PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
FACULTAD DE FÍSICA

COHERENCE AND ENTANGLEMENT IN A TWO-QUBIT SYSTEM

By

Maritza Lisette Hernández Gaete

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF PhD IN PHYSICS AT THE FACULTY OF PHYSICS, PONTIFICIA
UNIVERSIDAD CATÓLICA DE CHILE
SANTIAGO, CHILE
NOVEMBER 2009.

Research Supervisor	:	Dr. Miguel Orszag
External Examiner	:	Dr. Mario Molina
Examing Committee	:	Dr. Marcelo Loewe
		Dr. Hernán Chuaqui
		Dr. Sascha Wallentowitz

To my parents, Hector and Nenella.

To my sister Karin.

To my husband Daniel.

TABLE OF CONTENTS

Table of Contents v

Abstract *ix*

Acknowledgements *xi*

Introduction *1*

1 Quantum Entanglement **7**
- 1.1 Introduction . . . 7
- 1.2 Quantum Bits . . . 9
  - 1.2.1 The Bloch Sphere . . . 11
  - 1.2.2 Multiple Qubits . . . 12
- 1.3 Entanglement of pure states . . . 13
  - 1.3.1 The Schmidt decomposition . . . 16
  - 1.3.2 Measure of entanglement. Entropy of entanglement . . . 18
- 1.4 Entanglement of mixed states . . . 19
  - 1.4.1 Peres-Horodecki criteria. Positive partial transpose [15, 16] . . 20
  - 1.4.2 Entanglement of formation. Concurrence . . . 21

2 Quantum Decoherence **25**
- 2.1 Introduction . . . 25
- 2.2 The mechanism of Decoherence . . . 29
- 2.3 Decoherence Free Subspaces (DFS) . . . 31
- 2.4 Condition for DFS. The Lindblad semigroup Approach . . . 31

3 Dynamics of Entanglement **35**
- 3.1 Introduction . . . 35
- 3.2 Dynamics of Open Systems . . . 36
- 3.3 Time evolution of a two atom system. . . . 37

4 Entanglement in Independent Reservoirs **40**
4.1 Introduction 40
4.2 Vacuum Reservoir 40
4.3 Thermal Reservoir 43

5 Entanglement in a Common Reservoir **48**
5.1 Introduction 48
5.1.1 The model 49
5.1.2 Solution for initial states in DFS. 50
5.1.3 General Solution for vacuum reservoir. $N \longrightarrow 0$ 51
5.1.4 General Solution for $N \neq 0$ 55
5.1.5 Discussion 60

6 Creation of Quantum Entanglement **64**
6.1 Introduction 64

7 Conclusion **71**

Appendix **73**

A Solution of Master Equation I. Independent Reservoirs. **73**
A.1 Vacuum reservoirs 74
A.2 Thermal reservoirs 74
A.3 Squeezed reservoirs 76

B Solution of Master Equation II. A Common Reservoir **78**
B.1 Basis $\{\phi_i\}$ 78
B.2 Standard Basis 80
B.2.1 Vacuum reservoir 81

C A comparative study of the optical and circuit representation of the unambiguous quantum state discriminator [67] **82**
C.1 Introduction 83
C.2 Quantum Discriminator 83
C.3 Implementation of U_{AB} in a quantum circuit 85
C.3.1 Procedure to decompose a unitary matrix $U \in SU(4)$ 85
C.3.2 Decomposition of the unitary matrix U_{AB} 87
C.4 Feasibility of the circuit implementation using trapped ions 89
C.5 Linear optical implementation 91
C.6 Discussion 95
C.7 Proof of Lemma 1 in Ref. [80] 96

C.8 The decomposition of U 97

Bibliography *100*

ABSTRACT

Quantum entanglement is a physical resource, associated with the peculiar nonclassical correlations that are possible between separated quantum systems. Entanglement can be measured, transformed, and purified. A pair of quantum systems in an entangled state can be used as a quantum information channel to perform computational and cryptographic tasks that are impossible for classical systems.

The aim of this work is to study various aspects of quantum entanglement and coherence, illustrated by several examples. We relate the concepts of decoherence and disentanglement, via a model of two two-level atoms in different types of reservoir, including both cases of independent and common bath. Finally, we relate decoherence and disentanglement, by focussing on the sudden death of the entanglement and the dependence of the death time with the "distance" of our initial condition, from the decoherence free subspace. In particular, we study the sudden death of the entanglement, in a two-atom system with a common reservoir.

ACKNOWLEDGEMENTS

Foremost, I would like to thanks my God for being with me in every step that I take, for strengthening my heart, enlightening my mind and for having placed in my path those who have been my support and company throughout this study period.

I would like to thank my advisor, Prof. Michael Orszag, who shared with me a lot of his expertise and research insight. He quickly became for me the role model of a successful researcher in the field. I am deeply grateful to the examining committee for their important comments and contributions made to this work. Thanks to the National Commission of Scientific Research and Technology - CONICYT for the trust and support that they gave me in order to study my Ph.D. I would also like to express my gratitude to Juan, Carolina, Giliana and Liliana, the department staff, who always had excellent disposition to help. To my friend Daniela for her help in the printing and distribution of this work, and for making the time in the university so pleasant.

Wholehearted thanks to my parents Hector and Nenella, and my sister Karin, for their love, affection and understanding. For the company and support given forever. And in a special way I would like to thank to my husband Daniel for his support, understanding and love that allows me to feel able to achieve my goals. Thanks you for listening and for your advice (this is something you do very well). Thanks for being part of my life, you are the best thing that has happened to me.

Santiago, Chile

November 13, 2009

Maritza Hernández

INTRODUCTION

'Oh cat', she began rather timidly.
'Would you tell me please which way I ought to go from here?'.
'That depends a good deal on where you want to get to', said the Cat.
'I am not really sure where...,' began Alice.
'Then it doesn't matter which way you go', interrupted the Cat.
'But I have to decide between these two paths', said Alice.
'Now that is where you are wrong', mused the Cat. 'You do not have to decide, you can take all the paths'.

Robert Gilmore.
Alice in Quantumland. [1].

Quantum Mechanics has been an indispensable part of science ever since, and has been applied with enormous success, for example, to the structure of the atom, nuclear fusion stars, superconductors, the structure of DNA, and the elementary particles of Nature. Thus, Quantum Mechanics has had an enormous technological and societal impact. On the other hand, it is also easy to see the enormous impact of computers on everyday life. The importance of computers is such that it is appropriate to say that we are now living in the *information age*. This new information revolution becomes possible thanks to the relation between computer science and quantum physics. I am referring here to the fact that quantum mechanics can be used to process and transmit information.

Miniaturization provides us with an intuitive way of understanding why, in the

near future, quantum laws will become important for computation. The electronics industry for computers grows hand-in-hand with the decrease in size of integrated circuits. It is estimated that in a few years, the typical size of circuit shall reach the atomic size, at that point, quantum effects will become dominant unavoidable.

A quantum computer represents a challenge: *the aim is to build a machine based on quantum logic, that is, it processes the information and performs logic operations by exploiting the laws of quantum mechanics.*

The unit of quantum information is know as a *qubit* (the quantum counterpart of the classical *bit*) and a quantum computer may be viewed as a many-qubit system. Physically, a qubit is a two level system, like the two spin states of a spin-$\frac{1}{2}$ particle, the vertical and horizontal polarization states of a single photon or the ground and excited states of an atom, to name a few of the possibilities.

A quantum computation is composed of three basic steps: preparation of the input state, implementation of the desired unitary transformation acting on this state and measurement of the output state. The output of the measurement process is inherently probabilistic and the probabilities of the different possible outputs are set by the basic postulates of quantum mechanics.

The power of quantum computers is due to typical quantum phenomena, such as the *superposition* of quantum states and *entanglement*. There is an inherent quantum parallelism associated with the superposition principle. In simple terms, a quantum computer can process a large number of classical inputs in a single run, the same way that Alice can take the two paths at the same time

Quantum Entanglement was first viewed as a curiosity. However this point of view had gradually changed. In fact, entanglement plays an important role in most of the applications in the field of Quantum Information. Quantum entanglement refers to a property of a quantum state of a system of two or more objects in which the quantum states of the constituting objects are linked together thus one object can no longer be adequately described without full mention of its counterpart, even though the individual objects may be spatially separated.

The principal obstacle to the practical realization of a quantum computer is the Quantum Decoherence. Here the term decoherence denotes the decay of the quantum information stored in a quantum computer, due to the inevitable interaction of the quantum computer with the environment. Such interaction affects the performance of a quantum computer, introducing errors into the computation.

Although there is as yet no complete theory of entanglement or decoherence, some progress has been made in understanding these properties of quantum mechanics. Even, in the past years there has been a spectacular theoretical and experimental development of this field, which has allowed, sort to say, to dominate the quantum world. The different methods developed allow, at least in principle, to perform quantum computations.

Physical Implementations for Quantum Computation

There are very few systems in which one can implement a quantum computer. Many of the ideas in this respect come from the field of Quantum Optics, and Solid State.

Here I give a short overview of the first experimental implementations of quantum logic and few-qubit quantum processors. Given the generality of the requirements to build a quantum computer, many physical systems might be good candidates. Let me briefly discuss a few of them. This is not an exhaustive list, moreover, some of the proposals discussed below are less realistic than others. However, the purpose is present the variety of physical system that are under active investigation.

Optical system

A qubit can be realized with a singled photon in two optical modes, such as horizontal and vertical polarization states. Single-qubit gates can be implemented using linear-optics devices such as beam splitters and phase shifters. An interaction between photons is possible, but technically, as it must be mediated by atoms in a non-linear Kerr medium.

Cavity quantum electrodynamics

Using cavity quantum electrodynamics (QED) techniques, it has been possible to realize experiments in which a single atom interacts with a single mode or a few modes of the electromagnetic field inside a cavity. The two states of a qubit can be represented by the polarization states of a single photon or by two excited states of an atom. In the first case, the interaction between qubits is mediated by atoms, in the second case by photons. Cavity QED techniques have allowed the implementation of one and two qubit gates; however, it seems very difficult to perform a large number of operations or to address the scalability problem with these techniques. Nevertheless, it should be stressed that cavity QED experiments have been particulary successful in demonstrating basic features of quantum mechanics, such as Rabi oscillations and entanglement, or in exploring the effect of the decoherence and the transition from the quantum world to classical physics.

Ion traps

The quantum hardware is as follows: A string of ions is confined by a combination of static and oscillating electric fields in a linear trap (known as Paul trap). A qubit is a single ion and two long-lived states of the ion correspond to the two states of the qubit. The linear array of ions held in the trap is the quantum register. Singled qubit gates are obtained by addressing individual ions with laser Rabi pulses. The interactions between qubits, which are necessary to implement controlled two-qubit operations, are mediated by the collective vibrational motion of the trapped string of ions. The ion-trap technique has allowed the implementation of basic one and two qubit gates, the entanglement of four ions and, the demonstration of the Deutsch-Jozsa algorithm.

To build a scalable quantum computer, Cirac and Zoller envisaged a two dimensional array of independent ion traps and an independent ion (head) that moves about this plane, capable of approaching any particular ion. A suitable laser pulse could swap the state of the ion into the head and this would allow us to entangle well separated ions, the quantum communication between them being assured by the moving

head. It seems that there are no fundamental physical obstacles against this proposal, but a significant technological challenge remains.

Solid state proposals

Several proposals have been put forward to build a solid-state quantum computer. This is not surprising, since solid state physics has developed over the years a sophisticated technology, creating artificial structures and devices on nanoscale. Solid state physics is at the basis of the development of classical computer technology and therefore the scalability problem would find a natural solution in a solid state quantum computer. Indeed, such a quantum computer could benefit from the fabrication techniques of microelectronics. In the next, I shall briefly discuss a solid state proposals.

Quantum dots

Quantum dots are structures fabricated from semiconductor materials, in which electrostatic potentials confine electrons. The dot size is typically between 10 nanometers and 1 micron. The qubit is realized as the spin of an electron on a singled electron quantum dot and two qubit quantum operations are operated by purely electrical gating of the electrostatic tunnelling between neighboring quantum dots. Lowering (raising) this barrier correspond to switching on (off) the interaction between the two qubit. Scalability is in principle possible, since it is possible to produce arrays of quantum dots with present technology. However, the actual implementation of quantum gates and single spin measurements in such arrays constitutes a difficult experimental challenge. Furthermore, there are a great variety of possible decoherence processes in complex solid state devices and the knowledge of them is still very limited.

It has been my purpose to contribute to the study of these transcendental properties and facilitate the development of new applications.

This thesis is organized as follows. Chapter 1 is a review of the basic concepts of quantum computation. We discuss quantum entanglement and entanglement measures for pure and mixed states. Chapter 2 is devoted to the important subject of

decoherence, referring here to the dynamics of decoherence. The second part of the chapter is devoted to Decoherence Free Subspaces (DFS), namely corners in Hilbert space that are free from decoherence. Chapter 3 is a study of the entanglement evolution. At first is discussed a mathematical Model, based on a general Markovian Master Equation, that models a pair of qubits in various types of environments, like the vacuum, thermal or squeezed reservoirs. This general Master Equation includes both cases of independent and common reservoirs for both atoms. These two cases are treated separately in Chapter 4 and Chapter 5, respectively. Focussing on the recently observed phenomena of *entanglement sudden death* and *revival*. Chapter 6 is devoted to the effect of the *creation of the entanglement* and the *delayed sudden birth of entanglement*. The last Chapter is dedicated to discussion and conclusions.

The Appendices are organized as follows. In Appendix A, we present the solution of the master equation (3.3.1) in independent reservoirs. In Appendix B, the solution of the master equation (3.3.1) in a common reservoir. And, finally, in Appendix C we propose a quantum circuit implementation of the unambiguous quantum state discriminator.

CHAPTER 1

QUANTUM ENTANGLEMENT

1.1 Introduction

Quantum Correlations are among the most surprising effects in Quantum Mechanics. It was first pointed out by Schrödinger, who coined the term **'entanglement'** to describe this peculiar connection between quantum systems:

> *"When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled* [1]*".*

This property refers to a quantum systems composed of two or more parts or objects, in which the quantum states of these objects are such that each one of them cannot be described without referring to the other one, even in the case when both

[1] *E.Schrödinger,* Discussion of Probability Relations between Separated Systems, *Proc. Cambridge Philos. Soc.* ***31****, 555 (1935).*

objects are spatially far apart. This interaction leads to non-classical correlations for the physical properties of the systems.

However, the concept of quantum entanglement was considered a mere curiosity until it was pointed out by Einstein, Podolsky and Rosen, in their famous EPR paper [2], that quantum mechanics may not be a complete theory.

> *"In the quantum theory, the state of the system is characterized by the ψ-function which, in its turn, represents the solution of the Schrödinger equation. Each of these solutions (ψ-functions) has to be regarded within the sense of the theory, as a description of a physically possible state of the system. The question is* ***in what sense does the ψ*** **function describe the state of the system?**.
>
> *My assertion is this:*
> *The ψ-function cannot be regarded as a complete description of the system, only as an incomplete one.*
> *In other words: There are attributes of the individual system whose reality no-one doubts, but which the description by means of the ψ function does not include.* [1] *"*

Almost three decades later, when in 1964 John Bell [3] published his well known theorem, it was hardly noticed by the international community. Bell showed that, for a classical theory based on local hidden variables, certain correlations were upper bounded, known today as the Bell inequalities. This theory is based on the following assumptions:

- Measurement results are determined by the properties the particles carry prior to and independent of the measurements, **realism**.
- The results obtained at a particular location are independent of any actions performed in other spacial locations, **locality**.

[1] *M.Born, The Born-Einstein Letters 1916-1955, macmillan (2005).*

Bell proved that the above assumptions impose constraints on the statistical correlations in experiments involving bipartite systems, in the form of the well know Bell inequalities. Quantum Mechanics predicts that these inequalities can be violated, thus giving rise to possible experimental tests that could decide either in favor of the EPR argument or of quantum mechanics.

Experiments agree with quantum mechanics [4], although some researchers still find loopholes in the experimental assumptions[5].

More recently, entanglement has become an essential resource for various applications of quantum mechanics, such as quantum teleportation [6], quantum cryptography [7] and superdense coding [8]. So presently, quantum entanglement is not only subject of philosophical debate but a key resource for important applications, as the ones mentioned above.

Before discussing more extensively Quantum Entanglement, we shall briefly examine some basic ideas about quantum computation and quantum information.

1.2 Quantum Bits

The bit is the fundamental concept of classical computation and classical information. Quantum computation and quantum information are built upon an analogous concept, the *quantum bit*, or *qubit* for short. In this section we introduce the properties of single qubit, comparing and contrasting their properties to those of classical bits.

The qubits are described as *mathematical objects* with certain specific properties, and also as *physical objects*. Although, like bits, they are realized as actual physical systems, for the most part we treat qubits as abstract mathematical objects. The purpose of treating qubits as abstract entities is that it give us the freedom to construct a general theory of quantum computation and quantum information which does not depend upon a specific system for its realization.

A classical bit is a system that can exist in two distinct *states*, either 0 or 1. In contrast, a qubit is a two -dimensional complex Hilbert space. Two possible states

for a qubit are the states $|0\rangle$ and $|1\rangle$, which correspond to the states 0 and 1 for a classical bit.

$$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1.2.1}$$

The special states $|0\rangle$ and $|1\rangle$ are know as *computational basis states*, and form an orthonormal basis for this vector space.

The principal difference between bits and qubits is that a qubit can be in state *other than* $|0\rangle$ or $|1\rangle$. It is also possible to form *linear combination* of states, often called *superpositions*:

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1.2.2}$$

The numbers α and β are complex numbers, although for many purposes not much is lost by thinking of them as real numbers, they are constrained by the normalization condition,

$$|\alpha|^2 + |\beta|^2 = 1. \tag{1.2.3}$$

Due to the constrain (1.2.3), we may rewrite Equation (1.2.2) like a generic state as

$$|\Psi\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle), \tag{1.2.4}$$

where the numbers θ, φ, and γ are real numbers and with $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$. We can ignore the factor of $e^{i\gamma}$ out the front, because it has no observable effects, and we can write

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}. \tag{1.2.5}$$

Therefore, unlike the classical bit, which can only be set equal to 0 or 1, the qubit resides in a vector space, parameterized by the continuous variables α and β (or θ and ϕ). Thus, a continuum of states is allowed. This contradicts our "classical" way of thinking: according to our intuition, a system with states can only be in one state or in the other. However, as we have seen, quantum mechanics allows many other possibilities.

We can examine a bit to determine whether it is in the state 0 or 1. However we cannot examine a qubit to determine its quantum state, that is, the values of α and β. Instead, quantum mechanics tell us that we can only acquire much more restricted information about the quantum state. When we measure a qubit we get either the result 0, with probability $|\alpha|^2$, or the result 1 with probability $|\beta|^2$. The measurement *changes* the state of a qubit, collapsing it from its superposition of $|0\rangle$ and $|1\rangle$ to the specific state consistent with the measurement result. Why does this type of collapse occur?. Nobody knows, this behavior is simply one of the *fundamental postulates* of quantum mechanics.

1.2.1 The Bloch Sphere

The Bloch sphere representation is useful in thinking about qubits since it provides a geometric pictures of the qubit and the transformations that one can operate on the state of a qubit. Due to normalization condition (1.2.3), the qubit's state can be represented by a point on a sphere of unit radius, called *Bloch Sphere*. This sphere can be embedded in a three-dimensional space of Cartesian coordinates ($x = \cos\phi\sin\theta, y = \sin\phi\sin\theta, z = \cos\theta$). Thus, the state (1.2.5) can be written as

$$|\Psi\rangle = \begin{bmatrix} \sqrt{\frac{1+z}{2}} \\ \frac{x+iy}{\sqrt{2(1+z)}}, \end{bmatrix}. \tag{1.2.6}$$

By definition, a Bloch vector is a vector whose components (x, y, z) single out a point on the Bloch sphere. Therefore, each Bloch vector must satisfy the normalization condition $x^2 + y^2 + z^2 = 1$. The numbers θ and φ define a point on the unit three-dimensional sphere, as shown in Figure 1.1. Many operations on single qubits are described within the Bloch sphere picture. However, it must be kept in mind that this intuition is limited because there is no simple generalization of the Bloch sphere known for multiple qubits.

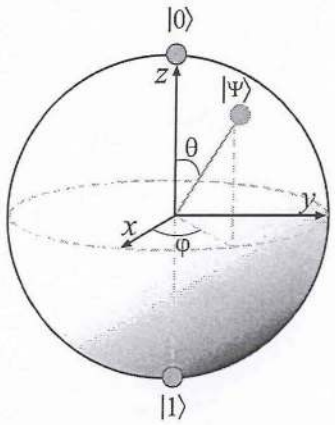

Figure 1.1: Bloch sphere representation of a qubit.

1.2.2 Multiple Qubits

A *composite system* is a system which consist of two or more parts. In the most general case, we may consider a system of n qubits. The computational basis states of this system are of the form $|x_1 x_2 \dots x_n\rangle$, and so a quantum state of such a system is specified by 2^n amplitudes.

Let us consider the simplest case of $n = 2$. If these were two classical bits, then there would be four possible states, $00, 01, 10, 11$. Correspondingly, a two qubit system has four *computational basis states* denoted $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. We call the two system A and B. Any state of each of the two systems can be written as

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A \qquad |\psi\rangle_B = \gamma|0\rangle_B + \delta|1\rangle_B, \tag{1.2.7}$$

with $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$. The composite state of the two systems is then simply the direct product of the two states.

$$|\Psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B. \tag{1.2.8}$$

Such a state is called a *product state*. If we let the two systems interact with each other, any superposition of product states is realizable. Thus, the quantum state of

two qubits involves associating a complex coefficient - called *amplitude* - with each computational basis state, such that the vector describing the two qubits is

$$|\Psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle, \tag{1.2.9}$$

where we have used $|ij\rangle$ as shorthand for $|i\rangle \otimes |j\rangle$. Similar to the case for a single qubit, the measurement result $x(= 00, 01, 10 \text{ or } 11)$ occurs with probability $|\alpha_x|^2$, with the state of the qubits after the measurement being $|x\rangle$. The condition that probabilities sum to one is therefore expressed by the normalization condition that $\sum |\alpha_x|^2 = 1$.

For bipartite qubit states, there are four important states, the singlet state

$$|\Psi^-\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{1.2.10}$$

and the three triplet states

$$\begin{aligned} |\Psi^+\rangle &\equiv \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\Phi^-\rangle &\equiv \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Phi^+\rangle &\equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \end{aligned} \tag{1.2.11}$$

They are known as the *Bell states*, or sometimes the *EPR states* or *EPR pairs*, after some of the people - Bell, and Einstein, Podolsky, and Rosen - who first pointed out the strange properties of these state. We will talk more about them in the next section. Together they form an orthonormal basis for the states space of two qubits, called *Bell basis*.

1.3 Entanglement of pure states

In the standard Hilbert-space formalization of quantum mechanics, *pure states* are described by state-vectors, $|\psi\rangle$, forming a linear state space attributed to the system in question.

Let us consider a system A in a state $|\psi\rangle_A$, and a second system B in state $|\psi\rangle_B$, with respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The Hilbert space of the composite system is given by $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Traditionally this systems are associated with two distant observers called Alice and Bob.

Definition If a pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as a product state

$$|\Psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B, \tag{1.3.1}$$

it is said to be separable. Otherwise we say that $|\Psi\rangle$ is an *Entangled State.*

Example 1

- **Separable State.** Let us consider the state $|1,0\rangle \in \mathcal{H}$, we can easily see that the system A is in the state $|1\rangle \in \mathcal{H}_A$, and the system B is in the state $|0\rangle \in \mathcal{H}_B$. Thus, the state $|1,0\rangle = |1\rangle_A \otimes |0\rangle_B$ is separable, therefore **non entangled.**

- **Nonseparable State.** Given two basis vector $\{|0\rangle_A, |1\rangle_A\} \in \mathcal{H}_A$ and two basis vectors $\{|0\rangle_B, |1\rangle_B\} \in \mathcal{H}_B$, let us now consider the singlet state

$$\frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B), \tag{1.3.2}$$

  it is impossible to attribute to either system A or system B a definite pure state. That is, the measurement outcomes are *correlated.* And therefore, the systems are **entangled.** The Bell states are maximally entangled and one can be converted in another by applying a unitary transformation on any of the subsystems.

Example 2

In this second example, suppose that we know that the two particles A and B are photons coming from a 2-photon source, with opposite polarization, horizontal H and

vertical V, without us knowing which is which. This situation can be described in a classical and in a quantum mechanical way:

$$\begin{aligned} \rho_{CM} &= \frac{1}{2}(|H_A V_B\rangle\langle H_A V_B|) + \frac{1}{2}(|V_A H_B\rangle\langle V_A H_B|), \\ \rho_{QM} &= |\psi^+\rangle\langle\psi^+|, \end{aligned} \tag{1.3.3}$$

with

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|H_A V_B\rangle + |V_A H_B\rangle).$$

If we observe, for example, an H polarization for photon A, expressed as

$$\frac{Tr_A\{\rho \mid H_A\rangle\langle H_A \mid\}}{Tr\{\rho \mid H_A\rangle\langle H_A \mid\}} = \mid V_B\rangle\langle V_B \mid, \tag{1.3.4}$$

this gives certain V-polarization of the photon B in *both cases.* As we shall see very soon, ρ_{CM} is not entangled, while ρ_{QM} is.

To appreciate the difference between the two cases, we write these two density matrices as rows and columns ordered as HH,HV,VH,VV. The result is:

$$\rho_{CM} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{QM} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.3.5}$$

We notice that the non-zero off-diagonal elements or "coherences", are only present in ρ_{QM}, in the mixed positions $HV - VH$ and $VH - HV$. This fact, however, has no consequence for the particles individually, thus for both cases, the reduced density matrix, say ρ_A yields

$$(\rho_{CM})_A = (\rho_{QM})_A = \frac{1}{2}I, \tag{1.3.6}$$

We being the identity matrix [9].

As we can see the off diagonal elements of ρ_{QM} does not have an impact in individual particles and therefore, we get the same reduced density matrix. The difference between the two cases shows up when doing violations of Bells inequalities, which are purely quantum effects [10, 11].

