 = 0$), consider the following modification to the problem (3.4.1), (3.4.2):

$$Q(\varepsilon, D)\,u(x) = f(x) + L^{-1}(\varepsilon, D) \sum_{1\leqslant j\leqslant r_2-j_0} a_j(\varepsilon, D)\big(\psi_j(x') \times \delta(x_n)\big)\,, \qquad x \in \mathfrak{R}^+_n \tag{3.4.9}$$

$$\pi_0\, b_j(\varepsilon, D)\, u(x') = \varphi_j(x')\,, \qquad j_0 - r_2 < j \leqslant r_2 + r_3 \tag{3.4.10}$$

where again $Q(\varepsilon, D)$ is elliptic singular perturbation of order $\nu \in \mathfrak{R}^3$, $L(\varepsilon, D)$ is elliptic singular perturbation of order $\sigma = \big(0, 0, 2(r_2 - j_0)\big) \in \mathfrak{R}^3$, the operators $a_j(\varepsilon, D)$, $b_j(\varepsilon, D)$ of orders $\alpha_j = (\alpha_{j2}, \alpha_{j2}, \alpha_{j3})$, $\mu_j = (\gamma_j, m_j, \varrho_j)$ respectively satisfy the coerciveness condition $(a)_2$-$(b)_2$, (i)$_2$-(iv)$_2$ in the Section 3.2 and the Poisson operators $L^{-1}(\varepsilon, D)\,a_j(\varepsilon, D)$ are defined as above.

THEOREM 3.4.3. – *The problem* (3.4.9), (3.4.10) *being elliptic singular coercive perturbation and the conditions* (3.2.114), (3.4.8) *being fulfilled, the following a priori estimate holds for its solution* u, $\{\psi_j\}$, $1 \leqslant j \leqslant r_2 - j_0$: (*for* ε_0 *small enough*)

$$(3.4.11)\qquad \|u\|^+_{(s)} + \sum_{1\leqslant j\leqslant r_2-j_0} [\psi_j]_{(\beta_j)} \leqslant C\Big(\|f\|^+_{(s-\nu)} + \sum_{m_j < s_2 - \frac12} [\varphi_j]_{(\tau_j)} + \sum_{m_j > s_2-\frac12} [\varphi_j]_{(\sigma_j)} + \|u\|^+_{(s')}\Big) \leqslant$$
$$\leqslant C_1\Big(\|u\|^+_{(s)} + \sum_{1\leqslant j\leqslant r_2-j_0} [\psi_j]_{(\beta_j)}\Big),$$

where $\tau_j, \sigma_j, \beta_j \in \mathfrak{R}^3$ *are the same as in* (3.2.61), (3.2.125), s' *is any vector in* $\partial_1 \Gamma_3$, $s' \prec s$, *and the constant* C, C_1 *depend only on* s, s'.

PROOF. – One argues like in the proof of the previous theorem using the Theorem 3.2.22.

REMARK 3.4.4. – If the boundary operators b_j, $1 \leqslant j \leqslant r_2 + r_3$ cannot be completed by b_j $j_0 - r_2 < j \leqslant 0$ with polynomial symbols in such a way as not to conflict with the coerciveness condition $(\mathrm{i})_2$-$(\mathrm{iv})_2$, then they may be given as pseudodifferential operators with rational symbols. As it was mentioned above, the coerciveness condition in this case is stated in terms of determinants which are supposed not to vanish uniformly with respect to the parameters ε, ξ', those conditions being similar to (i)-(iv) in the Section 3.3.

3.5. *Coercive singular perturbations in bounded domain.*

We state in this section two-sided a priori estimates for singular perturbations with smooth variable coefficients in a bounded domain with smooth boundary.

Let $U \subset \mathfrak{R}^n$ be bounded domain with boundary ∂U which is supposed to be C^∞ oriented manifold.

A singular perturbation $Q(x, \varepsilon, D) \in P_\nu$, $x \in U$, $\varepsilon \in (0, \varepsilon_0]$, is called *elliptic in* U if (2.4.3) holds for any $x \in \overline{U}$.

