Cohen-Macaulay Fiber Cones

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1 Introduction

Let (R, m) be a local ring and let I be an ideal of R. The *fiber cone* of I is the graded ring $F(I) := \oplus_{n \geq 0} I^n/mI^n$. In this paper we will give a necessary and sufficient condition, in terms of Hilbert series of $F(I)$, for it to be Cohen-Macaulay (CM). We apply this criterion to a variety of examples.

Cohen-Macaulay fiber cones have been studied by K. Shah [?] and T. Cortadellas and S. Zarzuela [?]. Shah characterized Cohen-Macaulay fiber cones terms of their Hilbert function. To recall his result let us introduce some notation and terminology. An ideal $J \subseteq I$ is called a reduction of I if $JI^n = I^{n+1}$ for some n. We say J is a minimal reduction of I if whenever $K \subseteq I$ and K is a reduction of I, then $K = J$. These concepts were introduced by Northcott and Rees in [NR]. They showed that when R/m is infinite, any minimal reduction of I is minimally generated by dim $F(I)$ elements. The Krull dimension of the fiber cone $F(I)$ is an important invariant of I and it is called the *analytic spread* of I. We denote it by $a(I)$. It is known [?] that ht $I \leq a(I) \leq \min(\mu(I), \dim R)$ where $\mu(I) = \dim I/mI$. Let ℓ denote the length of the module. We can now state the necessary condition found by Shah for $F(I)$ to be CM.

Theorem 1.1 (Shah) *Let (R, m) be a local ring and let I be an ideal in R of analytic spread a. Let J be a minimal reduction of I. If $F(I)$ is CM, then for all $n \geq 0$*

$$\mu(I^n) = \sum_{i \geq 0} \left[\mu(I^i) - \ell \left(\frac{JI^{i-1}}{JI^{i-1} \cap mI^i} \right) \right] \binom{n+a-i-1}{a-1}.$$

K. Shah conjectured that the converse is also true. In this paper we prove the converse. The proof is presented in section two. In section three we define ideals of minimal mixed multiplicity in CM local rings and calculate the Hilbert series of their fiber cone. It turns out that for such ideals the fiber cone is CM if and only if $r(I) \leq 1$. By using this result we construct zero dimensional ideals in a two dimensional regular local ring whose fiber cones are not CM. Section four is about fiber cones of ideals generated by quadratic sequences. In the final section we provide a few other examples of CM fiber cones.

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2 Cohen-Macaulay fiber cones

In this section we state K. Shah's result in terms of Hilbert series of $F(I)$. For the sake of completeness we give an alternate proof and prove the converse. We will denote the multiplicity of $F(I)$ by $e(F(I))$. We know that if $F(I)$ is CM then $e(F(I)) \geq \mu(I) - a + 1$ [**?**]. When equality holds, we say that $F(I)$ has *minimal multiplicity*. The reduction number of I with respect to a minimal reduction J of I is defined as

$$r_J(I) = \min\{n \geq 0 \mid JI^n = I^{n+1}\}.$$

The reduction number of I denoted by $r(I)$ is defined to be the minimum of $r_J(I)$ where J varies over all minimal reductions of I.

Theorem 2.1 *Let (R, m) be a local ring and let I be any ideal of R. Let J be any minimal reduction of I. Then the following are equivalent:*

1. *$F(I)$ is CM.*
2. *$H(F(I), t) = \dfrac{1}{(1-t)^a} \displaystyle\sum_{i=0}^{r} \ell \left(\frac{I^i}{JI^{i-1} + mI^i} \right) t^i$ where $r = r_J(I)$ and $a = \dim F(I)$.*
3. *$e(F(I)) = \displaystyle\sum_{n=0}^{r} \ell \left(\frac{I^n}{JI^{n-1} + mI^n} \right)$.*

Proof: (1) $\Rightarrow$ (2). Suppose that $F(I)$ is CM. Since $JF(I)$ is generated by a homogeneous system of parameters, it is generated by a regular sequence. Thus

$$H(F(I), t) = \frac{1}{(1-t)^a} H \left(\frac{F(I)}{JF(I)}, t \right).$$

Notice that

$$JF(I) = \bigoplus_{i=1}^{r} \left(\frac{JI^{i-1} + mI^i}{mI^i} \right) \bigoplus \left[\bigoplus_{i=r+1}^{\infty} \left(\frac{I^i}{mI^i} \right) \right].$$

Hence

$$\frac{F(I)}{JF(I)} = \frac{R}{m} \bigoplus \left[\bigoplus_{i=1}^{r} \left(\frac{I^i}{JI^{i-1} + mI^i} \right) \right].$$

This gives

$$H\left(\frac{F(I)}{JF(I)}, t \right) = \sum_{i=0}^{r} \ell \left(\frac{I^i}{JI^{i-1} + mI^i} \right) t^i.$$

which gives (2).

(2) $\Rightarrow$ (3). Put $t = 1$ in the numerator of $H(F(I), t)$.

(3) $\Rightarrow$ (1). Let $\mathcal{M}$ denote the maximal homogeneous ideal of $F(I)$. It is enough to show that $F(I)_{\mathcal{M}}$ is CM by [?]. Since $F(I)$ is homogeneous graded ring, the associated graded ring $gr(\mathcal{M}F(I)_{\mathcal{M}}, F(I)_{\mathcal{M}}) \cong F(I)$. Thus $e(F(I)) = e(F(I)_{\mathcal{M}})$. It is also clear that $\ell(F(I)/JF(I)) = \ell(F(I)_{\mathcal{M}}/JF(I)_{\mathcal{M}})$. Thus $F(I)_{\mathcal{M}}$ is CM and hence $F(I)$ is so.

Remark 2.2 Let (R, m) be a local ring and let I be any ideal of R. If $F(I)$ is Cohen-Macaulay , then by theorem **??**, for any minimal reduction J of I, $r_J(I)$ is the degree of the numerator of the Hilbert series of $F(I)$ in the reduced form. Hence the reduction number $r_J(I)$ is independent of the choice of the minimal reduction J of I. This observation is also proved, by different means, by Huckaba and Marley in [?].