1.3.1 The Schmidt decomposition

An interesting tool when working with bipartite pure state entanglement, i.e system A and system B, is the Schmidt decomposition, which is a diagonal decomposition in a bi-orthogonal basis [11].

If we have a pure state $|\psi_{AB}\rangle$, we can write it in terms of some product basis, orthonormal in both systems,

$$|\psi_{AB}\rangle = \sum_{i=1}^{d_A}\sum_{\mu=1}^{d_B} M(i,\mu) \mid i\rangle\otimes \mid \mu\rangle, \tag{1.3.7}$$

where $M(i,\mu)$ are the elements in a $d_A \times d_B$ matrix M

$$M = \sum_{i=1}^{d_A}\sum_{\mu=1}^{d_B} M(i,\mu) \mid i\rangle\langle\mu \mid . \tag{1.3.8}$$

One can always diagonalize M with a couple of unitary matrices $U(d_A \times d_A)$ and $V(d_B \times d_B)$ such that M can be now written as $M = UDV$, with D being a $d_A \times d_B$ matrix whose elements are zero except for the diagonal which are real and positive. Thus, M can be written as:

$$M(i,\mu) = \sum_{k=1}^{\min(d_A,d_B)} u_{ik} d_{kk} v_{k\mu}, \tag{1.3.9}$$

thus getting for the state $|\psi_{AB}\rangle$

$$\begin{aligned} |\psi_{AB}\rangle &= \sum_{i=1}^{d_A}\sum_{\mu=1}^{d_B}\sum_{k=1}^{\min(d_A,d_B)} u_{ik} d_{kk} v_{k\mu} \mid i\rangle\otimes \mid \mu\rangle, \\ &= \sum_{k=1}^{\min(d_A,d_B)} d_{kk} \left(\sum_{i=1}^{d_A} u_{ik} \mid i\rangle\right) \otimes \left(\sum_{\mu=1}^{d_B} v_{k\mu} \mid \mu\rangle\right). \end{aligned} \tag{1.3.10}$$

We now define a new couple of basis vectors

$$\begin{aligned} |a_k\rangle &= \left(\sum_{i=1}^{d_A} u_{ik}|i\rangle\right), \\ |b_k\rangle &= \left(\sum_{\mu=1}^{d_B} v_{k\mu}|\mu\rangle\right), \end{aligned} \tag{1.3.11}$$

and we have the *Schmidt decomposition*:

$$|\psi_{AB}\rangle = \sum_{k=1}^{\min(d_A,d_B)} d_{kk} \mid a_k\rangle \otimes \mid b_k\rangle. \tag{1.3.12}$$

What we have achieved is to write any state as a linear combination of $\min(d_A, d_B)$ product of bi-orthogonal basis vectors.

We notice that the original $|\psi_{AB}\rangle$ was written in terms of a double sum while the present "diagonal form" has been reduced to a single one.

In terms of the density operator, one can write

$$\rho = \sum_{i,k} \sqrt{\lambda_i \lambda_k} \mid a_i\rangle\langle a_k \mid \otimes \mid b_i\rangle\langle b_k|, \tag{1.3.13}$$

where $d_{kk} \equiv \sqrt{\lambda_k}$.

It is simple to see that the reduced density matrices are now diagonal:

$$\begin{aligned} \rho_A &= \sum_k \lambda_k \mid a_k\rangle\langle a_k|, \\ \rho_B &= \sum_k \lambda_k \mid b_k\rangle\langle b_k|. \end{aligned} \tag{1.3.14}$$

Since the λ_k are eigenvalues of the reduced density matrices and they obey the following conditions:

$$0 \leq \lambda_k \leq 1, \text{ and } \sum_k \lambda_k = 1. \tag{1.3.15}$$

In the case of a product state, there is only one Schmidt term different from zero and the corresponding eigenvalue is 1. Conversely, if we have a state with only one Schmidt coefficient, it must be a product state.

Thus, $|\psi_{AB}\rangle$ is a product state if and only if the corresponding reduced density matrices correspond to pure states. This implies that if we have an entangled state, the corresponding reduced density operators must correspond to a mixed state, with more than one Schmidt eigenvalue different from zero. We see that entanglement of a state is directly related to the mixedness of the reduced density operator.

A simple way of measuring the degree of entanglement in a two-partite pure state is via the "Schmidt number", defined as the reciprocal of the purity of the reduced density matrix [12]

$$K = \frac{1}{Tr_A \rho_A^2} = \frac{1}{Tr_B \rho_B^2} = \frac{1}{\sum_n \lambda_n^2}. \tag{1.3.16}$$

If only one eigenvalue is 1, and the rest are zero, then $K = 1$ and we have a product state. On the other hand, if all the λ_s are equal, implying that all the N terms are equally important in the Schmidt decomposition, then $\lambda_s = \frac{1}{N}$ and $K = N$.

So, if D is the dimension of the space, then

$$1 \leq K \leq D. \tag{1.3.17}$$

1.3.2 Measure of entanglement. Entropy of entanglement

One of the most popular measure of mixedness of the density operator is the *Von Neumann entropy* $S(\rho) = -tr(\rho \log_2 \rho)$. For a pure state, this entropy vanishes, and for a maximally mixed state, gives $log_2 d$, d being the dimension of the Hilbert space.

The entropy is a convex function, that is for $p \in [0, 1]$,

$$S[p\rho_1 + (1-p)\rho_2] \geq pS(\rho_1) + (1-p)S(\rho_2),$$

which implies that it always increases by further mixing. This motivates the next definition.

Definition Given a state $|\psi\rangle$, we define the *entropy of entanglement* $E(\psi)$ as the Von-Neumann entropy of the reduced density operator. So using the above discussion

$$E(\psi) = S(\rho_A) = S(\rho_B) = -\sum_{k=1}^{d} \lambda_k \log_2(\lambda_k), \tag{1.3.18}$$

thus, once more we see that the more mixed the reduced density operator is, the more entangled the original state is.

As we will see in the next section, this definition is only valid for pure states. For mixed states, the quantification of entanglement becomes more complex.

The entropy of entanglement depends on the Schmidt coefficients and it is independent from the basis. This means, that it is invariant under local unitary operations, in other words, if $|\psi'\rangle = U_A \otimes U_B|\psi\rangle$, then $E(\psi') = E(\psi)$.

Similarly, it is possible to show that it cannot increase on the average, by local operations. That is, if we perform independent measurements in A and B and obtain $|\psi_k\rangle$ with probability p_k, then we have that

$$E(\psi) \geq \sum_k p_k E(\psi_k).$$

We notice however, that the previous inequality does not imply that none of the $E(\psi_k)$ can be larger than $E(\psi)$, which is an interesting conclusion that will be used in the process of distillation.

1.4 Entanglement of mixed states

The states considered in the previous subsection are idealized ones, since real physical systems always interact with some environment. Thus, a mixed state may arise when an initially pure state interact, intentionally or unexpectedly, with this environment (other quantum degrees of freedom), resulting in a nonunitary evolution of the pure state into a mixed state. For a detailed discussion of this effect, we refer the reader to our next section on decoherence.

Though there are several criteria for separability of mixed states [13], the theory of mixed state entanglement is more complicated and less well understood than that of pure state entanglement.

A mixed state is called separable, if it can be prepared by the two-parties in a "classical way", which means agreeing by direct communication on the local preparation of states. The corresponding Density Matrix of a *separable* state should have only classical correlations, or mathematically should be of the form [14]

$$\rho = \sum_i p_i \mid a_i\rangle\langle a_i \mid \otimes \mid b_i\rangle\langle b_i|, \tag{1.4.1}$$

otherwise it is *entangled.*

Here, the coefficients p_i are probabilities with $0 \leq p_i \leq 1$, and $\sum_i p_i = 1$. This decomposition is not unique.

Example 3

An example of a mixed state that has classical but not quantum correlations is $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. Another example is the state $\rho = \frac{1}{2}(|\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-|)$ which is separable because it can be written as the previous example, as it can be seen from the definitions of the Bell states. However, we should warn the reader that a criteria for separability like Equation (1.4.1) is not easy to use. Or, in other words, finding for ρ a form like that or proving that it does not exist is not a simple task. Therefore, we must find a simpler criteria to detect entanglement.

1.4.1 Peres-Horodecki criteria. Positive partial transpose [15, 16]

In 1996, Asher Peres found a criterion for separability in a quantum system consisting of two subsystems. He conjectured his condition in terms on the partial transpose matrix, however that condition in general it is not sufficient. Posteriorly, Horodecki *et.al.* gave two necessary and sufficient conditions for separability of separable states for 2×2 and 2×3, completing the Peres'result.

The partial transpose of a composite density matrix is given by transposing only one of the two subsystems. For example, the partial transposition with respect to Alice is

$$\rho^{T_A}_{m\mu,n\nu} = \rho_{n\mu,m\nu}, \tag{1.4.2}$$

where, we are using Latin sub-indices for the Alice subsystem and Greek for Bobs.

Thus, for any separable state one can write the partial transpose as:

$$\rho^{T_A}_{sep} = \sum_i p_i (|\, a_i\rangle\langle a_i\,|)^T \otimes |\, b_i\rangle\langle b_i\,| \,. \tag{1.4.3}$$

Since the $(\mid a_i\rangle\langle a_i \mid)^T$ are again valid density matrices for Alice, one immediately finds that $\rho_{sep}^{T_A} \geq 0$ implying non negative eigenvalues.

The same holds for the partial transposition with respect to Bob. In summary, the partial transpose of a separable density matrix is positive. This means that it has only positive non-vanishing eigenvalues (or equivalently, a positive operator has a positive or zero expectation value *with any state*).

The converse, that is, if $\rho^{T_A} \geq 0$ then ρ is separable is true only for low dimensional systems, namely for composite systems 2×2 and 2×3. In these cases, the positivity of the partial transpose (PPT) is a necessary and sufficient condition for separability. For higher dimensions, the PPT condition is only necessary.

The partial transposition criterion, for detecting entanglement is simple: given a bipartite state ρ_{AB}, find the eigenvalues of any of its partial transpositions. A negative eigenvalue immediately implies that the state is entangled. Examples of such states include the singlet state.

The partial transposition criterion allows to detect in a simple way, all entangled states that are NPT (Negative partial transpose density matrices with at least a negative eigenvalue), which is a large class of states. However, in higher dimensions, there are PPT states which are not separable, called "bound entangled states", [17, 18].

1.4.2 Entanglement of formation. Concurrence

The *Entanglement of Formation* was originally proposed by Bennett *et. al.* in 1996 [19], and it is a direct generalization of entropy of entanglement applied to mixed states. They defined the entanglement of formation $E(\rho)$ of a mixed state ρ "as the least expected entanglement of any ensemble of pure state realizing ρ". It means, for a given mixed state ρ of a pair of systems A and B, such that,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

the entanglement of formation is the average entropy of entanglement for the states in this ensemble, minimized over all decompositions of ρ,

$$E(\rho) = \min \sum_i p_i E(\psi_i), \tag{1.4.4}$$

where the entanglement $E(\psi)$ is defined as the entropy of either of two subsystems A and B, i.e.

$$E(\psi) = -Tr(\rho_A \log \rho_A) = -Tr(\rho_B \log \rho_B). \tag{1.4.5}$$

Here ρ_A is the reduced density matrix obtained by partial trace of $|\psi\rangle\langle\psi|$ over the subsystem B, and similarly for ρ_B.

They found a lower bound on $E(\rho)$, specifically, for all states of a pair of spin-$\frac{1}{2}$ particles, $E(\rho) \geq h[f(\rho)]$. The function f is called "fully entangled fraction" of ρ, and is given by

$$f(\rho) = \max\langle e|\rho|e\rangle,$$

where, the function max is applied over all entangled states $|e\rangle$. The function $h(f)$ is defined by

$$h(f) = \begin{cases} H[\frac{1}{2} + \sqrt{f(1-f)}] & \text{for} f \geq \frac{1}{2} \\ 0 & \text{for} f < \frac{1}{2}, \end{cases} \tag{1.4.6}$$

$$H(x) = -x \log_2 x - (1-x) \log_2 (1-x) \tag{1.4.7}$$

is the binary entropy function. For mixtures of Bell states, $f(\rho)$ is simply the largest eigenvalue of ρ. A concept related to entanglement of formation is the concurrence, which is defined for a system of two qubits for Hill and Wootters in [20, 21]. The formula for concurrence use the spin flip transformation applied to each individual qubit.

Before writing the formula for the entanglement of mixed states, it is convenient to study the entanglement of a pure state of two qubits. For a pure state, the spin flip operation is defined by

$$|\tilde{\Psi}\rangle_{AB} = (\sigma_y^A \otimes \sigma_y^B)|\Psi^*\rangle_{AB},$$

where $|\Psi^*\rangle_{AB}$ is the complex conjugate of $|\Psi\rangle_{AB}$, σ_y^i, with $i = (A, B)$ is the Pauli operator, both taken in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

The entanglement for pure states defined in Equation (1.4.5), can be written as

$$E(\Psi) = \mathcal{E}(C(\Psi)),$$

where $C(\Psi)$ is called *"concurrence"*, and is defined as $C(\Psi) = |\langle\Psi|\tilde{\Psi}\rangle|$, and the function $\mathcal{E}$ is

$$\mathcal{E} = H[\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}], \tag{1.4.8}$$

with $H(x)$ defined in Eq.(1.4.7). The range for $\mathcal{E}$ and C is from 0 to 1.

For a general mixed state ρ_{AB} of two qubits, we define $\widetilde{\rho}$ to be the spin-flipped state

$$\widetilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y), \tag{1.4.9}$$

where ρ^* is the complex conjugate of ρ, and σ_y is again the Pauli matrix expressed in the same basis as

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.4.10}$$

The entanglement is given by

$$E(\rho_{AB}) = \mathcal{E}(C(\rho_{AB})). \tag{1.4.11}$$

The concurrence is defined as

$$C(\rho) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{1.4.12}$$

where $\{\lambda_i\}$ are the square roots, in decreasing order of the non-hermitian matrix $\rho\widetilde{\rho}$.

For separable qubits $C = 0$ and for maximally entangled ones $C = 1$.

The most common density matrix used is represented by "X-matrices". This class of density matrix only contains non-zero elements in an "X" formation, along the main diagonal and antidiagonal. In the standard two-qubit product basis $\{|1\rangle =$

$|11\rangle_{AB}, |2\rangle = |10\rangle_{AB}, |3\rangle = |01\rangle_{AB}, |4\rangle = |00\rangle_{AB}\}$ a X-matrix is given by

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{1.4.13}$$

where $\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1$.

It should be noticed that it includes pure Bell states as special cases. The concurrence for a "X-state" of the form given by Eq.(1.4.13) can be easily computed as

$$C = 2\max\{0, \sqrt{\rho_{23}\rho_{32}} - \sqrt{\rho_{11}\rho_{44}}, \sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}}\}. \tag{1.4.14}$$

The concurrence has the great advantage that is easily computable and is directly related to the entanglement of formation, providing an explicit formula for the entanglement of formation in the case of two qubits.

CHAPTER 2

QUANTUM DECOHERENCE

2.1 Introduction

Decoherence is the irreversible formation of quantum correlations of a system with the environment. These correlations show up in completely new properties as compared to the isolated system. When we consider observations related to only one of the two subsystems, system-environment, we notice immediately several consequences of this entanglement.

The first and most important effect is that the system no longer obeys Schrödinger's equation. The dynamics of our system becomes complicated, and in some approximations, it leads to a Master Equation for the density matrix of our system.

If we start from a pure state, the system has become mixed by the interaction with the reservoir. To put it in a different way, in an open system, the coherence leaks out of the system into the environment, and as a result, we have *decoherence* [22, 23, 24].

If the coupling to the environment becomes very strong, the internal dynamics of the system slows down or may even freeze, effect known as quantum Zeno effect.

On the other hand, the detailed dynamics depends on the type of interaction between the system and the environment. In many cases, this coupling is quite similar to a measurement process. Thus, it is appropriate to discuss here, some elements of quantum theory of measurement.

Niels Bohr [25] proposed that, according to the Copenhagen interpretation of

Quantum Mechanics, a classical apparatus was necessary to carry out the measurements, thus implying **a sharp borderline between the Classical and the Quantum world**. Traditionally, the Classical Systems are associated to the macroscopic and Quantum to the microscopic [22], but this distinction is actually not very adequate considering recently studied effects of macroscopic systems that behave quantum mechanically. We also have the non-classical squeezed states with large number of photons, etc.

As opposed to Bohr, Von Neumann [26] considered quantum measurements. Let us assume that we have a system with states $|a\rangle$ and $|b\rangle$ and a metre that can be in the states $|d_a\rangle$ and $|d_b\rangle$. If the detector is initially in the $|d_b\rangle$ state, we assume that it switches when the system is in the $|a\rangle$ state and does not change if the system is in the $|b\rangle$ state, that is

$$\begin{aligned} |a\rangle|d_b\rangle &\rightarrow |a\rangle|d_a\rangle, \\ |b\rangle|d_b\rangle &\rightarrow |b\rangle|d_b\rangle. \end{aligned} \tag{2.1.1}$$

If, on the other hand, we assume that the system is in a superposition state

$$|\psi_{initial}\rangle = \alpha|a\rangle + \beta|b\rangle, \tag{2.1.2}$$

with $|\alpha|^2 + |\beta|^2 = 1$, then

$$|\psi_{initial}\rangle = (\alpha|a\rangle + \beta|b\rangle)|d_b\rangle \stackrel{after\ meas}{\Longrightarrow} \alpha|a\rangle|d_a\rangle + \beta|b\rangle|d_b\rangle \equiv |\Psi^c\rangle, \tag{2.1.3}$$

where the state $|\Psi^c\rangle$ is a correlated one, and this process can be achieved, as we will see soon, just with Schrödinger's equation, with an appropriate interaction.

Thus, if the detector is in the $|\ d_a\rangle$ state, one can be certain that the system is in the $|\ a\rangle$ state.

However, we are ignorant about the quantum state of the system, and it is more realistic to approach the system in a statistical way, with the density matrix.

According to Von Neumann, besides the unitary evolution that rules the dynamics of the of the quantum phenomena, there is also a *non-unitary reduction of the*

wavefunction $|\Psi^c\rangle$ that takes the pure state density matrix $|\Psi^c\rangle\langle\Psi^c|$ and converts it into a mixed state, by eliminating the off-diagonal elements

$$\begin{aligned}\rho^c &= |\Psi^c\rangle\langle\Psi^c| \\ &= |\alpha|^2|a\rangle\langle a||d_a\rangle\langle d_a| + |\beta|^2|b\rangle\langle b||d_b\rangle\langle d_b| \\ &+ \alpha^*\beta|b\rangle\langle a||d_b\rangle\langle d_a| + \alpha\beta^*|a\rangle\langle b||d_a\rangle\langle d_b|\end{aligned}$$

$$\overset{Non}{unitary} \Longrightarrow \rho^r = |\alpha|^2|a\rangle\langle a||d_a\rangle\langle d_a| + |\beta|^2|b\rangle\langle b||d_b\rangle\langle d_b|. \tag{2.1.4}$$

The difference between the original ρ^c and the 'after the measurement' reduced density matrix ρ^r, is that because in the latter case, the off-diagonal elements are missing, one could safely describe the system with alternative states ruled by classical probabilities $|\alpha|^2$ and $|\beta|^2$.

On the other hand, in the quantum case (ρ^c), things are more complicated, because we may use a different basis, say

$$\begin{aligned}|c\rangle &= \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle), \\ |d\rangle &= \frac{1}{\sqrt{2}}(|a\rangle - |b\rangle),\end{aligned} \tag{2.1.5}$$

and choosing $\alpha = -\beta = \frac{1}{\sqrt{2}}$, we write

$$\begin{aligned}|\,\Psi^c\rangle &= \frac{1}{\sqrt{2}}(|\,a\rangle\,|\,\mathrm{d}_a\rangle - |\,b\rangle\,|\,\mathrm{d}_b\rangle) \\ &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|\,c\rangle + |\,\mathrm{d}\rangle)\,|\,\mathrm{d}_a\rangle - \frac{1}{\sqrt{2}}(|\,c\rangle - |\,\mathrm{d}\rangle)\,|\,\mathrm{d}_b\rangle\right] \\ &= \frac{1}{\sqrt{2}}\left[|\,c\rangle\,|\,\mathrm{d}_c\rangle + |\,\mathrm{d}\rangle\,|\,\mathrm{d}_d\rangle\right],\end{aligned} \tag{2.1.6}$$

where

$$\begin{aligned}|d_c\rangle &= \frac{1}{\sqrt{2}}(|d_a\rangle - |d_b\rangle), \\ |d_d\rangle &= \frac{1}{\sqrt{2}}(|d_a\rangle + |d_b\rangle).\end{aligned} \tag{2.1.7}$$

We see that the diagonal elements of ρ^c give us different alternatives. In the first basis

$$(\rho^c)_{\text{diag}} = \frac{1}{2} \mid a\rangle\langle a \mid\mid \mathrm{d}_a\rangle\langle \mathrm{d}_a \mid + \frac{1}{2} \mid b\rangle\langle b \mid\mid \mathrm{d}_b\rangle\langle \mathrm{d}_b \mid , \tag{2.1.8}$$

while in the second basis

$$(\rho^c)_{\text{diag}} = \frac{1}{2} \mid c\rangle\langle c \mid\mid \mathrm{d}_c\rangle\langle \mathrm{d}_c \mid + \frac{1}{2} \mid \mathrm{d}\rangle\langle \mathrm{d} \mid\mid \mathrm{d}_d\rangle\langle \mathrm{d}_d \mid \ . \tag{2.1.9}$$

The problem, once more, is that we do not know the quantum state of the system.

Now, as we mentioned before, the first step of the measurement is to obtain the correlated wavefunction $|\Psi^c\rangle$, which can be achieved via a unitary operator.

The second step, however, was the Von Neumann non-unitary reduction. *Can this step be achieved in a different way?* Perhaps, by another unitary operator?. The answer to this question is yes [22], and the way to do it is by coupling the system - detector pair to the environment, to dispose of the extra information.

We call the environment states $|\varepsilon\rangle$. Then

$$\begin{aligned} \mid \Psi^c\rangle \mid \varepsilon_0\rangle &= (\alpha \mid a\rangle \mid \mathrm{d}_a\rangle + \beta \mid b\rangle \mid \mathrm{d}_b\rangle) \mid \varepsilon_0\rangle \\ &\rightarrow (\alpha \mid a\rangle \mid \mathrm{d}_a\rangle \mid \varepsilon_a\rangle + \beta \mid b\rangle \mid \mathrm{d}_b\rangle \mid \varepsilon_b\rangle) = \mid \psi\rangle, \end{aligned} \tag{2.1.10}$$

where the correlation has been extended from the system–detector to system–detector–environment, getting a 'chain of states'.

If the environment states $\mid \varepsilon_a\rangle$ and $\mid \varepsilon_b\rangle$, corresponding to the detector states $\mid \mathrm{d}_a\rangle$ and $\mid \mathrm{d}_b\rangle$ respectively, are orthogonal, then we can trace (average) over the environment variables

$$\rho_{SD} = Tr_\varepsilon \mid \psi\rangle\langle\psi \mid = \sum_i \langle\varepsilon_i \mid \psi\rangle\langle\psi \mid \varepsilon_i\rangle = \rho^r \ , \tag{2.1.11}$$

getting precisely the Von Neumann reduced density matrix, but this time by only unitary transformations, without ad hoc assumptions.

2.2 The mechanism of Decoherence

In this section we will discuss the interaction between the "local" object, called *system of interest*, and the second system, that may be the *environment*. Since the interaction in many cases is strongly similar to a measurement, we will deal with the von Neumann measurement model.

In order to model the measurement process, we require an interaction which is diagonal in the eigenstates of the measured observable, and also be able to move the state of the measurement apparatus ("pointer state") [27].

We assume that the system is coupled to the environment (apparatus) by a Hamiltonian of the following form:

$$H_{\text{int}} = \hbar \sum_n \mid n\rangle\langle n \mid \mathbf{D}_n \,, \tag{2.2.1}$$

where $\mathbf{D}_n$ are n-dependent operators acting on the Hilbert space of the environment (apparatus) and $\mid n\rangle$ is an eigenstates of a system observable to be measured.

The environment (apparatus) acquires the information about the state $\mid n\rangle$, in the sense that changes according to

$$\begin{aligned} |n\rangle|\phi_0\rangle \xrightarrow{t} \exp\left(-i\frac{H_{\text{int}}}{\hbar}t\right)|n\rangle|\phi_0\rangle &= |n\rangle \exp(-iD_n t)|\phi_0\rangle \\ &= \mid n\rangle \mid \phi_n(t)\rangle. \end{aligned} \tag{2.2.2}$$

We notice that here, the 'measurement' is made not in the sense of Von Neumann but rather as a dynamical evolution of the joint system, according to Schrödinger's equation, with the appropriate coupling.

The resulting environment (apparatus) states $|\phi_n(t)\rangle$ are called 'pointer states'. In case the environment is interpreted as the measuring apparatus, they would correspond to particular apparatus states.

From the linearity of the Schrödinger's equation, one can also write

$$\sum_n F_n \mid n\rangle \mid \phi_0\rangle \rightarrow \sum_n F_n \mid n\rangle \mid \phi_n(t)\rangle \,, \tag{2.2.3}$$

that is, we get a correlated state representing the superposition of all the possible measured results. From the above expression, we can calculate the local or subsystem reduced density matrix, by tracing over the environment

$$\begin{aligned} \rho_S &= Tr_{\text{envir}} \sum_{n,m} F_n F_m^* \mid n\rangle\langle m \mid\mid \phi_n(t)\rangle\langle\phi_m(t) \mid \\ &= \sum_{n,m} F_n F_m^* \mid n\rangle\langle m \mid \langle\phi_n(t) \mid \phi_m(t)\rangle \ , \end{aligned} \tag{2.2.4}$$

and for orthogonal states

$$\langle\phi_n(t) \mid \phi_m(t)\rangle = \delta_{nm} \ , \tag{2.2.5}$$

the reduced density matrix for the system becomes diagonal

$$\rho_S \rightarrow \sum_n \mid F_n \mid^2 \mid n\rangle\langle n \mid \ . \tag{2.2.6}$$

During this evolution, the interference was destroyed and the system appears to be classical with respect to the quantum number n, which implies that the phase information has been destroyed. No interference between different $\mid n\rangle$ can be observed in the system .

