PHYSICAL REVIEW E **69**, 065102(R) (2004)

Coevolution of dynamical states and interactions in dynamic networks

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(Received 3 June 2003; published 11 June 2004)

We explore the coupled dynamics of the internal states of a set of interacting elements and the network of interactions among them. Interactions are modeled by a spatial game and the network of interaction links evolves adapting to the outcome of the game. As an example, we consider a model of cooperation in which the adaptation is shown to facilitate the formation of a hierarchical interaction network that sustains a highly cooperative stationary state. The resulting network has the characteristics of a small world network when a mechanism of local neighbor selection is introduced in the adaptive network dynamics. The highly connected nodes in the hierarchical structure of the network play a leading role in the stability of the network. Perturbations acting on the state of these special nodes trigger global avalanches leading to complete network reorganization.

DOI: 10.1103/PhysRevE.69.065102 PACS number(s): 89.75.Hc, 02.50.Le, 87.23.Ge, 89.65.−s

Recent studies on the structure of social, technological, and biological networks have shown that they share salient features that situate them far from being completely regular or random [1–3]. Most of the models proposed to construct these networks are grounded in a graph-theoretical approach, that is algorithmic methods to build graphs formed by elements (the nodes) and links that evolve according to prespecified rules. Despite the progress made, there are still several open questions [2]. An important issue to be considered among these questions is that networks are dynamical entities [4] that evolve and adapt driven by the actions of the elements that form a network.

The aim of this paper is to analyze a simple setting of such adaptive and evolving network, in which there is coevolution of the state of the elements in the nodes of the network and the interaction links defining the network. Interactions among elements are modeled with the aid of game theory [5], frequently applied in social, economic, and biological situations. This mathematical theory models an interaction involving two (or more) elements, each with two or more "strategies" or states, such that the outcome depends on the choices of all the interacting elements. The outcome is given in the form of a "utility" or payoff given to each element according to the selected action of the interacting elements. The introduction of spatial interactions lead to the development of "spatial games" [6–8], in which the elements are located in the nodes of a fixed network of interaction, displaying a rich spatiotemporal dynamics. Here, we go beyond these studies by introducing adaptation (*plasticity*) in the coupling between elements, so that the network of interaction evolves adapting to the outcome of the game. Our results include new asymptotic steady states, and the emergence of a hierarchical network structure that governs the global dynamics of the system.

The model. We consider a system composed of N elements whose interactions are specified by a network $\mathcal{N}$. The neighborhood of element i ($\mathcal{V}_i$) is composed of those elements directly connected to i by one link, and the size of $\mathcal{V}_i$ defines its *degree* k_i. The state of each element x_i can be $(1,0)$ or $(0,1)$. In each step (generation), every ith element interacts with all other elements inside its neighborhood $\mathcal{V}_i$, and accumulates a payoff $\Pi_i=\Sigma_{j\in\mathcal{V}_i} x_i\mathbf{J}x_j^T$, depending on the chosen states x_i and payoff matrix

$$\mathbf{J}=\begin{pmatrix}\pi_{00} & \pi_{01}\\ \pi_{10} & \pi_{11}\end{pmatrix}.$$

The ith element compares its own payoff with all $j\in\mathcal{V}_i$ and changes its state to the state of the site with the greatest payoff in $\{i\}\cup\mathcal{V}_i$ [7]. The *plasticity* of the network is introduced here as network dynamics in which existing links can be severed and replaced by new ones. We make the assumption that whether an interaction link is severed depends on the joint payoff, that is the total payoff by the pair of interacting elements: the interactions giving the lowest benefit will be removed.