Corollary 2.3 (Shah) *Let (R, m) be a local ring. Suppose I is an ideal of R of analytic spread a having a minimal reduction $J = (x_1, \ldots, x_a)$ with $JI = I^2$. If $x_1, \ldots, x_a$ is a regular sequence, then $F(I)$ is CM with minimal multiplicity.*

Proof: We can extend $x_1, \ldots, x_a$ to a minimal basis of I. So let $\mu(I) = a + p$ and $I = (x_1, \ldots, x_a, y_1, \ldots, y_p)$. Set $K = (y_1, \ldots, y_p)$. Then $I^n = (J^n, J^{n-1}K, \ldots, K^n)$. For $i \geq 2$, $J^{n-i}K^i \subseteq J^{n-i}I^i = I^n = J^{n-1}I$ since $JI = I^2$. Thus $I^n = (J^n, J^{n-1}K)$. Let $M_1, \ldots, M_r$ be all the monomials of degree n in $x_1, \ldots, x_a$ and let $m_1, \ldots, m_N$ be all the monomials in $x_1, \ldots, x_a$ of degree $n-1$. We claim that I^n is minimally generated by the elements

$$\{M_1, \ldots, M_r\} \cup \{m_i y_j \mid i = 1, \ldots, N;\ j = 1, \ldots, p\}.$$

Indeed, let them satisfy the relation

$$\sum_{i=1}^{r} a_i M_i + \sum_{j=1}^{p} \sum_{i=1}^{N} b_{ij} m_i y_j = 0.$$

Let $b_{\alpha\beta} \notin m$. Rewrite the above relation as

$$\sum_{i=1}^{r} a_i M_i + \sum_{j=1}^{p} \sum_{\substack{i=1 \\ i \neq \alpha}}^{N} b_{ij} m_i y_j + m_\alpha \sum_{j=1}^{p} b_{\alpha j} y_j = 0.$$

Hence

$$\sum_{j=1}^{p} b_{\alpha j} y_j \in (J^n, m_1, \ldots, \hat{m}_\alpha, \ldots, m_N) : m_\alpha$$

which gives

$$y_\beta \in (J, y_1, \ldots, \hat{y}_\beta, \ldots, y_p)$$

which gives a contradiction. Thus

$$\begin{aligned} H(F(I), t) &= \sum_{n=0}^{\infty} \mu(I^n) t^n \\ &= \sum_{n=0}^{\infty} \left[\binom{a-1+n}{a-1} + \binom{a-2+n}{a-1} p \right] t^n \\ &= \sum_{n=0}^{\infty} \binom{a-1+n}{a-1} t^n + p \sum_{n=0}^{\infty} \left[\binom{a-1+n}{a-1} - \binom{a-2+n}{a-2} \right] t^n \\ &= \frac{1}{(1-t)^a} + p \left[\frac{1}{(1-t)^a} - \frac{1}{(1-t)^{a-1}} \right] \\ &= \frac{1+pt}{(1-t)^a} \end{aligned}$$

It remains to show that $p = \ell(I/J + mI)$. In fact

$$\ell\left(\frac{I}{J+mI}\right) = \ell\left(\frac{I}{mI}\right) - \ell\left(\frac{J+mI}{mI}\right) = \mu(I) - \ell\left(\frac{J}{mJ}\right) = \mu(I) - \mu(J) = p..$$

Therefore $F(I)$ is CM. Since $e(F(I)) = 1+\mu(I)-a$, $F(I)$ is CM with minimal multiplicity.

3 Fiber cones of ideals with minimal mixed multiplicity

Let (R, m) be a local ring. Define the function $H : \mathbb{N}^g \to \mathbb{N}$ for g m-primary ideals $I_1, \ldots, I_g$ as follows:

$$H(r_1, \ldots, r_g) = \ell\left(\frac{R}{I_1^{r_1} \cdots I_g^{r_g}}\right).$$

It was proved in [?] that $H(r_1, \ldots, r_g)$ is given by a polynomial $P(r_1, \ldots, r_g)$ with rational coefficients for all large $r_1, \ldots, r_g$. The total degree of P is d and the terms of degree d in P can be written as

$$\sum_{i_1+\cdots+i_g=d} e(I_1^{[i_1]}|\cdots|I_g^{[i_g]})\binom{r_1+i_1}{i_1}\cdots\binom{r_g+i_g}{i_g}$$

for certain positive integers $e(I_1^{[i_1]}|\cdots|I_g^{[i_g]})$. These positive integers are called mixed multiplicities of $I_1, \ldots, I_g$. When $g = 2$ we adopt the notation

$$e_i(I_1|I_2) = e(I_1^{[d-i]}|I_2^{[i]}).$$

Rees [?] showed that $e_0(I_1|I_2) = e(I_1)$ and $e_d(I_1|I_2) = e(I_2)$. Other mixed multiplicities can also be interpreted as multiplicities of certain parameter ideals. Teissier and Risler [?] provided an interpretation of mixed multiplicities in terms of superficial elements. Rees [?] introduced joint reductions to study them. A set of elements $x_1, \ldots, x_d$ is a called a *joint reduction* of $I_1, \ldots, I_d$ if $x_i \in I_i$ for $i = 1, \ldots, d$ and there exists a positive integer n so that

$$\left[\sum_{j=1}^{d} x_j\, I_1 \cdots \hat{I_j} \cdots I_d\right] (I_1 \cdots I_d)^{n-1} = (I_1 \cdots I_d)^n.$$

Rees proved that if R/m is infinite, then joint reductions exist and

$$e(I_1^{[i_1]}|\cdots|I_g^{[i_g]}) = e(x_1, \ldots, x_d)$$

where $(x_1, \ldots, x_d)$ is any joint reduction of the set of ideals $I_1, \ldots, I_1, \ldots, I_g, \ldots, I_g$ where I_j is repeated i_j times for $j = 1, \ldots, g$..