The above evolution is viewed as a model of system–apparatus coupling. Unfortunately, the apparatus, being macroscopic, will invariably interact with the environment ε.

By the same mechanism, the information about the measurement is rapidly transferred to the environment, leading to a the following state

$$\sum_n F_n \mid n\rangle \mid \phi_n\rangle \mid \varepsilon_0\rangle \rightarrow \sum_n F_n \mid n\rangle \mid \phi_n\rangle \mid \varepsilon_n\rangle \ , \tag{2.2.7}$$

and if the environment states are orthogonal, then we obtain diagonal density matrix for the system-apparatus

$$\rho_{\text{system-apparatus}} = \sum_n \mid F_n \mid^2 \mid n\rangle\langle n \mid\mid \phi_n\rangle\langle\phi_n \mid \ . \tag{2.2.8}$$

Once more, we have defined the interaction of the apparatus with the environment by a Hamiltonian of the form given by (2.2.1), defining in this way, the pointer states $\mid \phi_n\rangle$.

2.3 Decoherence Free Subspaces (DFS)

As we have seen in the previous sections, decoherence is a consequence of the inevitable coupling of any quantum system to its environment, causing information loss from the system to the environment. In other words, we consider the decoherence as a non-unitary dynamics that is a consequence of the system–environment coupling. This includes both dissipative and dephasing contributions. Dissipation is a process by which the populations of the quantum states are modified by the interactions with the environment, while dephasing is a process of randomization of the relative phases of the quantum states. Both effects are caused by the entanglement of the system with the environment degrees of freedom, leading to the non-unitary dynamics of our system.

Lidar et al. [28] introduced the term *decoherence-free subspaces*, referring to robust states against perturbations, in the context of Markovian Master equations.

One uses the symmetry of the system–environment coupling to find a 'quiet corner' in the Hilbert Space not experiencing this interaction. A more formal definition of the DFS is as follows:

Definition *A system with a Hilbert space $\mathcal{H}$ is said to have a decoherence free subspace $\widetilde{\mathcal{H}} \subset \mathcal{H}$, if the evolution inside $\widetilde{\mathcal{H}}$ is purely unitary.*

2.4 Condition for DFS. The Lindblad semigroup Approach

We formulate now the existence of the DFS in terms of the Lindblad semigroup Master Equation. Lindblad has shown that the most general evolution of a system density matrix ρ_s is governed by the Master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H_s, \rho] + L_D(\rho), \tag{2.4.1}$$

with

$$L_D(\rho(t)) = \frac{1}{2}\sum_{\alpha,\beta=1}^{M} \mathrm{d}_{\alpha,\beta}\left(\left[F_\alpha, \rho F_\beta^\dagger\right] + \left[F_\alpha\rho, F_\beta^\dagger\right]\right) \tag{2.4.2}$$

where H_s is the system Hamiltonian, the F_α is a family of the 'Lindblad' operators in an M-dimensional space and $\mathrm{d}_{\alpha,\beta}$ are elements of a positive Hermitian matrix.

All the non-unitary, decohering dynamics is accounted for by L_D. That is, in the Lindblad form, there is a clear separation between the unitary and decohering dynamics.

Let $\left\{ \mid \widetilde{k}\rangle_{k=1}^{N} \right\}$ be a basis for an N-dimensional subspace

$$\widetilde{H}(DFS) \subseteq H(\text{TOTAL SYSTEM HILBERT SPACE}).$$

In this basis, we may express the density matrix as

$$\rho = \sum_{k,j=1}^{N} \rho_{kj} \mid \widetilde{k}\rangle\langle\widetilde{j} \mid \ . \tag{2.4.3}$$

Now, we consider the action of the Lindblad operators F_α on $\mid \widetilde{k}\rangle$,

$$F_\alpha \mid \widetilde{k}\rangle = \sum_{j=1}^{N} C_{kj}^{\alpha} \mid \widetilde{j}\rangle \ .$$

Substituting in (2.4.2), we find

$$\begin{aligned} L_D(\rho) \ &= \ \frac{1}{2}\sum_{\alpha,\beta=1}^{M} \mathrm{d}_{\alpha,\beta} \sum_{k,j,m,n=1}^{N} \rho_{kj}(2C_{jm}^{\beta *}C_{kn}^{\alpha} \mid \widetilde{n}\rangle\langle\widetilde{m} \mid \\ &- \ C_{mn}^{\beta *}C_{kn}^{\alpha} \mid \widetilde{m}\rangle\langle\widetilde{j} \mid -C_{jm}^{\beta *}C_{nm}^{\alpha} \mid \widetilde{k}\rangle\langle\widetilde{n} \mid) = 0. \end{aligned} \tag{2.4.4}$$

Notice that we have used the condition $L_D(\rho) = 0$, which is precisely the definition of the DFS.

The coefficients $\mathrm{d}_{\alpha,\beta}$ represent information about the bath, which we assume is uncontrollable. So we require that each term in the α, β sum vanishes separately.

Furthermore, we expect no dependence on the initial conditions, i.e., no dependence on ρ_{kj}, which implies that each term in the parenthesis vanishes separately.

This can be done if all the projectors are the same, requirement that is satisfied if

$$C^{\alpha}_{kn} = C^{\alpha}_{n}\delta_{k,n},$$

so the (2.4.4) becomes

$$\sum_{k,j=1}^{N} \rho_{kj} \mid \widetilde{k}\rangle\langle\widetilde{j} \mid (2C^{\beta *}_{j}C^{\alpha}_{k} - C^{\beta *}_{k}C^{\alpha}_{k} - C^{\beta *}_{j}C^{\alpha}_{j}) = 0. \tag{2.4.5}$$

Assuming that the $C^{\alpha}_{k} \neq 0$, this yields

$$(2C^{\beta *}_{j}C^{\alpha}_{k} - C^{\beta *}_{k}C^{\alpha}_{k} - C^{\beta *}_{j}C^{\alpha}_{j}) = 0\ , \tag{2.4.6}$$

or

$$2 = Z^{*} + Z^{-1}\ , \tag{2.4.7}$$

with Z$\equiv \frac{C^{\alpha}_{j}}{C^{\alpha}_{k}}$.

The (2.4.7) has the unique solution $Z = 1$. Thus, $C^{\alpha}_{j} = C^{\alpha}_{k} \equiv C^{\alpha}$

Thus, we proved the following Theorem.

Theorem 2.4.1. *The necessary and sufficient condition for a subspace* $\widetilde{H}(DFS) = \left\{ \mid \widetilde{k}\rangle_{k=1}^{N} \right\}$ *o be decoherence free is that the basis states* $\mid \widetilde{k}\rangle$ *are degenerate eigenstates of all Lindblad operators* F_{α}.

$$F_{\alpha} \mid \widetilde{k}\rangle = C^{\alpha} \mid \widetilde{k}\rangle \ for\ \forall\ \alpha, k\ . \tag{2.4.8}$$

The above condition can be also written as

$$[F_{\alpha}, F_{\beta}] \mid \widetilde{k}\rangle = 0. \tag{2.4.9}$$

If one can write

$$[F_{\alpha}, F_{\beta}] = \sum_{\gamma=1}^{M} f^{\gamma}_{\alpha,\beta}F_{\gamma},$$

with $f^{\gamma}_{\alpha,\beta} \neq 0$ and linearly independent (the Fs forming a 'semi-simple Lie algebra'). In this case, the condition (2.4.9) can be written as

$$\sum_{\gamma=1}^{M} f^{\gamma}_{\alpha,\beta}C^{\gamma} = 0 \tag{2.4.10}$$

that can only be satisfied for $C^{\gamma} = 0$.

Thus, the condition for a DFS, for the semisimple case, is that the set of states should be degenerate eigenstates of all Lindblad operators with zero eigenvalue.

$$F_\alpha \mid \widetilde{k}\rangle = 0, \ for \ \forall \ \alpha, k \ . \tag{2.4.11}$$

CHAPTER 3

DYNAMICS OF ENTANGLEMENT

3.1 Introduction

Decoherence and entanglement are closely related phenomena, mainly because decoherence is responsible for the fragility of the entanglement in systems interacting with reservoirs [29]. For this reason in the current decade, many papers have investigated extensively, the decoherence dynamics of entangled quantum systems under the influence of environmental noise by focusing mainly on the dynamical system of two parties.

Recently Yu and Eberly [30] investigated the dynamics of disentanglement of a bipartite qubit system due to spontaneous emission, where the two two level atoms (qubits) were coupled individually to two cavities (environments). They found that the quantum entanglement may vanish in a finite time, while local decoherence takes a infinite time. They called this phenomena *"Entanglement Sudden Death"* (ESD). In a previous work, Diósi [31] demonstrated, using Werner's criteria for separability that ESD can also occur in two-state quantum systems.

Since then, ESD has been examined in several model situations involving pairs of atomic, photonic and spin qubits [32, 33, 34], continuous Gaussian states [35] and spin chains [36]. Also, ESD has been examined for different environments including random matrix environments [37, 38], thermal reservoir [39, 40, 41] and squeezed reservoir [42]. ESD is not unique to systems of independent atoms. It can also occur for atoms coupled to a common reservoir, in which case we also observe the effect

of the revival of the entanglement that has already been destroyed [43]. The effect of global noise on entanglement decay may depend on whether the initial two-party state belongs to a decoherence free subspace (DFS) or not.

In a recent experiment by Almeida *et el* [44], they used correlated horizontally and vertically polarized photons to show evidence of sudden death of entanglement, under the influence of independent reservoirs. They created the initial state using a downconversion process.

As opposed to the ESD and against our intuition, it has been shown that under certain conditions, the process of spontaneous emission can entangle qubits that were initially unentangled [45], and in some cases the creation of entanglement can occur some time after the system-reservoir interaction has been turned on. The authors in [46] call this phenomenon *"delayed sudden birth of entanglement"*.

3.2 Dynamics of Open Systems

An open system, is a quantum system S which is coupled to another quantum system B called the *environment*. They represent a subsystem of the combined total system $S + B$. The state of the subsystem S, will change as a consequence of its dynamics and the interaction with the surroundings. The interaction leads to certain system-environment correlations such that the resulting state changes of S can no longer, in general, be represented in terms of unitary, Hamiltonian dynamics. The dynamics of the subsystem S induced by the Hamiltonian evolution of the total system is often referred to as *reduced system dynamics*, and S is also called the *reduced system*.

Let us denote by $\mathcal{H}_S$ the Hilbert space of the system and $\mathcal{H}_B$ the Hilbert space of the environment. The Hilbert space of the total system $S + B$ is the given by the tensor product space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. The total Hamiltonian $\mathcal{H}(t)$ may be taken to be the form

$$\mathcal{H}(t) = \mathcal{H}_S \otimes I_B + I_S \otimes \mathcal{H}_B + \mathcal{H}_I(t), \tag{3.2.1}$$

where $\mathcal{H}_S$ is the self-Hamiltonian of the open system S, $\mathcal{H}_B$ is the free Hamiltonian of

the environment B, and $\mathcal{H}_I(t)$ is the Hamiltonian describing the interaction between the system and the environment. A schematic picture of the typical situation under study is shown in Figure 3.1. The term *environment* is used, in general, to refer a

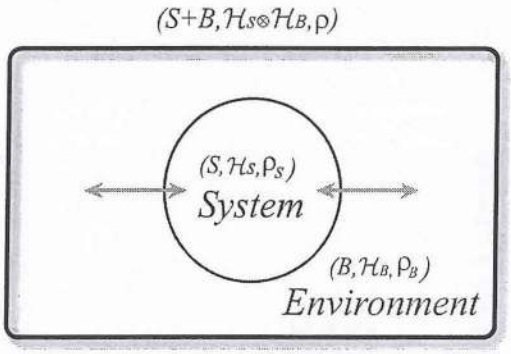

Figure 3.1: Schematic of an open quantum system.

system B coupled to an open system S. If this environment have a infinite number of degrees of freedom such that the frequencies of the environments modes form a continuum, we term it like *reservoir*. The term *bath* or *heat bath* will be used for a reservoir which is in a thermal equilibrium state.

3.3 Time evolution of a two atom system.

For quantum computation purposes, it is necessary to give some qubits a robust physical representation, and also a system in which they can evolve as desired. An attractive physical representation for qubits are two level atoms, or the optical photon among others. This section is concerned with two-atom systems. However, in real systems, they are not isolated and experience interactions with the outside world. In order to build useful quantum information systems, it is necessary to understand and control such noise processes.

An useful mathematical description for qubits (system of interest) is given by the density matrix. When the system is exposed to environmental noise, the density matrix will change in time. Such a time evolution is traditionally studied via a master

equation. In this approach, the dynamics are studied in terms of the reduced density operator $\hat{\rho}_s$ of the atomic system interacting with the quantized electromagnetic field regarded as a reservoir. The reservoirs have many possible realizations. The reservoir can be modelled as a vacuum field whose modes are ordinary vacuum states, or thermal states, or squeezed vacuum states. The major advantage of the master equation is that it allows us to consider the evolution of the atoms plus field system entirely in terms of average values of atomic operators.

We consider two situations, the first one consists of two two-level atoms initially entangled and coupled to uncorrelated reservoirs, and the second one of two two-level atoms initially entangled and interacting with a common reservoir. This coupling between the system and the reservoir originates the disentanglement.

We write now, a general master equation for the reduced density matrix in the interaction picture, assuming that the correlation time between the atoms and the reservoirs is much shorter than the characteristic time of the dynamical evolution of the atoms, so that the Markov approximation [1] is valid,

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} = \frac{\Gamma}{2}\sum_{i,j=1}^{2}[&(N+1)(2\sigma_i\hat{\rho}\sigma_j^\dagger - \sigma_i^\dagger\sigma_j\hat{\rho} - \hat{\rho}\sigma_i^\dagger\sigma_j)\\ &+ N(2\sigma_i^\dagger\hat{\rho}\sigma_j - \sigma_i\sigma_j^\dagger\hat{\rho} - \hat{\rho}\sigma_i\sigma_j^\dagger)\\ &- M(2\sigma_i^\dagger\hat{\rho}\sigma_j^\dagger - \sigma_i^\dagger\sigma_j^\dagger\hat{\rho} - \hat{\rho}\sigma_i^\dagger\sigma_j^\dagger)\\ &- M^*(2\sigma_i\hat{\rho}\sigma_j - \sigma_i\sigma_j\hat{\rho} - \hat{\rho}\sigma_i\sigma_j)],\end{aligned} \tag{3.3.1}$$

where Γ is the decay constant of the qubits, and $\sigma_i^+ = |1\rangle_i\langle 0|$ and $\sigma_i^- = |0\rangle_i\langle 1|$ are the raising $(+)$ and lowering $(-)$ operators of the $i$the atom. It should be pointed out that in Equation (3.3.1), the $i = j$ terms describe the interaction independently of the atoms with their local environments, while the $i \neq j$ terms denote the couplings between the modes induced by the common bath, see Figure 3.2. The Equation (3.3.1) have at least three possible realizations, these are vacuum reservoir, thermal reservoir, and squeezed reservoir.

[1] For non-Markovian effects, see [47, 48].

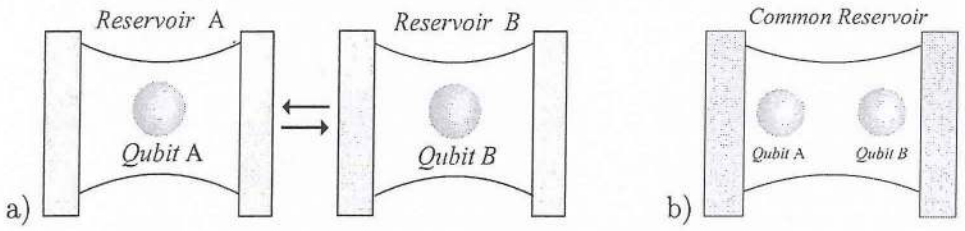

Figure 3.2: Schematic of an assembly of two qubits A and B, located in a) two independent and spatially separated, reservoirs. b) a common reservoir.

For a **vacuum reservoir** we have to make $N \rightarrow 0$ and $M \rightarrow 0$ in Equation (3.3.1).

For a **thermal reservoir** $N \rightarrow \bar{n}$ is the mean number of the thermal field (assumed to be the same for both qubits), and $M \rightarrow 0$. Additionally this reservoir can cause excitation of the qubits. Thus the first term on the right side of Equation (3.3.1) correspond to the depopulation of the atoms due to stimulated and spontaneous emission, and the second term describes the excitations caused by temperature.

For a **squeezed reservoir**, N represents the mean photon number of the reservoir and it is defined as $N = \sinh^2 r$, and M is a parameter related to the phase correlations of the squeezed reservoir defined as $M = -e^{i\theta} \sinh r \cosh r$. r is the squeezing parameter of reservoir and θ is the squeezing angle. The Heisenberg uncertainty relation imposes the constraint $|M|^2 \leq N(N+1)$, where the equality holds for a minimum-uncertainty squeezed state.

CHAPTER 4

ENTANGLEMENT IN INDEPENDENT RESERVOIRS

4.1 Introduction

In this chapter we consider two two-level atoms A and B that represent a two-qubit system, each one interacting independently with their local environments, as show in Figure 4.1. There is no direct interaction between the atoms. The correlation between the atoms results only from an initial quantum entanglement between them. The reservoirs considered here are: vacuum reservoir and thermal reservoir.

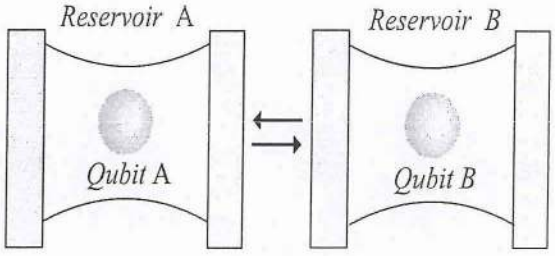

Figure 4.1: Schematic of an assembly of two qubits A and B, located in two independent.

4.2 Vacuum Reservoir

The vacuum environment can still have a noisy degrading effect through its quantum fluctuations. As a consequence, the atoms lose their excitation at a rate Γ. Thus, their stationary state is their ground state $|--\rangle$. In other words, the atoms experience a disentanglement process. This process can be completed in a finite-time (ESD) or

in an infinite time. These different behaviors exhibited in the disentanglement times depend on the initial state. Various publications have considered mixed and pure states [30, 49, 39, 50], as well as quantum recoil effects[51].

We solved analytically the master equation (3.3.1) under conditions of a vacuum reservoir, as shown in Appendix (A.1). However, for simplicity we will consider initial density matrices with a X-form:

$$\rho(0)\begin{pmatrix} a(0) & 0 & 0 & w(0) \\ 0 & b(0) & z(0) & 0 \\ 0 & z^*(0) & c(0) & 0 \\ w^*(0) & 0 & 0 & d^*(0) \end{pmatrix}. \tag{4.2.1}$$

We will use the measure of concurrence $C(\rho)$ in order to show the dynamics of the entanglement associated with the bipartite state. Considering as initial condition de matrix showed in (4.2.1), the concurrence is given by $C(\rho(t)) = 2\max\{0, C_1(t), C_2(t)\}$, where

$$\begin{aligned} C_1(t) &= 2e^{-\Gamma t}[|z_0| - \sqrt{a_0(d_0 + (b_0 + c_0)\omega^2 + a_0\omega^4)}], \\ C_2(t) &= 2e^{-\Gamma t}[|w_0| - \sqrt{b_0c_0 + a_0(b_0 + c_0)\omega^2 + a_0^2\omega^4}], \end{aligned} \tag{4.2.2}$$

with $\omega = \sqrt{1 - e^{-\Gamma t}}$. The respective disentanglement times when these functions vanish are,

$$t_{d-a} = \frac{1}{\Gamma}\ln\left(\frac{a_0(2a_0 + b_0 + c_0 + \sqrt{(b_0 + c_0)^2 - 4(a_0d_0 - |z_0|^2)})}{2(a_0 - |z_0|^2)}\right), \tag{4.2.3}$$

$$t_{d-b} = \frac{1}{\Gamma}\ln\left(\frac{a_0(2a_0 + b_0 + c_0 + \sqrt{(b_0 - c_0)^2 + 4|z_0|^2)})}{2(b_0c_0 + a_0(a_0 + b_0 + c_0) - |w_0|^2)}\right). \tag{4.2.4}$$

The ranges of the respective initial conditions are

$$\begin{aligned} &a_0 \leq |z_0|^2, \quad \text{The entanglement decay asimptotically;} \\ &a_0 > |z_0|^2, \quad \text{The entanglement decay in a finite time,} \end{aligned} \tag{4.2.5}$$

for the case Equation (4.2.3), and

$$\begin{aligned} &b_0c_0 + a_0(1 - d_0) \leq |w_0|^2, \quad \text{The entanglement decay asimptotically;} \\ &b_0c_0 + a_0(1 - d_0) > |w_0|^2, \quad \text{The entanglement decay in a finite time,} \end{aligned} \tag{4.2.6}$$

for Equation (4.2.4).

Let us show some examples.

1. • Consider first the initial state of the form $\alpha|00\rangle+\beta|11\rangle$, with $|\alpha|^2+|\beta|^2=1$. For this initial state, the concurrence is,

$$C = \max\{0, 2|\beta|e^{-\Gamma t}(|\alpha| - |\beta|(e^{-\Gamma t} - 1))\}, \tag{4.2.7}$$

and the disentanglement time (t_d) occur for

$$t_d = -\frac{1}{\Gamma}(1 - \frac{\alpha}{\beta}).$$

Only the states with $|\beta| > |\alpha|$ i.e, $|\beta| > \frac{1}{\sqrt{2}}$, have a finite disentanglement time.

• Consider now the initial state $\gamma|01\rangle + \delta|10\rangle$, with $|\gamma|^2 + |\delta|^2 = 1$. Its concurrence is

$$C = \max\{0, 2e^{-\Gamma t}\delta\gamma\}. \tag{4.2.8}$$

In this case the concurrence goes asymptotically to zero, for all values of δ.

2. The first and more traditionally case of ESD mentioned in the literature, contains the double excitation and the ground state components, and also includes the state in which one of the atoms is excited. This initial condition is

$$\rho(0) = \frac{1}{3}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix}. \tag{4.2.9}$$

Yu and Eberly studied the entanglement sudden death in the case $a = 1$. They showed that for $\frac{1}{3} \leq a \leq 1$, there is ESD. The concurrence is given by

$$C = \max\{0, \frac{2}{3}e^{-\Gamma t}(1 - \sqrt{a(3 - 2(1+a)e^{-\Gamma t} + ae^{-2\Gamma t})})\}, \tag{4.2.10}$$

and the time of disentanglement is ,

$$t_d = -\frac{1}{\Gamma}\ln(\frac{a+1-\sqrt{2-a+a^2}}{a}). \tag{4.2.11}$$

For the case $a = 1$ the results obtained in [30] are recovered. The time evolution

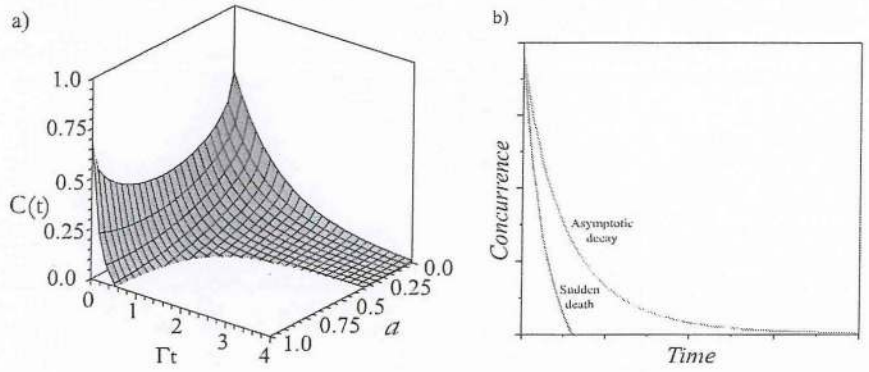

Figure 4.2: a) The time evolution of the concurrence in a vacuum reservoir when the atoms are initially in the entangled mixed state (Eq.4.2.9). For all values of a less than $\frac{1}{3}$, the half-life rule is obeyed, but for a between $\frac{1}{3}$ and 1, it is not. For those values, the curves show ESD. b) The figure shows the two typical behavior of disentanglement via concurrence: sudden death and asymptotic decay, for qubits interacting with two independent vacuum reservoirs.

of the concurrence for all values of a is shown in Figure 4.2

Clearly, to destroy the entanglement in a finite time, the spontaneous emission is not enough, and the sudden death of entanglement results from the decay of the mixed double excitation state component.

4.3 Thermal Reservoir

The interactions with a thermal reservoir lead typically to very rapid decoherence, thus one might to expect the destruction of quantum entanglement. But, How quickly

does it occur?. In a vacuum reservoir, the disentanglement time depends explicitly on the initial state of the atoms , specifically the ESD depends on the double excitation state $|++\rangle$. On the other hand, in the case of a thermal reservoir, the decay of the entanglement *always* occurs in finite time. This conjecture has been recently proposed for all atoms initially entangled in the form of two-qubit X-states, [39, 41].

