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COHEN-MACAULAY REES ALGEBRAS OF IDEALS HAVING ANALYTIC DEVIATION TWO

SHIRO GOTO AND YUKIO NAKAMURA[1]

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Abstract. For a certain class of ideals with analytic deviation two in a Gorenstein local ring, a criterion for the Rees algebra to be Cohen-Macaulay is given in terms of the Cohen-Macaulayness of the associated graded ring and the reduction number with respect to a minimal reduction for the ideal. Some consequences are discussed.

1. Introduction. The purpose of this paper is to prove the following, which is an analytic deviation two version of the main result in [GH] of the first author and Huckaba:

THEOREM 1.1. *Let I be an ideal in a Gorenstein local ring A and assume that* (i) *A/I is a Cohen-Macaulay ring,* (ii) *$s = \mathrm{ht}_A I \geq 1$ and IA_P is generated by an A_P-regular sequence of length s for all $P \in \operatorname{Spec} A$ such that $P \supseteq I$ and $\dim A_P \leq s+1$, and* (iii) *the analytic spread of I is equal to $s+2$. Suppose that the residue class field of A is infinite and choose a minimal reduction J of I. Then the Rees algebra $R(I)$ of I is a Cohen-Macaulay ring if and only if the associated graded ring $G(I) = R(I)/IR(I)$ is a Cohen-Macaulay ring and $I^{s+2} = JI^{s+1}$.*

As in [GH], the heart of this theorem is the calculation of the a-invariant $a(G(I))$ of $G(I)$ (see [GW, (3.1.4)] for the definition of the a-invariant) in terms of the reduction number $r_J(I)$ of I with respect to J and the assertion will be improved when the reduction number $r_J(I)$ is relatively small. Before entering into the details, however, we would like to fix the basic notation which we shall maintain throughout this paper.

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. We assume that the field $A/\mathfrak{m}$ is infinite. Let I ($\neq A$) be an ideal in A and $s = \mathrm{ht}_A I$. Let $R(I) = A[It] \subseteq A[t]$ (here t is an indeterminate over A) and $G(I) = R(I)/IR(I)$. Let J be a minimal reduction of I; hence $J \subseteq I$ and $I^{n+1} = JI^n$ for some $n \geq 0$. We put $r_J(I) = \min\{n \geq 0 \mid I^{n+1} = JI^n\}$ and call it the reduction number of I with respect to J. Let $\lambda(I)$ denote the analytic spread of I, that is, $\lambda(I) = \dim(A/\mathfrak{m}) \otimes_A G(I)$. Then as is well known, J is minimally generated by $\lambda(I)$-elements (cf. [NR]). Following [HH1], we put ad $I = \lambda(I) - s$ and call it the analytic deviation of I.

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The theory of ideals having small analytic deviation started from the researches [HH1, HH2] of Huckaba and Huneke. In [HH1] they explored certain ideals I with $\mathrm{ad}\, I = 1$ or 2 in a Cohen-Macaulay or Gorenstein local ring A and gave, for these ideals, a criterion for symbolic powers to equal the ordinary ones. In addition, they studied in [HH2] the Cohen-Macaulay property of $R(I)$ and showed especially, that $R(I)$ is a Cohen-Macaulay ring if $r_J(I) \leq 1$. Their research [HH2] was succeeded by [GH], in which the first author and Huckaba proved the following theorem:

THEOREM 1.2 (cf. [GH, 2.1]). *Let I be an ideal in a Cohen-Macaulay local ring A and assume that* (i) *$s = ht_A I \geq 0$ and IA_P is generated by an A_P-regular sequence of length s for all $P \in \mathrm{Min}_A A/I$ and* (ii) *$\lambda(I) = s+1$. Suppose that the residue class field of A is infinite and choose a minimal reduction J of I. Then $R(I)$ is a Cohen-Macaulay ring if and only if $s > 0$, $G(I)$ is Cohen-Macaulay ring, and $I^{s+1} = JI^s$.*

Theorem 1.2 was based on the calculation [GH, 2.4] of $a(G(I))$ in terms of $r_J(I)$ and both results [GH, 2.1 and 2.4] proved to be really useful in the research [GN1] on the Gorensteinness of $R(I)$ and $G(I)$ for the ideals I with $\mathrm{ad}\, I = 1$. Our Theorem 1.1 originally aimed at analytic deviation two versions of Theorem 1.2, which will be helpful for the further research [GN2] on $R(I)$ and $G(I)$ associated to ideals of analytic deviation two. The authors now believe that the assertions in Theorems 1.1 and 1.2 remain true, after suitable modifications if necessary, for the ideals with higher analytic deviation.

Let us cite two more results in this paper. We assume that our base ring A is a Gorenstein local ring and that our ideal I satisfies the conditions (i), (ii), and (iii) in Theorem 1.1. Let J be a minimal reduction of I. Then we have $\dim R(I) = \dim A + 1$, since $s = ht_A I > 0$ (cf. [V, 1.6]) and so by the theorem [TI, 1.1] of Trung and Ikeda, $R(I)$ is a Cohen-Macaulay ring if and only if $G(I)$ is a Cohen-Macaulay ring and $a(G(I)) < 0$. Therefore thanks to it, our Theorem 1.1 immediately follows from the next result, which is the heart of our argument:

THEOREM 1.3. *Suppose that $G(I)$ is a Cohen-Macaulay ring. Then $a(G(I)) = -s$ if $r_J(I) \leq 1$ and $a(G(I)) = r_J(I) - (s+2)$ if $r_J(I) \geq 2$.*

As was cited before, $R(I)$ is Cohen-Macaulay ring if $r_J(I) \leq 1$ (cf. [HH2, 4.1]). It seems natural to ask what happens when $r_J(I) = 2$. We would like to study this question also in this paper and in Section 4 we will prove the following:

THEOREM 1.4. *Suppose that $r_J(I) \leq 2$. Then the following conditions are equivalent*:

(1) *$R(I)$ is a Cohen-Macaulay ring.*

(2) *$G(I)$ is a Cohen-Macaulay ring.*

(3) $\mathrm{depth}\, A/I^2 \geq \dim A/I - 2$.

Hence $R(I)$ is a Cohen-Macaulay ring, if $\dim A/I = 2$ and $r_J(I) \leq 2$.

We close this section with a brief orientation for this paper. We shall prove Theorems 1.3 and 1.4 in Section 4, reducing these two assertions to those on ideals of analytic deviation one. The reduction basically depends on a certain canonical exact sequence of graded $R(I)$-modules, which we will explain in Section 3. Section 2 is devoted to the preliminaries for that. We will summarize some auxiliary results in the case where $\mathrm{ad}\, I=1$. Two examples will be explored in Section 4 in order to illustrate our theorems.

Throughout this paper let $(A, \mathfrak{m})$ denote a Noetherian local ring of $\dim A=d$. For simplicity, we always assume that the field $A/\mathfrak{m}$ is infinite. Let $I\,(\neq A)$ be an ideal in A and $s=ht_A I$. Let J be a minimal reduction of I and put $r=r_J(I)$. For a finitely generated A-module M we denote by $\mu_A(M)$ the number of elements in a minimal system of generators for M. Let $H^i_{\mathfrak{m}}(*)$ $(i\in \boldsymbol{Z})$ stand for the i^{th} local cohomology functor of A with respect to $\mathfrak{m}$.