Let $B_j(x, \varepsilon, D) \in P\mu_j$, $\mu_j = (\gamma_j, m_j, p_j)$ and denote by $b_{j0}(x', \varepsilon, D)$, principal non-homogeneous symbol of B_j, here $x' \in \partial U$, $1 \leqslant j \leqslant r_2 + r_3$, $2r_k = \nu_k$, $k = 1, 2$.

Assume that the orders m_j satisfy the condition (3.2.1).

As usual, let N and ξ' be respectively the inward normal and the cotangential variables at the point $x' \in \partial U$. We denote by $Q_0^+(x', \varepsilon, \xi' + \lambda N)$, $Q_{00}^+(x', \xi' + \lambda N)$ and $Q_0^{0+}(x', \xi' + \lambda N)$ respectively factors of the polynomials $\lambda \to Q_0(x', \varepsilon, \xi' + \lambda N)$, $\lambda \to$ $\to Q_{00}(x', \varepsilon, \xi' + \lambda N)$, $\lambda \to Q_0^0(x', \varepsilon, \xi' + \lambda N)$, which correspond to their zeros contained in the upper half of the complex λ-plane when $\xi' \in \mathfrak{R}^{n-1} \diagdown \{0\}$, $x' \in \partial U$, $\varepsilon \in (0, \varepsilon_0]$, where $Q_0(x', \varepsilon, \xi' + \lambda N)$, $Q_{00}(x', \xi' + \lambda N)$, $Q_0^0(x', \xi' + \lambda N)$ are respectively the principal non-homogeneous symbol of $Q(x', \varepsilon, \xi' + \lambda N)$, the principal homogeneous symbol of order $\nu_2 + \nu_3$ of $Q_0(x', 1, \xi' + \lambda N)$ and the reduced homogeneous symbol of order ν_2 for $Q_0(x', \varepsilon, \xi' + \lambda N)$ defined above in the Section 3.2.

The operators B_j, $1 \leqslant j \leqslant r_2 + r_3$, are supposed to satisfy the following *coerciveness condition*:

1) *The polynomials*

$$\lambda \to b^0_{j0}(x', \omega' + \lambda N), \quad 1 \leqslant j \leqslant r_2,$$

with $b^0_{j0}(x', \xi' + \lambda N)$ *reduced symbols for* B_j, *are linearly independent modulo* $Q_0^{0+}(x', \omega' + \lambda N)$, *for any* $\omega' \in \Omega_{n-1}$, *any* $x' \in \partial U$.

2) *The polynomials*

$$\lambda \to b_j(x', \lambda), \quad r_2 < j \leqslant r_2 + r_3,$$

with $b_j(x', \lambda) = b_{j0}(x', \lambda N)$ *are linearly independent modulo* $Q^+(x', \lambda)$, *for any* $x' \in \partial U$ *where* $Q^+(x', \lambda)$ *is the factor of the polynomial* $\lambda \to \lambda^{-\nu_2} Q_0(x', \lambda N)$, *which corresponds to its zeros contained in the upper half of the complex* λ*-plane.*

3) *The polynomials*

$$\lambda \to b_{j00}(x', \omega' + \lambda N), \quad 1 \leqslant j \leqslant r_2 + r_3.$$

are linearly independent modulo $Q^+_{00}(x', \omega' + \lambda N)$, *for any* $\omega' \in \Omega_{n-1}$, *any* $x' \in \partial U$.

4) *The polynomials*

$$\lambda \to b_{j0}(x', \varrho, \omega' + \lambda N), \quad 1 \leqslant j \leqslant r_2 + r_3,$$

are linearly independent modulo $Q_0^+(x', \varrho, \omega' + \lambda N)$ *for any* $\varrho \in (0, +\infty)$, *any* $\omega' \in \Omega_{n-1}$, *any* $x' \in \partial U$.

REMARK 3.5.1. – As a simple consequence of the Proposition 2.3.1, one gets the conclusion, that the coerciveness condition 1)-4) is invariant with respect to any local diffeomorphism which preserves the inward normal to the boundary ∂U.