We solved the master equation for a general initial condition (Appendix A.2). Let us consider the initial density matrix (4.2.1). The concurrence is given by $C(\rho(t)) = 2\max\{0, C_1(t), C_2(t)\}$, where $C_1(t)$ and $C_2(t)$ are given by

$$\begin{aligned} C_1(t) &= 2|z_0|e^{-(2n+1)t} \\ &- \frac{2}{(2n+1)^2}\{[((a_0-b_0-c_0+d_0)n^2+(2a_0-b_0-c_0)n+a_0)e^{-2(2n+1)t} \\ &+ 2n((a_0-d_0)n+\frac{1}{2}(2a_0+b_0+c_0))e^{-(2n+1)t}+n^2] \\ &\times [((a_0-b_0-c_0+d_0)n^2+(2a_0-b_0-c_0)n+a_0)e^{-2(2n+1)t} \\ &- 2(n+1)((a_0-d_0)n+\frac{1}{2}(2a_0+b_0+c_0))e^{-(2n+1)t}+(n+1)^2]\}^{\frac{1}{2}} \end{aligned} \tag{4.3.1}$$

$$\begin{aligned} C_2(t) &= 2|w_0|e^{-(2n+1)t} \\ &- \frac{2}{(2n+1)^2}\{[-((a_0-b_0-c_0+b_0)n^2+(2a_0-b_0-c_0)n+a_0)e^{-2(2n+1)t} \\ &+ (2(b_0-c_0)n^2+(a_0+2b_0-2c_0-d_0)n+a_0+b_0)e^{-(2n+1)t}+n(n+1)] \\ &\times [-((a_0-b_0-c_0+b_0)n^2+(2a_0-b_0-c_0)n+a_0)e^{-2(2n+1)t} \\ &+ (2(c_0-b_0)n^2+(a_0-2b_0+2c_0-d_0)n+a_0+c_0)e^{-(2n+1)t}+n(n+1)]\}^{\frac{1}{2}}, \end{aligned} \tag{4.3.2}$$

n being the number of average thermal photons.

Let us show three examples.

1. We consider the initial state

$$|\Psi(0)\rangle = \alpha|00\rangle + \beta|11\rangle, \quad \longrightarrow \rho_0 = \begin{pmatrix} \beta^2 & 0 & 0 & \alpha\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha\beta & 0 & 0 & \alpha^2 \end{pmatrix}. \tag{4.3.3}$$

where $|\alpha|^2 + |\beta|^2 = 1$, and for simplicity we consider α and β real.

In Figure 4.3, we plot the time evolution of the concurrence for $n = 0.01$ and $n = 1$. Notice that the entanglement sudden death occurs for all value of β. Also is observed that the death time decreases as the mean thermal photon number becomes large.

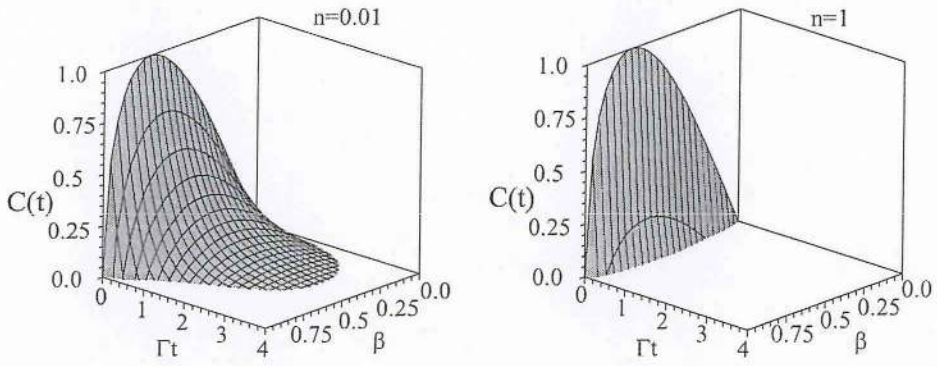

Figure 4.3: The figures show the entanglement evolution for the initial state $\alpha|00\rangle + \beta|11\rangle$ with $n = 0.01$ on the left and $n = 1$ on the right.

2. Let us consider again the initial state (4.2.9). We are interest in to observe the effect of the temperature in the disentanglement evolution. In [30], the authors have shown that for this initial state at zero temperature the entanglement is long lived for $0 \leq a \leq \frac{1}{3}$. In the Figure 4.4 we show that as soon as n becomes finite, the range vanishes and there is no asymptotic decay for any value of a. We also show two different values of n. We notice that when we increase the

value of n the death time decreases.

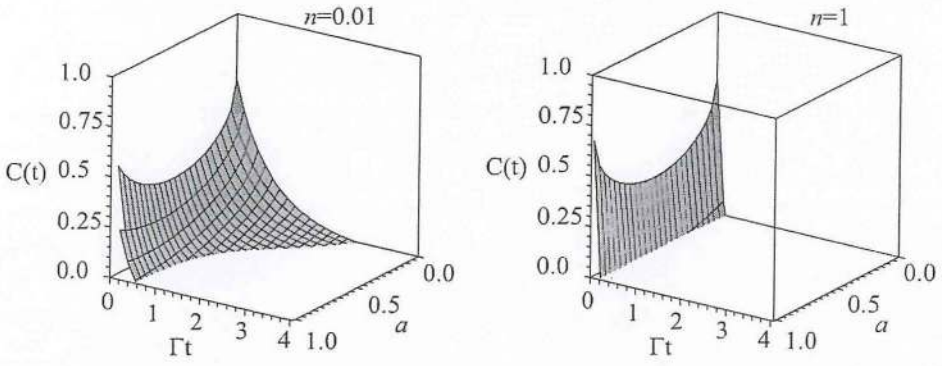

Figure 4.4: We plot the concurrence for the initial state (4.2.9) with $n = 0.01$ on the left and $n = 1$ on the right.

3. Also we considered the atoms initially entangled in a Werner state. In this case, the initial condition is:

$$\rho(0) = \frac{1}{4}\begin{pmatrix} 1-a & 0 & 0 & 0 \\ 0 & 1+a & -2a & 0 \\ 0 & -2a & 1+a & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix}. \tag{4.3.4}$$

In this state, the maximally entangled state $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ is mixed with the equally-weighted four possible states. For this initial state, we calculate the concurrence, and like in the previous example the sudden death occurs for all ranges of $0 < a < 1$ and as n becomes larger, the disentanglement time become shorter.

In the above examples, we see that when the thermal photon number is not zero the entanglement sudden death always happens no matter how small the nonzero

mean thermal photon number is and no matter which entangled state the atoms are initially. In the Figure 4.5 we show the time of disentanglement for different values of n. The main observed effect is that the death time of entanglement decreases with the mean thermal photon number.

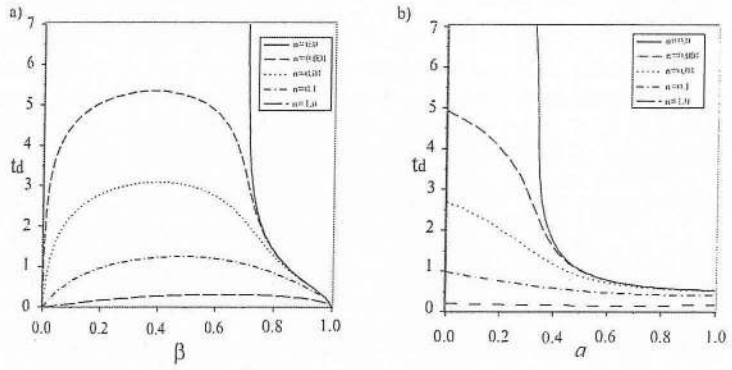

Figure 4.5: Death time for a) the initial state $\alpha|00\rangle \pm \beta|11\rangle$ and b) the initial state (4.2.9). We plotted the disentanglement time for different values of the parameter n. In the case of $n = 0$, there exists a range in the initial conditions for which the ESD is not permitted, decaying asymptotically. For larger values of n, the disentanglement time decreases, i.e. the sudden death occurs faster.

As we can see, for the various cases of initial mixed states, the phenomena of entanglement sudden death is always present, at finite (non-zero) temperatures.

CHAPTER 5

ENTANGLEMENT IN A COMMON RESERVOIR

5.1 Introduction

In this chapter, we will explore the relation between the sudden death (and revival) of the entanglement between the two two-level atoms in a squeezed bath and the normal decoherence via the *decoherence free subspace* (DFS), which in this case is a two-dimensional plane [42]. We will also look at the special case of the vacuum reservoirs.

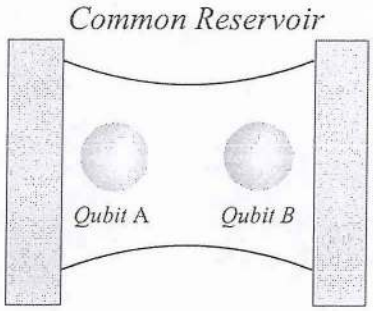

Figure 5.1: Schematic of an assembly of two qubits A and B, located in a common reservoir.

We should point out, that since we have a common bath, it would imply that the atoms are rather close to each other, that is, at a distance which is smaller that the correlation length of the reservoir. This would mean that we cannot neglect some direct interaction between them, like a dipole or Ising-type coupling. However, one can show that this type of coupling does not damage the decoherence free subspace, in this case a plane [52].

The case we present next, is a good example, where we are able to study in detail, the effects of decoherence and disentanglement. Furthermore, we can relate the two phenomena, by studying the effect of the *distance* from the decoherence free subspace on the sudden death and revival times.

5.1.1 The model

We consider, two two-level atoms that interact with a *common* squeezed reservoir, and we will focus on the evolution of the entanglement between them, using as a basis, the Decoherence Free Subspace states, as defined in [52, 53].

The master equation in the interaction picture, for two two-level system in a broadband squeezed vacuum bath is given in Eq.(3.3.1). It is simple to show that this master equation can also be written in the Lindblad form with a single Lindblad operator S

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\Gamma(2S\rho S^\dagger - S^\dagger S\rho - \rho S^\dagger S), \tag{5.1.1}$$

with

$$\begin{aligned} S &= \sqrt{N+1}(\sigma_1+\sigma_2) - \sqrt{N}e^{i\Psi}(\sigma_1^\dagger+\sigma_2^\dagger) \\ &= \cosh(r)(\sigma_1+\sigma_2) - \sinh(r)e^{i\Psi}(\sigma_1^\dagger+\sigma_2^\dagger), \end{aligned} \tag{5.1.2}$$

As was explained in Sec.(3.3), $\sigma^\dagger, \sigma$ are the usual Pauli raising and lowering matrices

$$\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.1.3}$$

and γ is the spontaneous emission rate. The squeeze parameters are Ψ, and $N = \sinh^2 r$, here we will consider $M = \sqrt{N(N+1)}$.

The Decoherence Free Subspace consists of the eigenstates of S with zero eigenvalue. The states defined in this way, form a two-dimensional plane in Hilbert Space. Two orthogonal vectors in this plane are:

$$|\phi_1\rangle = \frac{1}{\sqrt{N^2+M^2}}(N|++\rangle + Me^{-i\Psi}|--\rangle), \tag{5.1.4}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|-+\rangle - |+-\rangle). \tag{5.1.5}$$

We can also define the states $|\phi_3\rangle$ and $|\phi_4\rangle$ orthogonal to the $\{|\phi_1\rangle\ ,\ |\phi_2\rangle\}$ plane:

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle), \tag{5.1.6}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{N^2+M^2}}(M|++\rangle - Ne^{-i\Psi}|--\rangle). \tag{5.1.7}$$

We solve analytically the master equation by using the $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$ basis, however, we use the standard basis to calculate the concurrence. For simplicity we will consider $\Gamma = 1$ throughout this section.

5.1.2 Solution for initial states in DFS.

a) Consider $|\Phi_1(0)\rangle = |\phi_1\rangle$ as the initial state.

The solution of master equation is $\rho_1(t) = |\phi_1\rangle\langle\phi_1|$. This corresponds to an invariant state, and its concurrence is

$$C(\rho_1(t)) = \frac{2\sqrt{N(N+1)}}{2N+1},$$

which is a constant in time. The concurrence only depends of N.

For $N = 0$, $|\Phi_1(0)\rangle = |--\rangle$ we have a factorized state at all times, but as we increase N, we get a maximally entangled state in the large N limit.

b) If we now consider $|\Phi_2(0)\rangle = |\phi_2\rangle$ as the initial state.

The solution of the master equation is $\rho_2(t) = |\phi_2\rangle\langle\phi_2|$. This initial state is also an invariant state and its concurrence is independent of time,

$$C(\rho_2(t)) = 1.$$

5.1.3 General Solution for vacuum reservoir. $N \longrightarrow 0$

The solution of the master equation for a general initial density matrix spanned by $\{\phi_i\}$ basis is given by

$$\rho(t) = \begin{pmatrix} 1-\rho_{22}-2\rho_{44}te^{-2t}-(\rho_{33}+\rho_{44})e^{-2t} & \rho_{12} & (\rho_{13}+2\rho_{34})e^{-t}-2\rho_{34}e^{-2t} & \rho_{14}e^{-t} \\ \rho_{21} & \rho_{22} & \rho_{23}e^{-t} & \rho_{24}e^{-t} \\ (\rho_{31}+2\rho_{43})e^{-t}-2\rho_{43}e^{-2t} & \rho_{32}e^{-t} & \rho_{33}e^{-2t}+2\rho_{44}te^{-2t} & \rho_{34}e^{-2t} \\ \rho_{41}e^{-t} & \rho_{42}e^{-t} & \rho_{43}e^{-2t} & \rho_{44}e^{-2t} \end{pmatrix}. \tag{5.1.8}$$

In the following, we will consider the two states orthogonal in the DFS plane, and we will calculate the concurrence in the standard basis.

c) The third initial state considered is $|\Phi_3(0)\rangle = |\phi_3\rangle$.

Initially its concurrence is: $C(\rho_3(0)) = 1$. It corresponds to a maximally entangled state.

The solution of master equation for this initial condition and $N = 0$ is given by

$$\rho_3(t) = \begin{pmatrix} (e^{2t}-1)e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1.9}$$

Since the matrix $\rho_3(t)\widetilde{\rho_3}(t)$ has only one nonzero eigenvalue, in this case we use the separability criterion [15]. According to this criterion, the necessary condition for separability is that a matrix ρ^{PT}, obtained by partial transposition of ρ, should have only non-negative eigenvalues. In this particular case, we observe a negative eigenvalue for all times, thus the state stays entangled.

d) Consider the initial state $|\Phi_4(0)\rangle = |\phi_4\rangle$.

When $N = 0$, $|\Phi_4(0)\rangle = |++\rangle$ and $C(\rho_4(0)) = 0$, since is a factorized state.

The solution of master equation for this initial condition is given by:

$$\rho_4(t) = \begin{pmatrix} (-1-2t+e^{2t})e^{-2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2te^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix}, \tag{5.1.10}$$

and its concurrence is $C(\rho_4(t)) = 0$.

In the following, we will consider superpositions with one component in the DFS and an orthogonal one (to the DFS), of the form:

$$|\Psi_a\rangle = \varepsilon|\phi_1\rangle + \sqrt{1-\varepsilon^2}|\phi_4\rangle, \tag{5.1.11}$$

$$|\Psi_b\rangle = \varepsilon|\phi_2\rangle + \sqrt{1-\varepsilon^2}|\phi_3\rangle, \tag{5.1.12}$$

where ε varies from zero to one. The idea is to increase ε and study the effect of having an increased component in the DFS on the death time of the entanglement.

1) We consider an initial superposition of $|\phi_1\rangle$ and $|\phi_4\rangle$, i.e. $|\Psi_a\rangle$ given by Eq.(5.1.11).

For $N = 0$ we have:

$$|\Psi_a(0)\rangle = \varepsilon|--\rangle + \sqrt{1-\varepsilon^2}|++\rangle, \tag{5.1.13}$$

and its initial concurrence is $C(\Psi_a(0)) = 2\varepsilon\sqrt{1-\varepsilon^2}$.

The solution of master equation for this initial condition is given by:

$$\rho_a(t) = \begin{pmatrix} \frac{(-2t-1+2t\varepsilon^2+\varepsilon^2+e^{2t})}{e^{2t}} & 0 & 0 & \frac{\varepsilon\sqrt{1-\varepsilon^2}}{e^t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2t(1-\varepsilon^2)}{e^{2t}} & 0 \\ \frac{\varepsilon\sqrt{1-\varepsilon^2}}{e^t} & 0 & 0 & \frac{(1-\varepsilon^2)}{e^{2t}} \end{pmatrix}, \tag{5.1.14}$$

and the corresponding concurrence is given by:

$$C(\rho_a) = \max\{0, 2((\varepsilon\sqrt{1-\varepsilon^2})e^{-t} - te^{-2t}(1-\varepsilon^2))\} \tag{5.1.15}$$

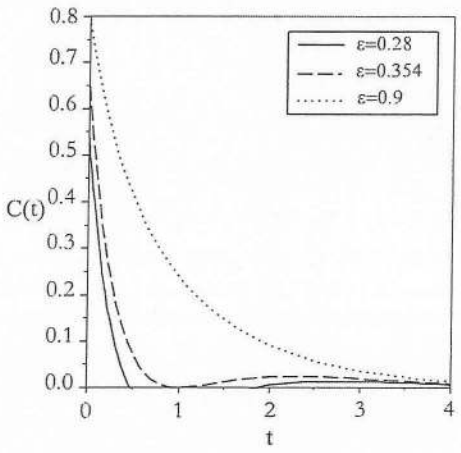

Figure 5.2: Time evolution of the concurrence for initial $|\Psi_a(0)\rangle$ with : $\varepsilon = 0.28$ (solid line), $\varepsilon = 0.345$ (dashed line), $\varepsilon = 0.9$ (dotted line),

which is shown in Fig.5.2 for various values of ε: For $\varepsilon > 0$, the initial entanglement decreases in time, and the system becomes disentangled (sudden death) at a time satisfying the relation:

$$te^{-t} = \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}. \tag{5.1.16}$$

- For $\varepsilon = 0$ and $\varepsilon = 1$, the concurrence is zero, therefore we have a non-entangled state.
- For $0 < \varepsilon < 0.34525$ the equation (5.1.16) has two solutions, namely, t_d when the system becomes separable, and $t_r \geq t_d$ when the entanglement revives. It should be noted that there is a critical ε for which $t_d = t_r$.
- For $0.34525 < \varepsilon < 1$ the above equation has no solution and the concurrence vanishes asymptotically in time.

Thus, when we are "not far" from $|\phi_4\rangle$ we observe a sudden death and revival, but when we get "near" $|\phi_1\rangle$ this phenomenon disappears. Fig.5.3 shows the behavior of the death and revival time as function of ε.

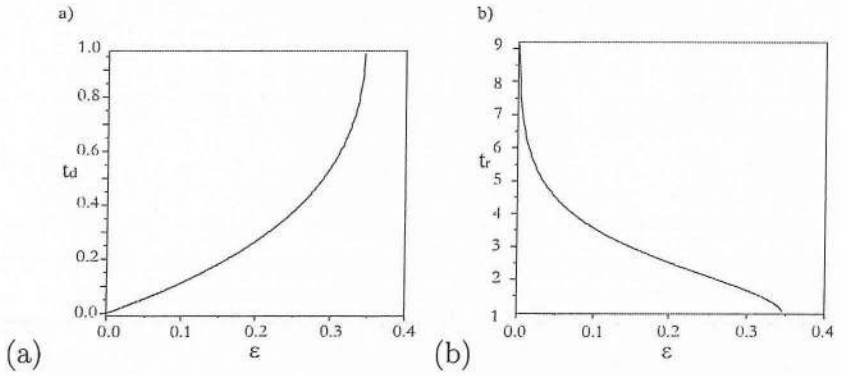

Figure 5.3: **a)** The death time **b)** The revival time of the entanglement as a function of ε, with initial $|\Psi_a\rangle$.

2) Finally, we consider an initial superposition of $|\phi_2\rangle$ and $|\phi_3\rangle$: $|\Psi_b(0)\rangle = \varepsilon\,|\phi_2\rangle + \sqrt{1-\varepsilon^2}\,|\phi_3\rangle$, (5.1.12) which is independent of N. Here, like in the pervious cases, as we increase ε, starting from $\varepsilon = 0$, we increase the initial projection onto the DFS. For $\varepsilon = 1$ the initial state is in the DFS plane.

For $N = 0$ we have

$$|\Psi_b(0)\rangle = \frac{1}{\sqrt{2}}[(\varepsilon + \sqrt{1-\varepsilon^2})|-+\rangle - (\varepsilon - \sqrt{1-\varepsilon^2})|+-\rangle], \qquad (5.1.17)$$

and its initial concurrence is $C(\Psi_b(0)) = |2\varepsilon^2 - 1|$.

The solution of master equation for this initial condition is given by:

$$\rho_b(t) = \begin{pmatrix} \frac{(e^{2t}-\varepsilon^2 e^{2t}-1+\varepsilon^2)}{e^{2t}} & 0 & 0 & 0 \\ 0 & \varepsilon^2 & \frac{\varepsilon\sqrt{1-\varepsilon^2}}{e^t} & 0 \\ 0 & \frac{\varepsilon\sqrt{1-\varepsilon^2}}{e^t} & \frac{(1-\varepsilon^2)}{e^{2t}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad (5.1.18)$$

and the corresponding concurrence is:

$$C(\rho_b(t)) = \max\{0, e^{-2t}|\varepsilon^2 e^{2t} - 1 + \varepsilon^2|\}, \qquad (5.1.19)$$

which is shown in Fig.5.4(a).

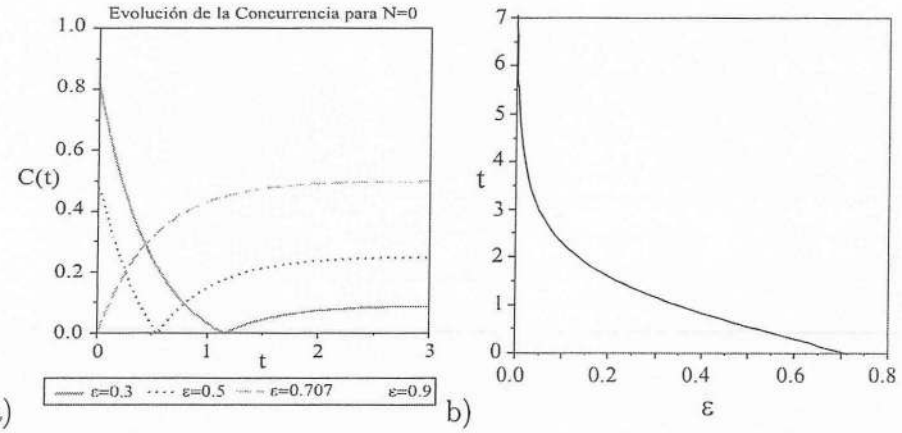

Figure 5.4: a) Time evolution of the concurrence with initial $|\Psi_b\rangle$, for: $\varepsilon = 0.3$ (solid line), $\varepsilon = 0.5$ (dotted line), $\varepsilon = 0.707$ (dashed line), $\varepsilon = 0.9$ (dash dotted line). b) Death-revival time as given by Eq.(5.1.20), versus ε.

For $0 < \varepsilon < 0.707$ the initial entanglement decreases in time, and the system becomes disentangled at a time given by (See Fig.5.4(b)):

$$t = \frac{1}{2}\ln(\frac{1-\varepsilon^2}{\varepsilon^2}), \tag{5.1.20}$$

however, at the same time, the entanglement revives reaching asymptotically its stationary value. It is mean that the sudden death and revival happen simultaneously. The phenomena of one or periodical revivals have been obtained before, but always in the context of one single reservoir connecting both atoms, like in the present case [54, 56, 55].

When we approach the decoherence free subspace this phenomenon disappears.

Next, we treat the cases with $N > 0$.

5.1.4 General Solution for $N \neq 0$

In general, for both $|\Psi_a\rangle$ and $|\Psi_b\rangle$ as initial states, the evolution of the concurrence $C(\rho(t))$ is calculated in the standard basis but written in terms of density matrix $\rho'(t)$

in the $\{\phi_i\}$ basis as $C(\rho'(t)) = max\{0, C_1(\rho'(t)), C_2(\rho'(t))\}$, where

$$C_1(\rho'(t)) = |\rho'_{33}(t) - \rho'_{22}(t)| - 2\sqrt{\frac{(N(\rho'_{11}(t) + \rho'_{44}(t)) + \rho'_{44}(t) + 2\rho'_{14}(t)\sqrt{N(N+1)})}{2N+1}} \times \sqrt{\frac{(N(\rho'_{11}(t) + \rho'_{44}(t)) + \rho'_{11}(t) - 2\rho'_{14}(t)\sqrt{N(N+1)})}{2N+1}}). \quad (5.1.21)$$

$$C_2(\rho'(t)) = \frac{2}{2N+1}|\sqrt{N(N+1)}(\rho'_{11}(t) - \rho'_{44}(t)) + \rho'_{14}(t)| - \sqrt{(\rho'_{22}(t) - 2\rho'_{23}(t) + \rho'_{33}(t))(\rho'_{22}(t) + 2\rho'_{23}(t) + \rho'_{33}(t))}. \quad (5.1.22)$$

where $\rho'_{ij}(t)$ are the density matrix elements in the $\{\phi_i\}$ basis.

a) Next, we consider again the case for initial $|\phi_3\rangle$. In Fig.5.5, we show the evolution of the concurrence for various values of N. We always observe sudden death in a finite time, then the concurrence remains zero for a period of time until the entanglement revives, and the concurrence reaches asymptotically its stationary value. Notice that this time period increases with N. In the fig.5.6

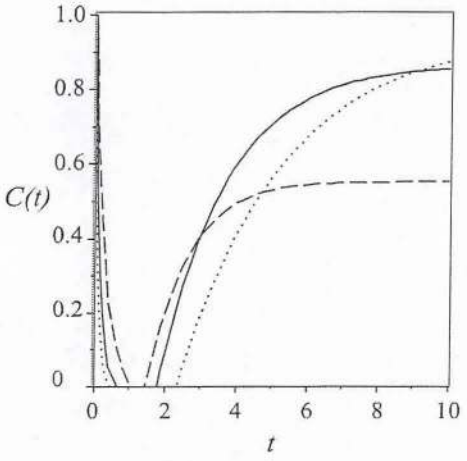

Figure 5.5: Time evolution of concurrence for initial $|\phi_3\rangle$, with: N=0.1 (dashed line), N=0.5 (solid line), N=1 (dotted line).

we show the death and revival times versus N. They decrease and increase with N respectively.