In the remainder, for the sake of concreteness, we will address the case of the Prisoner's Dilemma (PD) game, which has been widely used as a model displaying complex behavior emerging from the competition between cooperative and selfish behavior [6]. In its simplest form, two elements may either choose to cooperate $[C,x_{\mathrm{C}}=(1,0)]$, or defect $[D,x_{\mathrm{D}}=(0,1)]$. If both elements choose C, each gets a payoff π_{00}; if one defects while the other cooperates, the former gets payoff $\pi_{10}>\pi_{00}$, while the latter gets the "suckers" payoff $\pi_{01}<\pi_{00}$; if both defect, each gets π_{11}. Under the standard restrictions $\pi_{10}+\pi_{01}<2\pi_{00}$, $\pi_{10}>\pi_{00}>\pi_{11}>\pi_{01}$, defection is the best choice in a one-shot game resulting in a Nash equilibrium in which both elements defect. Following previous studies [7,8], we consider a simplified version of the game given by the interaction matrix $\pi_{00}=1$, $\pi_{10}=b$, $\pi_{11}=\epsilon$, $\pi_{01}=0$, in the limit $\epsilon=0$ [9].

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In this context, the dynamical rule proposed for local neighborhood adaptation—*plasticity*—is defined by analyzing the joint benefit obtained by each of the possible pairwise interactions: C-C, C-D, and D-D. Thus, according to the payoff obtained, the worst interaction is clearly observed in a D-D situation, in which both elements will be better off by searching for a new partner. Given this simplistic analysis, taking into account that we are considering undirected links, and assuming that the probability to rewire a C-D interaction is much smaller than to rewire a D-D interaction, our implementation of plasticity will allow Defectors to exchange (probabilistically) a D-neighbor by another randomly chosen element.

Thus, the game is divided in three stages. (i) Each element i plays the PD game with the same current state with all its neighbors, and collects an aggregate payoff Π_i. (ii) Each element i updates its current state by comparing its payoff with its neighbors and *imitates* the state of the wealthiest element. An element is said to be *satisfied* if its own payoff is the highest among its neighbors (otherwise it is unsatisfied). (iii) Unsatisfied D-elements that imitate a Defector, replace this link with probability p by a new one pointing to a randomly chosen element.

The plasticity parameter p leads to a time evolution of the local connectivity of the network, leaving the average degree $\langle k_i \rangle$ constant. The parameter p sets a time scale for the evolution of the network with respect to the state update. In general we expect $p \ll 1$, so that the state update evolves in a much faster time scale than the network evolution, while $p=1$ represents the limit of simultaneous update of interactions and states.

We have characterized numerically the model using $N=10\ 000$ elements, averaged over 100 different random initial conditions, with an initial population of $0.6N$ Cooperators randomly distributed in the network [10]. The initial network is generated by randomly distributing $N\langle k_i \rangle/2$ links. A prototype value of $\langle k_i \rangle = 8$ was chosen in order to secure an initial large connected component. The game is played *synchronously*; that is, elements decide their state in advance and they all play at the same time.

Stationary states. To characterize the macroscopic behavior of the system, we introduce the fraction $\rho_C(t)$ of cooperators at a given time. We define the order parameter ρ_C as the average over realizations of the stationary Cooperators' density. In the case of random mixing (i.e., in the absence of an interaction network), population dynamics gives [11] $\dot{\rho}_C = \rho_C^2(1-\rho_C)(1-b)$. Thus, for $b>1$, the only stable solution corresponds to a fully defective population. For fixed networks ($p=0$), a typical time evolution shows in general that the order parameter fluctuates around an average value that decreases as the incentive b to defect increases (Fig. 1). At $b \simeq 2$, the Defectors dominate the network [12]. For fixed networks, the precise value for this transition has been studied in detail [7,12,13]. In contrast to random mixing, *context preservation* (fixed interactions) *sustains partial cooperation* [14].