2. Preliminaries from the case where $\mathrm{ad}\, I=1$. The purpose of this section is to gather some auxiliary results, which we later need to prove Theorems 1.1 and 1.4.

LEMMA 2.1. *Suppose that A is a Gorenstein ring and A/I is a Cohen-Macaulay ring. Assume that* (i) $\lambda(I)\geq s+2$ *and* (ii) *IA_P is generated by an A_P-regular sequence of length s for all $P\in \operatorname{Spec} A$ such that $P\supseteq I$ and* $\dim A_P\leq \lambda(I)-1$. *Then* $s\geq 1$.

PROOF. Assume $s=0$ and put $L=I+((0):I)$. Then $I\cap((0):I)=(0)$ and $\dim A/L<d=\dim A$. Thus both A/I and $A/(0):I$ are Cohen-Macaulay rings of dimension d (cf. [PS]). Apply the depth lemma (cf. [HH1, Remark 1]) to the exact sequence

$$0\to A\to A/I\oplus A/(0):I\to A/L\to 0$$

of A-modules. Then we have depth $A/L\geq d-1$. Thus $\dim A/L=d-1$. Choose $P\in \operatorname{Spec} A$ so that $P\supseteq L$ and $\dim A_P=1$. Then since $\dim A_P\leq \lambda(I)-1$ by the assumption (i), we have $IA_P=(0)$ by the assumption (ii), which contradicts the fact that $P\supseteq (0):I$. Hence $s\geq 1$. q.e.d.

Let $B=A[[X]]$ be the formal power series ring in one variable X over A. We put $\mathfrak{n}=\mathfrak{m}B+XB$, $I^*=IB+XB$, and $J^*=JB+XB$. Let $B[t]$ denote a polynomial ring. We put $T=Xt \bmod I^*R(I^*)$. Then as is well-known, $G(I^*)=G(I)[T]$ and T is transcendental over $G(I)$. Hence $G(I^*)$ is a Cohen-Macaulay (resp. Gorenstein) ring if and only if $G(I)$ is a Cohen-Macaulay (resp. Gorenstein) ring. When this is the case, we have $a(G(I^*))=a(G(I))-1$ (cf. [GW, (3.1.6)]). We furthermore have the following results, which enable us to assume that $s=\mathrm{ht}_A I$ is arbitrarily high. Hence the results given by [HH1, HH2] on ideals I with $\mathrm{ad}\, I=1$ or 2 are still true also in the case where $\mathrm{ht}_A I=\mathrm{ad}\, I-1$, although Huckaba and Huneke throughout assume $\mathrm{ht}_A I\geq \mathrm{ad}\, I$ in their theorems.

LEMMA 2.2. (1) $\mathrm{ht}_B I^*=s+1$.

(2) $\lambda(I^*)=\lambda(I)+1$.

(3) *J^* is a minimal reduction of I^* and $r_{J^*}(I^*)=r_J(I)$.*

(4) *Let $Q\in \operatorname{Spec} B$ such that $Q\supseteq I^*$. Let $P=Q\cap A$. Then I^*B_Q is generated by a*

B_Q-regular sequence of length $s+1$ if and only if IA_P is generated by an A_P-regular sequence of length s.

(5) *$I^{*(n)}=I^{*n}$ for all $n\in \boldsymbol{Z}$ if and only if $I^{(n)}=I^n$ for all $n\in \boldsymbol{Z}$, where $I^{*(n)}$ and $I^{(n)}$ denote the symbolic powers.*

PROOF. Let $Q\in \operatorname{Spec} B$ such that $Q\supseteq I^*$. We put $P=Q\cap A$. Then $Q=PB+XB$ so that $\dim B_Q=\dim A_P+1$. Hence $\operatorname{ht}_B I^*=s+1$. Since $\mathfrak{n}G(I^*)=\mathfrak{m}G(I^*)$, we have $B/\mathfrak{n}\otimes_B G(I^*)=A/\mathfrak{m}\otimes_A G(I)[T]$, whence the equality $\lambda(I^*)=\lambda(I)+1$ follows. Thus J^* is a minimal reduction of I^*. Since $I^{*n+1}=XI^{*n}+I^{n+1}B$ $(n\geq 0)$, we have $r_{J^*}(I^*)=r_J(I)$. Because $\mu_{B_Q}(I^*B_Q)=\mu_{A_P}(IA_P)+1$ and $\operatorname{grade}(I^*B_Q, B_Q)=\operatorname{grade}(IA_P, A_P)+1$, we get the assertion (4). The assertion (5) follows from the equality $I^{*(n)}=\sum_{i=0}^{n} X^i\cdot I^{(n-i)}B$ $(n\geq 1)$. q.e.d.

Our proof of Theorems 1.3 and 1.4 that we will give in Section 4 is based on the reduction, modulo certain super-regular sequences, to the Cohen-Macaulayness of the associated graded ring $G(I)$ in the case where $\operatorname{ad} I=1$ and the base ring A is one-dimensional and Cohen-Macaulay. In this section we shall summarize some preliminary results for that case. Now, for the rest of this section, let A be a Cohen-Macaulay local ring of $\dim A=1$. Let

$$(0)=\bigcap_{\mathfrak{p}\in \operatorname{Ass} A} I(\mathfrak{p})$$

be the primary decomposition of (0) in A. Let F be a non-empty subset of $\operatorname{Ass} A$ and assume that $F\neq \operatorname{Ass} A$. We put $I=\bigcap_{\mathfrak{p}\in F} I(\mathfrak{p})$. Then A/I is a Cohen-Macaulay ring of $\dim A/I=1$ and $\lambda(I)=1$. Let $b\in I$ be such that $I^{n+1}=bI^n$ for some $n\geq 0$. We put $J=bA$ and $\mathfrak{a}=(0):b$. Then since $\mathfrak{a}=\bigcap_{\mathfrak{p}\in \operatorname{Ass} A\setminus F} I(\mathfrak{p})$, we have $I\cap \mathfrak{a}=(0)$. Since $=\sqrt{\mathfrak{a}+I}=\mathfrak{m}$, we see that $\mathfrak{a}$ contains an element y such that $y+b$ is A-regular. Hence y is A/I-regular and b is $A/\mathfrak{a}$-regular. We put $\boldsymbol{R}=R(I)$, $\boldsymbol{G}=G(I)$, and $\mathfrak{M}=\mathfrak{m}\boldsymbol{G}+\boldsymbol{G}_+$. Let $f=y+bt\in \boldsymbol{R}$. We note:

LEMMA 2.3 (cf. [GN, (3.3)]). $\mathfrak{M}=\sqrt{f\boldsymbol{G}}$. *Hence $\boldsymbol{G}$ is a Cohen-Macaulay ring if and only if f is $\boldsymbol{G}$-regular.*

PROOF. Let $Q\in \operatorname{Spec} \boldsymbol{R}$ be such that $Q\supseteq I\boldsymbol{R}+f\boldsymbol{R}$. Then since $y\equiv -bt \bmod Q$, we have $y^2=-ybt\equiv 0 \bmod Q$ so that $y, bt\in Q$. Hence $Q\supseteq \boldsymbol{R}_+$ since $J=bA$ is a (minimal) reduction of I. Because $y+b\in Q$, we get $Q\supseteq \mathfrak{m}$. Thus $Q=\mathfrak{m}\boldsymbol{R}+\boldsymbol{R}_+$, whence $\mathfrak{M}=\sqrt{f\boldsymbol{G}}$. The second assertion follows from the first. q.e.d.