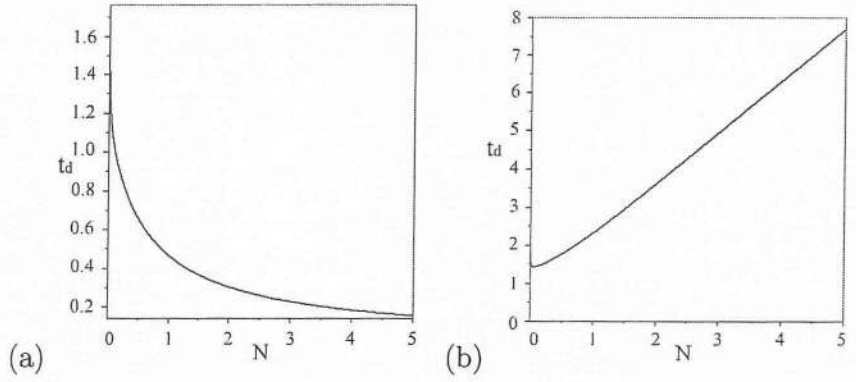

Figure 5.6: a) Death time b) Revival time versus N for the initial state $|\phi_3\rangle$.

b) We consider $|\phi_4\rangle$ as an initial state. The behavior of concurrence is similar as in $|\phi_3\rangle$ case. The initial entanglement quickly decays to zero, getting disentanglement for a finite time interval, then the entanglement revives and asymptotically it reaches its stationary value. However, unlike the case with initial state $|\phi_3\rangle$, the death time first increases reaching a maximum for $N = 0.421$, and subsequently it decreases, as shown in Fig 5.7. The revival time has the same behavior as in $|\phi_3\rangle$.

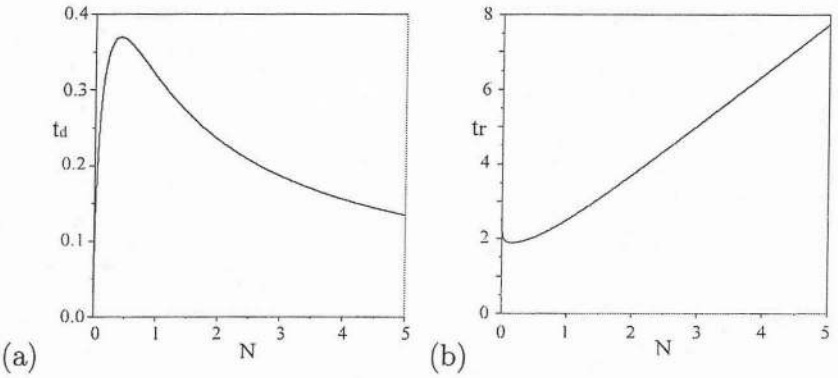

Figure 5.7: (a) Death, and (b) Revival times versus N for the initial state $|\phi_4\rangle$.

c) In the following case, we consider the superposition

$$|\Psi_a(0)\rangle = \varepsilon|\phi_1\rangle + \sqrt{1-\varepsilon^2}|\phi_4\rangle, \tag{5.1.23}$$

as the initial state. The solution of master equation for this initial condition depends on ε and N and also its concurrence. In this case, $\rho'_{23}(t) = \rho'_{22}(t) = 0$ thus, the concurrence is given by $C(\rho'_a(t)) = \max\{0, C_2(\rho'_a(t))\}$, where $C_2(\rho'_a(t))$ given in Eq.(5.1.22)

Initially, the concurrence is given by

$$C(\rho'_a(0)) = \frac{|2\varepsilon\sqrt{1-\varepsilon^2} + 4\sqrt{N(N+1)}(\varepsilon^2 - \frac{1}{2})|}{2N+1}, \tag{5.1.24}$$

we see that for certain pairs of N and ε, our initial state will be a non-separable one.

In the Fig. 5.8 we show the time evolution of the concurrence for $N = 0.1$ and several values of ε.

For $\varepsilon = 0$ and $\varepsilon = 1$ we retrieve $|\phi_4\rangle$ and $|\phi_1\rangle$ respectively.

For $0 < \varepsilon < 0.5$ the concurrence dies in a finite time, stays zero for a time interval and subsequently revives, going asymptotically to its stationary value.

For values larger than $\varepsilon = 0.5$, there is no more sudden death, since we are getting "close" to the DFS, and the concurrence goes asymptotically to its stationary value.

The Fig.5.9(a) shows the death times versus ε for $N = \{0, 0.1, 0.2\}$. In the case $N = 0$, we notice a steady increase of the death time up to some critical value of ε, where the death time becomes infinite. There is a curious effect, that for $N \neq 0$, as we increase ε, the death time first decreases up to the value $\varepsilon = \sqrt{\frac{N}{2N+1}}$, and subsequently it behaves "normally", by increasing with ε. In the Fig.5.9(b) we show the revival time as a function of ε for the same values of N. In all cases the revival time decreases with ε.

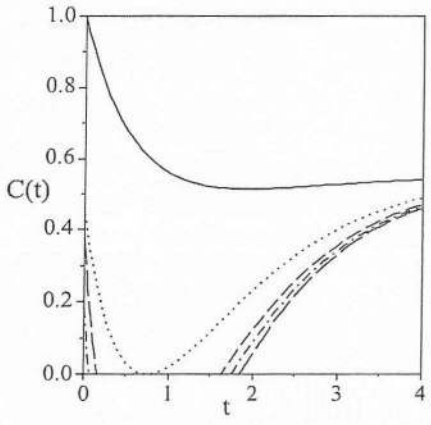

Figure 5.8: Time evolution of Concurrence for $|\Psi_a(t)\rangle$ as initial state and $N = 0.1$: $\varepsilon = 0.1$ (long dashed line), $\varepsilon = 0.2$ (dash dotted line), $\varepsilon = 0.29$ (dashed line), $\varepsilon = 0.5$ (dotted line), $\epsilon = 0.9$ (solid line).

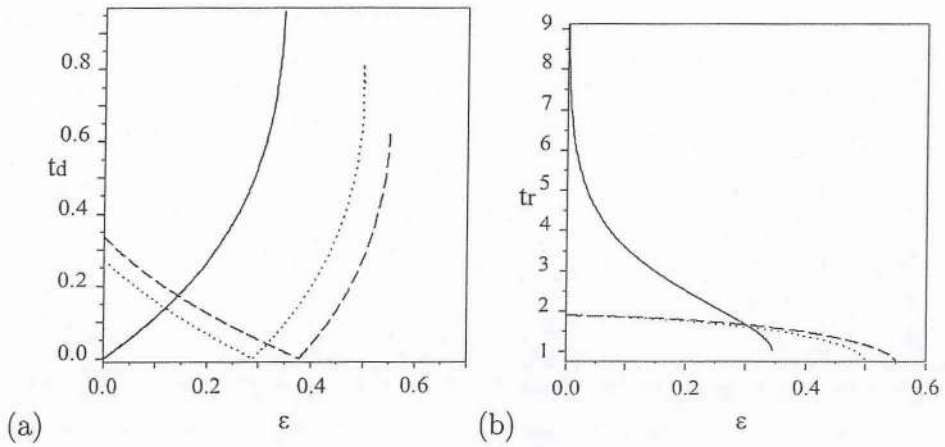

Figure 5.9: a) Death time and b) Revival time, with initial state $|\Psi_a\rangle$ and: $N = 0$ (solid line), $N = 0.1$ (dotted line), $N = 0.2$ (dashed line).

d) Finally, we consider the case with initial

$$|\Psi_b(0)\rangle = \varepsilon|\phi_2\rangle + \sqrt{1-\varepsilon^2}|\phi_3\rangle. \tag{5.1.25}$$

Its concurrence is: $C(\rho_b'(t)) = \max\{0, C_1(\rho_b'(t)), C_2(\rho_b'(t))\}$, with C_1 and C_2 defined in (5.1.21,5.1.22), and its initial value: $C(\rho_b'(0)) = |2\varepsilon^2 - 1|$. In the Fig. 5.10, we show the time evolution of the concurrence with $N = 0.1$ for several values of ε.

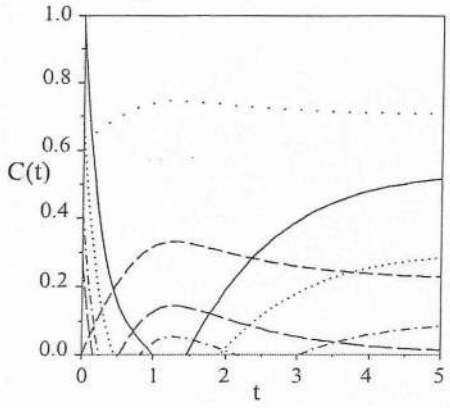

Figure 5.10: Time evolution of the Concurrence for $|\Psi_2(t)\rangle$ as initial state and $N = 0.1$: $\varepsilon = 0$ (solid line), $\varepsilon = 0.4$ (dotted line), $\varepsilon = 0.54$ (dash dotted line), $\varepsilon = 0.6$ (long dashed line), $\varepsilon_c = 0.707$ (dashed line), $\varepsilon = 0.9$ (space dotted line).

As we can see from the Fig. 5.10, this case is more complex, since there are more than one death and revival before reaching the critical value of ε. Such a situation has been described previously [54, 56]. Like in the previous cases above a certain critical ε, when we get "close" to the DFS, these effects disappear and $C(t)$ goes asymptotically to its stationary value.

5.1.5 Discussion

The first observation is that if we start from an initial state that is in the DFS plane, the local and non-local coherences are not affected by the environment, thus

it experiences no decoherence and the concurrence stays constant in time.

In the case of initial $|\phi_1\rangle$, the concurrence does increase with the squeeze parameter N, getting maximum entanglement for $N \to \infty$. So this reservoir is not acting as a thermal one, in the sense that introduces randomness. On the contrary, a common squeezed bath tends to enhance the entanglement, as we increase the parameter N.

This is clear if we observe that for $N \to \infty$:

$$|\phi_1\rangle \to \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle),$$

which is a Bell state.

On the other hand, if we start with the initial state $|\phi_2\rangle$, this state is independent of N and it is also maximally entangled, so $C = 1$ for all times and all $N's$.

Now, we consider other situations with initial states outside the DFS. We consider as initial states the superpositions given in (5.1.11) and (5.1.12), where we vary ε between 0 and 1 for a fixed value of the parameter N. It is observed:

- When $\varepsilon = 0$ we recovered $|\phi_4\rangle$ and $|\phi_3\rangle$. In the $|\phi_4\rangle$ case, the population of the $|++\rangle$ goes down with N, meaning that the interaction with the reservoir goes also down with N and therefore, the death time will necessarily increase with N, which describes qualitatively the first part of the curve(fig 5.7-a). Furthermore, as we increase the average photon number N, other processes like the two photon absorption will be favored, and since there will be more photons and the $|--\rangle$ population tends to increase with N, this will enhance the system-bath interaction and therefore the death of the entanglement will occur faster, or the death time will decrease.

  In the $|\phi_3\rangle$ case, initially there is no $|++\rangle$ component, thus we expect a higher initial death time. However this case is different from the previous one in the sense that the state is independent of N, so there is no initial increase. However, as the state evolves in time, the $|++\rangle$ and $|--\rangle$ components will build up and the argument for the decrease of the death time with N follows the same logic as in the previous case. See Fig.5.6(a).

- For the interval $0 \leq \varepsilon < \varepsilon_c$ the initial entanglement decays to zero in a finite time, t_d. After a finite period of time during which the concurrence stay null, it revives a time t_r reaching asymptotically its steady-state value. This death and revival cycle happens may occur once for the initial state $|\Psi_a\rangle$, and for the initial state $|\Psi_b\rangle$ may occur twice.

  For $|\Psi_a\rangle$ as initial state, when ε is equal to the critical value ε_c, the entanglement dies and revives simultaneously and eventually goes to its steady-state value. For the initial state $|\Psi_b\rangle$, the critical value of ε is $\varepsilon_c = \frac{1}{\sqrt{2}}$, and, unlike to the $|\Psi_a\rangle$ case, it is independent of N.

- When we get *near* to the DFS ($\varepsilon_c < \varepsilon \leq 1$), the system shows no disentanglement and this phenomenon of sudden death and revival disappears.

The squeezed vacuum has only non-zero components for even number of photons, so the interaction between our system and the reservoir goes by pairs of photons. Now, for very small N, the average photon number is also small, so the predominant interaction with the reservoir will be the doubly excited state that would tend to decay via two photon spontaneous emission.

Let us consider again $|\Psi_a\rangle$, but now in terms of the standard basis

$$|\Psi_a\rangle = k_1|++\rangle + k_2|--\rangle,$$

with

$$k_1 = \frac{\varepsilon N + M\sqrt{1-\varepsilon^2}}{\sqrt{N^2+M^2}}, \quad k_2 = \frac{\varepsilon M - N\sqrt{1-\varepsilon^2}}{\sqrt{N^2+M^2}}, \tag{5.1.26}$$

We plotted k_1 versus ε for N between 0 to 2, Fig.5.11. Initially, k_1 increases with ε, thus favoring the coupling with the reservoir, or equivalently, producing a decrease in the death time. This is up to $\varepsilon = \sqrt{\frac{N}{2N+1}}$, where the curve shows a maxima. Beyond this point, k_1 starts to decrease and therefore our system is slowly decoupling from the bath and therefore the death time shows a steady increase.

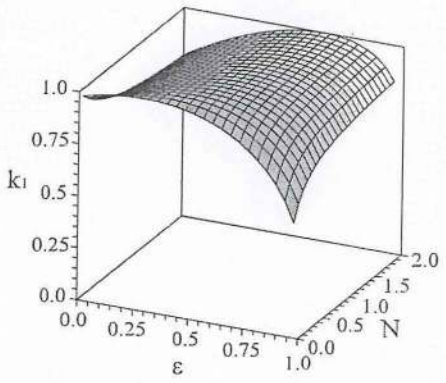

Figure 5.11: k_1 versus ε for N between 0 to 2.

CHAPTER 6

CREATION OF QUANTUM ENTANGLEMENT

6.1 Introduction

In some cases, if two systems do not interact directly, but share a common heath bath in thermal equilibrium, entanglement can be created, some time after the interaction is turned on. Furthermore, this entanglement may persist in time. So, contrary to intuition that spontaneous emission should have a destructive effect on the entanglement, it has been shown by several authors that under certain conditions, this irreversible process can even entangle initially unentangled qubits [57, 58, 59, 60, 61, 62, 63, 64, 65].

Also, more recently, it was shown that a "sudden" feature in the creation of entanglement exists, in the sense that it takes some finite time after the system is connected to the bath, for the creation of entanglement to take place [66].

In the following we considered the simplest composite quantum system of two qubits. This qubits are coupled to a common thermal reservoir at zero temperature. The qubits do not interact directly with each other but through the common environment, in that case there is a probability that a photon emitted by one atom will be absorbed by the other, and as a consequence of this photon exchange process can produce entanglement between atoms, which is larger than the decoherence caused by spontaneous emission. According to this assumption, is crucial to have one atom

excited and the other in the ground state to create entanglement. The dynamics of the system is given by the Eq.(3.3.1) with N and M equal to zero.

For a given initial state $\rho(0) = \rho_{ij}(0)$ the solution of the master equation (3.3.1) in the $\{e_1 = |11\rangle, e_2 = |10\rangle, e_3 = |01\rangle, e_4 = |00\rangle\}$ basis, is given by the following matrix elements

$$
\begin{aligned}
\rho_{11}(t) &= \rho_{11}e^{-2\Gamma t} \\
\rho_{22}(t) &= \frac{1}{4}(\rho_{22}+\rho_{33}-\rho_{23}-\rho_{32}+2(\rho_{22}-\rho_{33})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{33}(t) &= \frac{1}{4}(\rho_{22}+\rho_{33}-\rho_{23}-\rho_{32}+2(\rho_{33}-\rho_{22})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{44}(t) &= \frac{1}{2}(\rho_{23}+\rho_{32}+\rho_{11}+\rho_{44}+1-(4\rho_{11}t+2\rho_{11}+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{12}(t) &= \frac{1}{2}((\rho_{12}-\rho_{13})e^{-\Gamma t}+(\rho_{12}+\rho_{13})e^{-2\Gamma t}) \\
\rho_{13}(t) &= \frac{1}{2}(-(\rho_{12}-\rho_{13})e^{-\Gamma t}+(\rho_{12}+\rho_{13})e^{-2\Gamma t}) \\
\rho_{14}(t) &= \rho_{14}e^{-\Gamma t} \\
\rho_{22}(t) &= \frac{1}{4}(\rho_{22}+\rho_{33}-\rho_{23}-\rho_{32}+2(\rho_{22}-\rho_{33})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{23}(t) &= \frac{1}{4}(\rho_{23}+\rho_{32}-\rho_{22}-\rho_{33}+2(\rho_{23}-\rho_{32})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{24}(t) &= \frac{1}{2}(\rho_{24}-\rho_{34}-2(\rho_{12}+\rho_{13})e^{-2\Gamma t}+(2\rho_{12}+2\rho_{13}+\rho_{34}+\rho_{24})e^{\Gamma t}) \\
\rho_{34}(t) &= \frac{1}{2}(\rho_{34}-\rho_{24}-2(\rho_{12}+\rho_{13})e^{-2\Gamma t}+(2\rho_{12}t+2\rho_{13}+\rho_{34}+\rho_{24})e^{-\Gamma t})
\end{aligned}
\tag{6.1.1}
$$

For simplicity, we write each matrix element $\rho_{ij}(0)$ as ρ_{ij} and take $\Gamma = 1$. The remaining matrix elements can be obtained from $\rho_{ij} = \rho^*_{ji}$.

In order to obtain a condition that determines when a state that was initially separable becomes non-separable after a certain period of time, we will study the stationary state. In the limit $t \to \infty$, the density matrix is given by

$$\rho_{st} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22}+\rho_{33}-(\rho_{23}+\rho_{32}) & \rho_{23}+\rho_{32}-(\rho_{22}+\rho_{33}) & 2(\rho_{24}-\rho_{34}) \\ 0 & \rho_{23}+\rho_{32}-(\rho_{22}+\rho_{33}) & \rho_{22}+\rho_{33}-(\rho_{23}+\rho_{32}) & 2(\rho_{34}-\rho_{24}) \\ 0 & 2(\rho_{42}-\rho_{43}) & 2(\rho_{43}-\rho_{42}) & 2(\rho_{23}+\rho_{32})+2(1+\rho_{11}+\rho_{44}) \end{pmatrix}$$

Following the procedure to get the concurrence, see section [?], we calculate the matrix $R_{st} = \rho_{st}(\sigma_y \otimes \sigma_y)\rho^*_{st}(\sigma_y \otimes \sigma_y)$. For real matrix elements, R_{st} has only one nonzero eigenvalue:

$$\lambda = \frac{1}{4}(\rho_{22}+\rho_{33}-2Re(\rho_{23}))^2, \tag{6.1.2}$$

Thus, the condition to have a disentangled steady state is

$$\rho_{22}+\rho_{33} = 2\rho_{23}. \tag{6.1.3}$$

Now, we will consider some examples of different initial separable states.

1. Let us first have two atoms in excited state $|\Psi(0)\rangle = |11\rangle$. The concurrence is

$$C(t) = \max\{0, 2te^{-2t} - 2\sqrt{e^{-2t}(1-(1+2t)e^{-2t})}\} = 0. \tag{6.1.4}$$

   Thus, this state is separable for all times. In the stationary state, both atoms go to the ground state. Thus its concurrence is $C(t) = 0$.

2. For the case when one atom is in an excited state $|1\rangle$ and the other one in the ground state $|0\rangle$, the concurrence is

$$C(t) = \max\{0, \frac{1}{2}|e^{-2t}-1|\}, \tag{6.1.5}$$

   this result being the same for $|10\rangle$ and $|01\rangle$. The initial value of the concurrence increases to its steady state value of $\frac{1}{2}$.

3. When both atoms are initially in the ground state, $|\Psi(0)\rangle = |00\rangle$, the system remains in its ground state for all times, and the concurrence is $C(t) = 0$.

As mentioned before, the only way to generate entanglement (for the vacuum reservoir) is when one atom is in the ground state and the other one in the excited state, since this combination makes it possible for photon exchanges between the atoms. In the cases when there is no photon exchange, the entanglement generation never occurs.

In order to study this effect we consider linear combinations of the $|0\rangle$ and $|1\rangle$ states, for one of the atoms, and the ground or the excited state for the other one, respectively

1. Consider first

$$|\Psi(0)\rangle = |0\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) = \alpha|01\rangle + \beta|00\rangle, \tag{6.1.6}$$

with $\alpha^2 + \beta^2 = 1$. The Fig.6.1 (a) shows that the entanglement reaches a higher value when α is greater.

2. The second possible initial condition is

$$|\Psi(0)\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes |1\rangle = \alpha|01\rangle + \beta|11\rangle, \tag{6.1.7}$$

with $\alpha^2 + \beta^2 = 1$. The Fig.6.1 (b) shows the time evolution of the concurrence for different values of α. As in the previous case, when α decreases, the maximum value of the entanglement decreases, but unlike the previous case, the entanglement creation has a time delay.

3. Consider a more general initial condition

$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle). \tag{6.1.8}$$

According to the Eq.(6.1.2), the condition to get a completely disentangled stationary state is

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}, \tag{6.1.9}$$

implying that when both qubits are in the same state, the created entanglement is eventually destroyed, at large times. We show this effect in the Fig.(6.2),

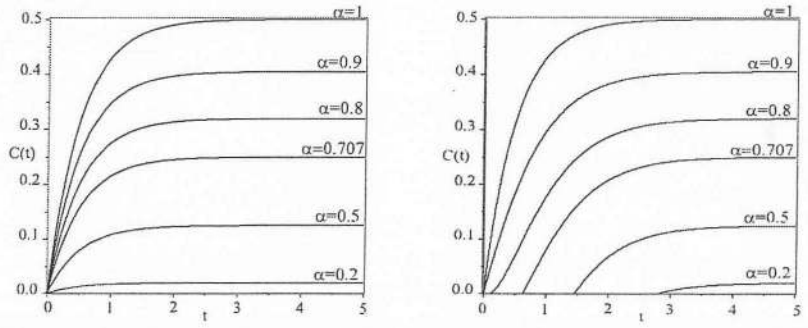

Figure 6.1: (a) Time evolution of concurrence for the initial $|0\rangle \otimes (\alpha|1\rangle + \beta|0\rangle)$ with different values of α. (b) Time evolution of concurrence for the initial $(\alpha|0\rangle+\beta|1\rangle)\otimes|1\rangle$ with different values of α. The reservoir is at T=0

where we plotted the evolution of the concurrence for the various initial conditions in Eq.(6.1.8), when $\alpha_1 = \alpha_2 \equiv \alpha$.

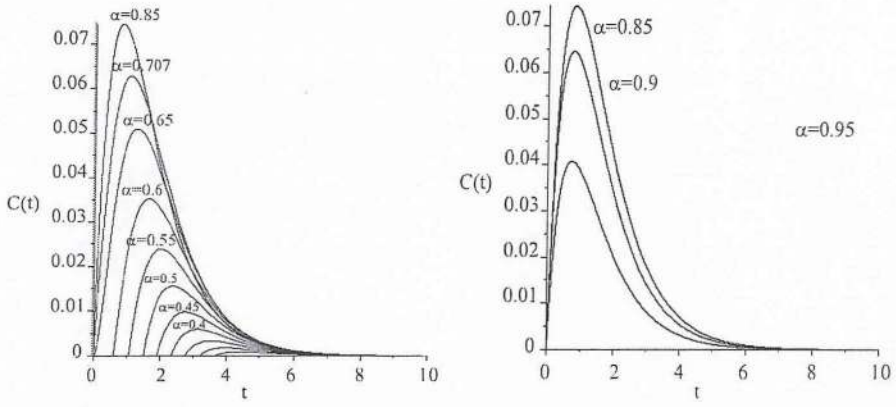

Figure 6.2: Concurrence versus time for different initial conditions $\alpha_1 = \alpha_2 \equiv \alpha$. The reservoir is at T=0.

Finally, we analyze the effects of having a reservoir with a finite (non zero) temperature.

1. In a vacuum reservoir the initial states $|\Psi(0)\rangle = |11\rangle$ and $|\Psi(0)\rangle = |00\rangle$ are

kept separable at all times. However it is noted that the concurrence increases with the temperature, since thermal excitation may help in the generation of entanglement, see Fig. 6.3. We observe in both cases, an entanglement that is

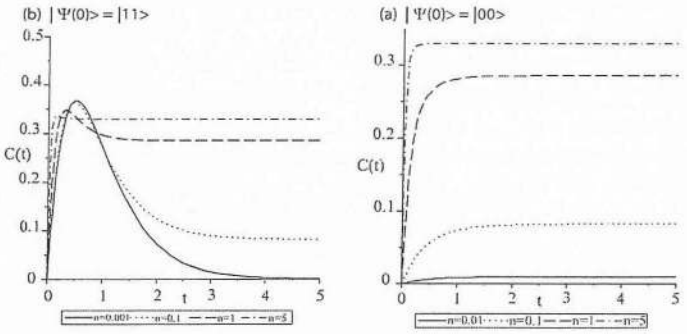

Figure 6.3: Evolution of the concurrence for the initial conditions a)$|11\rangle$, and b) $|00\rangle$, and different values of n.

actually enhanced by increasing the reservoir temperature, even the minimum temperature is enough to generate entanglement.

2. The opposite is true for the cases $|\Psi(0)\rangle = |10\rangle$ and $|\Psi(0)\rangle = |01\rangle$, see Fig. 6.4, where the one photon exchange, necessary for the entanglement in the case of the vacuum reservoir, is disturbed by the presence of thermal photons.

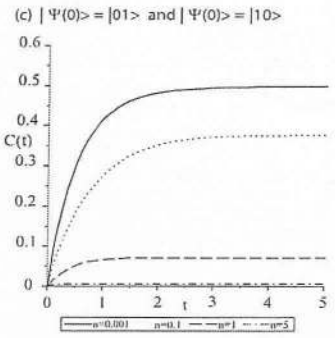

Figure 6.4: Concurrence versus time for the initial conditions $|\Psi(0)\rangle = |10\rangle$ and $|\Psi(0)\rangle = |01\rangle$. The parameter of each curve is the reservoir temperature (average thermal photon number).