This picture changes when the elements turn on their plasticity behavior ($p>0$) (see Fig. 1). Extensive numerical simulations show that the system either reaches a stationary configuration with $\rho_C > 0$ (where the states and the network do not change in time), or an absorbing state with all elements being Defectors $\rho_C = 0$. The *cooperative phase* —the stationary states with a large value of ρ_C— is formed by a set of solutions corresponding to different network configurations and distribution of Cooperators. In Fig. 1, we characterize these states showing that $\rho_C > 0.8$, a value always much larger than in the nonadaptive case. Slight variations exist for different $\langle k \rangle \geq 4$. The crucial difference is the disappearance of the behavior observed in the case $p=0$ in which, upon increasing b, the large majority of the realizations reaches a configuration with a very low fraction of Cooperators. The plasticity parameter p changes the time it takes to reach the stationary state: smaller p produce longer transients.

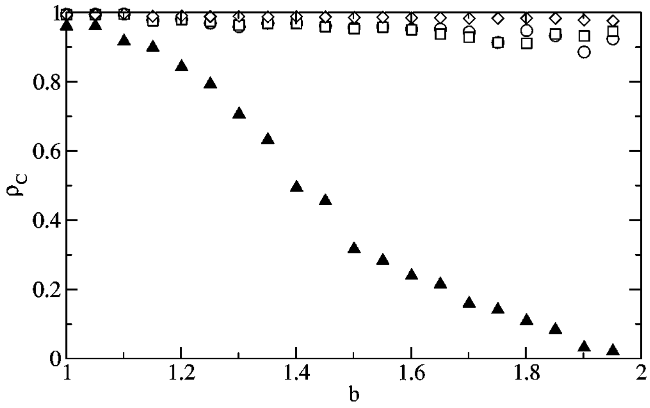

FIG. 1. Average fraction of Cooperators ($\langle \rho_C \rangle$) as a function of b and p in the stationary regime. The defective phase $\rho_C=0$ is not included in the averages of ρ_C. ($p=0$: full triangles; $p=0.01$: circles; $p=0.1$; squares; $p=1$: diamonds).

Network structure. In order to understand how such a highly cooperative structure can be sustained, we analyze the implications of the proposed dynamical rules in the network structure. Consider that element i updates its state imitating the state of element j; we define the correspondence $l:\mathcal{N} \to \mathcal{N}$ such that $l(i)=j$. Focusing only on those links, we identify the *imitation network* as the subnetwork composed of directed links $i \to l(i)$ (Fig. 2). *Necessary and sufficient* conditions for a stationary state ($\rho_C > 0$, $p \neq 0$) are (a) there are no links between two Defectors, and (b) each C-neighbor i of a Defector γ satisfies the payoff relation

$$\Pi_j > \Pi_\gamma > \Pi_i, \quad j = l(i). \tag{1}$$

In other words, in a stationary state, *all* Defectors become satisfied interacting only with Cooperators, while Cooperators can be unsatisfied while imitating other Cooperators. These steady state conditions naturally imply that the element with largest payoff in a stationary configuration is a *satisfied Cooperator*. In Fig. 2, we show a partial view (the nodes in the lowest level are not shown) of an imitation network, where the nodes in a layer imitate those elements in an upper layer indicated by the directed edges. At the top of the figure lie the nodes whose action is imitated by a chain of Cooperators.

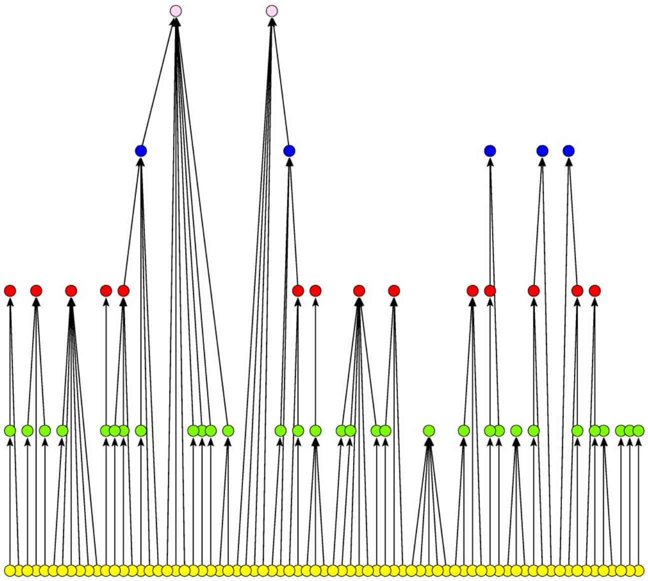

FIG. 2. (Color online) Partial view of a sample imitation network in a steady state. Elements on a lower layer imitate the state of elements in an upper layer.