PROPOSITION 2.4. *The following conditions are equivalent*:

(1) *$\boldsymbol{G}$ is a Cohen-Macaulay ring.*

(2) *$bI\cap I^n=bI^{n-1}$ for all $n\geq 3$.*

PROOF. (1) $\Rightarrow$ (2) Let $x\in I$ be such that $bx\in I^n$ and assume $bx\notin bI^{n-1}$. Choose $i\geq 1$ so that $x\in I^i$ but $x\notin I^{i+1}$. Then $i\leq n-2$ since $x\notin I^{n-1}$, whence $bx\in I^{i+2}$. Therefore

$(y+bt)\cdot xt^i = bxt^{i+1} \in IR$; so we have $xt^i \in IR$ because $f = y+bt$ is G-regular by (2.3). Thus $x \in I^{i+1}$, which is the required contradiction.

(2) ⇒ (1) Let $g = \sum_{i=0}^{n} g_i t^i$ ($g_i \in I^i$) and assume that $fg \in IR$. Then since $fg = yg_0 + \sum_{i=0}^{n} bg_i t^{i+1}$, we have $yg_0 \in I$ and $bg_i \in I^{i+2}$ ($1 \le i \le n$). Hence $g_0 \in I$, because y is A/I-regular. Since $bg_i \in bI \cap I^{i+2}$, we see $bg_i \in bI^{i+1}$ so that $g_i \in I^{i+1}$ for all $1 \le i \le n$ (recall that $I \cap \mathfrak{a} = (0)$). Thus $g \in IR$, which proves f is G-regular. Hence G is a Cohen-Macaulay ring by Lemma 2.3. q.e.d.

COROLLARY 2.5 (cf. [GN, (1.5)]). *G is a Cohen-Macaulay ring if $r_J(I) \le 2$.*

PROOF. Since $I^n = bI^{n-1}$ for all $n \ge 3$, we have $bI \cap I^n = bI^{n-1}$. Hence G is a Cohen-Macaulay ring by Proposition 2.4. q.e.d.

The next result is due to [GH]. Notice that in the proof given by [GH, 2.4] for the case where $\dim A = 1$, we do not utilize the assumption that G is Cohen-Macaulay.

PROPOSITION 2.6 (cf. [GH, 2.4]). *$a(G) = 0$ if $I = J$, while $a(G) = r_J(I) - 1$ if $I \ne J$.*

We put $\bar{A} = A/\mathfrak{a}$, $\bar{I} = I\bar{A}$ and $\bar{J} = J\bar{A}$. Let $\bar{\mathfrak{m}} = \mathfrak{m}\bar{A}$. Then $\bar{I}$ is an $\bar{\mathfrak{m}}$-primary ideal in $\bar{A}$ and $\bar{J}$ is a minimal reduction of $\bar{I}$. If $\bar{I}^{n+1} = \bar{J}\bar{I}^n$ for some $n \ge 0$, we have $I^{n+1} \subseteq JI^n + \mathfrak{a}$, so that $I^{n+1} = JI^n$ since $I \cap \mathfrak{a} = (0)$. Thus we get $r_{\bar{J}}(\bar{I}) = r_J(I)$. Let $\varphi: G \to G(\bar{I})$ denote the canonical epimorphism of the associated graded rings. We put $L = \operatorname{Ker} \varphi$. Let L_n denote the homogeneous component of L of degree n. Then since $I \cap \mathfrak{a} = (0)$, we have $L_n = (0)$ ($n \ne 0$) and $L_0 = (I + \mathfrak{a})/I \cong \mathfrak{a}$ as A-modules. Hence we get an exact sequence

$$(2.7)\quad \text{(cf. [GN, (2.3)])} \qquad 0 \longrightarrow {}_{\rho}\mathfrak{a} \longrightarrow G \xrightarrow{\varphi} G(\bar{I}) \longrightarrow 0$$

of graded R-modules, where ${}_{\rho}\mathfrak{a}$ denotes $\mathfrak{a}$ which is regarded, via the canonical projection $\rho: R \to A$, as a graded R-mocule concentrated at degree 0.

PROPOSITION 2.8. *The following conditions are equivalent:*

(1) *$G(\bar{I})$ is a Cohen-Macaulay ring.*

(2) *G is a Cohen-Macaulay ring and $I^2 : b = I + \mathfrak{a}$.*

(3) *$bA \cap I^n = bI^{n-1}$ for all $n \in \mathbb{Z}$.*

PROOF. (1) ⇒ (2) Since ${}_{\rho}\mathfrak{a}$ is a Cohen-Macaulay R-module of dimension one, we get by (2.7) that G is Cohen-Macaulay (use the depth lemma). Let $x \in I^2 : b$. Then since $b\bar{A} \cap \bar{I}^2 = b\bar{I}$ (cf. [VV, 2.3], recall that bt is $G(\bar{I})$-regular), we have $bx \in bI + \mathfrak{a}$. Hence $bx \in bI$ since $I \cap \mathfrak{a} = (0)$; thus $x \in I + \mathfrak{a}$.

(2) ⇒ (3) We may assume $n \ge 2$. Then $bA \cap I^n = b(I^n : b) \subseteq b(I^2 : b)$, and we have $bA \cap I^n \subseteq bI$. Hence by Proposition 2.4, $bA \cap I^n = bI^{n-1}$.

(3) ⇒ (1) Let $n \ge 1$ be an integer. Then since $I \cap \mathfrak{a} = (0)$, we have $bA \cap (I^n + \mathfrak{a}) = bA \cap I^n$ so that $b\bar{A} \cap \bar{I}^n = b\bar{I}^{n-1}$. Hence bt is $G(\bar{I})$-regular (cf. [VV, 2.3]). Thus $G(\bar{I})$ is a Cohen-Macaulay ring. q.e.d.

COROLLARY 2.9. *Suppose that $G(\bar{I})$ is a Cohen-Macaulay ring. Then* $r_J(I) = \max\{n \geq 0 \mid I^n \not\subseteq J\}$.

PROOF. Let $n = \max\{n \geq 0 \mid I^n \not\subseteq J\}$. Then since $I^{n+1} \subseteq J$, we have by Proposition 2.8 that $I^{n+1} = JI^n$. Hence $n \geq r_J(I)$. The opposite inequality is obvious. q.e.d.

EXAMPLE 2.10. Let $R = k[[X, Y, Z]]$ be the formal power series ring in three variables over an infinite field k. Let $r \geq 0$ be an integer and put $A = R/((X, Y) \cap (Y, Z)^{r+1})$. Let $I = (X, Y)A$. Then $G(\bar{I})$ is a Cohen-Macaulay ring, $a(G) = 0$ if $r = 0$, while $a(G) = r - 1$ if $r \geq 1$.