CHAPTER 7

CONCLUSION

In this thesis, we gave an overall picture of decoherence and disentanglement, first, in the context of a general discussion, and later, giving particular examples related to two qubits in contact with different types of reservoirs.

Entanglement is at the heart of quantum Mechanics. Also, besides its conceptual relevance, it is also crucial in many applications in Quantum communications and quantum computing. On the other hand, it is a subtle and delicate effect that can be easily altered or destroyed. In the various cases studying the dynamics of entanglement, we encountered different situations, described in Figure (7.1), where we show *trajectories* between the PPT (Partial Positive Transpose) and the NPT (Negative Partial Transpose) areas. The first case corresponds to a system that has both the initial and final states entangled. In the second case, the initial entanglement goes asymptotically to zero. In the third and fourth cases, we observe sudden death and death with revival respectively. Case number five corresponds to a periodic Hamiltonian system (for example, the Jaynes-Cummings Model), where death and revival appear periodically. Death and revival are intriguing effects that basically depend on the initial conditions as well as the nature of the reservoir. Since, for many applications, the sudden death is an undesired effect, one can try to protect these states, using for example, the distillation procedures.

Finally, in cases six and seven, we have generation of entanglement with and without time delay respectively, starting from a separable state. These are very interesting effects, where, in spite of the fact that there is no coupling between the

atoms, the common reservoir acts as an effective coupling, with particularly good results, in the case of a vacuum reservoir, when the initial atomic state combines an excited atom with the other one in the ground state, thus allowing, via a common reservoir, a photon exchange. When the reservoir is at a finite temperature, this effect is reduced.

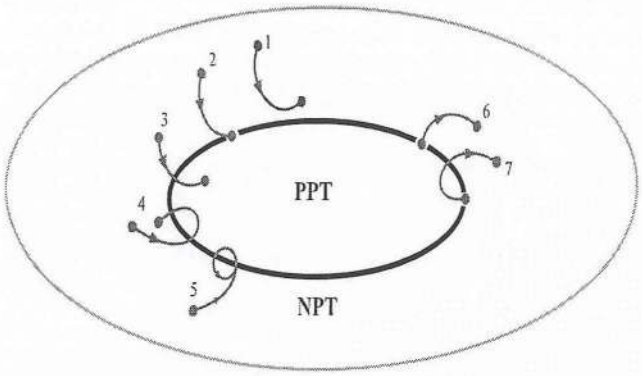

Figure 7.1: "Trajectories" between the positive partial transpose(separable) and the negative partial transpose (entangled) areas: 1: Initial and final states are both entangled, 2: Initial entanglement going asymptotically to zero, 3: Initial entanglement with sudden death, 4: Initial entanglement with sudden death and revival, 5: Periodic death and revival, 6: Entanglement generation starting from a separable state, 7: Time delayed generation of entanglement starting from a separable state.

APPENDIX A

SOLUTION OF MASTER EQUATION I. INDEPENDENT RESERVOIRS.

The solution of master equation (3.3.1) depends on the initial state. We will consider a general initial condition,

$$\rho(0) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \tag{A.0.1}$$

spanned by two-qubit product states $|1\rangle = |11\rangle, |2\rangle = |10\rangle, |3\rangle = |01\rangle, |4\rangle = |00\rangle$, called "*Standard Basis*". For simplicity, we write each matrix element $\rho_{ij}(0)$ as ρ_{ij} and $\Gamma = 1$. In the following, we list the solutions of (3.3.1).

A.1 Vacuum reservoirs

In a vacuum reservoir we have to make $N \to 0$ and $M \to 0$ in Eq.(3.3.1). Under these conditions, the solution is given by,

$$\begin{aligned}
\rho_{11}(t) &= \rho_{11}e^{-2t}, \\
\rho_{12}(t) &= \rho_{12}e^{\frac{-3}{2}t}, \\
\rho_{13}(t) &= \rho_{13}e^{\frac{-3}{2}t}, \\
\rho_{14}(t) &= \rho_{14}e^{-t}, \\
\rho_{21}(t) &= \rho_{21}e^{\frac{-3}{2}t}, \\
\rho_{22}(t) &= (\rho_{11}+\rho_{22})e^{-t} - \rho_{11}^{-2t}, \\
\rho_{23}(t) &= \rho_{23}e^{-t}, \\
\rho_{24}(t) &= (\rho_{13}+\rho_{24})e^{\frac{-1}{2}t} - \rho_{13}e^{\frac{-3}{2}t}, \\
\rho_{31}(t) &= \rho_{31}e^{\frac{-3}{2}t}, \\
\rho_{32}(t) &= \rho_{32}e^{-t}, \\
\rho_{33}(t) &= (\rho_{11}+\rho_{33})e^{-t} - \rho_{11}e^{-2t}, \\
\rho_{34}(t) &= (\rho_{12}+\rho_{34})e^{\frac{-1}{2}t} - \rho_{12}e^{\frac{-3}{2}t}, \\
\rho_{41}(t) &= \rho_{41}e^{-t}, \\
\rho_{42}(t) &= (\rho_{31}+\rho_{42})e^{\frac{-1}{2}t} - \rho_{31}e^{\frac{-3}{2}t}, \\
\rho_{43}(t) &= (\rho_{21}+\rho_{43})e^{\frac{-1}{2}t} - \rho_{21}e^{\frac{-3}{2}t}, \\
\rho_{44}(t) &= \rho_{11}e^{-2t} - (2\rho_{11}+\rho_{22}-\rho_{33})e^{-t} + 1.
\end{aligned} \tag{A.1.1}$$

A.2 Thermal reservoirs

In a thermal reservoir we have to make $N \to \bar{n}$, it is the mean number of the thermal field (assumed to be the same for both qubits), and $M \to 0$. Under these conditions,

the solution is given by,

$$
\begin{aligned}
\rho_{11}(t) &= \frac{1}{(2n+1)^2}\{[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})n^2+(2\rho_{11}-\rho_{22}-\rho_{33})n+\rho_{11}]e^{-2(2n+1)t} \\
&\quad +[2(\rho_{11}-\rho_{44})n^2+(2\rho_{11}+\rho_{22}+\rho_{33})n]e^{-(2n+1)t}+n^2\}, \\
\rho_{12}(t) &= \frac{1}{2n+1}[(\rho_{12}+(\rho_{12}-\rho_{34})n)e^{\frac{-3}{2}(2n+1)t}+(\rho_{12}+\rho_{34})ne^{\frac{-1}{2}(2n+1)t}], \\
\rho_{13}(t) &= \frac{1}{2n+1}[(\rho_{13}+(\rho_{13}-\rho_{24})n)e^{\frac{-3}{2}(2n+1)t}+(\rho_{13}+\rho_{24})ne^{\frac{-1}{2}(2n+1)t}], \\
\rho_{14}(t) &= \rho_{14}e^{-(2n+1)t}, \\
\\
\rho_{21}(t) &= \frac{1}{2n+1}[(\rho_{21}+(\rho_{21}-\rho_{43})n)e^{\frac{-3}{2}(2n+1)t}+(\rho_{21}+\rho_{43})ne^{\frac{-1}{2}(2n+1)t}], \\
\rho_{22}(t) &= \frac{1}{(2n+1)^2}\{-[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})n^2+(2\rho_{11}-\rho_{22}-\rho_{33})n+\rho_{11}]e^{-2(2n+1)t} \\
&\quad +[2(\rho_{22}-\rho_{33})n^2+(\rho_{11}+2\rho_{22}-2\rho_{33}-\rho_{44})n+\rho_{11}+\rho_{22}]e^{-(2n+1)t}+n(n+1)\}, \\
\rho_{23}(t) &= \rho_{23}e^{-(2n+1)t}, \\
\rho_{24}(t) &= \frac{1}{2n+1}[((\rho_{24}-\rho_{13})n-\rho_{13})e^{\frac{-3}{2}(2n+1)t}+(\rho_{13}+\rho_{24})(n+1)e^{\frac{-1}{2}(2n+1)t}], \\
\\
\rho_{31}(t) &= \frac{1}{2n+1}[(\rho_{31}+(\rho_{31}-\rho_{42})n)e^{\frac{-3}{2}(2n+1)t}+(\rho_{31}+\rho_{42})ne^{\frac{-1}{2}(2n+1)t}], \\
\rho_{32}(t) &= \rho_{32}e^{-(2n+1)t}, \\
\rho_{33}(t) &= \frac{1}{(2n+1)^2}\{-[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})n^2+(2\rho_{11}-\rho_{22}-\rho_{33})n+\rho_{11}]e^{-2(2n+1)t} \\
&\quad +[2(\rho_{33}-\rho_{22})n^2+(\rho_{11}-2\rho_{22}+2\rho_{33}-\rho_{44})n+\rho_{11}+\rho_{33}]e^{-(2n+1)t}+n(n+1)\}, \\
\rho_{34}(t) &= \frac{1}{2n+1}[((\rho_{34}-\rho_{12})n-\rho_{12})e^{\frac{-3}{2}(2n+1)t}+(\rho_{12}+\rho_{34})(n+1)e^{\frac{-1}{2}(2n+1)t}], \\
\\
\rho_{41}(t) &= \rho_{41}e^{-(2n+1)t}, \\
\rho_{42}(t) &= \frac{1}{2n+1}[((\rho_{42}-\rho_{31})n-\rho_{31})e^{\frac{-3}{2}(2n+1)t}+(\rho_{31}+\rho_{42})(n+1)e^{\frac{-1}{2}(2n+1)t}], \\
\rho_{43}(t) &= \frac{1}{2n+1}[-((\rho_{21}-\rho_{43})n-\rho_{21})e^{\frac{-3}{2}(2n+1)t}+(\rho_{21}+\rho_{43})(n+1)e^{\frac{-1}{2}(2n+1)t}], \\
\rho_{44}(t) &= \frac{1}{(2n+1)^2}\{[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})n^2+(2\rho_{11}-\rho_{22}-\rho_{33})n+\rho_{11}]e^{-2(2n+1)t} \\
&\quad +2(n+1)[(\rho_{44}-\rho_{11})n-\frac{1}{2}(2\rho_{11}+\rho_{22}+\rho_{33})]e^{-(2n+1)t}+(n+1)^2\}. \quad \text{(A.2.1)}
\end{aligned}
$$

A.3 Squeezed reservoirs

In the case of a squeezed reservoir, we consider N and M. Under these conditions, the solution is given by,

$$\begin{aligned}
\rho_{11}(t) &= \frac{1}{(2N+1)^2}\{[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})N^2+(2\rho_{11}-\rho_{22}-\rho_{33})N+\rho_{11}]e^{-2(2N+1)t} \\
&+[2(\rho_{11}-\rho_{44})N^2+(2\rho_{11}+\rho_{22}+\rho_{33})N]e^{-(2n+1)t}+N^2\}, \\
\rho_{12}(t) &= \frac{1}{2(2N+1)}\{[(\rho_{12}-\rho_{21}+\rho_{43}-\rho_{34})N+\rho_{12}-\rho_{21}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&+[(\rho_{12}+\rho_{21}-\rho_{34}-\rho_{43})N+\rho_{12}+\rho_{21}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&+N[\rho_{12}+\rho_{21}+\rho_{34}+\rho_{43}]e^{\frac{-1}{2}(2N+1+2M)t}+N[\rho_{12}-\rho_{21}+\rho_{34}-\rho_{43}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\rho_{13}(t) &= \frac{1}{2(2N+1)}\{[(\rho_{13}-\rho_{31}+\rho_{42}-\rho_{24})N+\rho_{13}-\rho_{31}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&+[(\rho_{13}+\rho_{31}-\rho_{24}-\rho_{42})N+\rho_{13}+\rho_{31}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&+N[\rho_{13}+\rho_{31}+\rho_{24}+\rho_{42}]e^{\frac{-1}{2}(2N+1+2M)t}+N[\rho_{13}-\rho_{31}+\rho_{24}-\rho_{42}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\rho_{14}(t) &= \frac{1}{4}\{2[\rho_{14}-\rho_{41}]e^{-(2N+1)t}+[\rho_{14}+\rho_{41}+\rho_{23}+\rho_{32}]e^{-(2N+1+2M)t} \\
&+[\rho_{14}+\rho_{41}-\rho_{23}-\rho_{32}]e^{-(2N+1-2M)t}\}, \\
\rho_{21}(t) &= \frac{1}{2(2N+1)}\{[-(\rho_{12}-\rho_{21}+\rho_{43}-\rho_{34})N+\rho_{12}-\rho_{21}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&+[(\rho_{12}+\rho_{21}-\rho_{34}-\rho_{43})N+\rho_{12}+\rho_{21}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&+N[\rho_{12}+\rho_{21}+\rho_{34}+\rho_{43}]e^{\frac{-1}{2}(2N+1+2M)t}-N[\rho_{12}-\rho_{21}+\rho_{34}-\rho_{43}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\rho_{22}(t) &= \frac{1}{(2N+1)^2}\{[2(\rho_{22}-\rho_{33})N^2+(\rho_{11}+2\rho_{22}-2\rho_{33}-\rho_{44})N+\rho_{11}+\rho_{22}]e^{-(2N+1)t} \\
&-[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})N^2+(2\rho_{11}-\rho_{22}-\rho_{33})N-\rho_{11}]e^{-2(2N+1)t}+N(N+1)\}, \\
\rho_{23}(t) &= \frac{1}{4}\{[\rho_{23}+\rho_{32}-\rho_{14}-\rho_{41}]e^{-(2N+1-2M)t}+[\rho_{23}+\rho_{32}+\rho_{14}+\rho_{41}]e^{-(2N+1+2M)t} \\
&+2[\rho_{23}-\rho_{32}]e^{-(2N+1)t}\}, \\
\rho_{24}(t) &= \frac{1}{2(2N+1)}\{-[(\rho_{13}-\rho_{31}-\rho_{24}+\rho_{42})N+\rho_{13}-\rho_{31}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&-[(\rho_{13}+\rho_{31}-\rho_{24}-\rho_{42})N+\rho_{13}+\rho_{31}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&+(N+1)[\rho_{13}+\rho_{31}+\rho_{24}+\rho_{42}]e^{\frac{-1}{2}(2N+1+2M)t} \\
&+(N+1)[\rho_{13}-\rho_{31}+\rho_{24}-\rho_{42}]e^{\frac{-1}{2}(2N+1-2M)t}\},
\end{aligned}$$

$$
\begin{aligned}
\rho_{31}(t) &= \frac{1}{2(2N+1)}\{[-(\rho_{13}-\rho_{31}+\rho_{42}-\rho_{24})N-\rho_{13}+\rho_{31}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&\quad +[(\rho_{13}+\rho_{31}-\rho_{24}-\rho_{42})N+\rho_{13}+\rho_{31}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&\quad +N[\rho_{13}+\rho_{31}+\rho_{24}+\rho_{42}]e^{\frac{-1}{2}(2N+1+2M)t}-N[\rho_{13}-\rho_{31}+\rho_{24}-\rho_{42}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\rho_{32}(t) &= \frac{1}{4}\{[\rho_{23}+\rho_{32}-\rho_{14}-\rho_{41}]e^{-(2N+1-2M)t}+[\rho_{23}+\rho_{32}+\rho_{14}+\rho_{41}]e^{-(2N+1+2M)t} \\
&\quad -2[\rho_{23}-\rho_{32}]e^{-(2N+1)t}\}, \\
\rho_{33}(t) &= \frac{1}{(2N+1)^2}\{[(\rho_{11}-2\rho_{22}+2\rho_{33}-\rho_{44})N+\rho_{11}+\rho_{33}]e^{-(2N+1)t} \\
&\quad +[-(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})N^2-(2\rho_{11}-\rho_{22}-\rho_{33})N-\rho_{11}]e^{-2(2N+1)t}+N(N+1)\}, \\
\rho_{34}(t) &= \frac{1}{2(2N+1)}\{-[(\rho_{12}-\rho_{21}-\rho_{34}+\rho_{43})N+\rho_{12}-\rho_{21}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&\quad -[(\rho_{12}+\rho_{21}-\rho_{34}-\rho_{43})N+\rho_{12}+\rho_{21}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&\quad +(N+1)[\rho_{12}+\rho_{21}+\rho_{34}+\rho_{43}]e^{\frac{-1}{2}(2N+1+2M)t} \\
&\quad +(N+1)[\rho_{12}-\rho_{21}+\rho_{34}-\rho_{43}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\\
\rho_{41}(t) &= \frac{1}{4}\{2[\rho_{41}-\rho_{14}]e^{-(2N+1)t}+[\rho_{14}+\rho_{41}+\rho_{23}+\rho_{32}]e^{-(2N+1+2M)t} \\
&\quad +[\rho_{14}+\rho_{41}-\rho_{23}-\rho_{32}]e^{-(2N+1-2M)t}\}, \\
\rho_{42}(t) &= \frac{1}{2(2N+1)}\{[(\rho_{13}-\rho_{31}-\rho_{24}+\rho_{42})N+\rho_{13}-\rho_{31}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&\quad -[(\rho_{13}+\rho_{31}-\rho_{24}-\rho_{42})N+\rho_{13}+\rho_{31}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&\quad +(N+1)[\rho_{13}+\rho_{31}+\rho_{24}+\rho_{42}]e^{\frac{-1}{2}(2N+1+2M)t} \\
&\quad +(N+1)[\rho_{13}-\rho_{31}+\rho_{24}-\rho_{42}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\rho_{43}(t) &= \frac{1}{2(2N+1)}\{[(\rho_{12}-\rho_{21}-\rho_{34}+\rho_{43})N+\rho_{12}-\rho_{21}]e^{\frac{-1}{2}(3(2N+1)-2M)t} \\
&\quad -[(\rho_{12}+\rho_{21}-\rho_{34}-\rho_{43})N+\rho_{12}+\rho_{21}]e^{\frac{-1}{2}(3(2N+1)+2M)t} \\
&\quad +(N+1)[\rho_{12}+\rho_{21}+\rho_{34}+\rho_{43}]e^{\frac{-1}{2}(2N+1+2M)t} \\
&\quad -(N+1)[\rho_{12}-\rho_{21}+\rho_{34}-\rho_{43}]e^{\frac{-1}{2}(2N+1-2M)t}\}, \\
\rho_{44}(t) &= \frac{1}{(2N+1)^2}\{[(\rho_{11}-\rho_{22}-\rho_{33}+\rho_{44})N^2+(2\rho_{11}-\rho_{22}-\rho_{33})N+\rho_{11}]e^{-2(2N+1)t} \\
&\quad -(N+1)[2(\rho_{11}-\rho_{44})N+2\rho_{11}+\rho_{22}+\rho_{33}]e^{-(2N+1)t}+(N+1)^2\}. \qquad \text{(A.3.1)}
\end{aligned}
$$

APPENDIX B

SOLUTION OF MASTER EQUATION II. A COMMON RESERVOIR

The solution of the master equation (5.1.1) depends on the initial state. We will consider a general initial condition, but spanned by two different basis ϕ_i and standard basis.

B.1 Basis $\{\phi_i\}$

The basis $\phi_i = \{|1\rangle = |\phi_1\rangle, |2\rangle = |\phi_2\rangle, |3\rangle = |\phi_3\rangle, |4\rangle = |\phi_4\rangle\}$ is defined by

$$\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{N^2+M^2}}(N|++\rangle + Me^{-i\Psi}|--\rangle), \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}(|-+\rangle - |+-\rangle), \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle), \\
|\phi_4\rangle &= \frac{1}{\sqrt{N^2+M^2}}(M|++\rangle - Ne^{-i\Psi}|--\rangle).
\end{aligned}$$

The general initial condition considered spanned by this basis is given by,

$$\rho(0) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \tag{B.1.1}$$

For simplicity, we write each matrix element $\rho_{ij}(0)$ as ρ_{ij} and $\Gamma = 1$. We will consider the general case to have a squeezed reservoir and for a minimum uncertainty squeezed state, it is mean, we use the constraint $M = \sqrt{N(N+1)}$. For $N \to 0$ we recover the vacuum reservoir. In the following, we list the solutions of (3.3.1) under the above conditions.

$$
\begin{aligned}
\rho_{11}(t) &= \frac{-1}{\sqrt{N(N+1)}(2N+1)}\{ \\
&\times[\frac{1}{2}((2N+1)\sqrt{N(N+1)}+2N^2+2N)(\rho_{33}+\rho_{44})+\frac{1}{4}\rho_{44}]e^{-2(2N+1-2\sqrt{N(N+1)})t} \\
&+[\frac{1}{2}((2N+1)\sqrt{N(N+1)}-2N^2-2N)(\rho_{33}+\rho_{44})-\frac{1}{4}\rho_{44}]e^{-2(2N+1+2\sqrt{N(N+1)})t} \\
&-\sqrt{N(N+1)}(2N+1)(\rho_{11}+\rho_{33}+\rho_{44})\}, \\
\rho_{12}(t) &= \rho_{12}, \\
\rho_{13}(t) &= \frac{e^{-(2N+1)t}}{12N^2+12N-1}\{ \\
&[2N+1+4\sqrt{N(N+1)}](\rho_{34}-\rho_{43})e^{-(2N+1-4\sqrt{N(N+1)})t} \\
&+[2N+1-4\sqrt{N(N+1)}](\rho_{34}+\rho_{43})e^{-(2N+1+4\sqrt{N(N+1)})t} \\
&+(12N^2+12N-1)\rho_{13}-2(2N+1)\rho_{34}+8\sqrt{N(N+1)}\rho_{43}\}, \\
\rho_{14}(t) &= \frac{-e^{-(2N+1)t}}{(2N+1)(12N^2+12N-1)}\{ \\
&(2N+1+4\sqrt{N(N+1)})(2\rho_{33}+(2N+1)\rho_{44})e^{-(2N+1-4\sqrt{N(N+1)})t} \\
&(2N+1-4\sqrt{N(N+1)})(2\rho_{33}-(2N+1)\rho_{44})e^{-(2N+1+4\sqrt{N(N+1)})t} \\
&-(2N+1)[(12N^2+12N-1)\rho_{14}+4\sqrt{N(N+1)}(2\rho_{44}+\rho_{33})]\}, \\
\rho_{21}(t) &= \rho_{21}, \\
\rho_{22}(t) &= \rho_{22}, \\
\rho_{23}(t) &= \rho_{23}e^{-(2N+1)t}, \\
\rho_{24}(t) &= \rho_{24}e^{-(2N+1)t},
\end{aligned}
$$

$$
\begin{aligned}
\rho_{31}(t) &= \frac{e^{-(2N+1)t}}{12N^2+12N-1}\{ \\
&\quad -[2N+1+4\sqrt{N(N+1)}](\rho_{34}-\rho_{43})e^{-(2N+1-4\sqrt{N(N+1)})t} \\
&\quad +[2N+1-4\sqrt{N(N+1)}](\rho_{34}+\rho_{43})e^{-(2N+1+4\sqrt{N(N+1)})t} \\
&\quad +(12N^2+12N-1)\rho_{31}-2(2N+1)\rho_{43}+8\sqrt{N(N+1)}\rho_{34}\}, \\
\rho_{32}(t) &= \rho_{32}e^{-(2N+1)t}, \\
\rho_{33}(t) &= \frac{1}{4\sqrt{N(N+1)}}\{[(2N+1)\rho_{44}+2\sqrt{N(N+1)}\rho_{33}]e^{-2(2N+1-2\sqrt{N(N+1)})t} \\
&\quad -[(2N+1)\rho_{44}-2\sqrt{N(N+1)}\rho_{33}]e^{-2(2N+1+2\sqrt{N(N+1)})t}\}, \\
\rho_{34}(t) &= \frac{1}{2}\{(\rho_{34}-\rho_{43})e^{-2(2N+1-2\sqrt{N(N+1)})t}+(\rho_{34}+\rho_{43})e^{-2(2N+1+2\sqrt{N(N+1)})t}\}, \\
\rho_{41}(t) &= \frac{-e^{-(2N+1)t}}{(2N+1)(12N^2+12N-1)}\{ \\
&\quad (2N+1+4\sqrt{N(N+1)})(2\rho_{33}+(2N+1)\rho_{44})e^{-(2N+1-4\sqrt{N(N+1)})t} \\
&\quad (2N+1-4\sqrt{N(N+1)})(2\rho_{33}-(2N+1)\rho_{44})e^{-(2N+1+4\sqrt{N(N+1)})t} \\
&\quad -(2N+1)[(12N^2+12N-1)\rho_{41}+4\sqrt{N(N+1)}(2\rho_{44}+\rho_{33})]\}, \\
\rho_{42}(t) &= \rho_{42}e^{-(2N+1)t}, \\
\rho_{43}(t) &= \frac{1}{2}\{(\rho_{43}-\rho_{34})e^{-2(2N+1-2\sqrt{N(N+1)})t}+(\rho_{43}+\rho_{34})e^{-2(2N+1+2\sqrt{N(N+1)})t}\}, \\
\rho_{44}(t) &= \frac{1}{4N+2}\{((2N+1)\rho_{44}+2\sqrt{N(N+1)}\rho_{33})e^{-2(2N+1-2\sqrt{N(N+1)})t} \\
&\quad +((2N+1)\rho_{44}-2\sqrt{N(N+1)}\rho_{33})e^{-2(2N+1+2\sqrt{N(N+1)})t}\}. \qquad \text{(B.1.2)}
\end{aligned}
$$

B.2 Standard Basis

The standard basis is defined as $\{e_1 = |11\rangle, e_2 = |10\rangle, e_3 = |01\rangle, e_4 = |00\rangle\}$, in this basis, we consider a general initial density matrix given by

$$
\rho(0) = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \qquad \text{(B.2.1)}
$$

For simplicity, we write each matrix element $\rho_{ij}(0)$ as ρ_{ij} and we will take $\Gamma = 1$. The solution of the master equation (3.3.1) with respect to the standard basis and for the above initial condition is

B.2.1 Vacuum reservoir

$$
\begin{aligned}
\rho_{11}(t) &= \rho_{11}e^{-2\Gamma t} \\
\rho_{22}(t) &= \frac{1}{4}(\rho_{22}+\rho_{33}-\rho_{23}-\rho_{32}+2(\rho_{22}-\rho_{33})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{33}(t) &= \frac{1}{4}(\rho_{22}+\rho_{33}-\rho_{23}-\rho_{32}+2(\rho_{33}-\rho_{22})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{44}(t) &= \frac{1}{2}(\rho_{23}+\rho_{32}+\rho_{11}+\rho_{44}+1-(4\rho_{11}t+2\rho_{11}+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{12}(t) &= \frac{1}{2}((\rho_{12}-\rho_{13})e^{-\Gamma t}+(\rho_{12}+\rho_{13})e^{-2\Gamma t}) \\
\rho_{13}(t) &= \frac{1}{2}(-(\rho_{12}-\rho_{13})e^{-\Gamma t}+(\rho_{12}+\rho_{13})e^{-2\Gamma t}) \\
\rho_{14}(t) &= \rho_{14}e^{-\Gamma t} \\
\rho_{22}(t) &= \frac{1}{4}(\rho_{22}+\rho_{33}-\rho_{23}-\rho_{32}+2(\rho_{22}-\rho_{33})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{23}(t) &= \frac{1}{4}(\rho_{23}+\rho_{32}-\rho_{22}-\rho_{33}+2(\rho_{23}-\rho_{32})e^{-\Gamma t}+(4\rho_{11}t+\rho_{22}+\rho_{33}+\rho_{23}+\rho_{32})e^{-2\Gamma t}) \\
\rho_{24}(t) &= \frac{1}{2}(\rho_{24}-\rho_{34}-2(\rho_{12}+\rho_{13})e^{-2\Gamma t}+(2\rho_{12}+2\rho_{13}+\rho_{34}+\rho_{24})e^{\Gamma t}) \\
\rho_{34}(t) &= \frac{1}{2}(\rho_{34}-\rho_{24}-2(\rho_{12}+\rho_{13})e^{-2\Gamma t}+(2\rho_{12}t+2\rho_{13}+\rho_{34}+\rho_{24})e^{-\Gamma t})
\end{aligned}
\tag{B.2.2}
$$

The remaining matrix elements can be obtained from $\rho_{ij} = \rho^*_{ji}$.