Abhyankar [?] showed that for a CM local ring (R, m), $\mu(m) - \dim R + 1 \leq e(R)$. We extend this to m-primary ideals.

Lemma 3.1 *Let (R, m) be a CM local ring of dimension d. Then*

$$e_{d-1}(m|I) \geq \mu(I) - d + 1.$$

Proof: By passing to the ring $R(X) = R[X]_{mR[X]}$ we may assume that R/m is infinite. Let $(x, a_1, \ldots, a_{d-1})$ be a joint reduction of $m, I, \ldots, I$ where I is repeated $d-1$ times. Consider the map

$$f : \frac{R}{I} \bigoplus \left(\frac{R}{m}\right)^{d-1} \longrightarrow \frac{(x, a_1, \ldots, a_{d-1})}{xI + (a_1, \ldots, a_{d-1})m}$$

defined as $f(a', b_1', \ldots, b_{d-1}') = (xa + a_1b_1 + \cdots + a_{d-1}b_{d-1})'$ where primes denote the residue classes. Let $K = \text{Kernel}\, f$. Then

$$\ell(K) + \ell\left(\frac{R}{xI + (a_1, \ldots, a_{d-1})m}\right) - e_{d-1}(m|I) = \ell\left(\frac{R}{I}\right) + d - 1.$$

Hence

$$\begin{aligned} e_{d-1}(m|I) &= \ell(K) + \ell\left(\frac{R}{Im}\right) - \ell\left(\frac{R}{I}\right) - d + 1 - \ell\left(\frac{R}{Im}\right) + \ell\left(\frac{R}{xI + (a_1, \ldots, a_{d-1})m}\right) \\ &= \ell(K) + \mu(I) - d + 1 + \ell\left(\frac{Im}{xI + (a_1, \ldots, a_{d-1})m}\right). \end{aligned}$$

Hence

$$e_{d-1}(m|I) \geq \mu(I) - d + 1.$$

Corollary 3.2 [cf. Sa, pg 49] *Let (R, m) be a one-dimensional CM local ring. Then for any m-primary ideal I of R*

$$\mu(I) \leq e(R).$$

Proof: When $d = 1$, $e_{d-1}(m|I) = e_0(m|I) = e(R)$.

Corollary 3.3 *Let (R, m) be a d-dimensional CM local ring. Then*

$$e(m) \geq \mu(m) - d + 1.$$

Proof: Put $I = m$ and use the fact that $e_i(m|m) = e(m)$ for all i.

Definition 3.4 Let (R, m) be a CM local ring, I an m-primary ideal of R. We say that I has *minimal mixed multiplicity* if

$$e_{d-1}(m|I) = \mu(I) - d + 1.$$

We now find the Hilbert series of $F(I)$ when I is an m-primary ideal with minimal mixed multiplicity in a d-dimensional CM local ring. We begin with one dimensional local rings.

Proposition 3.5 *Let (R, m) be a CM local ring of dimension one with $\mu(I) = e(R) := e$ for an m-primary ideal I of R. Then*

$$H(F(I), t) = \frac{1 + (e-1)t}{1 - t}.$$

Proof: By passing to the ring $R(X)$ we may assume that R/m is infinite. Let xR be a minimal reduction of m. Then for any $n \geq 1$,

$$e(R) = \ell\left(\frac{R}{xR}\right) = \ell\left(\frac{R}{xR}\right) + \ell\left(\frac{xR}{xI^n}\right) - \ell\left(\frac{R}{I^n}\right) = \ell\left(\frac{I^n}{xI^n}\right).$$

Since $\ell(I/mI) = e(R) = \ell(I/xI)$, it follows that $mI = xI$. Thus $xI^n = mI^n$ which gives that $\mu(I^n) = e(R)$ for all $n \geq 1$. Hence

$$H(F(I), t) = 1 + \sum_{n=1}^{\infty} e\, t^n = 1 - e + \frac{e}{1-t} = \frac{1 + (e-1)t}{1-t}.$$

Theorem 3.6 *Let (R, m) CM local ring of dimension $d \geq 2$. Let I be an m-primary ideal having minimal mixed multiplicity. Then*

$$H(F(I), t) = \frac{1 + (\mu(I) - d)t}{(1-t)^d}$$

Proof: By passing to the ring $R(X)$ we may assume that R/m is infinite. Let $x, a_1, \ldots, a_{d-1}$ be a joint reduction of $(m, I, \ldots, I)$ where I is repeated $d-1$ times. Then $e_{d-1}(m|I) = \ell(R/(x, a_1, \ldots, a_{d-1}))$ and $\mu(I) = d - 1 + \ell(R/(x, a_1, \ldots, a_{d-1}))$. We will show that for all $n \geq 1$,

$$\mu(I^n) = \binom{n+d-2}{d-2} + \binom{n+d-2}{d-1} e_{d-1}(m|I).$$

Fix n. Put $r = \binom{n+d-2}{d-2}$. Let $M_1, \ldots, M_r$ denote all the monomials of degree n in $a_1, \ldots, a_{d-1}$. Consider the map

$$f : \frac{R}{I^n} \oplus \left(\frac{R}{m}\right)^r \longrightarrow \frac{(x, (a_1, \ldots, a_{d-1})^n)}{xI^n + (a_1, \ldots, a_{d-1})^n m}$$

given by $f(y', z_1', \ldots, z_r') = (yx + \sum_{i=1}^r z_i M_i)'$ where primes denote the residue classes. If $f(y', z_1', \ldots, z_r') = 0$, then $yx + \sum_{i=1}^r z_i M_i \in xI^n + (a_1, \ldots, a_{d-1})^n m$. Hence there exist $b \in I^n$ and $c_1, \ldots, c_r \in m$ such that