PROOF. We put $x = X \bmod (X, Y) \cap (Y, Z)^{r+1}$. Then $F = \{I\}$ and $I^{r+1} = xI^r$. Since $\mu_A(I^j) = j + 1$ $(0 \leq j \leq r)$, we get $I^j \neq xI^{j-1}$ for $1 \leq j \leq r$; hence $r_J(I) = r$ where $J = xA$. Thus we have the assertion on $a(G)$ (cf. Proposition 2.6). Let $n \geq 1$ be an integer. Then we have

$$\begin{aligned} XR \cap ((X, Y)^n + (X, Y) \cap (Y, Z)^{r+1}) &\subseteq X \cdot (X, Y)^{n-1} + XR \cap (Y^n R + (X, Y) \cap (Y, Z)^{r+1}) \\ &\subseteq X \cdot (X, Y)^{n-1} + XR \cap (Y^n R + Y \cdot (Y, Z)^r + XZ^{r+1} R) \\ &= X \cdot (X, Y)^{n-1} + XZ^{r+1} R + XY^n R + XY \cdot (Y, Z)^r \\ &\subseteq X \cdot (X, Y)^{n-1} + (X, Y) \cap (Y, Z)^{r+1}. \end{aligned}$$

Hence $G(\bar{I})$ is a Cohen-Macaulay ring by Proposition 2.8. q.e.d.

3. Auxiliary results for the reduction to the case where ad $I = 1$. Throughout this section let I be an ideal in a Gorenstein local ring A and assume that A/I is a Cohen-Macaulay ring. Let $s = \mathrm{ht}_A I$ and put $F = \{P \in \operatorname{Spec} A \mid P \supseteq I \text{ and } \dim A_P \leq s + 1\}$. We assume that $\lambda(I) = s + 2$ and IA_P is generated by an A_P-regular sequence of length s for all $P \in F$. Hence $s \geq 1$ by Lemma 2.1. We put $\boldsymbol{R} = R(I)$, $\boldsymbol{G} = G(I)$, and $\mathfrak{M} = \mathfrak{m}\boldsymbol{G} + \boldsymbol{G}_+$. Let J be a minimal reduction of I.

PROPOSITION 3.1. *The ideal J contains a system $a_1, a_2, \ldots, a_s, b, c$ of generators satisfying the following conditions, where $K = (a_1, a_2, \ldots, a_s)$ and $L = K + bA$:*

(1) *$a_1, a_2, \ldots, a_s$ is an A-regular sequence and b is A-regular.*
(2) *$IA_P = KA_P$ for all $P \in \mathrm{Ass}_A A/I$.*
(3) *$a_1, a_2, \ldots, a_{s-1}$ form a part of a minimal system of generators of IA_P for all $P \in F$.*
(4) *$K : b = K : I$ and $(K : I) \cap I = K$.*
(5) *$(L : c) \cap I = L$.*
(6) *$\mathrm{ht}_A(I + (K : I)) \geq s + 1$ and $\mathrm{ht}_A(I + (L : I)) \geq s + 2$.*

For the proof see [HH2, 3.3].

Let $a_1, a_2, \ldots, a_s, b, c$ be throughout as in Proposition 3.1 above. Then we have:

LEMMA 3.2. *$L \cap I^2 = LI$ and $K \cap I^2 = KI$.*

PROOF. Consider the following four exact sequences of A-modules:

(1) $0 \to K/KI \to A/KI \to A/K \to 0$,

(2) $0 \to A/bK \to A/KI \oplus A/bI \to A/LI \to 0$,

(3) $0 \to K \to A \to A/bK \to 0$,

(4) $0 \to I \to A \to A/bI \to 0$,

where the second one follows from the equality $KI \cap bI = bK$ (use Proposition 3.1 (4)). Then since $K/KI \cong (A/I)^s$, by (1) A/KI is a Cohen-Macaulay ring of dimension $d-s$ (use the depth lemma). Now assume $L \cap I^2 \neq LI$ and choose $P \in \mathrm{Ass}_A(L \cap I^2)/LI$. Then $P \in \mathrm{Ass}_A A/LI$ and $P \supseteq I + (L:I)$. Hence $\dim A_P \geq s+2$ by Proposition 3.1 (6). Therefore the depth lemma implies, by (3) and (4), that depth $A_P/bK_P \geq 2$ and depth $A_P/bI_P \geq 2$; so by (2) we get $P \in \mathrm{Ass}_A A/KI$, because $P \in \mathrm{Ass}_A A/LI$. Hence $\dim A_p = s$, since A/KI is a Cohen-Macaulay ring of dimension $d-s$. This is impossible. Thus $L \cap I^2 = LI$. Similarly, if $P \in \mathrm{Ass}_A(K \cap I^2)/KI$, we get $P \supseteq I + (K:I)$ whence $\mathrm{ht}_A P \geq s+1$ by Proposition 3.1 (6), while we have $\mathrm{ht}_A P = s$ because $P \in \mathrm{Ass}_A A/KI$. This is absurd. Thus $K \cap I^2 = KI$ as claimed. q.e.d.

Because $\mathrm{ht}_A(I+(K:I)) \geq s+1$ and $\mathrm{ht}_A(I+(L:I)) \geq s+2$, we may choose a system $\{x_1, x_2, \dots, x_{d-s-2}, y, z\}$ of parameters for the ring A/I with $y \in K:I$ and $z \in L:I$. Let $\mathfrak{a} = (a_1 t, \dots, a_s t, x_1, \dots, x_{d-s-2}, y+bt, z+ct)\boldsymbol{R}$.

Lemma 3.3. *$\mathfrak{M} = \sqrt{\mathfrak{a}\boldsymbol{G}}$. Hence $a_1 t, \dots, a_s t, x_1, \dots, x_{d-s-2}, y+bt$ form a $\boldsymbol{G}$-regular sequence, if $\boldsymbol{G}$ is a Cohen-Macaulay ring.*

Proof. Let $Q \in \mathrm{Spec}\, R$ and assume that $Q \supseteq I\boldsymbol{R} + \mathfrak{a}$. Then since $y \equiv -bt \mod Q$, we have $y^2 \equiv -ybt \mod Q$. Hence $y, bt \in Q$, because $yb \in K$. Similarly $z, ct \in Q$, whence $Q \supseteq Jt$. Thus $Q \supseteq \boldsymbol{R}_+$, since J is a reduction of I. Because $Q \supseteq I + (x_1, \dots, x_{d-s-2}, y, z)$, we have $Q \supseteq \mathfrak{m}$. Hence $Q = \mathfrak{m}\boldsymbol{R} + \boldsymbol{R}_+$, so that we have $\mathfrak{M} = = \sqrt{\mathfrak{a}\boldsymbol{G}}$. The second assertion follows from the first. q.e.d.

We put $\bar{A} = A/(K:I)$, $\bar{I} = I\bar{A}$, and $\bar{J} = J\bar{A}$. Then $\bar{A}$ is a Cohen-Macaulay ring of dimension $d-s$ (cf. [PS]). We have the following theorem, which claims that $\bar{I}$ is generically a complete intersection in $\bar{A}$ and $\mathrm{ad}\, \bar{I} = 1$, that is, the ideal $\bar{I}$ satisfies the conditions (i) and (ii) stated in Theorem 1.2.