Consider the following singularly perturbed boundary value problem:

$$Q(x, \varepsilon, D)\, u(x) = f(x), \quad x \in U, \tag{3.5.1}$$

$$\lim_{x \to x'} B_j(x, \varepsilon, D)\, u(x) = \varphi_j(x'), \quad x' \in \partial U,\ 1 \leqslant j \leqslant r_2 + r_3. \tag{3.5.2}$$

THEOREM 3.5.2. – *Singular perturbation* (3.5.1), (3.5.2) *being elliptic and coercive, the conditions* (3.2.1), (3.2.46), (3.2.69) *being fulfilled, the solutions to the problem* (3.5.1), (3.5.2) *satisfy the following a priori estimates*:

$$\|u\|_{(s)} \leqslant C\Big(\|f\|_{(s-\nu)} + \sum_{m_j < s_2 - \frac{1}{2}} [\varphi_j]_{(\tau_j)} + \sum_{m_j > s_2 - \frac{1}{2}} [\varphi_j]_{(\sigma_j)} + \|u\|_{(s')}\Big) \leqslant C' \|u\|_{(s)} \tag{3.5.3}$$

provided that $\varepsilon \in (0, \varepsilon_0]$ *with* ε_0 *sufficiently small; here* s' *is any vector in* $\partial_1\Gamma_s$, *the vectorial subscripts* $\tau_j, \sigma_j \in \mathfrak{R}^3$ *are the same as in* (3.2.61) *and the constants* C, C' *depend only on* s, s' *and* ε_0.

PROOF. – Using the standard argument by partition of unity for $\overline{U}$ and also the estimates (2.4.23) and (3.4.3) one gets (3.5.3), given that both (2.4.23) and (3.4.3) are two-sided a priori estimates. ■

Now we drop the assumption: $m_{r_2} < m_{r_2+1}$. Let $s_2 > m_{r_2+1} + \frac{1}{2}$ and denote by j_0, $j_0 > r_2$, such a subscript that (3.4.4) holds. Consider the following modification to the problem (3.5.1), (3.5.2):

$$(3.5.4)\qquad Q(x, \varepsilon, D)\,u(x) = f(x) + L^{-1}(x, \varepsilon, D) \sum_{1 \leqslant j \leqslant j_0 - r_2} a_j(x, \varepsilon, D)\big(\psi_j(x') \times \delta(x_n)\big)\,, x \in U\,,$$

$$(3.5.5)\qquad \lim_{x \to x'} B_j(x, \varepsilon, D)\,u(x) = \varphi_j(x')\,,\quad 1 \leqslant j \leqslant r_3 + j_0\,,\ x' \in \partial U\,,$$

where $L(x, \varepsilon, D)$ is elliptic order $\sigma = \big(0, 2(j_0 - r_2), 0\big)$, $L(x, \varepsilon, \xi) \neq 0$, $\forall x \in \overline{U}$, $\forall \varepsilon \in (0, \varepsilon_0]$, $\forall \xi \in \mathfrak{R}^n$, $a_j(x, \varepsilon, \xi' + \lambda N) \in \mathscr{P}_{\alpha_j}$, $\alpha_j \in \mathfrak{R}^3$, and $B_j(x, \varepsilon, \xi' + \lambda N) \in \mathscr{P}_{\mu_j}$ satisfy the *coerciveness condition* $(a)_1$-$(c)_1$, (i)$_1$-(iv)$_1$ at any point $x' \in \partial U$. The solution $u(x)$, $\{\psi_j(x')\}$, $1 \leqslant j \leqslant j_0 - r_2$, is sought in $\mathcal{H}_{(s)}(U) \times \prod_{1 \leqslant j \leqslant j_0 - r_2} \mathcal{H}_{(\Delta_j)}(\partial U)$ with Δ_j defined in (3.2.118).

THEOREM 3.5.3. – *Singular perturbation* (3.5.4), (3.5.5) *being elliptic and coercive, the conditions* (3.2.114), (3.4.4) *being fulfilled, the following a priori estimates hold for its solutions*:

$$(3.5.6)\qquad \|u\|_{(s)} + \sum_{1 \leqslant j \leqslant j_0 - r_2} [\psi_j]_{(\Delta_j)} \leqslant C\Big(\|f\|_{(s-\nu)} + \sum_{m_j < s_2 - \frac{1}{2}} [\varphi_j]_{(\tau_j)} + \sum_{m_j > s_2 - \frac{1}{2}} [\varphi_j]_{(\sigma_j)} + \|u\|_{(s')}\Big) \leqslant$$
$$\leqslant C'\Big(\|u\|_{(s)} + \sum_{1 \leqslant j \leqslant j_0 - r_2} [\psi_j]_{(\Delta_j)}\Big)\,,$$

provided that $\varepsilon \in (0, \varepsilon_0]$ *with* ε_0 *sufficiently small; here* s' *is any vector in* $\partial_1\Gamma_s$, τ_j, σ_j, Δ_j *are the same as defined above, and the constants* C, C' *depend only on* s, s' *and* ε_0.