Appendix C

A comparative study of the optical and circuit representation of the unambiguous quantum state discriminator [67]

We propose a quantum circuit implementation of the unambiguous quantum state discriminator. The circuit is made entirely of standard logical quantum gates, and provides an optimal implementation of the Positive Operator Valued Measurement (POVM) for the unambiguous discrimination of quantum states. We also propose an actual experimental setup of this device using the vibrational degrees of freedom of one- or two-dimensional ion traps. We compare this implementation to the one that has been used exclusively so far in experiments, which is based on single-photon interferometry, and discuss their relative advantages.

C.1 Introduction

The discrimination of quantum states is a basic task in quantum information and quantum communication systems as well as in quantum cryptography [68]. A particularly clear example for the role of state discrimination in quantum cryptography is provided by the B92 QKD protocol [69]. A recent review on state discrimination can be found in Ref. [70]. Two popular strategies for optimal results are: the minimum error (ME) discrimination [71, 72], where each measurement outcome selects one of the possible states and the error probability is minimized, and the unambiguous quantum state discrimination (USD) for linearly independent states [73], where we are not permitted to make an erroneous identification of the state, but we can get inconclusive results from the measurement. The goal is to minimize the fraction of the inconclusive results.

For the case of two pure states, with equal a priori probabilities, the optimum USD measurement was found more than a decade ago by Ivanovic, Dieks, and Peres [74, 75, 76]. The physical methods that have been previously proposed to implement USD include linear optical systems [77], ion trap architecture [78] and nuclear magnetic resonance [79]. In this work we shall deal with the unambiguous state discrimination, following the Peres model [76] and making use of POVM's for two known states. In particular, we propose a quantum circuit implementation of this device with logical quantum gates, and an experimental setup, employing trapped ions. We also investigate a different physical implementation which is based on single-photon optical interferometry and compare the relative strengths of these two different approaches.

C.2 Quantum Discriminator

We want to discriminate unambiguously between two nonorthogonal states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ that span a two-dimensional subspace of the complex Hilbert space .

$$|\Psi_0\rangle = \sqrt{a}|0\rangle + \sqrt{1-a}|1\rangle, \tag{C.2.1}$$

$$|\Psi_1\rangle = \sqrt{a}|0\rangle - \sqrt{1-a}|1\rangle, \tag{C.2.2}$$

where $a > \frac{1}{2}$.

We consider an additional ancilla qubit in the state $|0\rangle$. The bases are labeled by A for the system and B for the ancilla. Our initial state is

$$\begin{aligned} |\Psi_i\rangle_{AB}^{in} &= |\Psi_i\rangle_A \otimes |0\rangle_B \\ &= \sqrt{a}|00\rangle_{AB} \pm \sqrt{1-a}|10\rangle_{AB}, \end{aligned} \tag{C.2.3}$$

where $\pm$ signs correspond to initial states $|\Psi_0\rangle_A$ and $|\Psi_1\rangle_A$, the second qubit is the ancilla, and $i = 0, 1$.

Making use of the Peres choice, we define a unitary operator U that yields the following final state, when applied to $|\Psi_i\rangle_{AB}^{in}$

$$\begin{aligned} |\Psi_i\rangle_{AB}^{out} &= U_{AB}|\Psi_i\rangle_{AB}^{in} \\ &= \sqrt{1-a}(|0\rangle_A \pm |1\rangle_A)|0\rangle_B \\ &\quad +\sqrt{2a-1}|1\rangle_A|1\rangle_B \\ &= \sqrt{2(1-a)}|\pm\rangle_A|0\rangle_B \\ &\quad +\sqrt{2a-1}|1\rangle_A|1\rangle_B, \end{aligned} \tag{C.2.4}$$

where $|\pm\rangle_A \equiv (|0\rangle_A \pm |1\rangle_A)/\sqrt{2}$.

If we perform a measurement on the ancilla (system B) and get the state $|0\rangle_B$, we can make a further von Neumann measurement on the system A and unambiguously determine the input state. If the final state is the $|+\rangle_A$ state, then $|\Psi_i\rangle_A^{in} = |\Psi_0\rangle$, and if the final state is $|-\rangle_A$ state, then $|\Psi_i\rangle_A^{in} = |\Psi_1\rangle$, and the probability of the successful discrimination is $2(1-a) \equiv 1 - \cos(2\theta)$. If the measurement on the ancilla gives the state $|1\rangle_B$, then the A qubit will always be in the state $|1\rangle_A$, so we get an inconclusive result with probability $2a - 1 \equiv \cos(2\theta)$, where we introduced $a \equiv \cos^2(\theta)$.

We can write the unitary operator U_{AB}, in the basis $\{|00\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}, |11\rangle_{AB}\}$,

as

$$U_{AB} = \begin{pmatrix} \sqrt{\frac{1-a}{a}} & 0 & 0 & -\sqrt{\frac{2a-1}{a}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\frac{2a-1}{a}} & 0 & 0 & \sqrt{\frac{1-a}{a}} \end{pmatrix}. \tag{C.2.5}$$

In the next section we shall find a quantum circuit representation of this two-qubit unitary operator.

C.3 Implementation of U_{AB} in a quantum circuit

One of the main tools for the development of quantum computing is the construction of quantum circuits capable of realizing the processing and manipulation of the quantum information. In this context, it is necessary to decompose any unitary transformation in a sequence of quantum gates, resulting in a corresponding quantum circuit.

In this section, we review an algorithm to decompose an arbitrary unitary transformation $U \in SU(4)$ [80] in order to express the unitary transformation by a quantum circuit. Then we apply this procedure to the particular case of the unambiguous discriminator of known states (C.2.5).

C.3.1 Procedure to decompose a unitary matrix $U \in SU(4)$

The main decomposition is the Cartan decomposition [81], in the "KAK" form from the Lie theory. Several cases have been studied, like the two-qubit *magic decomposition* [80, 82, 83, 84, 85], the *cosine-sine decomposition* [86, 87, 88], and the *demultiplexing decomposition* [88]. The first one is for two qubits, and the last two decompositions hold for n-qubit operators.

For any operator $U \in SU(4)$ there exist local unitary operators U_A, U_B, V_A, V_B and a nonlocal unitary operator U_d such that

$$U = (U_A \otimes U_B)U_d(V_A \otimes V_B) \tag{C.3.1}$$

C. A COMPARATIVE STUDY OF THE OPTICAL AND CIRCUIT REPRESENTATION OF THE UNAMBIGUOUS QUANTUM STATE DISCRIMINATOR [67]

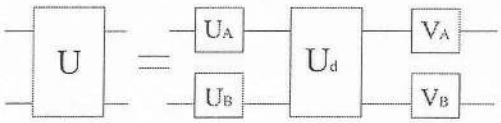

Figure C.1: Decomposition of operator $U \in SU(4)$.

where

$$\begin{aligned} U_d &= e^{-i\vec{\sigma_A}^T d\vec{\sigma_B}} \\ &= e^{-i(\alpha_x\sigma_x\otimes\sigma_x+\alpha_y\sigma_y\otimes\sigma_y+\alpha_z\sigma_z\otimes\sigma_z)} \\ &= e^{-iH}. \end{aligned} \tag{C.3.2}$$

Here $\vec{\sigma}^T$ is the transpose of Pauli operators denoted by $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ in the computational basis, $\{|0\rangle, |1\rangle\}$, the subscripts specify on which system the operator is acting, and d is a diagonal matrix whose diagonal elements are denoted by α_x,α_y,α_z. Due to the periodicity and symmetry of the entanglement generated by U_d one has the following restriction [80]:

$$\frac{\pi}{4} \geq \alpha_x \geq \alpha_y \geq |\alpha_z|. \tag{C.3.3}$$

It is easy to show that the operator U_d is diagonal in the magic basis $\{|\Phi_k\rangle\}$, defined as

$$\begin{aligned} |\Phi_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_2\rangle = \frac{i}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Phi_3\rangle &= \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned} \tag{C.3.4}$$

Therefore, U_d in the magic basis can be written as

$$U_d = \sum_{k=1}^{4} e^{-i\lambda_k} |\Phi_k\rangle\langle\Phi_k|, \tag{C.3.5}$$

where the λ_k's are given by

$$\begin{aligned}\lambda_1 &= \alpha_x - \alpha_y + \alpha_z,\\ \lambda_2 &= -\alpha_x + \alpha_y + \alpha_z,\\ \lambda_3 &= \alpha_x + \alpha_y - \alpha_z,\\ \lambda_4 &= -\alpha_x - \alpha_y - \alpha_z.\end{aligned} \tag{C.3.6}$$

In order to find the matrices U_A, U_B, V_A, V_B and the phases λ_k in the decomposition (C.3.1), we follow the procedure given in [80].

The procedure goes as follows:

a) Calculate U in the magic basis, then obtain the eigensystem of the product $U^T U$, denoting the eigenvalues and eigenstates by $e^{2i\epsilon_k}$ and $|\Psi_k\rangle$, respectively.

b) Following the lemma 1 of Ref. [80] (Appendix A), choose V_A, V_B and the phases ξ_k, such that

$$V_A \otimes V_B e^{i\xi_k}|\Psi_k\rangle = |\Phi_k\rangle, \tag{C.3.7}$$

where $|\Psi_k\rangle$ corresponds to our eigenstates calculated in a), and $|\Phi_k\rangle$ is the magic basis (C.3.4).

c) According to the eigenvalues and eigenstates obtained in a), calculate

$$|\widetilde{\Psi}_k\rangle = e^{-i\epsilon_k} U|\Psi_k\rangle. \tag{C.3.8}$$

d) Using the lemma 1 again, choose the U_A, U_B and λ_k such that

$$U_A^\dagger \otimes U_B^\dagger e^{i(\lambda_k+\xi_k+\epsilon_k)}|\widetilde{\Psi}_k\rangle = |\Phi_k\rangle. \tag{C.3.9}$$

C.3.2 Decomposition of the unitary matrix U_{AB}

Following the above procedure, for our particular unitary matrix (C.2.5), we readily find the elements in the decomposition U_A, U_B, V_A,V_B (see Appendix B):

$$V_A = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}, \qquad V_B = \begin{pmatrix} 0 & 1 \\ e^{-\frac{i\pi}{4}} & 0 \end{pmatrix},$$
$$U_A = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{pmatrix}, \qquad U_B = \begin{pmatrix} 0 & e^{\frac{i\pi}{4}} \\ 1 & 0 \end{pmatrix}, \tag{C.3.10}$$

and the diagonal elements of U_d:

$$\begin{aligned} e^{-i\lambda_1} &= 1, \\ e^{-i\lambda_2} &= 1, \\ e^{-i\lambda_3} &= \sqrt{\frac{1-a}{a}} - i\sqrt{\frac{2a-1}{a}}, \\ e^{-i\lambda_4} &= \sqrt{\frac{1-a}{a}} + i\sqrt{\frac{2a-1}{a}}. \end{aligned} \tag{C.3.11}$$

If we now compare the equations (C.3.6) and (C.3.11), we readily find $\alpha_z = 0, \alpha_x = \alpha_y \equiv \alpha$, and defining $\cos\phi = \frac{\sqrt{1-a}}{a^{1/2}}$, we can write $\alpha = -\frac{\phi}{2}$, and $e^{-i\lambda_3} = e^{2i\alpha}$, $e^{-i\lambda_4} = e^{-2i\alpha}$.

We have all elements in the decomposition (C.3.1), namely U_A,U_B,V_A,V_B and U_d. But we still can decompose U_d. Following the reference [83], U_d can be further decomposed in terms of C-NOT and single qubit gates, as shown in Fig. C.2. In the

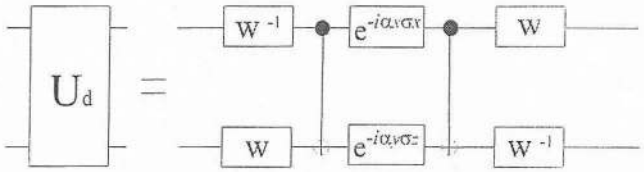

Figure C.2: Quantum circuit for U_d in terms in terms of a C-NOT and single qubit gates when $\alpha_z = 0$.

figure w is defined by

$$w = \frac{1 + i\sigma_x}{\sqrt{2}} = R_x(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{C.3.12}$$

and

$$e^{-i\alpha_x\sigma_x} = e^{-i\alpha\sigma_x} = e^{i\frac{\phi}{2}\sigma_x} = R_x(-\phi), \tag{C.3.13}$$

$$e^{-i\alpha_y\sigma_z} = e^{-i\alpha\sigma_z} = e^{-i\frac{\phi}{2}\sigma_z} = R_z(\phi). \tag{C.3.14}$$

Finally, the complete circuit for the unambiguous discriminator in terms of C-Not gates and single qubit rotations is shown in Fig. C.3. In Fig. C.3, $u_A = WV_A$ and

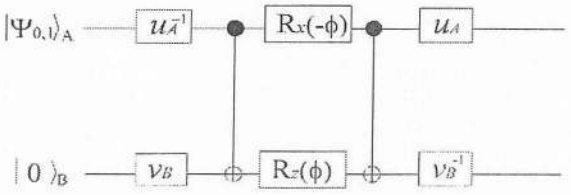

Figure C.3: Complete quantum circuit for the unambiguous discriminator in terms of C-NOT gates and single qubit rotations.

$v_B = U_B W$.

Usually V_A is known as the $\pi/8$ gate (denoted T) [89]. Thus $V_A = T = e^{i\pi/8}R_z(\pi/4)$, and $U_A = T^\dagger = e^{-i\pi/8}R_z(-\pi/4)$. On the other hand, U_B and V_B can also be represented as rotations, $V_B = e^{-i\pi/8}(e^{i\pi/2}R_x(\pi))R_z(\pi/4)$,$U_B = e^{i\pi/8}(e^{i\pi/2}R_x(\pi))R_z(\pi/4)$. Therefore, it is possible to write every single qubit operation as a product of rotations:

$$\begin{aligned} u_A &= e^{i\frac{\pi}{8}}R_z(\frac{\pi}{4})R_x(\frac{-\pi}{2}), \\ v_B &= e^{i\frac{\pi}{8}}e^{i\frac{\pi}{2}}R_x(\pi)R_z(\frac{\pi}{4})R_x(\frac{-\pi}{2}) \end{aligned} \tag{C.3.15}$$

C.4 Feasibility of the circuit implementation using trapped ions

The circuit proposed in the previous section uses several gates, namely, C-Not gates and single qubit rotations. These gates can be implemented using, for example, trapped ions.

C. A COMPARATIVE STUDY OF THE OPTICAL AND CIRCUIT REPRESENTATION OF THE UNAMBIGUOUS QUANTUM STATE DISCRIMINATOR [67]

Single quantum bit operations for quantum information processing with trapped ions have been shown to be experimentally feasible, where individual addressing of single ions in a linear string with a laser beam is required [90]. Specifically, single-qubit rotations of trapped ion qubits have been implemented by turning laser beams on and off for a duration appropriate to achieve a certain rotation angle on the Bloch sphere. Arbitrary rotations on the Bloch sphere can be implemented by also controlling the relative phase of the laser beam, [91, 92]. Cirac and Zoller were the first to propose the use of trapped ions for quantum computing in 1995, a 2-qubit C-NOT gate, where a selected mode of motion is cooled to the ground state and the ground and first excited state of this mode are used as "bus-qubit". The spin qubit of an ion can be mapped onto the bus-qubit using a laser sharply focused onto that ion. In this way, a gate operation can be performed between the motional qubit and a second ion, thus effectively performing a C-NOT gate operation between the first and second ion.

On the other hand, Mølmer and Sørensen [93] also proposed a two-qubit gate, using two fields at different frequencies, which are non-resonant with the atomic transition, but when combined, produce a two-qubit transition. This scheme has certain advantages over the Cirac-Zoller scheme, namely, it consists of a one step process and most importantly, it does not require individual ion-laser addressing, or in other words, both ions are equally illuminated.

The experimental realization of the Cirac Zoller C-NOT gate was achieved by Schmidt *et al.* in 2003 [94]. In their experiment, two $^{40}Ca^{+}$ ions are held in a linear Paul trap and are individually addressed using focused laser beams; the qubits are represented by superpositions of two long-lived electronic states.

C.5 Linear optical implementation

We obtain a very different physical implementation of the discriminator when we employ techniques from linear optical quantum interferometry combined with a generalization of the so-called dual-rail representation of a qubit (Chuang and Yamamoto, Ref. [95, 96]). The generalized measurement can then be realized by utilizing linear optical elements and photodetectors, based on the proposal of [97].

In order to accommodate the three possible returns from the optimal state discriminating measurement, we need a Hilbert space with a minimum dimensionality of three. We will span this Hilbert space by a single photon entering one of the three input ports of the six-port interferometer in Fig. C.4, so that the appropriate interferometer could be implemented for any desired discrimination problem.

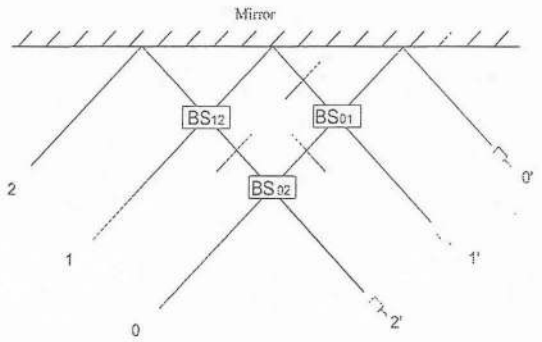

Figure C.4: A general six-port optical interferometer. Suitable beamsplitters are placed at each crossing of two optical rails and suitable phase shifters (denoted by short lines crossing the corresponding rails) are placed at each internal arm before the beamsplitters to realize any desired 3×3 unitary transformation on the input states. To realize the optimal unambiguous discrimination of two known quantum states, i.e. to realize U_{AB} of Eq. (C.5.3), only two beamsplitters are needed, BS_{02} and BS_{01}, with their parameters given in Eqs. (C.5.4) and (C.5.5). A detection in rail 2' corresponds to an inconclusive result and a detection in rails 0' (or 1') corresponds to state $|\psi_0\rangle$ (or $|\psi_1\rangle$).

C. A COMPARATIVE STUDY OF THE OPTICAL AND CIRCUIT REPRESENTATION OF THE UNAMBIGUOUS QUANTUM STATE DISCRIMINATOR [67]

The basis states of this interferometer, the so-called rails, are given by

$$\begin{aligned} |0\rangle_{AB} &= |100\rangle, \\ |1\rangle_{AB} &= |010\rangle, \\ |2\rangle_{AB} &= |001\rangle. \end{aligned} \tag{C.5.1}$$

Here a 1 in the first position refers to one photon entering the first port, while zero in the second and third position refers to vacuum entering the corresponding port, and similarly for the other states. With this representation the states to be discriminated, as given in Eqs. (C.2.1) and (C.2.2), can be expressed in terms of a single photon split between the first and second input ports. Thus, the first two ports represent the system, while the third port is always empty at the input and it serves as the ancilla. The subscript AB refers to the basis of the joint system-ancilla Hilbert space which is three-dimensional.

For the purposes of optical interferometry, we slightly modify the Peres choice to define a unitary operator U_{AB}, entangling the system with the ancilla. It yields the following final state, when applied to $|\Psi_i\rangle_{AB}^{in}$ (i=0,1),

$$\begin{aligned} |\psi_i\rangle_{AB}^{out} &= U_{AB}|\psi_i\rangle_{AB}^{in} \\ &= \sqrt{2-2a}|i\rangle_{AB} + \sqrt{2a-1}|2\rangle_{AB}, \end{aligned} \tag{C.5.2}$$

where $|i\rangle_{AB}$ (i=0,1) were defined in Eq. (C.5.1).

Any $N \times N$-dimensional discrete unitary operation can be implemented by an appropriate multi-path optical interferometer, using beamsplitters and phase shifters only, as was shown in general in [98] and [99]. Based on this general result an all optical implementation was proposed in the particular context of state discrimination in [97]. We start by reviewing this proposal and address the actual determination and implementation of the unitary operator U_{AB}.

In the basis given by Eq. (C.5.1) we have

$$U_{AB} = \begin{pmatrix} \sqrt{\frac{1-a}{2a}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{2a-1}{2a}} \\ \sqrt{\frac{1-a}{2a}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{2a-1}{2a}} \\ \sqrt{\frac{2a-1}{a}} & 0 & -\sqrt{\frac{1-a}{a}} \end{pmatrix}. \tag{C.5.3}$$

The key point in the implementation of this unitary is the observation that it can be written as $U_{01}U_{02}$, where

$$U_{02} = \begin{pmatrix} \sqrt{\frac{1-a}{a}} & 0 & \sqrt{\frac{2a-1}{a}} \\ 0 & 1 & 0 \\ \sqrt{\frac{2a-1}{a}} & 0 & -\sqrt{\frac{1-a}{a}} \end{pmatrix}, \tag{C.5.4}$$

and

$$U_{01} = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{C.5.5}$$

Both of these unitaries correspond to beamsplitters. U_{02} is a beamsplitter placed at the intersection of beams 0 and 2 (BS_{02} in Fig. C.4) with a transmission coefficient of $t = (\frac{1-a}{a})^{1/2} = \tan\theta$ and reflection coefficient of $r = (\frac{2a-1}{a})^{1/2} = \sqrt{1-\tan^2\theta}$. Thus, BS_{02} is related to the parameters of the input states in a very simple way. U_{01} is simply a fifty-fifty a beamsplitter placed at the intersection of beams 0 and 1 (BS_{01} in Fig. C.4) with transmission and reflection coefficients of $t = r = 1/\sqrt{2}$. In fact, U_{02} is essentially identical to the unitary given in Eq. (C.2.5) (after discarding the unused degree of freedom) and accomplishes the exact same task. The coefficient of the $|0\rangle$ state in both input states is the same, $\sqrt{a}$. By transmitting only a portion of $\sqrt{\frac{1-a}{a}}$ of the $|0\rangle$ state, the components of the two inputs that remain in the original system Hilbert space become orthogonal and transform into the $|\pm\rangle$ states, so they can, in principle be discriminated by standard von Neumann measurements. The role of the second beamsplitter is to transform the $|\pm\rangle$ states into the $|0\rangle$ and $|1\rangle$ states, respectively, so they can be discriminated by projective measurements in the

computational basis. The third beamsplitter, BS_{12}, and the phase shifters are not needed in this implementation.

Now that we have seen how one can implement the above unitary with the six-port interferometer, we discuss how this implementation works. We can place single-photon detectors in each of the output ports of the six-port interferometer. If the detector at output port $0'$ clicks it unambiguously identifies the input as $|\Psi_0\rangle$, if the one at $1'$ clicks it identifies the input as $|\Psi_1\rangle$. The probability of the successful discrimination is $2(1-a) \equiv 1-\cos(2\theta)$, same as before. If the detector at $2'$ clicks the result is inconclusive. The probability of the inconclusive outcome is $2a-1 \equiv \cos(2\theta)$, again same as before, so this is an implementation of the optimal POVM.

After the explanation of how this implementation performs, three remarks are in order here. First, there is a remarkable simplicity inherent in this implementation as it requires only two beamsplitters. Second, the implementation requires one extra dimension only, not two as the previous implementation, so it works in 3D. This could be an advantage (although it should be mentioned that qubit ancillas are very cheap and readily available on demand). The fact that this particular implementation requires one extra dimension only originates in the fact that optical elements can address individual degrees of freedom of the ancilla, of which only one is used. The third advantage of this particular implementation is that the final measurement is carried out in the computational basis and not in the $|\pm\rangle$ basis which is conceptually simpler and, in our rail representation, it corresponds to the detection of a single photon in the appropriate rail.