Theorem 3.4. (1) *$\mathrm{ht}_{\bar{A}} \bar{I} = 1$ and $\bar{A}/\bar{I}$ is a Cohen-Macaulay ring of dimension $d-s-1$.*

(2) *b is $\bar{A}$-regular and $I\bar{A}_P = b\bar{A}_P$ for all $P \in \mathrm{Ass}_{\bar{A}} \bar{A}/\bar{I}$.*

(3) *$\lambda(\bar{I}) = 2$ and $\bar{J}$ is a minimal reduction of $\bar{I}$.*

(4) *$r_{\bar{J}}(\bar{I}) = r_J(I)$, if $K \cap I^n = KI^{n-1}$ for all $n \in \boldsymbol{Z}$.*

Proof. Since $I \cap (K:I) = K$ by Proposition 3.1 (4), we have an exact sequence

$$0 \to A/K \to A/I \oplus A/K:I \to A/(I+(K:I)) \to 0$$

of A-modules. Hence depth $A/(I+(K:I)) \geq d-s-1$, so that $\bar{A}/\bar{I}$ is a Cohen-Macaulay ring of dimension $d-s-1$ (recall that $\mathrm{ht}_A(I+(K:I)) \geq s+1$ by Proposition 3.1 (6)).

Therefore $\mathrm{ht}_{\bar{A}}\bar{I}=1$. Since $bA\cap(K:I)\subseteq K$ and $K:b=K:I$, b is certainly $\bar{A}$-regular. Let $P\in\mathrm{Ass}_A A/(I+(K:I))$. Then since $P\supseteq I+(K:I)$ and $\dim A_P=s+1$, we have by Proposition 3.1 (6) that $IA_P=LA_P$. Hence $I\bar{A}_P=b\bar{A}_P$. Because $\bar{J}$ is a reduction of $\bar{I}$, we have $\lambda(\bar{I})\leq 2$. If $\lambda(\bar{I})<2$, then $\lambda(\bar{I})=\mathrm{ht}_{\bar{A}}\bar{I}=1$, so that by [CN, Theorem] we find $\bar{I}=x\bar{A}$ for some $x\in I$. Then since $I=xA+(K:I)\cap I$, we have by Proposition 3.1 (4) the equality $I=xA+K$. Hence $\mu_A(I)\leq s+1$, which is impossible because $\lambda(I)=s+2$. Thus $\lambda(\bar{I})=2$, whence $\mathrm{ad}\,\bar{I}=1$. Suppose that $K\cap I^n=KI^{n-1}$ for all $n\in \boldsymbol{Z}$. If $\bar{I}^{n+1}=\bar{J}\bar{I}^n$ for some $n\geq 0$, then $I^{n+1}=JI^n+(K:I)\cap I^{n+1}$. Since $(K:I)\cap I^{n+1}\subseteq K$ by Proposition 3.1 (4), we have that $I^{n+1}=JI^n+KI^n=JI^n$. Thus $r_{\bar{J}}(\bar{I})\geq r_J(I)$. Because the opposite inequality is obvious, this completes the proof of Theorem 3.4. q.e.d.

LEMMA 3.5. $\operatorname{depth}\bar{A}/\bar{I}^2\geq\min\{\operatorname{depth}A/I^2, d-s-1\}$.

PROOF. Since $(K:I)\cap I^2\subseteq K$ by Proposition 3.1 (4), we have $(K:I)\cap I^2=KI$ by Lemma 3.2. So we get an exact sequence

$$0\to A/KI\to A/I^2\oplus A/K:I\to\bar{A}/\bar{I}^2\to 0$$

of A-modules. Because both A/KI and A/K:I are Cohen-Macaulay rings of dimension $d-s$ (cf. the proof of Lemma 3.2), the depth lemma will guarantee the required inequality. q.e.d.

Let $\varphi: G(I/K)\to G(\bar{I})$ be the canonical epimorphism of the associated graded rings. Then since $(K:I)\cap I=K$, we get $\mathrm{Ker}\,\varphi=[\mathrm{Ker}\,\varphi]_0$ and $[\mathrm{Ker}\,\varphi]_0=(I+(K:I))/I\cong(K:I)/K$ as A-modules. Hence we have an exact sequence

$$0\longrightarrow {}_\rho[(K:I)/K]\longrightarrow G(I/K)\xrightarrow{\ \varphi\ } G(\bar{I})\longrightarrow 0 \tag{3.6}$$

of graded $\boldsymbol{R}$-modules, where ${}_\rho[(K:I)/K]$ denotes $(K:I)/K$ which is regarded, via the canonical projection $\rho: \boldsymbol{R}\to A$, as a graded $\boldsymbol{R}$-module concentrated at degree 0.

Let $\bar{a}$ denote, for each $a\in A$, the reduction of $a \bmod K$. We put $B=A/K$. Let $B[t]$ be the polynomial ring. We put $\boldsymbol{S}=R(I/K):=B[(I/K)t]$ $(\subseteq B[t])$.

LEMMA 3.7. $\mathrm{Ker}\,\varphi\cap(\bar{y}+\bar{b}t)G(I/K)=(\bar{y}+\bar{b}t)\cdot\mathrm{Ker}\,\varphi$.

PROOF. Let $f\in\mathrm{Ker}\,\varphi\cap(\bar{y}+\bar{b}t)G(I/K)$ and choose $g\in G(I/K)$ so that $f=(\bar{y}+\bar{b}t)g$. We write $f=\bar{x} \bmod I\boldsymbol{S}$ with $x\in K:I$ and $g=\sum_{i=0}^n\bar{c}_it^i \bmod I\boldsymbol{S}$ with $c_i\in I^i$ $(0\leq i\leq n)$. Then since $y\in K:I$, we have $\bar{x}\equiv\overline{yc_0}+\sum_{i=0}^n\overline{bc_i}t^{i+1} \bmod I\boldsymbol{S}$. Hence $x\equiv yc_0 \bmod I$ and $bc_0\in K+I^2$. Because $x, y\in K:I$ and $(K:I)\cap I=K$ (cf. Proposition 3.1 (4)), we have $x\equiv yc_0 \bmod K$. On the other hand, writing $bc_0=k+v$ with $k\in K$ and $v\in I^2$, we get $v\in L\cap I^2=LI$ (cf. Lemma 3.2). Let $v=k_1+bi$ with $k_1\in K$ and $i\in I$. Then since $bc_0=(k+k_1)+bi$, we have $c_0-i\in K:I$ by Proposition 3.1 (4), whence $\varphi(\bar{c}_0 \bmod I\boldsymbol{S})=0$. Because $\bar{x}=(\bar{y}+\bar{b}t)\bar{c}_0-\overline{bc_0}\,t\equiv(\bar{y}+\bar{b}t)\bar{c}_0 \bmod I\boldsymbol{S}$, we have $f\in(\bar{y}+\bar{b}t)\cdot\mathrm{Ker}\,\varphi$ as claimed. q.e.d.

PROPOSITION 3.8. *Suppose that* $\boldsymbol{G}$ *is a Cohen-Macaulay ring. Then* bt *is* $G(\bar{I})$-*regular.*

PROOF. By Lemma 3.3, $a_1t, a_2t, \ldots, a_st, y+bt$ form a $\boldsymbol{G}$-regular sequence. Hence by [VV, 2.1] we have a canonical isomorphism $\boldsymbol{G}/(a_1t, a_2t, \ldots, a_st)\boldsymbol{G} \cong G(I/K)$, so that $y+bt$ is $G(I/K)$-regular. Therefore by Lemma 3.7 we see that $y+bt$ is $G(\bar{I})$-regular. Thus bt is $G(\bar{I})$-regular too, because $y \in K:I$. q.e.d.

