

## COGROWTH OF REGULAR GRAPHS

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**ABSTRACT.** Let  $\mathcal{G}$  be a  $d$ -regular graph and  $T$  the covering tree of  $\mathcal{G}$ . We define a cogrowth constant of  $\mathcal{G}$  in  $T$  and express it in terms of the first eigenvalue of the Laplacian on  $\mathcal{G}$ . As a corollary, we show that the cogrowth constant is as large as possible if and only if the first eigenvalue of the Laplacian on  $\mathcal{G}$  is zero. Grigorchuk's criterion for amenability of finitely generated groups follows.

In this note, we shall relate the first eigenvalue of the Laplacian on a connected regular graph to the size of the kernel of the universal covering map. The main results have been proven in [C, G, P]. The proof presented here appears simpler; it depends on the explicit formula for minimal positive solutions of  $\Delta F + \varepsilon F = -I$ .

Let  $\mathcal{G}$  be a connected simple graph with constant vertex degree  $d \geq 3$ ,  $T$  be the universal covering tree of  $\mathcal{G}$ , and  $\theta$  the covering map (i.e.,  $\theta$  is a vertex surjection of  $T$  on  $\mathcal{G}$  that preserves adjacency and vertex degree). We let  $T$  and  $\mathcal{G}$  denote the vertex sets of the corresponding graphs. Note that  $T$  has constant vertex degree  $d$ . Since  $T$  is connected,  $T$  may be considered a metric space with the usual graph metric  $\delta$  ( $\delta(x, y)$  is the length of the shortest path connecting  $x$  and  $y$ ). For  $x \in T$  and  $n \geq 0$ , let  $[x] = \theta^{-1}(\theta(x))$  and  $S_n(x) = \{y: \delta(x, y) = n\}$ . For  $x, y \in T$ , note that

$$\limsup_{n \rightarrow \infty} |[y] \cap S_n(x)|^{1/n} = \inf \left\{ \lambda > 0: \sum_{z \in [y]} \lambda^{-\delta(x, z)} < \infty \right\}$$

and is thus independent of  $x$  and  $y$ . We call this number,  $\text{cogr}(T, \mathcal{G})$ , the cogrowth constant of  $\mathcal{G}$  in  $T$ .

For  $x, y$  vertices of a graph, we write  $xEy$  if  $x$  and  $y$  are connected by an edge. For  $x, y \in T$ , let

$$q(x, y) = \begin{cases} \frac{1}{d} & \text{if } xEy, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $q$  is the transition matrix of the simple random walk on  $T$ . Let  $q^{(n)}$  denote the  $n$ th power of  $q$ . For  $a, b \in \mathcal{G}$  and  $x \in \theta^{-1}(a)$ , since  $\theta$  takes the

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simple random walk on  $T$  to the simple random walk on  $\mathcal{G}$ ,

$$(1) \quad p^{(n)}(a, b) = \sum_{y \in \theta^{-1}(b)} q^{(n)}(x, y)$$

where  $p$  is the transition matrix of the simple random walk on  $\mathcal{G}$ . We define  $\Delta \equiv p - 1$ ,  $G \equiv \sum_{n \geq 0} p^{(n)}$ , and for  $\varepsilon \geq 0$ ,  $G^\varepsilon \equiv \sum_{n \geq 0} p^{(n)} / (1 - \varepsilon)^{n+1}$ . Similarly, we define  $\Delta_T, F$ , and  $F^\varepsilon$  as above with  $p$  replaced by  $q$ .

$\Delta$  and  $\Delta_T$  are the Laplacians on  $\mathcal{G}$  and  $T$ , respectively. We call  $\lambda_{\mathcal{G}}$  and  $\lambda_T$  the first eigenvalues of  $\Delta$  and  $\Delta_T$ , respectively. Since  $T$  is  $d$ -regular,

$$(2) \quad \lambda_T = 1 - 2(d - 1)^{1/2} / d$$

(see [DK]). Also,

$$(3) \quad \lambda_{\mathcal{G}} = \sup\{\lambda: \exists f > 0: \Delta f + \lambda f \leq 0\}$$

(see [DK] or [N]). It is true that  $\lambda_{\mathcal{G}} \leq \lambda_T$  (see [N]).