$$yx + \sum_{i=1}^r z_i M_i = xb + \sum_{i=1}^r c_i M_i. \tag{1}$$

Thus $x(y - b) = \sum_{i=1}^r M_i(c_i - z_i)$. Since $(x, a_1, \ldots, a_{d-1})$ is a regular sequence, $y - b \in (M_1, \ldots, M_r) = (a_1, \ldots, a_{d-1})^n$. Hence $y = b + \sum_{i=1}^r d_i M_i$ for some $d_1, \ldots, d_r \in R$. Substituting for y in (**??**), we get $\sum_{i=1}^r M_i(z_i - c_i + xd_i) = 0$. Since $a_1, \ldots, a_{d-1}$ is a regular sequence, $z_i - c_i + xd_i \in (a_1, \dots, a_{d-1})$ for all $i = 1, \dots, r$. Hence $z_i \in m$ for all $i = 1, \dots, r$. Thus f is an isomorphism which implies that

$$\ell\left(\frac{R}{xI^n + (a_1, \dots, a_{d-1})^n m}\right) - \ell\left(\frac{R}{(x, (a_1, \dots, a_{d-1})^n)}\right) = \ell\left(\frac{R}{I^n}\right) + r.$$

Put $n = 1$ to get

$$\ell\left(\frac{R}{xI + (a_1, \dots, a_{d-1})m}\right) - \ell\left(\frac{R}{I}\right) = r + e_{d-1}(m|I) = \mu(I) = \ell\left(\frac{R}{mI}\right) - \ell\left(\frac{R}{I}\right).$$

Hence $Im = xI + (a_1, \dots, a_{d-1})m$. Thus for all $n \geq 1$, $I^n m = xI^n + (a_1, \dots, a_{d-1})^n m$ by an easy induction on n. Hence for all $n \geq 1$,

$$\mu(I^n) = r + \binom{n+d-2}{d-1} e_{d-1}(m|I).$$

Therefore

$$\begin{aligned}
H(F(I), t) &= \sum_{n=0}^{\infty} \left[r + \binom{n+d-2}{d-1} e_{d-1}(m|I) \right] t^n \\
&= \sum_{n=0}^{\infty} \left[r(1 - e_{d-1}(m|I)) + \binom{n+d-1}{d-1} e_{d-1}(m|I) \right] t^n \\
&= \frac{1 - e_{d-1}(m|I)}{(1-t)^{d-1}} + \frac{e_{d-1}(m|I)}{(1-t)^d} \\
&= \frac{1 + [e_{d-1}(m|I) - 1]t}{(1-t)^d} \\
&= \frac{1 + [\mu(I) - d]t}{(1-t)^d}.
\end{aligned}$$

Theorem 3.7 *Let I be an m-primary ideal in a CM local ring (R, m). Suppose I has minimal mixed multiplicity. Then $F(I)$ is CM if and only if $r(I) \leq 1$.*

Proof: If $r(I) \leq 1$, then $F(I)$ is CM by corollary **??**. If $F(I)$ is CM, then

$$r(I) = \text{ the degree of the numerator of } H(F(I), t) \leq 1.$$

We now consider fiber rings of contracted ideals in two dimensional regular local rings. An ideal I of a two dimensional regular local ring (R, m) is called *contracted* if there is an $x \in m \setminus m^2$ such that $IR[m/x] \cap R = I$ [ZS, App. 5]. The order of I, $o(I)$ is by definition $\max\{n \,|\, I \subseteq m^n\}$. I is contracted if and only if $\mu(I) = 1 + o(I)$ [HS, Theorem 2.1]. If I is contracted from $S = R[m/x]$, then for all n, I^n is also contracted from S. Any m-primary integrally closed ideal is contracted [ZS, App. 5].

Corollary 3.8 *Let (R,m) be a two dimensional regular local ring. Let I be an m-primary contracted ideal of order r. Then*

$$H(F(I),t) = \frac{1+(r-1)t}{(1-t)^2}.$$

Proof: By [V1, Theorem 4.1], $e_1(m|I) = r = o(I)$. Since I is an ideal of minimal mixed multiplicity, we can apply Theorem **??**.

In view of the above theorem **??**, to produce a non CM fiber cone it is enough to find an ideal having minimal mixed multiplicity whose reduction number exceeds one. We do so in the next example.

Example 3.9 We construct contracted ideals of reduction number greater than one which have non CM fiber cones. Consider the ideals

$$I = (x^r, xy^{r+1}(x,y)^{r-2}, y^{2r+1}).$$

of $k[[x,y]]$ where k is a field. Since $\mu(I) = r+1 = 1+o(I)$, I is a contracted ideal and by corollary **??**,

$$H(F(I),t) = \frac{1+(r-1)t}{(1-t)^2}.$$

If $F(I)$ were CM, then by corollary **??**, $r(I) \leq 1$. We now show that $r(I) > 1$ for $r \geq 3$. By [V2, Theorem 3.2], $r(I) \leq 1$ if and only if $e(I) = \ell(R/I^2) - 2\ell(R/I)$. Since I is integral over (x^r, y^{2r+1}), (x^r, y^{2r+1}) is a minimal reduction of I and $e(I) = r(2r+1)$. Now

$$I^2 = (x^{2r}, x^{r+1}y^{r+1}(x,y)^{r-2}, x^r y^{2r+1}, x^2 y^{3r+1}(x,y)^{r-3}, xy^{4r}, y^{4r+2}).$$

Therefore

$$\begin{aligned} \ell\left(\frac{R}{I^2}\right) &= \sum_{i=0}^{r-2}(r+1+i) + 2r+1 + \sum_{i=0}^{r-3}(3r+1+i) + 4r + 4r + 2 \\ &= (r+1)(r-1) + \binom{r-1}{2} + 2r + 1 + (3r+1)(r-2) + \binom{r-2}{2} + 8r + 2 \\ &= \binom{r-1}{2} + \binom{r-2}{2} + 4r^2 + 5r \end{aligned}$$

and

$$\ell\left(\frac{R}{I}\right) = \sum_{i=0}^{r-2}(r+1+i) + 2r + 1$$

$$
\begin{aligned}
&= (r+1)(r-1) + \binom{r-1}{2} + 2r + 1 \\
&= \binom{r-1}{2} + r^2 + 2r.
\end{aligned}
$$