We close this section with the following two lemmas:

LEMMA 3.9. *Suppose that* $\boldsymbol{G}$ *is a Cohen-Macaulay ring and* $\dim A > \lambda(I) = s+2$. *Then* $\mathfrak{m}$ *contains an element* x *satisfying the following conditions, where* $C = A/xA$:

(1) x *is* $\boldsymbol{G}$-*regular.*

(2) C *is a Gorenstein local ring of dimension* $d-1$ and C/IC *is a Cohen-Macaulay ring of dimension* $d-s-1$.

(3) $\mathrm{ht}_C IC = s$ *and* $\lambda(IC) = s+2$.

(4) JC *is a minimal reduction of* IC *and* $r_{JC}(IC) = r_J(I)$.

(5) IC_P *is generated by a* C_P-*regular sequence of length* s *for all* $P \in \operatorname{Spec} C$ *such that* $P \supseteq IC$ *and* $\dim C_P \le s+1$.

(6) $G(IC)$ *is a Cohen-Macaulay ring and* $a(G(IC)) = a(\boldsymbol{G})$.

PROOF. Let $F_1 = \bigcup_{n \ge 1} \mathrm{Ass}_A A/I^n$ and $F_2 = \{P \in \mathrm{Supp}_A \bigwedge^{s+1} I \mid \dim A_P = s+2\}$. Then F_1 is a finite set (cf. [Br]). Since $F_2 \subseteq \mathrm{Min}_A \bigwedge^{s+1} I$, we see that F_2 is finite too. Notice that depth $A/I^n > 0$ for all $n \ge 1$ by Lemma 3.3, because $\boldsymbol{G}$ is a Cohen-Macaulay ring and $d > s+2$. Hence $\mathfrak{m} \notin F_1 \cup F_2$. Choose $x \in \mathfrak{m}$ so that $x \notin P$ for any $P \in F_1 \cup F_2$. Then x is $\boldsymbol{G}$-regular, since x is A/I^n-regular for all $n \ge 1$. Therefore since $\boldsymbol{G}/x\boldsymbol{G} \cong G(IC)$ by [VV, 2.1], we see that $G(IC)$ is a Cohen-Macaulay ring, $\lambda(IC) = \lambda(I)$, and $a(G(IC)) = a(\boldsymbol{G})$ (cf. [GW, (3.1.6)]). Because x is regular on both A and A/I, we have the assertion (2); hence $\mathrm{ht}_C IC = s$. If $(IC)^{n+1} = JC \cdot (IC)^n$ for some $n \ge 0$, then $I^{n+1} = JI^n + (xA \cap I^{n+1})$. Since $xA \cap I^{n+1} = xI^n$ ([VV, 2.3]), we have $I^{n+1} = JI^n$ by Nakayama's lemma, which proves $r_{JC}(IC) = r_J(I)$. The assertion (4) follows from the fact that $x \notin P$ for any $P \in F_2$. q.e.d.

LEMMA 3.10. *Suppose* $\boldsymbol{G}$ *is a Cohen-Macaulay ring and let* $C = A/a_1A$. *Then we have*:

(1) C *is a Gorenstein local ring of dimension* $d-1$ *and* C/IC *is a Cohen-Macaulay ring of dimension* $d-s$.

(2) $\mathrm{ht}_C IC = s-1$ *and* $\lambda(IC) = s+1$.

(3) JC *is a minimal reduction of* IC *and* $r_{JC}(IC) = r_J(I)$.

(4) *If* $s \ge 2$, *then* IC_P *is generated by a* C_P-*regular sequence of length* $s-1$ *for all* $P \in \operatorname{Spec} C$ *such that* $P \supseteq IC$ *and* $\dim C_P \le s$.

(5) $G(IC)$ *is a Cohen-Macaulay ring and* $a(G(IC)) = a(\boldsymbol{G}) + 1$.

PROOF. Since a_1t is $\boldsymbol{G}$-regular by Lemma 3.3, we have $G(IC) \cong \boldsymbol{G}/(a_1t)\boldsymbol{G}$ (cf. [VV, 2.1]). Hence $G(IC)$ is a Cohen-Macaulay ring and $a(G(IC)) = a(\boldsymbol{G}) + 1$ (cf. [GW, (3.1.6)]).

Since a_1 is A-regular too (cf. [VV, 2.3]), we have the assertion (1). Clearly, $\mathrm{ht}_C IC = s-1$ and $\lambda(IC)=\lambda(I)-1$. If $(IC)^{n+1}=JC\cdot(IC)^n$ for some $n\geq 0$, we have $I^{n+1}=JI^n+(a_1A\cap I^{n+1})$. Hence $I^{n+1}=JI^n$, because $a_1A\cap I^{n+1}=a_1I^n$ (cf. [VV, 2.3]). Thus $r_{JC}(IC)=r_J(I)$. The assertion (4) follows from Proposition 3.13 (3). q.e.d.

4. Proofs of Theorems 1.1 and 1.4. We shall maintain the same notation as in Section 3. The purpose of this section is to prove Theorems 1.1 and 1.4.

PROOF OF THEOREM 1.3. By Lemma 3.9 and Lemma 3.10 we may assume that $\dim A=3$ and $\mathrm{ht}_A I=1$. Let $a=a_1$, b, c be the system of generators for J given by Proposition 3.1. Let $K=aA$ and $L=(a, b)$. Consider the exact sequence

$$(4.1)\qquad 0\longrightarrow {}_\rho[(K:I)/K]\longrightarrow G(I/K)\xrightarrow{\varphi} G(\bar{I})\longrightarrow 0$$

given in (3.6), where $\bar{A}=A/(K:I)$ and $\bar{I}=I\bar{A}$. Then at is $\boldsymbol{G}$-regular by Lemma 3.3 and $G(I/K)$ is a Cohen-Macaulay ring with $a(G(I/K))=a(\boldsymbol{G})+1$ by Lemma 3.10. Let $H^i_{\mathfrak{M}}(*)$ $(i\in Z)$ denote the i^{th} local cohomology functor of $\boldsymbol{G}$ relative to $\mathfrak{M}$. Then applying $H^i_{\mathfrak{M}}(*)$ to (4.1), we have an exact sequence

$$(4.2)\qquad 0\to H^1_{\mathfrak{M}}(G(\bar{I}))\to H^2_{\mathfrak{M}}({}_\rho[(K:I)/K])\to H^2_{\mathfrak{M}}(G(I/K))\to H^2_{\mathfrak{M}}(G(\bar{I}))\to 0$$

of graded $\boldsymbol{G}$-modules. Recall that the A-module $(K:I)/K$ is (Cohen-Macaulay and) of dimension 2. Then by [GH, 2.2] we get

$$H^2_{\mathfrak{M}}({}_\rho[(K:I)/K])={}_\rho[H^2_{\mathfrak{m}}((K:I)/K)]\neq(0).$$

Hence by (4.2) we have $H^1_{\mathfrak{M}}(G(\bar{I}))=[H^1_{\mathfrak{M}}(G(\bar{I}))]_0$, that is, $H^1_{\mathfrak{M}}(G(\bar{I}))$ is concentrated at degree 0. We put $r=r_J(I)$. If $r\leq 1$, then since $r_{\bar{J}}(\bar{I})\leq r$, we see that $G(\bar{I})$ is a Cohen-Macaulay ring by Theorem 3.4 and [HH1, 2.9], so that $a(G(\bar{I}))=-1$ by [GH, 2.4]. Because $H^1_{\mathfrak{M}}(G(\bar{I}))=(0)$, we get by the exact sequence (4.2) that $a(G(I/K))=0$. Hence $a(\boldsymbol{G})=-1$.