Assume now that $s_2 < m_{r_2} + \frac{1}{2}$ and let j_0 be such a subscript that (3.4.8) holds. Consider the following modification to the problem (3.5.1), (3.5.2):

$$(3.5.7)\qquad Q(x, \varepsilon, D)\,u(x) = f(x) + L^{-1}(x, \varepsilon, D) \sum_{1 \leqslant j \leqslant r_2 - j_0} a_j(x, \varepsilon, D)\big(\psi_j(x') \times \delta(x_n)\big)\,,$$

$$(3.5.8)\qquad \lim_{x \to x'} B_j(x, \varepsilon, D)\,u(x) = \varphi_j(x')\,,\quad j_0 - r_2 < j \leqslant r_2 + r_3\,,\ x' \in \partial U\,,$$

where $L(x, \varepsilon, D)$ is elliptic singular perturbation of order $\sigma = \big(0, 0, 2(r_2 - j_0)\big)$,

$$L(x, \varepsilon, \xi) \neq 0\,,\quad \forall x \in \overline{U},\ \forall \varepsilon \in (0, \varepsilon_0],\ \forall \xi \in \mathfrak{R}^n,$$

and

$$a_j(x, \varepsilon, \xi' + \lambda N) \in \mathscr{P}_{\alpha_j}\,,\qquad \alpha_j \in \mathfrak{R}^3,\qquad B_j(x, \varepsilon, \xi' + \lambda N) \in \mathscr{P}_{\mu_j}\,,$$

satisfy the *coerciveness condition* $(a)_2$-$(c)_2$, (i)$_2$-(iv)$_2$ at any point $x' \in \partial U$.

THEOREM 3.5.4. – *Singular perturbation* (3.5.7), (3.5.8) *being elliptic and coercive, the conditions* (3.2.114), (3.4.8) *being fulfilled, the following a priori estimates hold for its solutions*:

$$(3.5.9)\qquad \|u\|_{(s)} + \sum_{1\leqslant j\leqslant r_2-j_0}[\psi_j]_{(\beta_j)} \leqslant C\Big(\|f\|_{(s-\nu)} + \sum_{m_j<s_2-\frac{1}{2}}[\varphi_j]_{(\tau_j)} + \sum_{m_j>s_2-\frac{1}{2}}[\varphi_j]_{(\sigma_j)} + \|u\|_{(s')}\Big) \leqslant C'\Big(\|u\|_{(s)} + \sum_{1\leqslant j\leqslant r_2-j_0}[\psi_j]_{(\beta_j)}\Big),$$

provided that $\varepsilon\in(0,\varepsilon_0]$ *with* ε_0 *sufficiently small*; *here* $s'\in\partial_1\Gamma_s$, β_j, τ_j, σ_j *are the same as above and the constants* C, C' *depend only on* s, s' *and* ε_0.

The proof of (3.5.6), (3.5.9) can be done using the same partition of unity argument as above, given that the estimates (2.4.23), (3.4.7) and (3.4.11) are two-sided.

REMARK 3.5.5. – Using the partition of unity for $\overline{U}$ and the estimates for singular perturbations with constant coefficients in $\mathfrak{R}^n$ and $\mathfrak{R}^n_+$ one can prove by the standard argument existence of left and right- quasi-inverse operators to singular perturbations (3.5.1)-(3.5.2), (3.5.4)-(3.5.5) and (3.5.7)-(3.5.8) respectively. Of course, as a simple consequence of the existence of quasi-inverse operator one gets again the a priori estimates (3.5.3), (3.5.6) and (3.5.9) respectively.

3.6. *Necessity of coerciveness condition.*

We show here that the coerciveness condition 1)-4) is necessary for (3.5.3) to hold.