Thus, a single-photon representation of the input and output states, a multirail optical network (in our case a six-port) for performing the unitary transformation U_{SA}, Eq. (C.5.3), and photodetectors at each of output ports to carry out the required nonunitary transformation accomplish the required optimal discrimination task. For the case of two nonorthogonal states, $\{|\psi_0\rangle, |\psi_1\rangle\}$, living in a 2-D Hilbert space, a six-port optical interferometer can be constructed to perform transformations in the 3-D system plus ancilla space. All beam splitters in this interferometer were designed

to be optimal for given input states. By using beam splitters to send one photon into some linear superposition of the first two rails, we can generate arbitrary quantum states in this two-dimensional Hilbert space, represented as $|\psi\rangle_{in} = \sum_{j=0}^{1} c_j |j\rangle$, where $\sum_{j=0}^{1} |c_j|^2 = 1$, and $|j\rangle$ is the j^{th} optical rail state. Note that the third rail, which acts as the ancilla, never contains a photon at the input. The interferometer is designed to perform the unitary operation U_{AB} which optimizes state discrimination. It maps the input states $|\psi_i\rangle$ into output states given by Eq. (C.5.2). A photon in mode $2'$ now indicates an inconclusive result. On the other hand, a photon in rail $0'$ or $1'$ unambiguously indicates that the initial state was $|\psi_0\rangle$ or $|\psi_1\rangle$, respectively.

C.6 Discussion

Following Peres [76] proposal for an unambiguous quantum discriminator of known states, we derived a four dimensional unitary transformation and describe step by step an algorithm that converts the unitary into a quantum circuit involving only standard one- and two-qubit gates. We have shown how one can implement the discriminator in terms of trapped ions. Both the required C-NOT and single qubit rotation gates have been recently implemented, using ions in a linear trap, with the qubits corresponding to the atomic sub-levels and the bus, in the case of the two qubit gate, corresponding to a quantized vibrational motion state of the ions [94, 91]. Furthermore, we have also discussed a linear optical implementation, using techniques of single-photon interferometry, that gives an alternative physical realization of the optimal discriminator. The circuit implementation uses readily available, off-the shelf elements only but the algorithm required to convert a given unitary into a quantum circuit can be quite involved and, even for a relatively simple task, the number of necessary elements can be quite large. The optical implementation, on the other hand, is conceptually much simpler for simple tasks and usually requires a few beamsplitters and phase shifters only. In fact, until now all experiments in this area have been based on some variant of the optical implementation [100]. This simplicity, however, can be quite

misleading. The complexity of the multiport interferometer increases exponentially with the size of the problem and alignment of the beams as well as the number of optical elements make this implementation impractical for larger systems, whereas the complexity of the circuit implementation, being based on standard elements, will increase only polynomially with the size of the problem and becomes preferable for larger systems.

C.7 Proof of Lemma 1 in Ref. [80]

In order to understand the procedure used to choose the elements in the decomposition of U (C.3.1), we use the constructive proof of the lemma 1 in [80]. According to that lemma, it is always possible to write

$$O_A \otimes O_B e^{i\zeta}|\varphi_k\rangle = |\Phi_k\rangle, \tag{C.7.1}$$

where $\{|\varphi_k\rangle\}$ is any maximally entangled basis, ζ_k are some appropriate phases and O_A and O_B local unitaries.

For the proof of (C.7.1) let us consider some properties of the concurrence.

i) A state $|\phi\rangle$ written in the magic basis, is maximally entangled if and only if its coefficients are real, except for a global phase.

ii) A state $|\phi\rangle$ written in the magic basis, is completely disentangled, i.e. a product state, if and only if the sum of the squares of its expansion coefficients is zero.

According to these two properties if $|\phi\rangle$ and $|\phi^\perp\rangle$ are real in the magic basis (and therefore they are maximally entangled), then the state $|\phi\rangle \pm i|\phi^\perp\rangle$ is a product state.

Thus, we can always write $|\varphi_k\rangle = e^{\gamma_k}|\overline{\varphi}_k\rangle$, where $|\overline{\varphi}_k\rangle$ is real in the magic basis. Likewise we can consider two different states $|\overline{\varphi}_k\rangle$ and $|\overline{\varphi}_l\rangle$, then the combination $\frac{1}{\sqrt{2}}(|\overline{\varphi}_k\rangle + i|\overline{\varphi}_l\rangle) = |e, f\rangle$ and $\frac{1}{\sqrt{2}}(|\overline{\varphi}_k\rangle - i|\overline{\varphi}_l\rangle) = |e^\perp, f^\perp\rangle$ are product states. Thus we

write

$$|\overline{\varphi}_1\rangle = \frac{1}{\sqrt{2}}(|e,f\rangle + |e^\perp,f^\perp\rangle),$$
$$|\overline{\varphi}_2\rangle = \frac{-i}{\sqrt{2}}(|e,f\rangle - |e^\perp,f^\perp\rangle). \tag{C.7.2}$$

Following the above arguments, $|\overline{\varphi}_{3,4}\rangle$ can be written as

$$|\overline{\varphi}_3\rangle = \frac{-i}{\sqrt{2}}(e^{i\delta}|e,f^\perp\rangle + e^{-i\delta}|e^\perp,f\rangle),$$
$$|\overline{\varphi}_4\rangle = \pm\frac{1}{\sqrt{2}}(e^{i\delta}|e,f^\perp\rangle - e^{-i\delta}|e^\perp,f\rangle), \tag{C.7.3}$$

for some δ. Finally, by choosing

$$O_A = |0\rangle\langle e| + |1\rangle\langle e^\perp|e^{i\delta}, \tag{C.7.4}$$
$$O_B = |0\rangle\langle f| + |1\rangle\langle f^\perp|e^{-i\delta}, \tag{C.7.5}$$

and the phases ζ_k appropriately, we obtain eq. (C.7.1)

C.8 The decomposition of U

In this appendix, we show the derivation of the results following the procedure described in Section III.

a) In the first step we obtain U^TU in the magic basis

$$U^TU = \frac{1}{a}\begin{pmatrix} 2-3a & 2i\sqrt{2a-1}\sqrt{1-a} & 0 & 0 \\ 2i\sqrt{2a-1}\sqrt{1-a} & 2-3a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \tag{C.8.1}$$

and its eigensystem. The eigenvalues, $e^{2i\epsilon_k}$, are given by

$$e^{2i\epsilon_k} = \{1, 1, (\frac{\sqrt{1-a} \pm i\sqrt{2a-1}}{\sqrt{a}})^2\}. \tag{C.8.2}$$

C. A COMPARATIVE STUDY OF THE OPTICAL AND CIRCUIT REPRESENTATION OF THE UNAMBIGUOUS QUANTUM STATE DISCRIMINATOR [67]

The eigenstates $\{|\Psi_k\rangle\}$ are

$$\begin{aligned}
|\Psi_1\rangle &= |\Phi_3\rangle = \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\Psi_2\rangle &= |\Phi_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\
|\Psi_3\rangle &= \frac{1}{\sqrt{2}}(|\Phi_1\rangle + |\Phi_2\rangle) \\
&= \frac{1}{\sqrt{2}}(\frac{1+i}{\sqrt{2}}|00\rangle + \frac{1-i}{\sqrt{2}}|11\rangle), \\
|\Psi_4\rangle &= \frac{1}{\sqrt{2}}(|\Phi_1\rangle - |\Phi_2\rangle) \\
&= \frac{-i}{\sqrt{2}}(\frac{1+i}{\sqrt{2}}|00\rangle - \frac{1-i}{\sqrt{2}}|11\rangle).
\end{aligned} \tag{C.8.3}$$

This set of states is a maximally entangled basis.

ii) Then, we choose V_A, V_B and the phases ξ_k by lemma 1. Since $|\Psi_k\rangle$ is real in the magic basis, we can find by comparison of (C.7.2,C.7.3) with (C.8.3): $e^{i\delta} = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$, and $|e\rangle = |0\rangle, |f\rangle = |1\rangle$, therefore

$$V_A = |0\rangle\langle 0| + e^{i\frac{\pi}{4}}|1\rangle\langle 1|, \tag{C.8.4}$$

$$V_B = |0\rangle\langle 1| + e^{-i\frac{\pi}{4}}|1\rangle\langle 0|, \tag{C.8.5}$$

and the phases ξ_k are easily found to be $\left\{e^{i\xi_1} = -i, e^{i\xi_2} = i, e^{i\xi_3} = i, e^{i\xi_4} = -i\right\}$.

iii) In the next step we calculate a second maximally entangled basis $\{|\widetilde{\Psi}\rangle\}$. It is easy to show that in our case $|\widetilde{\Psi}_k\rangle = |\Psi_k\rangle$.

iv) We apply the operator U given by Eq. (C.3.1) to $|\Psi_k\rangle$, getting

$$\begin{aligned}
U|\Psi\rangle &= (U_A \otimes U_B)U_d(V_A \otimes V_B)|\Psi_k\rangle, \\
e^{i\epsilon_k}|\widetilde{\Psi}_k\rangle &= (U_A \otimes U_B)(e^{-i\lambda_k}|\Phi_k\rangle\langle\Phi_k|) \\
&\quad (V_A \otimes V_B)|\Psi_k\rangle, \\
e^{i\epsilon_k}|\Psi\rangle &= (U_A \otimes U_B)e^{-i\lambda_k}|\Phi_k\rangle \\
&\quad \langle\Phi_k|e^{-i\xi_k}|\Phi_k\rangle, \\
e^{i\epsilon_k}|\Psi\rangle &= (U_A \otimes U_B)e^{-i\lambda_k}e^{-i\xi_k}|\Phi_k\rangle,
\end{aligned} \tag{C.8.6}$$

where in the last three steps we used the equations (C.3.5), (C.3.7) and (C.3.8). The above equation can also be written as

$$(U_A^\dagger \otimes U_B^\dagger) e^{i\lambda_k} e^{i\xi_k} e^{i\epsilon_k} |\Psi\rangle = |\Phi_k\rangle. \tag{C.8.7}$$

Now, similarly to step two, we can choose the operators U_A, U_B, and the phases λ_k, getting

$$U_A = |0\rangle\langle 0| + e^{-i\frac{\pi}{4}} |1\rangle\langle 1|, \tag{C.8.8}$$

$$U_B = e^{i\frac{\pi}{4}} |0\rangle\langle 1| + |1\rangle\langle 0|, \tag{C.8.9}$$

and the phases λ_k, which are given in Eqs. (C.3.11).

BIBLIOGRAPHY

[1] Robert Gilmore, *Alice in Quantumland. An allegory of quantum physics*, edited by Springer-Verlag, New York,(1995).

[2] A.Einstein, B.Podolosky y N.Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys.Rev, **47**, 777 (1935).

[3] J.S.Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics **1**, 195-200 (1964).

[4] A.Aspect, J.Dalibard, G.Roger, Phys.Rev.Lett, *Experimental Test of Bell's Inequalities Using Time- Varying Analyzers*, **49**, 1804 (1982).

[5] M.Horodecki, P.Horodecki, R.Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys.Lett. A, **223**, 1, (1996).

[6] C.H.Bennett, G.Brassard, C.Crepeau, R.Jozsa, A.Peres, W.K.Wootters, *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*,Phys.Rev.Lett, **70,** 1895 (1993).

[7] A.K.Ekert, *Quantum cryptography based on Bell's theorem*, Phys.Rev.Lett, **67**, 661 (1991); D.Deutsch, A.K.Ekert,R.Jozsa, C.Macchiavello, S.Popescu, A.Sanpera, Phys.Rev.Lett, *Quantum Privacy Amplification and the Security of Quantum Cryptography over Noisy Channels*, **77**, 2818 (1996).

[8] C.H.Bennett, S.J.Wiesner, *Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states*, Phys.Rev.Lett, **69**, 2881 (1992).

[9] J.H.Eberly, *Schmidt Analysis of Pure-State Entanglement*,quant-ph/0508019.

[10] S.J.Freedman, J.F.Clauser, *Experimental Test of Local Hidden-Variable Theories*, Phys.Rev.Lett, **28**, 938 (1972).

[11] J.H.Eberly, *Bell Inequalities in Quantum Mechanics*, Am.J.Phys,**70**, 276 (2002).

[12] R.Grobe, K.Rzazewski, J.H.Eberly, *Measure of electron-electron correlation in atomic physics*,J.Phys.B, **27**, L503 (1994).

[13] D.Bruss, *Characterizing entanglement*, J.Math.Phys. **43**, 4237 (2002).

[14] R.Werner, *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys.Rev.A, **40**, 4277(1989).

[15] A.Peres, *Separability Criterion for Density Matrices*, Phys.Rev.Lett, **77**, 1413 (1996).

[16] M.Horodecki, P.Horodecki and R.Horodecki *Separability of mixed states: necessary and sufficient conditions*, Phys.Lett. A, **223**, 1, (1996).

[17] P.Horodecki, *Separability criterion and inseparable mixed states with positive partial transposition*, Phys.Lett. A, **232**, 333, (1997).

[18] A.Sen, U.Sen, M.Lewestein, A.Sampera, *The separability versus entanglement problem*, quant-phys/0508032.

[19] C.Bennett, D.DiVicenzo, J.Smolin, W.Wootters, *Mixed-state entanglement and quantum error correction*, Phys.Rev.A, **54**, 3824 (1996).

[20] S.Hill, W.Wootters, *Entanglement of a Pair of Quantum Bits*, Phys.Rev.Lett, **78**, 5022 (1997).

[21] W.Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*, Phys.Rev.Lett, **80**, 2245 (1998).

[22] W.H.Zurek, *Decoherence and the transition from quantum to classical*, Physics Today, **44**, (10), 36-44, (1991).

[23] W.H.Zurek, *Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?*, Phys.Rev.D, **24**, 1516 (1981).

[24] W.H.Zurek, *Environment-induced superselection rules*, Phys.Rev.D, **26**, 1862 (1982).

[25] Niels Bohr, Nature, **121**, 580 (1928).

[26] John Von Neumann, *Mathematische Grundlagen der Quanten mechanik*, Springer, Berlin (1932).

[27] E. Joos, H.D. Zeh, C. Kiefer, D.Giulini, J. Kupsch, I.-O.Stamatescu, *Decoherence and the Appearance of Classical World in Quantum Theory*. Springer, Berlin (1997).

[28] D.A.Lidar, I.L.Chuang, K.B.Whaley, *Decoherence-Free Subspaces for Quantum Computation* ,Phys.Rev.Lett., **81**, 2594 (1998).

[29] W.H.Zurek, *Decoherence, einselection, and the quantum origins of the classical*, Rev. Mod. Phys. **75**, 715 (2003).

[30] Ting Yu and J.H.Eberly, *Finite-Time Disentanglement Via Spontaneous Emission*, Phys.Rev.Lett. **93**,140404 (2004).

[31] L.Diósi, *Progressive Decoherence and Total Environmental Disentanglement*, Lect. Notes Phys. 622, 157 (2003).

[32] P.Marek, J.Lee, M.S.Kim, *Vacuum as a less hostile environment to entanglement*, Phys.Rev.A, **77**, 032302 (2008).

[33] Y.X. Gong, Yong-Sheng Zhang, Yu-Li Dong, Xiao-Ling Niu, Yun-Feng Huang, Guan-Can Guo, *Dependence of the decoherence of polarization states in phase-damping channels on the frequency spectrum envelope of photons*, Phys.Rev.A. **78**, 042103 (2008).

[34] D.Tolkunov, V.Privman, P.K.Aravind, *Decoherence of a measure of entanglement*, Phys.Rev.A. **71**, 060308(R) (2005).

[35] Juan Pablo Paz and Augusto J. Roncaglia, *Dynamical phases for the evolution of the entanglement between two oscillators coupled to the same environment*, Phys.Rev.A. **79**,032102 (2009).

[36] Cecilia Cormick and Juan Pablo Paz, *Decoherence of Bell states by local interactions with a dynamic spin environment* , Phys.Rev.A. **78**, 012357 (2008).

[37] T.Gorin, C.Pineda and T.H.Seligman, *Decoherence of an n-Qubit Quantum Memory*, Phys.Rev.Lett. **99**,240405 (2007).

[38] C.Pineda, T. Gorin, and T. H. Seligman, *Decoherence of two-qubit systems: a random matrix description*, New J. Phys. **9,** 106 (2007).

[39] M.Ikram, Fu-li Li and M.Zubairy, *Disentanglement in a two-qubit system subjected to dissipation environments* , Phys.Rev.A. **75**, 062336 (2007).

[40] Kuan-Liang Liu and Hsi-Sheng Goan *et.al*, *Non-Markovian entanglement dynamics of quantum continuous variable systems in thermal environments*, Phys.Rev.A. **76**, 022312 (2007).

[41] Asma Al-Qasimi and Daniel F. V. James, *Sudden death of entanglement at finite temperature*, Phys.Rev.A. **77**,012117 (2008).

[42] Maritza Hernandez and Miguel Orszag, *Decoherence and disentanglement for two qubits in a common squeezed reservoir* , Phys.Rev.A. **78**, 042114 (2008).

[43] Z.Ficek and R.Tanás *Dark periods and revivals of entanglement in a two qubit system*, Phys.Rev.A. **74**,024304 (2006).

[44] M.P.Almeida, F.de Melo, M.Hor-Meyll, A.Salles, S.P.Walborn, P.H.Souto-Ribeiro, L.Davidovich, *Environment induced death of entanglemen*t, Science, **316**, 579 (2007); A.Salles,F.de Melo, M.P.Almeida, M.Hor-Meyll, S.P.Walborn,

P.H.Souto-Ribeiro, L.Davidovich, *Experimental investigation of the dynamics of entanglement:sudden death, complementarity and continuous monitoring of the environment.* Phys.Rev.A. **78**, 022322 (2008).

[45] D.Braun, *Creation of Entanglement by Interaction with a Common Heat Bath*, Phys.Rev.Lett. **89**, 277901; Z.Ficek and R.Tanas, *Entanglement induced by spontaneous emission in spatially extended two-atom systems* J.Mod. Opt. **50**, 2765 (2003); F.Benatti, R.Floreanini and M.Piani, *Environment Induced Entanglement in Markovian Dissipative Dynamics* Phys.Rev.Lett. **91**, 070402 (2003).

[46] Z.Ficek and R.Tanás, *Delayed sudden birth of entanglement*, Phys.Rev.A. **77**, 054301 (2008).

[47] B.Bellomo, R. Lo Franco, G.Compagno, *Non-Markovian Effects on the Dynamics of Entanglement*, Phys.Rev.Lett, **99**, 160502 (2007)

[48] S.Maniscalco, S.Olivares, M.G.A.Paris, *Entanglement Oscillations in non-Markovian quantum channels*, Phys.Rev.A, **75**, 062119 (2007); See also the related work: L.Mazzola, S.Maniscalco, J.Piilo, K-A. Suominen, B.M.Garraway, *Sudden death and sudden birth of entanglement in common structured reservoirs*, Phys.Rev.A, **79**, 042302 (2009); B.Bellomo, R.Lo Franco, S.Maniscalco, G.Compagno, *Entanglement trapping in structured environments*, Phys.Rev.A, **78**, 060302 (2008), and for multipartite systems: C.E.Lopez, G.Romero, F.Lastra, E.Solano, J.C.Retamal, *Sudden birth versus sudden death of Entanglement in Multipartite systems*, Phys.Rev.Lett, 101, 080503 (2008)

[49] M.Franca-Santos, P.Milman, L.Davidovich, N.Zagury, *Direct Measurement of finite-time disentanglement induced by a reservoir*, Phys.Rev.A, **73**, 040305 (2006)

[50] T.Yu, J.H.Eberly, *Evolution from entanglement to decoherenceof standard bipartite mixed states*, arxiv:quant-ph/0503089(2006)

[51] F.Lastra, S.Wallentowitz, M.Orszag, M.Hernandez, *Quantum recoil effects in finite-time disentanglement of two distinguishable atoms*, J.Phys.B, At.Mol.Opt.Phys, **42**, 065504(2009)

[52] D.Mundarain and M. Orszag, *Decoherence-free subspace and entanglement by interaction with a common squeezed bath*, Phys.Rev.A **75**, 040303(R) (2007).

[53] D.Mundarain and M. Orszag, J.Stephany, *Total quantum Zeno effect and intelligent states for a two-level system in a squeezed bath*, Phys.Rev.A **74**, 052107 (2006).

[54] Z.Ficek, R.Tanas, *Dark periods and revivals of entanglement in a two qubit system,* Phys.Rev.A, **74**, 024304 (2006).

[55] R.Tanas, Z.Ficek, *Stationary two-atom entanglement induced by non-classical two-photon correlations*, J.Opt.B: Quantum.Semiclass, **6**, S610 (2004).

[56] T.Yu, J.H.Eberly, *Negative entanglement measure and what it implies*, e-print arXiv:quant-ph/0703083.

[57] A.Beige et al, *Entangling atoms and ions in dissipative environments*, J.Mod.Opt, **47**, 2583(2000).

[58] D.Braun, *Creation of entanglement by interaction with a common heath bath*, Phys.Rev.Lett, **89**, 277901 (2002).

[59] A.M.Basharov, *Entanglement of atomic states upon collective radiative decay*, JETP Lett, **75**, 123 (2002).

[60] L.Jacobczyk, *Entangling two qubits by dissipation*, J.Phys.A, **35**, 6383 (2002).

[61] Z,Fizek, R.Tanas, *Entanglement induced by spontaneous emission in spatially extended two atom systems*, J.Mod.Opt, **50**, 2765 (2003).

[62] F,Benatti, R.Floreanini, M.Piani, *Environment induced Entanglement in Markovian dissipative dynamics*, Phys.Rev. Lett, **91**, 070402 (2003).

[63] M.Paternostro, S.M.Tame, G.M.Palma, M.S.Kim, *Entanglement generation and protection by detuning modulation*, Phys.Rev.A, **74**, 052317 (2006).

[64] S.Natali, Z.Ficek, Phys.Rev.A, *Temporal and diffraction effects in entanglement creation in an optical cavity*, **75**, 042307 (2007).

[65] L.Derkacz, L.Jacobczyk, *Vacuum induced stationary entanglement in radiatively coupled three-level atoms*, e-print arXiv: 0710.5048.

[66] Z.Fizek, R.Tanas, Phys.Rev.A, *Delayed sudden birth of entanglement,* **77**, 054301 (2008).

[67] Maritza Hernández, Miguel Orszag and János Bergou, Journal of Modern Optics, *A comparative study of the optical and circuit representation of the unambiguous quantum state discriminator, Journal of Modern Optics.* **56**, iFirst. (2009).

[68] N.Gisin, G.Ribordy, W.Tittel, and H.Zbinden, *Rev.Mod.Phys.* **74**, 145 (2002).

[69] C.H.Bennett, *Phys.Rev.Lett* **68**, 3121 (1992).

[70] J.A.Bergou, U.Herzog, and M.Hillery, *Lecture Notes in Phys.* Springer: Berlin, 2004; **649**, p417 .

[71] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[72] A. S. Holevo, *Probabilistic and Quantum Aspects of Quantum Theory* (North-Holland, Amsterdam,1982).

[73] A. Chefles, *Phys. Lett. A* **239**, 339 (1998).

[74] I. D. Ivanovic, *Phys. Lett. A.* **123**, 257 (1987).

[75] D. Dieks, *Phys. Lett. A* **126**, 303 (1988).

[76] A. Peres, *Phys. Lett. A* **128**, 19 (1988).

[77] J. A. Bergou, M. Hillery, and Y. Sun, *J. Mod. Opt* **47**, 487 (2000).

[78] L. Roa, J. C. Retamal, and C. Saavedra, *Phys.Rev.A* **66**, 012103 (2003).

[79] T. Gopinath, R. Das, and A. Kumar, *Phys.Rev.A* **71**, 042307 (2005).

[80] B. Kraus and J. I. Cirac, *Phys.Rev.A* **63**, 062309 (2001).

[81] A. W. Knapp, *Lie Groups Beyond an Introduction, Progress in Mathematics*, vol. 140, *Birkhäuser*,1996.

[82] Yu. G. Makhlin, *QIP* **1**, 243 (2002).

[83] G. Vidal and C. M. Dawson, *Phys.Rev.A* **69**, 010301(R) (2004).

[84] V. V. Shende, I. L. Markov, and S. S. Bullock, *Phys.Rev.A.* **69**, 062321 (2004).

[85] Y. S. Zhang, M. Y. Ye, and G. C. Guo, *Phys.Rev.A.* **71**, 062331 (2005).

[86] N. Khaneja and S. J. Glaser, *J. Chem. Phys.* **267**, 11-23 (2001).

[87] M. Möttönen, J. J. Vartiainen, V. Bergholm, and M. M. Salomaa,*Phys.Rev.Lett.* **93**, 130502 (2004).

[88] V. V. Shende, S. S. Bullock, and I. L. Markov, *IEEE Trans. on CAD* **25**, 1000 (2006).

[89] M. A. Nielsen and I.L.Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, 2000).

[90] H. C. Nägerl, D. Leibfried, H. Rhode, G. Thalhammer, J. Eschner, F. Schmidt-Kaler, R. Blatt, *Phys.Rev.A* **60**, 145 (1999).

[91] D. J. Wineland, C. Monroe, W. M. Itano, D. Leibfried, B. E. King, and D. M. Meekhof, *J. Res. Natl. Inst. Stand. Technol.* **103**, 259 (1998).

[92] D. Leibfried, E. Knill, C. Ospelkaus, and D. J. Wineland, *Phys.Rev.A* **76**, 032324 (2007).

[93] A. Sørensen and K. Mølmer, *Phys.Rev.A* **62**, 022311 (2000).

[94] F. Schmidt-Kaler, H. Hafner, M. Riebe, S. Gulde, G. P. T. Lancaster, T. Deuschle, C. Becher. C. F. Roos, J. Eschner, and R. Blatt, *Nature* **422**, 408 (2003).

[95] I. L. Chuang and Y. Yamamoto, *Phys.Rev.A* **52**, 3489 (1995).

[96] I. L. Chuang and Y. Yamamoto, *Phys.Rev.Lett.* **76**, 4281 (1996).

[97] Y. Sun, M. Hillery, and J. A. Bergou, *Phys.Rev.A* **64**, 022311 (2001); J. A. Bergou, M. Hillery, and Y. Sun, J. Mod. Opt. **47**, 487 (2000).

[98] M. Reck and A. Zeilinger, *Phys.Rev.Lett.* **73**, 58 (1994).

[99] M. Zukowski, A. Zeilinger and M.A. Horne, *Phys.Rev.A* **55**, 2564 (1997).

[100] S. M. Barnett and S. Croke, *arXiv: quant-ph 0810.1970 (to appear in Advances in Optics and Photonics).* **2009**, 1, 238-278