Therefore if $r \geq 3$, then

$$
\begin{aligned}
\ell\left(\frac{R}{I^2}\right) - 2\ell\left(\frac{R}{I}\right) &= 4r^2 + 5r + \binom{r-2}{2} - 2\left[r^2 + 2r + \binom{r-1}{2}\right] \\
&= 2r^2 + r - (r-2) \\
&< 2r^2 + 2r \\
&= e(I)
\end{aligned}
$$

Example 3.10 [cf. [HJLS, (6.3)],[HM, Corollary (3.5) and Example (3.9)] There exist examples of m-primary ideals in $R = k[[x, y]]$ of arbitrary reduction number and whose fiber ring is CM. Let $I = (x^r, x^{r-1}y, y^r)$, $r \geq 1$. Then $F(I)$ is CM for all $r \geq 1$. Since I is generated by homogeneous polynomials of equal degree in R, $F(I) \cong k[x^r, x^{r-1}y, y^r]$. It is easy to see that $k[x^r, x^{r-1}y, y^r] \cong k[t_1, t_2, t_3]/(t_2^r - t_1^{r-1}t_3)$. Hence $F(I)$ is CM and its Hilbert series is

$$
H(F(I), t) = \frac{1 + t + \cdots + t^{r-1}}{(1-t)^2}.
$$

It follows from theorem **??** that the reduction number of I is independent of the reduction chosen and is equal to $r - 1$.

Example 3.11 Let $R = k[[x, y, z]]$, $I = (x^3, y^3, z^3, xy, xz, yz)$. This ideal was studied by Huneke and Lipman [?]. We claim that I has minimal mixed multiplicity. It is easy to see that

1. $J = (x^3 + yz, y^3 + z^3 + xz, xz + xy)$ is a minimal reduction of I and $JI = I^2$.
2. $(yz, y + z, x)$ is a joint reduction of the set of ideals (I, m, m) and $Im^2 = yzm^2 + (y + z)Im + xIm$.
3. $(yz, xy + xz, x + y + z)$ is a joint reduction of the set of ideals (I, I, m) and $I^2m = yzIm + (xy + xz)Im + (x + y + z)I^2$.

Hence

$$
\begin{aligned}
e(I) &= e(J) = 11 \\
e_1(m|I) &= e(yz, y + z, x) = 2 \\
e_2(m|I) &= e(yz, xy + xz, x + y + z) = 4.
\end{aligned}
$$

Since $\mu(I) = 6 = e_2(m|I) + 2$, I has minimal mixed multiplicity. Hence by corollary **??**, $F(I)$ is CM.

4 Cohen-Macaulayness of fiber cones of ideals generated by quadratic sequences.

Let R be a standard graded ring over a field, let M denote the unique graded maximal ideal of R, and let I be an ideal generated by a homogeneous quadratic sequence. Under some technical assumptions on the quadratic sequence, we show that a certain deformation of the fiber cone $F(I) := R[It]/MR[It]$ has a nice form. From this we can deduce the Cohen-Macaulayness of $F(I)$ and also a formula for its Hilbert series. (The Hilbert series can also be calculated from the formula in [RV] for the bigraded Hilbert series of $R[It]$—just set $X = 1$ in that formula.) These results apply to the examples treated in [RV], namely, straightening-closed ideals in graded algebras with straightening law, Huckaba-Huneke ideals of analytic deviation 1 and 2, and Moralès-Simis ideals defining the homogeneous co-ordinate rings of certain projective space curves.

The main result of this section is stated in terms of stable linearizations of quadratic sequences. We recall the relevant definitions from [RS]. A subset Λ of a finite poset $(\Omega, \leq)$ is an *ideal* if

$$\lambda \in \Lambda, \quad \omega \in \Omega, \quad \text{and} \quad \omega \leq \lambda \qquad \Longrightarrow \qquad \omega \in \Lambda.$$

If Λ is an ideal of Ω and $\omega \in \Omega \backslash \Lambda$ is such that $\lambda \in \Lambda$ for every $\lambda \lneq \omega$, then (Λ, ω) is a *pair* of Ω. Given a set $\{x_\omega \mid \omega \in \Omega\}$ of elements of a ring R and $\Lambda \subseteq \Omega$, denote by X_Λ the ideal $(x_\lambda \mid \lambda \in \Lambda)$ of R ($X_\Lambda = 0$ if Λ is empty) and by I the ideal $X_\Omega = (x_\omega \mid \omega \in \Omega)$.

Definition 4.1 A set $\{x_\omega \mid \omega \in \Omega\} \subseteq R$ is a *quadratic sequence* if for every pair (Λ, ω) of Ω there exists an ideal Θ of Ω such that

1. $(X_\Lambda : x_\omega) \cap I = X_\Theta$.
2. $x_\omega X_\Theta \subseteq X_\Lambda I$.

Such an ideal Θ is said to be *associated* to the pair (Λ, ω). This association need not be unique—the set $\{x_\omega \mid \omega \in \Omega\}$ of generators of I may not be unshortenable—but X_Θ is unique by 1.

Definition 4.2 A *linearization* of a poset Ω of cardinality n is a bijective map $\# : \Omega \longrightarrow [1,n] := \{1,\ldots,n\}$ such that $\omega \leq \omega' \Longrightarrow \#(\omega) \leq \#(\omega')$.