A first characterization of how the structure of the cooperative stationary network configurations changes as a function of b and p is obtained by measuring the normalized degree variance $\sigma_n^2=(\langle k_i^2\rangle-\langle k_i\rangle^2)/\langle k_i\rangle$ (Fig. 3). We find that the degree distribution departs significantly from the initial Poisson distribution ($\sigma_n^2=1$) only for large values of the plasticity parameter p. For increasing b, the tail of the degree distribution expands and approaches an exponential form, indicating some elements become more connected than others (hubs).

We now address the question of whether the structure generated in our dynamical model has the characteristics of a *small world* network [1]. The clustering coefficient c measures the fraction of neighbors of a node that are connected among them, averaged over all the nodes in the network. In our simulations we find (Fig. 4) that the clustering coefficient increases very mildly with respect to a fixed random network $c_{rand}=\langle k_i\rangle/N$ [15]. Thus, even though the average path length is similar to a random network, in order to account for the

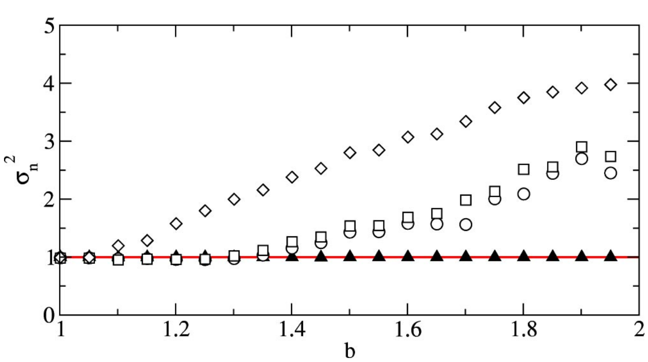

FIG. 3. Normalized variance $\sigma_n^2=(\langle k_i^2\rangle-\langle k_i\rangle^2)/\langle k_i\rangle$ as a function of b. The solid line ($\sigma_n^2=1$) corresponds to the fixed random network with a Poisson distribution of degree. Parameter values as in Fig. 1.

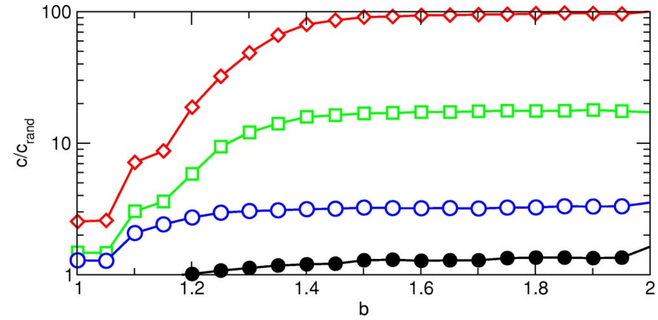

FIG. 4. Normalized clustering coefficient c/c_{rand} as a function of b. For $p=0$, we recover the random value. Open symbols for $q=0$ (diamonds: $p=0.01$; squares: $p=0.1$; circles: $p=1$); filled circles correspond to $p=1$ and $q=1$.

high clustering, we need to introduce "local" neighbor selection [16]. This mechanism is easily implemented introducing a parameter q that modifies step (iii), so that with probability q, the new neighbor is selected among the neighbors of the neighbors; otherwise, with probability $1-q$, the random neighbor is chosen. We find that, while most of our results previously discussed are qualitatively independent of the value of q, the clustering coefficient reaches a very large value even for a small value of q. For instance, just 1% ($q=0.01$) of local neighbor selection is enough to increase c a hundred times, being the clustering largest for a slow evolution of the network ($p\ll 1$). In addition, the clustering coefficient decreases slightly with system size, an indication of a decay slower than the N^{-1} decay expected for random graphs. All together, our results indicate that local neighbor selection is needed in order to generate a small world network.