Assume that $r\geq 2$. Then since bt is $G(\bar{I})$-regular by Proposition 3.8, we have an exact sequence $0\to G(\bar{I})(-1)\xrightarrow{bt} G(\bar{I})\to G(\bar{I}/b\bar{A})\to 0$. Apply the functors $H^i_{\mathfrak{M}}(*)$ to it. Then since $H^1_{\mathfrak{M}}(G(\bar{I}))=[H^1_{\mathfrak{M}}(G(\bar{I}))]_0$, we get an exact sequence

$$(4.3)\quad 0\longrightarrow H^1_{\mathfrak{M}}(G(\bar{I}))\longrightarrow H^1_{\mathfrak{M}}(G(\bar{I}/b\bar{A}))\xrightarrow{\psi} H^2_{\mathfrak{M}}(G(\bar{I}))(-1)\xrightarrow{bt} H^2_{\mathfrak{M}}(G(\bar{I}))\longrightarrow 0$$

of graded $\boldsymbol{G}$-modules. Notice that thanks to Theorem 3.4, the ideal $\bar{I}/b\bar{A}$ has analytic deviation one and is generically a complete intersection in $\bar{A}/b\bar{A}$. Then we have $a(G(\bar{I}/b\bar{A}))=r-1\geq 1$ by Proposition 2.6, because $r_{\bar{J}}(\bar{I})=r_J(I)$ by Theorem 3.4 and $r_{\bar{J}/b\bar{A}}(\bar{I}/b\bar{A})=r_{\bar{J}}(\bar{I})$ (see the proof of Lemma 3.10). Hence by (4.3) we get $(0)\neq \psi([H^1_{\mathfrak{M}}(G(\bar{I}/b\bar{A}))]_{r-1})\subseteq[H^2_{\mathfrak{M}}(G(\bar{I}))]_{r-2}$, since $H^1_{\mathfrak{M}}(G(\bar{I}))=[H^1_{\mathfrak{M}}(G(\bar{I}))]_0$. Thus $a(G(\bar{I}))\geq r-2=a(G(\bar{I}/b\bar{A}))-1$. Since $a(G(\bar{I}))\leq a(G(\bar{I}/b\bar{A}))-1$ in general, we have $a(G(\bar{I}))=r-2$. Hence $a(\boldsymbol{G})=r-3$ as claimed. This completes the proof of Theorem 1.1 as well as the proof of Theorem 1.3. q.e.d.

Let $\{a_1, a_2, \dots, a_s, b, c\}$ be the system of generators for J given by Proposition 3.1. We put $K=(a_1, a_2, \dots, a_s)$ and $L=K+bA$. To prove Theorem 1.4 we need the following two results:

LEMMA 4.4. *Suppose that $r_J(I)\leq 2$. Then $L\cap I^n=LI^{n-1}$ for all $n\in \mathbf{Z}$.*

PROOF. By Lemma 3.2 we may assume $n\geq 3$. Hence $I^n=JI^{n-1}$, so that we have $L\cap I^n=LI^{n-1}+L\cap cI^{n-1}$. Since $L\cap cI^{n-1}=c((L:c)\cap I^{n-1})$, we get by Proposition 3.1 (5) that $L\cap cI^{n-1}=c(L\cap I^{n-1})$. Thus by induction on n we get $L\cap cI^{n-1}=cLI^{n-2}$. Hence $L\cap I^n=LI^{n-1}$. q.e.d.

PROPOSITION 4.5. *Suppose that $r_J(I)\leq 2$. Then $K\cap I^n=KI^{n-1}$ for all $n\in \mathbf{Z}$.*

PROOF. We may assume $n\geq 2$. Since $K\cap I^n\subseteq LI^{n-1}$ by Lemma 4.4, we have $K\cap I^n=KI^{n-1}+K\cap bI^{n-1}$. Because $K\cap bI^{n-1}=b((K:b)\cap I^{n-1})$, we get by Proposition 3.1 (4) that $K\cap bI^{n-1}=b(K\cap I^{n-1})$. Thus by induction on n we have $K\cap I^n=KI^{n-1}$. q.e.d.

PROOF OF THEOREM 1.4. (1)$\Rightarrow$(2) is proved in [H, Proposition 1.1].

(2)$\Rightarrow$(3) Choose a system $\{x_1, x_2, \dots, x_{d-s-2}, y, z\}$ of parameters for A/I so that $y\in K:I$ and $z\in L:I$. Then by Lemma 3.3, $\{x_1, x_2, \dots, x_{d-s-2}\}$ forms a $\mathbf{G}$-regular sequence. Hence it forms a regular sequence for both A/I and I/I^2, so that $\{x_1, x_2, \dots, x_{d-s-2}\}$ is an A/I^2-regular sequence too. Thus depth $A/I^2\geq \dim A/I-2$.

(3)$\Rightarrow$(1) By Theorem 1.1 it suffices to show that $\mathbf{G}$ is a Cohen-Macaulay ring. Recall that depth $\bar{A}/\bar{I}^2\geq d-s-2=\dim \bar{A}/\bar{I}-1$ by Lemma 3.5. Then since $r_{\bar{J}}(\bar{I})\leq r_J(I)\leq 2$, we see that $G(\bar{I})$ is a Cohen-Macaulay ring by [GN1, 1.3]. Hence the exact sequence (3.6) guarantees that $G(I/K)$ is a Cohen-Macaulay ring, because $(K:I)/K$ is a Cohen-Macaulay A-module of $\dim d-s$. Thus $\mathbf{G}$ is a Cohen-Macaulay ring, since $\{a_1t, a_2t, \dots, a_st\}$ is a $\mathbf{G}$-regular sequence and $G(I/K)\cong \mathbf{G}/(a_1t, a_2t, \dots, a_st)\mathbf{G}$ by Proposition 4.5 (cf. [VV, 2.1 and 2.3]). This completes the proof of Theorem 1.4. q.e.d.

If $r_J(I)\leq 1$, then since $r_{\bar{J}}(\bar{I})\leq 1$, we get by Theorem 3.4 and [HH1, 2.9] that $G(\bar{I})$ is a Cohen-Macaulay ring. Hence by (3.6) and Proposition 4.5 $\mathbf{G}$ is Cohen-Macaulay. Thus by Theorem 1.1 we have:

COROLLARY 4.6 (cf. [HH2, 4.1]). ***R** is a Cohen-Macaulay ring if $r_J(I)\leq 1$.*

The next example is fairly well-known. We would like to explore it in our context.