Let $\{x_\omega \mid \omega \in \Omega\}$ be a quadratic sequence and $\# : \Omega \longrightarrow [1,n]$ be a fixed linearization. Identify Ω with $[1,n]$ via $\#$. Then $([1,j-1],j)$ is a pair of Ω for every $j \in [1,n]$. Let

$$\begin{aligned} \Theta_j &= \text{ an ideal of } \Omega \text{ associated to } ([1,j-1],j) \\ I_j &= ((x_1,\ldots,x_{j-1}) : x_j) \\ \Delta &= \{(j,k) \mid 1 \leq j \leq k \leq n, \quad x_j \cdot x_k \in (x_1,\ldots,x_{j-1})\}. \end{aligned}$$

For $k \in [1,n]$, set

$$\Psi_k = [1,k-1] \bigcup_{\substack{j\leq k \\ (j,k)\notin\Delta}} \Theta_j \quad \text{and} \quad \mathfrak{A}_k = X_{\Psi_k}.$$

Note that Ψ_k is an ideal of Ω and that $\mathfrak{A}_k$ is independent of the choices of Θ_j.

Definition 4.3 A linearization $\# : \Omega \longrightarrow [1,n]$ of the indexing poset Ω of a quadratic sequence is *stable* if $I_k = (\mathfrak{A}_k : x_k)$ for every k, $1 \leq k \leq n$.

We can now state the main result of this section:

Proposition 4.4 *Let R be a standard graded algebra over a field k, that is, $R = \oplus_{j\geq 0} R_j = R_0[R_1]$ with $R_0 = k$. Let $\{x_\omega | \omega \in \Omega\} \subseteq R$ be a quadratic sequence consisting of homogeneous elements of R, let $\# : \Omega \to \{1,\ldots,n\}$ be a stable linearization, and suppose that*

$$\deg(x_1) \leq \cdots \leq \deg(x_n).$$

Further assume that $x_1,\ldots,x_n$ form an unshortenable set of generators of the ideal $I := (x_1,\ldots,x_n)$, that is, $I \neq (x_1,\ldots,\widehat{x_j},\ldots,x_n)$. Let $M := \oplus_{j>0} R_j$ denote the irrelevant maximal ideal of R. Then

$$k[\Delta] := k[T_1,\ldots,T_n]/(T_jT_k | (j,k) \in \Delta)$$

is a deformation of the fiber cone $F(I) := R[It]/MR[It]$ of I. In particular (1) $F(I)$ is Cohen-Macaulay if $k[\Delta]$ is so, and (2) $F(I)$ has the same Hilbert series as $k[\Delta]$.

Proof. We borrow the notation and set-up of §1 of [RS] (see page 541). Let J denote the kernel of the presentation map from the polynomial ring $\mathcal{A} := R[T_1,\ldots,T_n]$ onto the Rees ring $R[It]$. This map is defined by $T_i \mapsto x_i t$. The presentation ideal of the fiber cone $F(I)$ as a quotient of $\mathcal{A}$ is then (J, M). Let $\mathcal{F}$ be the filtration on $\mathcal{A}$ specified on page 541 of [RS]. This filtration $\mathcal{F}$ is a refinement of the $(M, T_1, \ldots, T_n)$-adic filtration on $\mathcal{A}$. Theorem 1.4 of [RS] asserts that

$$(*) \qquad \mathrm{gr}_{\mathcal{F}}(J) = (I_1T_1, \ldots, I_nT_n, T_jT_k|(j,k) \in \Delta),$$

where $I_j := ((x_1, \ldots, x_{j-1}) : x_j)$. So it is natural to expect the following:

$$(\dagger) \qquad \mathrm{gr}_{\mathcal{F}}(J, M) = (M, T_jT_k|(j,k) \in \Delta).$$

The theorem clearly follows from (†) and we prove (†) below.

Since M is homogeneous with respect to $\mathcal{F}$ and $T_jT_k \in \mathrm{gr}_{\mathcal{F}}(J)$ from (*), it follows that $\mathrm{gr}_{\mathcal{F}}(J, M) \supseteq (M, T_jT_k|(j,k) \in \Delta)$. To prove the other inclusion, it suffices to prove the statement (‡) below. Fix notation as in 3.2 of [RS]. Let F be a form of degree d in J. Let Q be the subset of $P(n,d)$ such that $F = \sum_{p\in Q} a_pT^p$ where $0 \neq a_p \in R$ for $p \in Q$. Assume further that the a_p are homogeneous. Let q be the such element of Q that the initial form F_* of F with respect to $\mathcal{F}$ equals a_qT^q, and let s be the such element of Q that $a_s \notin M$ and the term $F_s = a_sT^s$ has the least degree in $\mathbb{N}_0^{n+1}$ among those with $a_p \notin M$. We claim that

$$(\ddagger)\ T_jT_k \text{ divides } T^s \text{ for some } (j,k) \in \Delta.$$

We prove (‡) by "induction on $s - q$". If $s = q$, then from (*) we know that either $a_s \notin I_j$ for some j or (‡) holds. But $I_j \subseteq M$ by unshortenability, so $a_s \notin I_j$ by the definition of s, so (‡) holds. Now suppose that $\deg(s) > \deg(q)$. By (*), either $a_q \in I_j$ for some j, or T_jT_k divides T^q for some $(j,k) \in \Delta$. Accordingly there are two cases.

First assume that $a_q \in I_j$ for some j. Then there exists a linear form $F' = a_qT_j - a_1T_1 - \cdots - a_{j-1}T_{j-1}$ in J with $\deg(a_q) + \deg(x_j) = \deg(a_1) + \deg(x_1) = \cdots = \deg(a_{j-1}) + \deg(x_{j-1})$. By our assumption that $\deg(x_1) \leq \ldots \leq \deg(x_n)$, we have $\deg(a_1) \geq \ldots \geq \deg(a_{j-1}) \geq \deg(a_q)$. Since $a_q \in M$ by assumption, it follows that $a_1, \ldots, a_{j-1}$ are also in M. Consider $G = F - F'T^q/T_j$. Since $F'_* = a_qT_j$, it follows that $F_* = (F'T^q/T_j)_*$, so that $\deg(G_*) < \deg(F_*)$. Since $F'T^q/T_j$ belongs to $M[T_1, \ldots, T_n]$, it follows that G_s is the term of least degree in G with coefficient not in M. By induction, we have T_jT_k divides T^s for some (j,k) in Δ. We are done.