It is worth noting that an evolutionary model based on the PD game with a more complex strategy representation also shows, in the absence of local neighbor selection, that the increase of the clustering coefficient can be related to the change of the degree distribution [15]. In contrast with Ref. [15], we do not observe a power law degree distribution.

Dynamics: global avalanches and network stability. The hierarchical structure of the network is of fundamental importance to the dynamics on the system. A closer look at the evolution towards a stationary state indicates the presence of avalanches (Fig. 5). The transient dynamics is characterized in general by large oscillations in $\rho_C(t)$. When the payoff of

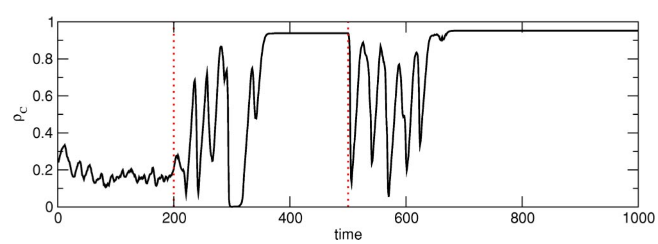

FIG. 5. Time evolution of ρ_C. The evolution starts in a fixed random network ($p=0$) up to time $t=200$ when network dynamics is switched on, so that $p=1$ for $t>200$. At time $t=500$, the state of the node with largest payoff is forced from C to D. Parameter $b=1.7$.

any D element increases above the upper limit of Eq. (1), an avalanche towards defection is triggered. This D element will be imitated by C neighbors, and each will initiate an avalanche of replication of D state through all those elements connected by the imitation network. During the avalanche, recovery of cooperation is possible through those satisfied Cooperators, which rebuild the hierarchical topology [17].

The description in terms of the imitation network also indicates the vulnerability of the structure to stochastic fluctuations. Figure 5 illustrates the sensitivity of the stationary network structure to perturbations acting on the highly connected nodes, which reflects their key role in sustaining cooperation. At time $t=500$, the most connected node is externally forced to change state from C to D, triggering an avalanche. Notice the large oscillations in ρ_C, reproducing the transient dynamics in which the system searches for a new stationary globally cooperative structure.

Conclusion. We have addressed the general question of network formation from the perspective of coevolution between the dynamics of the elements' state and the interactions network. Our model of cooperation with network plasticity leads to hierarchical topologies [18], the emergence of global cascades, [19,20] and vulnerability to attacks acting on specific targets [21]. The hierarchical interaction network is reached as a stationary network starting from a random network of interactions. The network appears structured from a few highly connected elements easily identified through the imitation network. Such a network has the characteristics of a small world when a mechanism of local neighbor selection is introduced in the adaptive dynamics of the network. The hierarchical structure supports a stationary, highly cooperative state for general situations in which, for a fixed network, the system would not settle in a stationary state and in which the cooperation level would be much smaller. The stability of the network is very sensitive to changes in state of the few highly connected nodes: external perturbations acting on these nodes trigger global avalanches, leading to transient dynamics in which the network completely reorganizes itself searching for a new, highly cooperative stationary state. Future work should explore the robustness of these results in slightly different settings. For instance, we have checked that the same qualitative results are obtained with asynchronous update regarding Fig. 1, and that adding continuous noise weakens the cooperative phase by the spontaneous occurrence of avalanches. Work along these lines is in progress.

We acknowledge financial support from MCyT (Spain) and FEDER (EU) through Projects CONOCE and BFM2002-04474-C02-01.

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[9] We checked, as in Ref. [12], that considering $0.1 > \epsilon \geqslant 0$, no noticeable difference is found in this game.

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