EXAMPLE 4.7. Let $R=k[X_{1j}, X_{2j}\,|\,j=1, 2, 3, 4]$ be the polynomial ring in eight variables over an infinite field k. Let P denote the ideal of R generated by the maximal minors of the matrix

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \end{bmatrix}.$$

We put $M=(X_{1j}, X_{2j} \mid j=1, 2, 3, 4)R$. Let $A=R_M$ and $\mathfrak{p}=PA$. Then $\lambda(\mathfrak{p})=5$ and $\mathrm{ht}_A \mathfrak{p}=3$. $R(\mathfrak{p})$ is a Cohen-Macaulay ring and $a(G(\mathfrak{p}))=-3$.

PROOF. $A/\mathfrak{p}$ is a Cohen-Macaulay normal ring and $\mathrm{ht}_A \mathfrak{p}=3$ (cf. [HE]). Let

$$[i\ j]=\det\begin{bmatrix} X_{1i} & X_{1j} \\ X_{2i} & X_{2j} \end{bmatrix} \quad \text{for} \quad 1 \le i < j \le 4 .$$

Let $J=([1\ 2], [2\ 3], [3\ 4], [1\ 4], [1\ 3]-[2\ 4])A$. Then since $[1\ 2][3\ 4]-[1\ 3][2\ 4]+[1\ 4][2\ 3]=0$, we get $\mathfrak{p}^2=J\mathfrak{p}$. Because $\lambda(\mathfrak{p})=5$ by [CN, p. 221, Proposition], J is a minimal reduction of $\mathfrak{p}$ and $r_J(\mathfrak{p})=1$. Since $A/\mathfrak{p}$ is a normal ring and $\mathrm{ht}_A \mathfrak{p}=3$, $\mathfrak{p}A_Q$ is generated by an A_Q-regular sequence of length 3 for all $Q \in \operatorname{Spec} A$ such that $Q \supseteq \mathfrak{p}$ and $\dim A_Q \le 4$. Thus $R(\mathfrak{p})$ is a Cohen-Macaulay ring by Corollary 4.6. We have $a(G(\mathfrak{p}))=-3$ by Theorem 1.3. q.e.d.

As for the next example we do not know whether $G(\mathfrak{p})$ is a Cohen-Macaulay ring or not.

EXAMPLE 4.8. Let $R=k[X, Y, Z, W, V, U]$ be the polynomial ring in six variables over an infinite field k. Let P be the ideal of R generated by the maximal minors of the matrix

$$\Omega=\begin{bmatrix} X & Y & Z & W & V \\ Y & Z & W & V & U \end{bmatrix}.$$

We put $M=(X, Y, Z, W, V, U)R$. Let $A=R_M$ and $\mathfrak{p}=PA$. Then $\lambda(\mathfrak{p})=6$, $\mathrm{ht}_A \mathfrak{p}=4$, and $r_J(\mathfrak{p})=3$ for any minimal reduction J of $\mathfrak{p}$.

PROOF. Since $A/\mathfrak{p}$ is a normal ring of dimension two, we have $\mathrm{ht}_A \mathfrak{p}=4$. Let $G=G(\mathfrak{p})$ and $\bar{G}=G/MG$. We denote by $[i\ j]$ the determinant of the matrix consisting of the i^{th} and j^{th} columns of Ω. Let $S=k[X_{ij} \mid 1 \le i < j \le 5]$ be the polynomial ring in ten variables over k and let $\Psi: S \to \bar{G}$ denote the homomorphism of k-algebras defined by $\Psi(X_{ij})=[i\ j]t \bmod MG$ $(1 \le i < j \le 5)$, where t is an indeterminate over A. Let $\mathfrak{a}$ be the ideal of S generated by the Pfaffians of order four in the alternating matrix

$$\begin{bmatrix} 0 & X_{12} & X_{13} & X_{14} & X_{15} \\ -X_{12} & 0 & X_{23} & X_{24} & X_{25} \\ -X_{13} & -X_{23} & 0 & X_{34} & X_{35} \\ -X_{14} & -X_{24} & -X_{34} & 0 & X_{45} \\ -X_{15} & -X_{25} & -X_{35} & -X_{45} & 0 \end{bmatrix}.$$

Then a direct checking shows that $\operatorname{Ker} \Psi \supseteq \mathfrak{a}+fS$, where $f=X_{24}^2-X_{12}X_{45}+X_{13}X_{35}-$

$X_{14}X_{34}-X_{23}X_{25}-X_{23}X_{34}$. Since $S/\mathfrak{a}$ is a Gorenstein UFD (cf. [BE]) and $f \bmod \mathfrak{a}$ is a prime element in $S/\mathfrak{a}$, we see that $\mathfrak{a}+fS$ is a prime ideal in S of height 4. Hence $\lambda(\mathfrak{p})=\dim \bar{G} \leq 6$, so that $\mathrm{ad}\,\mathfrak{p} \leq 2$. Notice that $\mathfrak{p}A_Q$ is generated by an A_Q-regular sequence of length 4 for all $Q \in \operatorname{Spec} A \setminus \{MA\}$ such that $Q \supseteq \mathfrak{p}$ (since $A/\mathfrak{p}$ is normal and $\dim A/\mathfrak{p}=2$). Therefore, if $\mathrm{ad}\,\mathfrak{p}<2$, by [CN, Theorem] we have $\mathrm{ad}\,\mathfrak{p}=1$ because $\mathfrak{p}$ is not a complete intersection in A. Hence $\mathfrak{p}^{(2)}=\mathfrak{p}^2$ by [HH1, 2.5]. This is impossible, since

$$\Delta=\det\begin{bmatrix} X & Y & Z \\ Y & Z & W \\ Z & W & V \end{bmatrix} \in \mathfrak{p}^{(2)}$$

but $\Delta \notin \mathfrak{p}^2$ (cf. [G, (7.5)]). Thus $\mathrm{ad}\,\mathfrak{p}=2$, whence $\lambda(\mathfrak{p})=\dim \bar{G}=6$. Therefore we have $\operatorname{Ker} \Psi=\mathfrak{a}+fS$. Consequently $\bar{G}$ is a Gorenstein ring with $a(\bar{G})=-3$, since $a(\bar{G})=a(S/\mathfrak{a})+2$ by [GW, (3.1.6)] and since $a(S/\mathfrak{a})=-5$. Let $J=(a_1, a_2, \ldots, a_6)A$ be a minimal reduction of $\mathfrak{p}$. Then since $\{a_1t, a_2t, \ldots, a_6t\}$ forms a homogeneous system of parameters for $\bar{G}$, we get $a(\bar{G}/(a_1t, a_2t, \ldots, a_6t)\bar{G})=a(\bar{G})+6=3$ (cf. [GW, (3.1.6)]). Hence $G_4 \subseteq (a_1t, a_2t, \ldots, a_6t)G+MG$, which implies, by Nakayama's lemma, that $\mathfrak{p}^4=J\mathfrak{p}^3$. Since $G_3 \not\subseteq (a_1t, a_2t, \ldots, a_6t)G+MG$, we have $\mathfrak{p}^3 \neq J\mathfrak{p}^2$. Thus $r_J(\mathfrak{p})=3$. Therefore, if G is a Cohen-Macaulay ring, then so is the ring $R=R(\mathfrak{p})$ by Theorem 1.1. q.e.d.

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Department of Mathematics
Meiji University
School of Science and Technology
Higashi-mita 1–1–1, Tama-ku
Kawasaki-shi 214
Japan

Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1–1
Hachioji-shi 192-03
Japan