Now assume that T_jT_k divides T^q for some (j,k) in Δ. By the definition of Δ and axiom 2 in the definition of quadratic sequence, there exists a 2-form F' in J with $F'_* = T_jT_k$. Now consider $G = F - a_qF'T^q/T_jT_k$, and argue just as in the first case.

Example 4.5 (Defining ideals of monomial projective space curves lying on the quadric $xw - yz = 0$ as in [MS]) This example is treated in §2 of [RS] (see pages 558, 559) and in §3.3 of [RV]. We have

$$k[\Delta] = k[T_1, \dots, T_{b-c+2}]/(T_3, \dots, T_{b-c+1})^2$$

so that $k[\Delta]$ is Cohen-Macaulay and its Hilbert series is given by

$$H(k[\Delta]; t) = \frac{1 + (b-c-1)t}{(1-t)^3}.$$

Example 4.6 (Huckaba-Huneke ideals of analytic deviation 1 and 2.) This example is treated in [RV] (see §3.4 of that paper). We have

$$k[\Delta] = k[T_1, \dots, T_n]/(T_{m+1}, \dots, T_n)^2$$

so that $k[\Delta]$ is Cohen-Macaulay and its Hilbert series is given by

$$H(k[\Delta]; t) = \frac{1 + (n-m)t}{(1-t)^m}$$

Here m is the analytic spread of the ideal and n is the minimal number of generators of the ideal.

Example 4.7 (Straightening-closed ideals in graded algebras with straightening law.) These examples are treated in §2 of [RS] and in §3.2 of [RV]. Let R be a graded algebra with straightening law on a finite poset Π over a field k and let M denote the graded maximal ideal of R. If Ω is a straightening-closed ideal of Π which admits a linearization $\# : \Omega \to \{1, \dots, n\}$ satisfying

$$\deg(x_1) \leq \cdots \leq \deg(x_n),$$

then $k[\Delta]$ is the face ring of the poset Ω. Thus $F(I)$ has the same Hilbert series as the face ring of Ω. And $F(I)$ is Cohen-Macaulay if Ω is a Cohen-Macaulay poset. While these conclusions are well-known for most particular examples of straightening-closed ideals, what appears to be new is the conclusion from Remark **??** that the reduction number is independent of the minimal reduction and can be read off the Hilbert series of $F(I)$. We illustrate this last conclusion by means of an example: Let $(Z_{ij} | 1 \leq i \leq 2, 1 \leq j \leq n)$ be a generic $2 \times n$ matrix ($n \geq 3$), $R = k[Z_{ij}]$ the polynomial ring over a field k in these $2n$ indeterminates, Ω the poset of 2×2 minors of the matrix Z, and I the ideal of R generated by the elements of Ω. The Hilbert series of $F(I)$ is (see [**?**])

$$H(F(I), t) = \frac{h_0 + h_1 t + \cdots + h_{n-3} t^{n-3}}{(1-t)^{2n-3}}$$

where

$$h_i = \binom{n-2}{i}^2 - \binom{n-3}{i-1}\binom{n-1}{i+1}.$$

Thus the reduction number of I is $n-3$.

5 Examples

In this section we will use the fact that if I is generated by homogeneous polynomials of equal degree in a polynomial ring over a field k, then the fiber cone of I is isomorphic to $k[I]$.

Example 5.1 [**?**, Example 10.27] Let $R = k[x, y, z]_m$, where k is a field and $m = (x, y, z)$. Let $I = ((x^2, y^2)^3, x(x^2, y^2)z^3, z^6)$. Then $F(I)$ is CM, $r(I) = 2$ and

$$H(F(I), t) = \frac{1 + 4t + t^2}{(1-t)^3}.$$

We prove by induction on n, that

$$I^n = ((x^2, y^2)^{3i} z^{6(n-i)}; i = 0 \ldots, n) + (x(x^2, y^2)^{3i+1} z^{6(n-1-i)+3}; i = 0, \ldots, n-1).$$

For $n = 1$, it is easy to verify. Assume that $n \geq 1$. Then

$$\begin{aligned}
& I^{n+1} \\
= \; & ((x^2, y^2)^{3(i+1)} z^{6(n-i)}, (x^2, y^2)^{3i} z^{6(n+1-i)}, x(x^2, y^2)^{3(i+1)} z^{6(n-i)+3}; i = 0, \ldots n) \\
& +(x(x^2, y^2)^{3i+4} z^{6(n-1-i)+3}, x(x^2, y^2)^{3i+1} z^{6(n-i)+3}, x^2(x^2, y^2)^{3i+2} z^{6(n-i)}; i = 0, \ldots n-1) \\
= \; & ((x^2, y^2)^{3i} z^{6(n+1-i)}; i = 0 \ldots, n+1) + (x(x^2, y^2)^{3i+2} z^{6(n-i)+3}; i = 0, \ldots, n)
\end{aligned}$$

since $x^2(x^2, y^2)^{3i+2} z^{6(n-i)} \subseteq (x^2, y^2)^{3i+3} z^{6(n-i)}$ for all $i = 0, \ldots, n-1$. Therefore

$$\begin{aligned}
\mu(I^n) &= \sum_{i=0}^{n} (3i+1) + \sum_{i=0}^{n-1} (3i+2) \\
&= \sum_{i=0}^{n} (6n+3) - (3n+2) \\
&= 6\binom{n+1}{2} + 1 \\
&= 6\binom{n+2}{2} - 6\binom{n+1}{1} + 1.
\end{aligned}$$

Hence

$$\sum_{i=0}^{\infty} \mu(I^n)t^n = \frac{6}{(1-t)^3} - \frac{6}{(1-t)^2} + \frac{1}{(1-t)} = \frac{1+4t+t^2}{(1-t)^3}.$$