THEOREM 3.6.1. – *If the a priori estimate* (3.5.3) *holds for the singular perturbation* (3.5.1), (3.5.2), *then it is elliptic coercive singular perturbation, i.e.* (3.5.1), (3.5.2) *satisfies the ellipticity condition* (2.4.3) *for any* $x\in\overline{U}$ *and the coerciveness condition* 1)-4) *for any* $x'\in\partial U$.

PROOF. – Taking in (3.5.3) the function u to be with compact support in U, one gets immediately, using the Theorem 2.4.11, that $Q(x,\varepsilon,\xi)$ has to be elliptic singular perturbation for any $x\in U$.

Now, let $x_0'\in\partial U$, and U_j be a small neighborhood of x_0 in $\mathfrak{R}^n$ such that there exists a diffeomorphism, mapping $U_j\cap\overline{U}$ onto a half ball $\overline{V}_r^+=\{y\in\overline{\mathfrak{R}^n_+},\ |y|<r\}$ and the point x_0' into the origin: $y=0$. Here $y=(y',y_n)$ are local coordinates related with the point x_0', such that the portion of the boundary $\partial U\cap U_j$ is given by the equation: $y_n=0$.

First assume that the ellipticity condition (*) is violated at the point $x_0'\in\partial U$. If the condition (i) is not fulfilled then taking in the a priori estimate (3.5.3) ε to be 1, one gets a contradiction given that the ellipticity condition for $Q_{00}(x,\xi)$ is necessary for the a priori estimate (3.5.3) with $\varepsilon=1$ to hold, as it follows from the classical elliptic theory. It follows also from the classical elliptic theory, without small parameter that the condition 3) should also be satisfied if (3.5.3) holds for some

(*) See page 57.

$\varepsilon > 0$. Now, assuming that the reduced symbol $Q_0^0(x_0', \xi)$ does not satisfy the condition (ii); taking in (3.5.3) u to be: $u(x, \varepsilon) = \varepsilon^{s_1} w(x)$ with $w(x) \in C^\infty(\overline{U})$ and letting $\varepsilon \to 0$, the a priori estimate (3.5.3) becomes usual a priori inequality for the reduced problem to (3.5.1), (3.5.2).

Therefore, the condition (ii) should necessarily hold for $Q_0^0(x, \xi)$ along with the condition 1). Assume now that the condition (iii) is violated at the point $x_0' \in \partial U$. Using the local coordinates y, denote by V_r the ball $\{|y| < r\}$ and let $v(y) \in C_0^\infty(V_r)$ such that $\|v\|_{(0)} > 0$. With η_0 real zero of the equation:

(3.6.1)
$$Q(x_0', 1, \eta_0) = 0\,,$$

we introduce the functions:

(3.6.2)
$$u(\varepsilon, y) = \exp(i\varepsilon^{-1} y \cdot \eta_0)\, v(y)\,.$$

Substituting $u(\varepsilon, y)$ into the estimate (3.5.3) one finds easily, given (3.6.1) that

(3.6.3)
$$\begin{aligned} &\|u\|_{(s)} \sim \varepsilon^{-s_1 - s_2} \|v\|_{(0)} \\ &\|Qu\|_{(s-\nu)} \sim \varepsilon^{-s_1 - s_2 + 1} \|v\|_{(0)} \\ &[B_j u]_{(\tau_j)} \sim \varepsilon^{-s_1 - s_2 + \frac{1}{2}} [\pi_0 v]_{(0)} \\ &[B_j u]_{(\sigma_j)} \sim \varepsilon^{-s_1 - s_2 + \frac{1}{2}} [\pi_0 v]_{(0)} \end{aligned}$$

as $\varepsilon \to 0$, where $\|v\|_{(0)}$ and $[\pi_0 v]_{(0)}$ are just L^2-norms of $v(y)$ and $\pi_0 v = v(y', 0)$ respectively in $\mathfrak{R}^n$ and $\mathfrak{R}^{n-1}$.

The formulas (3.6.3) lead to a contradiction with (3.5.3) as $\varepsilon \to 0$.