For  $\varepsilon \leq \lambda_T$ , let  $a(\varepsilon) = d(1 - \varepsilon) / (d - 1)$ ,  $b = 1 / (d - 1)$ , and  $\sigma(\varepsilon) \leq \tau(\varepsilon)$  be the (real) roots of  $t = a(\varepsilon) - b/t$ . Note that

$$\begin{aligned} \sigma(\varepsilon) &= \{d(1 - \varepsilon) - [d^2(1 - \varepsilon)^2 - 4d + 4]^{1/2}\} / 2(d - 1), \\ \tau(\varepsilon) &= \{d(1 - \varepsilon) + [d^2(1 - \varepsilon)^2 - 4d + 4]^{1/2}\} / 2(d - 1). \end{aligned}$$

In particular,  $\sigma(\varepsilon)$  is increasing and  $\tau(\varepsilon)$  is decreasing on  $[0, \lambda_T)$ .

**Lemma.** For  $\varepsilon \in [0, \lambda_T)$ ,  $F^\varepsilon(x, y) = \sigma(\varepsilon)^{\delta(x,y)} / (1 - \varepsilon - \sigma(\varepsilon))$ .

*Proof.* Let  $\varepsilon \in [0, \lambda_T)$ . For  $\lambda \in (\varepsilon, \lambda_T)$ , there exists a function  $f > 0$  such that  $\Delta_T f + \lambda f \leq 0$ . Let  $v = -(\Delta_T f + \varepsilon f) / (1 - \varepsilon)$  and  $r = q / (1 - \varepsilon)$ . Note that  $v > 0$  and  $f = v + qf$ . By induction,  $f = \sum_{0 \leq k \leq n} r^{(k)} v + r^{(n+1)} f \geq \sum_{0 \leq k \leq n} r^{(k)} v$  since  $f > 0$ . Letting  $n \rightarrow \infty$ ,  $f \geq \sum_{k \geq 0} r^{(k)} v = (1 - \varepsilon) F^\varepsilon v$ . Since  $v > 0$ ,  $F^\varepsilon$  exists.

By the symmetry of  $T$ , there exists a sequence  $\gamma_0, \gamma_1, \dots$  such that for any  $x$ , if  $\delta(x, y) = n$  then  $F^\varepsilon(x, y) = \gamma_n$ . Since  $(\Delta_T + \varepsilon) F^\varepsilon = -I$ , it follows that  $\gamma_{k+2} = a\gamma_{k+1} - b\gamma_k$  and  $\gamma_1 - (1 - \varepsilon)\gamma_0 = -1$ . Let  $r_k = \gamma_{k+1} / \gamma_k$ ,  $\mu = a/2$ , and  $\nu = [a^2/4 - b]^{1/2}$ . By the addition of angle formulae for hyperbolic functions, it is easy to verify that for all  $r_0$  there exists  $\theta$ , so that

$$r_n = \begin{cases} \mu + \nu \tanh(\theta + \rho n) & \text{if } r_0 \in (\sigma, \tau), \\ \mu + \nu \coth(\theta + \rho n) & \text{if } r_0 \notin [0, \tau], \\ r_0 & \text{if } r_0 \in \{\sigma, \tau\}, \end{cases}$$

where  $\rho = \tanh(\nu/\mu)$ .

Clearly, if  $r_0 \neq \sigma$  then  $r_n \rightarrow \tau$ , and thus  $\lim_{n \rightarrow \infty} \gamma_n^{1/n} = \tau$ . It is easy to verify that  $\lim_{n \rightarrow \infty} \gamma_n^{1/n}$  is increasing as a function of  $\varepsilon$  (since  $p^{(n)} \geq 0$ ). Therefore  $r_n \equiv \sigma$  since  $\tau$  is decreasing. It follows that  $F^\varepsilon(x, y) = c\sigma^{\delta(x,y)}$  and, since  $\tau_1 - (1 - \varepsilon)\gamma_0 = -1$ ,  $c = 1/[1 - \varepsilon - \sigma]$ .  $\square$

**Theorem.** (a)  $\text{cogr}(T, \mathcal{G}) \leq \{d(1 - \lambda_{\mathcal{G}}) + [d^2(1 - \lambda_{\mathcal{G}})^2 - 4d + 4]^{1/2}\} / 2$ ,

(b) If  $\lambda_{\mathcal{G}} \neq \lambda_T$  then  $\text{cogr}(T, \mathcal{G}) = \{d(1 - \lambda_{\mathcal{G}}) + [d^2(1 - \lambda_{\mathcal{G}})^2 - 4d + 4]^{1/2}\} / 2$ .