Let $J = (x^6, y^6, z^6)$. We will show that $JI^2 = I^3$. Now,

$$\begin{aligned}
I^2 &= ((x^2,y^2)^6, x(x^2,y^2)^4z^3, (x^2,y^2)^3z^6, x(x^2,y^2)z^9, z^{12}) \\
I^3 &= ((x^2,y^2)^9,\ x(x^2,y^2)^7z^3,\ (x^2,y^2)^6z^6,\ x(x^2,y^2)^4z^9,\ (x^2,y^2)^3z^{12},\ x(x^2,y^2)z^{15},\ z^{18}) \\
&= z^6I^2 + (x^6,y^6)((x^2,y^2)^6, x(x^2,y^2)^4z^3) \\
&\subseteq JI^2.
\end{aligned}$$

Hence $JI^2 = I^3$, J is a minimal reduction of I and $\ell(I/J + mI) = \mu(I) - \mu(J) = 4$. We will show that $\ell(I^2/JI + mI^2) = 1$. Now,

$$\begin{aligned}
JI &= (x^6,y^6,z^6)((x^2,y^2)^3, xz^3(x^2,y^2), z^6) \\
&= ((x^2,y^2)^6, (x^7,xy^6)(x^2,y^2)z^3, (x^2,y^2)^3z^6, x(x^2,y^2)z^9, z^{12}) \\
mI^2 &= ((x,y)^{13}, (x^2,y^2)^6z, x(x,y)^9z^3, x(x^2,y^2)^4z^4, (x,y)^7z^6, (x^2,y^2)^3z^7, \\
&\quad x(x,y)^3z^9, x(x^2,y^2)z^{10}, (x,y)z^{12}, z^{13}) \\
JI + mI^2 &= JI + xm(x^4y^4)z^3.
\end{aligned}$$

Hence

$$\begin{aligned}
\ell\left(\frac{I^2}{JI+mI^2}\right) &= \ell\left(\frac{R}{JI+mI^2}\right) - \ell\left(\frac{R}{I^2}\right) \\
&= \left[\sum_{i=0}^{11}\binom{n+2}{2} + 73 + 48 + 20 + 4 + 2\right] \\
&\quad - \left[\sum_{i=0}^{11}\binom{n+2}{2} + 72 + 48 + 20 + 4 + 2\right] \\
&= 1.
\end{aligned}$$

In view of Theorem **??**, $F(I)$ is CM.

Example 5.2 [**?**, Remark 1] Let $R = k[x_1, x_2, y_1, \ldots, y_s]$ $(s \geq 2)$ and let $I = (x_iy_j : i = 1,2; 1 \leq j \leq s)$ be an ideal of R. Then $F(I) \cong k[x_iy_j : i = 1,2; 1 \leq j \leq s] \cong k[T_{ij} : 1 \leq i \leq 2; 1 \leq j \leq s]/I_2$, where I_2 is the ideal generated by the 2×2 minors of the matrix (T_{ij}) [**?**, Remark 1]. It is known that $F(I)$ is CM [**?**, Theorem 1^0].. We show that $F(I)$ is CM by using Theorem **??**. First we will show that $r(I) \leq 1$. Put

$$J = (x_1y_i + x_2y_{i+1} : 1 \leq i \leq s-1; x_1y_s, x_2y_1).$$

We will prove that $JI = I^2$. Now $I^2 = (x_1y_i\, x_1y_j, x_1y_i\, x_2y_j, x_2y_i\, x_2y_j; 1 \leq i \leq j \leq s)$. Obviously $x_1y_i\, x_1y_s, \in JI$ for $1 \leq i \leq s$. Let $j < s$. Then

$$x_1y_1\, x_1y_j \quad = \quad x_1y_1(x_1y_j + x_2y_{j+1}) - x_2y_1\, x_1y_{j+1} \in JI.$$

If $i \geq 2$, then by induction hypothesis,

$$x_1y_i\, x_1y_j \quad = \quad x_1y_i(x_1y_j + x_2y_{j+1}) - x_1y_{j+1}(x_1y_{i-1} + x_2y_i) + x_1y_{j+1}\, x_1y_{i-1} \in JI.$$

Now,

$$x_1y_1\, x_2y_j = x_1y_j\, x_2y_1 \in JI$$

for $1 \leq j \leq s$. If $i \geq 2$, then

$$x_1y_i\, x_2y_j \quad = \quad x_1y_i(x_1y_{j-1} + x_2y_j) - x_1y_i\, x_1y_{j-1} \in JI$$

for $1 \leq j \leq s$. Since $x_2y_1 \in J$, $x_2y_1\, x_2y_j \in JI$ for $1 \leq j \leq s$. Hence for $2 \leq i \leq j \leq s$,

$$x_2y_i\, x_2y_j \quad = \quad x_2y_i(x_1y_{j-1} + x_2y_j) - x_2y_i\, x_1y_{j-1} \in JI.$$

This shows that $JI = I^2$. We will now compute $H(F(I), t)$. Note that I^n is generated by monomials $x_1^{h_1}x_2^{h_2}y_1^{l_1}\ldots y_s^{l_s}$ where $h_1 + h_2 = n$ and $l_1 + \cdots + l_s = n$. Hence,

$$\begin{aligned}
\sum_{n\geq 0} \mu(I^n)t^n &= \sum_{n\geq 0} \left[\binom{n+1}{1}\binom{n+s-1}{s-1}\right] t^n \\
&= \sum_{n\geq 0} \left[s\binom{n+s-1}{s} + \binom{n+s-1}{s-1}\right] t^n \\
&= \sum_{n\geq 0} \left[s\binom{n+s}{s} - (s-1)\binom{n+s-1}{s-1}\right] t^n \\
&= \frac{s-(s-1)(1-t)}{(1-t)^{s+1}} \\
&= \frac{1+(s-1)t}{(1-t)^{s+1}}.
\end{aligned}$$

This shows that $a(I) = s+1$ and hence J is a minimal reduction of I. Since $e(F(I)) = s = \mu(I) - \dim F(I) + 1$, $F(I)$ is CM with minimal multiplicity.

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