Now, assume that the condition 4) is violated at a point $x_0' \in \partial U$, and let $\varrho_0 > 0$, $\omega_0' \in \Omega_{n-1}$ be such that the polynomials $\lambda \to b_{j0}(x_0', \varrho_0, \omega_0' + \lambda N)$, $1 \leqslant j \leqslant r_2 + r_3$, are linearly dependent modulo $Q_0^+(x_0', \varrho_0, \omega_0' + \lambda N)$. Denote by $b_{j0}(\varrho_0, \omega_0', \lambda)$, $Q_0^+(\varrho_0, \omega_0', \lambda)$ these symbols rewritten in the local coordinates y.

Given the assumption above, the boundary value problem for ordinary differential equation on the half-line $\mathfrak{R}_+$:

(3.6.4)
$$\begin{aligned} &Q_0(\varrho_0, \omega_0', D_t)\, w(t) = 0\,, \qquad t > 0 \\ &\pi_0 b_{j0}(\varrho_0, \omega_0', D_t)\, w = 0\,, \qquad 1 \leqslant j \leqslant r_2 + r_3 \end{aligned}$$

has a non-trivial decreasing at $+\infty$ solution.

Introduce

(3.6.5)
$$u(\varepsilon, y) = \exp(i\varrho_0 \varepsilon^{-1} \omega_0' \cdot y')\, w(\varrho_0 \varepsilon^{-1} y_n)\, v(y', 0)$$

where again $v \in C_0^\infty(V_r)$, $[\pi_0 v]_0 > 0$, and $w(t)$ is non-trivial solution to (3.6.4).

Substituting $u(\varepsilon, y)$ defined by (3.6.5) into (3.5.3) and using (3.6.4) one gets contradiction as $\varepsilon \to 0$, given that the left hand side in (3.5.3) is of order $O(\varepsilon^{-s_2-s_1+\frac{1}{2}})$ at least while the right hand side is $O(\varepsilon^{-s_2-s_1+1})$ at most.

Finally, assuming that the condition 2) is violated, and denoting by $b_j(\lambda)$, $r_2 < j \leqslant r_2 + r_3$, $Q(\lambda)$ the symbols of corresponding ordinary differential operators rewritten in the local coordinate $y = (y', y_n)$ (with cotangent coordinate $\eta' = 0$), one can find a non-trivial decreasing at $+\infty$ solution to the problem on the half-line $\mathcal{R}_+$:

$$Q(D_t)\, w(t) = 0 \,, \qquad t > 0 \tag{3.6.6}$$

$$\pi_0 b_j(D_t)\, w = 0 \,, \qquad r_2 < j \leqslant r_2 + r_3 \,.$$

Define

$$u(\varepsilon, y) = w(\varepsilon^{-1} y_n)\, v(y', 0) \tag{3.6.7}$$

with $w(t)$ solution to (3.6.6) and $v(y', 0)$ as above.

Substituting (3.6.7) into (3.5.3) one gets once more a contradiction as $\varepsilon \to 0$, and that ends the proof of the Theorem 3.6.1. ■

REMARK 3.6.2. – Similar argument making use of asymptotic solutions to the singularly perturbed boundary value problems (3.5.4), (3.5.5) and (3.5.7), (3.5.8) shows that the coerciveness condition $(a)_1$-$(c)_1$, (i)$_1$-(iv)$_1$ and $(a)_2$-$(c)_2$, (i)$_2$-(iv)$_2$ are necessary for the a priori estimate (3.5.6) and (3.5.9) to hold uniformly with respect to $\varepsilon \in (0, \varepsilon_0]$.

3.7. *Examples.*

Some examples of singular perturbations are discussed in this section.

EXAMPLE 1. – Consider the singular perturbation:

$$(\varepsilon^2 \Delta^2 - \Delta)\, u = f(x) \,, \qquad x \in U \,, \tag{3.7.1}$$

$$u(x') = \varphi_1(x') \,, \qquad \frac{\partial u}{\partial N}(x') = \varphi_2(x') \,, \qquad x' \in \partial U \,. \tag{3.7.2}$$