*Proof.* (a) If  $\lambda_{\mathcal{G}} = 0$  then  $\text{cogr}(T, \mathcal{G}) \leq \limsup_{n \rightarrow \infty} |S_n(x)|^{1/n} = d - 1 = 1/\sigma(\lambda_{\mathcal{G}})$ .

Let  $\lambda_{\mathcal{G}} > 0$  and  $\varepsilon \in (0, \lambda_{\mathcal{G}})$ . As in the proof of the lemma,  $G^\varepsilon$  exists. Since

$$\sum_{z \in [y]} \sigma^{\delta(x, z)} = c \sum_{z \in [y]} F^\varepsilon(x, z) = cG^\varepsilon(\theta(x), \theta(y)) < \infty,$$

$\text{cogr}(T, \mathcal{G}) \leq 1/\sigma(\varepsilon)$ . Since  $\sigma(\varepsilon)$  is increasing, the result follows by letting  $\varepsilon$  approach  $\lambda_{\mathcal{G}}$ .

(b) Let  $\varepsilon \in [0, \lambda_T)$ . If  $\text{cogr}(T, \mathcal{G}) < 1/\sigma(\varepsilon)$ , then

$$G^\varepsilon(\theta(x), \theta(y)) = \sum_{z \in [y]} F^\varepsilon(x, z) = \sum_{z \in [y]} \sigma^{\delta(x, z)} < \infty$$

and thus  $G^\varepsilon$  exists. Fix  $g \in \mathcal{G}$  and let  $f(x) = G^\varepsilon(g, x)$ . Clearly  $\Delta_T f + \varepsilon f \leq 0$  and  $f > 0$  and, therefore,  $\varepsilon \leq \lambda_{\mathcal{G}}$ . Assume  $\lambda_{\mathcal{G}} \neq \lambda_T$  (and thus  $\lambda_{\mathcal{G}} < \lambda_T$ ). If  $\lambda_{\mathcal{G}} < \lambda_{\mathcal{G}} + \kappa \leq \lambda_T$ , then  $\text{cogr}(T, \mathcal{G}) \geq 1/\sigma(\lambda_{\mathcal{G}} + \kappa)$ . Since  $1/\sigma$  is decreasing on  $[0, \lambda_{\mathcal{G}}]$ ,  $\text{cogr}(T, \mathcal{G}) \geq 1/\sigma(\lambda_{\mathcal{G}})$ .  $\square$

**Corollary 1.** *Let  $\mathcal{G}$  be connected and  $d$ -regular. Then  $\text{cogr}(T, \mathcal{G}) = d - 1$  iff  $\lambda_{\mathcal{G}} = 0$ .*

Let  $A$  be a finitely generated discrete group with  $k$  generators,  $F$  the free group with  $k$  generators,  $\phi$  the canonical mapping of  $F$  onto  $A$ , and  $K = \ker \theta$ .

The map  $\phi$  induces a covering map  $\theta$  from  $T$  onto  $\mathcal{G}$  where  $T$  and  $\mathcal{G}$  are the Cayley graphs of  $F$  and  $A$  respectively. As is well known,  $A$  is amenable iff  $\lambda_{\mathcal{G}} = 0$  (see [K, DK, DG]).

By [P],  $\lim_{n \rightarrow \infty} |K \cap S_{2n}|^{1/2n}$  exists.

**Corollary 2.**  *$A$  is amenable iff  $\lim_{n \rightarrow \infty} |K \cap S_{2n}|^{1/2n} = 2k - 1$ .*

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