One finds immediately that it is an elliptic coercive singular perturbation. Here $r_2 = r_3 = 1$, $\mu_1 = (0, 0, 0)$, $\mu_2 = (0, 1, 0)$ and $\nu = (0, 2, 2)$. The Theorem 3.5.2 asserts that for $s = (s_1, s_2, s_3)$ with $s_2 + s_3 > \frac{3}{2}$ and s_2 satisfying the inequalities:

$$\tfrac{1}{2} < s_2 < \tfrac{3}{2} \tag{3.7.3}$$

the following a priori estimate holds for the solutions of (3.7.1), (3.7.2):

$$\|u\|_{(s)} \leqslant C(\|f\|_{(s-\nu)} + [\varphi_1]_{(\tau)} + [\varphi_2]_{(\sigma)} + \|u\|_{(s')}) \tag{3.7.4}$$

with $s' < s$, $s' \in \partial_1 \Gamma_s$, and

$$(3.7.5) \quad \tau = s - \tfrac{1}{2} e_2 , \qquad \sigma = s - \tfrac{1}{2} e_2 + (s_2 - \tfrac{1}{2}) e , \qquad e = (1, -1, 1) , \qquad e_2 = (0, 1, 0) .$$

Actually, the term $\|u\|_{(s')}$ can be deleted, given that the solution to (3.7.1), (3.7.2) is unique.

If s_2 does not satisfy (3.7.3) there is a need to introduce Poisson operators with unknown densities to have a well posed problem uniformly stable with respect to $\varepsilon \in (0, \varepsilon_0]$.

EXAMPLE 2. – The problem

$$(3.7.6) \qquad (-\varepsilon^2 \Delta^2 - \Delta) u(x) = f(x) , \qquad x \in U$$

$$(3.7.7) \qquad u(x') = \varphi_1(x') , \qquad \frac{\partial u}{\partial N}(x') = \varphi_2(x') , \qquad x' \in \partial U$$

is not an elliptic singular perturbation, given that

$$Q_0(1, \xi) = - |\xi|^4 + |\xi|^2$$

vanishes for $|\xi| = 1$.

As a consequence of this fact even the estimate

$$(3.7.8) \qquad \|u\|_1 \leqslant C(\|f\|_{-1} + \|u\|_0)$$

for solutions to (3.7.6), (3.7.7) with $\varphi_1 = \varphi_2 = 0$ does not hold with constant C which does not depend on ε.

Indeed, substituting into (3.7.8) $u(\varepsilon, x)$ of the form

$$(3.7.9) \qquad u(\varepsilon, x) = \exp(ix \cdot \eta / \varepsilon)\, \psi(x)$$

with $\eta \in \mathfrak{R}^n$, $|\eta| = 1$, and $\psi \in C_0^\infty(U)$, $\|\psi\|_0 = 1$ one gets a contradiction as $\varepsilon \to 0$, given that the left hand side in (3.7.8) grows at least like ε^{-1} while the right hand side is bounded when $\varepsilon \to 0$. Here $\|\cdot\|_k$ is usual Sobolev norm in $H_k(U)$.

EXAMPLE 3. – Consider the singular perturbation:

$$(3.7.10) \qquad -\Delta u(x) = f(x) , \qquad x \in U$$

$$(3.7.11) \qquad \left(-\varepsilon \frac{\partial}{\partial N} + 1\right) u(x') = \varphi(x') , \qquad x' \in \partial U .$$

Here $\nu = (0, 2, 0)$, $r_2 = 1$, $r_3 = 0$, $\mu = (0, 0, 1)$.

One checks easily that it is an elliptic coercive singular perturbation and for $s = (s_1, s_2, s_3)$ with $s_2 + s_3 > \frac{3}{2}$, and s_2 satisfying the inequality:

$$s_2 > \tfrac{1}{2} \tag{3.7.12}$$

the following a priori estimate holds for the solution u to (3.7.10), (3.7.11):

$$\|u\|_{(s)} \leqslant C(\|f\|_{(s-\nu)} + [\varphi]_{(\tau)} + \|u\|_{(s')}) \tag{3.7.13}$$

where

$$\tau = s - \mu - \tfrac{1}{2} e_2 ,$$

and $s' < s$, $s' \in \partial_1 \Gamma_s$.

Again, actually the term $\|u\|_{(s')}$ can be removed given that the problem (3.7.10), (3.7.11) has unique solution.

EXAMPLE 4. – Consider the singular perturbation:

$$-\Delta u(x) = f(x) , \qquad x \in U \tag{3.7.14}$$

$$\left(\varepsilon \frac{\partial}{\partial N} + 1\right) u(x') = \varphi(x') , \qquad x' \in \partial U . \tag{3.7.15}$$

It is not coercive: the condition 4 fails to be fulfilled for $\varrho = 1$, $\omega' \in \Omega_{n-1}$.

Even an a priori estimate of the type

$$\|u\|_1 \leqslant C(\|f\|_{-1} + [\varphi]_{\frac{1}{2}} + \|u\|_0) \tag{3.7.16}$$

cannot hold for the solutions to (3.7.14), (3.7.15) with a constant C which does not depend on ε.

Indeed, assuming (3.7.16) and substituting there for $u(\varepsilon, x)$ the functions:

$$u(\varepsilon, x) = \exp(-x_n/\varepsilon + ix' \cdot \eta'/\varepsilon)\, \psi(x') , \qquad x_2 \geqslant 0 , \tag{3.7.17}$$

with (x', x_n) local coordinates in the neighborhood of a point $x_0' \in \partial U$ and $\psi(x') \in C_0^\infty(\mathcal{R}^{n-1})$, $[\psi]_0 > 0$, one gets contradiction as $\varepsilon \to 0$.

EXAMPLE 5. – It is easy to check that the singular perturbation:

$$(-\varepsilon^2 \Delta + 1)\, u(x) = f(x) , \qquad x \in U \tag{3.7.18}$$

$$\left(\frac{\partial}{\partial N} + \varepsilon \Delta'\right) u(x') = \varphi(x') , \qquad x' \in \partial U \tag{3.7.19}$$

with Δ' Laplace-Beltrami operator on the manifold ∂U, is coercive singular perturbation, while the same equation (3.7.18) with the boundary condition

$$\left(\frac{\partial}{\partial N} - \varepsilon\Delta'\right)u(x') = \varphi(x'), \qquad x' \in \partial U \tag{3.7.20}$$

is not (again, the condition 4 fails to be fulfilled in this case).

Here $\nu = (0, 0, 2)$, $r_2 = 0$, $r_3 = 1$, $\mu = (0, 1, 1)$.

EXAMPLE 6. – The singular perturbation

$$(\varepsilon^2\Delta^2 - \Delta)\,u(x) = f(x), \qquad x \in U \tag{3.7.21}$$

$$u(x') = \varphi_1(x'), \qquad \Delta u(x') = \varphi_2(x'), \qquad x' \in \partial U \tag{3.7.22}$$

is elliptic coercive with $\nu = (0, 2, 2)$, $r_2 = r_3 = 1$, $\mu_1 = (0, 0, 0)$, $\mu_2 = (0, 2, 0)$.

For $s_2 + s_3 > \frac{5}{2}$, $\frac{1}{2} < s_2 < \frac{5}{2}$ the following a priori estimate holds for its solutions:

$$\|u\|_{(s)} \leq C(\|f\|_{(s-\nu)} + [\varphi_1]_{(\tau)} + [\varphi_2]_{(\sigma)} + \|u\|_{(s')}) \tag{3.7.23}$$

where

$$\tau = s - \mu_1 - \tfrac{1}{2}e_2, \qquad \sigma = s - \mu_2 - \tfrac{1}{2}e_2 + (s_2 - \tfrac{5}{2})e$$

and $s' < s$, $s' \in \partial_1\Gamma_s$.

If s_2 is not included between $\frac{1}{2}$ and $\frac{5}{2}$ then there is a need to introduce Poisson type operators with unknown densities to have a uniformly stable problem with respect to $\varepsilon \in (0, \varepsilon_0]$.

EXAMPLE 7. – The singular perturbation:

$$(\varepsilon^2\Delta^2 - \Delta)\,u(x) = f(x), \qquad x \in U \tag{3.7.24}$$

$$u(x') = \varphi_1(x'), \qquad \left(\Delta - \varepsilon^2\frac{\partial^4}{\partial N^4}\right)u(x') = \varphi_2(x'), \qquad x' \in \partial U \tag{3.7.25}$$

is non coercive: the condition 2 fails to be fulfilled.

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