Coherence Resonance in a Noise-Driven Excitable System

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We study the dynamics of the excitable Fitz Hugh–Nagumo system under external noisy driving. Noise activates the system producing a sequence of pulses. The coherence of these noise-induced oscillations is shown to be maximal for a certain noise amplitude. This new effect of coherence resonance is explained by different noise dependencies of the activation and the excursion times. A simple one-dimensional model based on the Langevin dynamics is proposed for the quantitative description of this phenomenon. [S0031-9007(97)02349-1]

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The response of dynamical systems to noise has attracted large attention recently. There are many examples demonstrating that noise can lead to more order in the dynamics. To be mentioned here are the effects of noise-induced order in chaotic dynamics [1], synchronization by external noise [2], and stochastic resonance [3–5]. Also, noise has been shown to play a stabilizing role in ensembles of coupled oscillators and maps [6]. Especially interesting is the phenomenon of stochastic resonance, which appears when a nonlinear system is simultaneously driven by noise and a periodic signal. At a certain noise amplitude the periodic response is maximal; this has been confirmed by numerous experimental studies (cf. [7,8]).

In this paper we study the effect of noise on the autonomous excitable oscillator—the famous Fitz Hugh–Nagumo system. We demonstrate that a characteristic correlation time of the noise-excited oscillations has a maximum for a certain noise amplitude, and present a theory of this effect. Contrary to the usual setup of stochastic resonance, no external periodic driving is assumed, so the coherence appears as a nonlinear response to purely noisy excitation. The phenomenon considered is also different from stochastic resonance without periodic force reported recently in Ref. [9], where the effect of noise on a limit cycle at a bifurcation point was studied.

The Fitz Hugh–Nagumo model is a simple but representative example of excitable systems that occur in different fields of application ranging from kinetics of chemical reactions and solid-state physics to biological processes [10]. Originally it was suggested for the description of nerve pulses [11]; it was also widely used for modeling of spiral waves in a two-dimensional excitable medium. Different aspects of the dynamics of this and similar excitable models in the presence of noise have been discussed in Refs. [12–16]. The equations of motion are

$$\varepsilon \frac{dx}{dt} = x - \frac{x^3}{3} - y\,, \tag{1}$$

$$\frac{dy}{dt} = x + a + D\xi(t)\,. \tag{2}$$

Here $\varepsilon \ll 1$ is a small parameter allowing one to separate all motions in the fast (only x changes) and slow ($y \approx x - x^3/3$) ones. The parameter a governs the character of solutions: For $|a| > 1$ the only attractor is a stable fixed point, and for $|a| < 1$ a limit cycle appears. This cycle consists of two pieces of slow motion connected with fast jumps. For $|a|$ slightly larger than one the system is excitable; i.e., small but finite deviations from the fixed point produce large pulses. Indeed, if the perturbation brings the system to the border of the slow branch on which the stable fixed point lies, the jump to another slow branch happens and the system returns to the stable fixed point only after a large excursion. This highly nonlinear response to perturbations makes the dynamics of the forced Fitz Hugh–Nagumo system nontrivial. Finally, the parameter D governs the amplitude of the noisy external force ξ which we assume to be Gaussian delta-correlated with zero mean: $\langle \xi(t)\xi(t')\rangle = \delta(t - t')$ [17].

We integrate system (1), (2) numerically using Euler's method [18] for the parameters $\varepsilon = 0.01$, $a = 1.05$, and different noise amplitudes. The results reported in Fig. 1 show that for both small and large noise amplitudes, the noise-excited oscillations appear to be rather irregular, while for moderate noise relatively coherent oscillations are observed. This phenomenon, which we call *coherence resonance,* resembles the well-known stochastic resonance [3–5]. The stochastic resonance appears if both periodic and noisy forces drive a nonlinear system, with the periodic response having a maximum at some noise amplitude. In our case there is, however, no periodic force (cf. [9,19]) and no discrete component appears in the spectrum, but at some noise amplitude the regularity of the process is, nevertheless, maximal.

To characterize this ordering quantitatively, we compute the normalized autocorrelation function

$$C(\tau) = \frac{\langle \tilde{y}(t)\tilde{y}(t + \tau)\rangle}{\langle \tilde{y}^2\rangle}, \qquad \tilde{y} = y - \langle y\rangle\,. \tag{3}$$

One can see from Fig. 2 that the correlations are indeed much more pronounced for the moderate noise. To

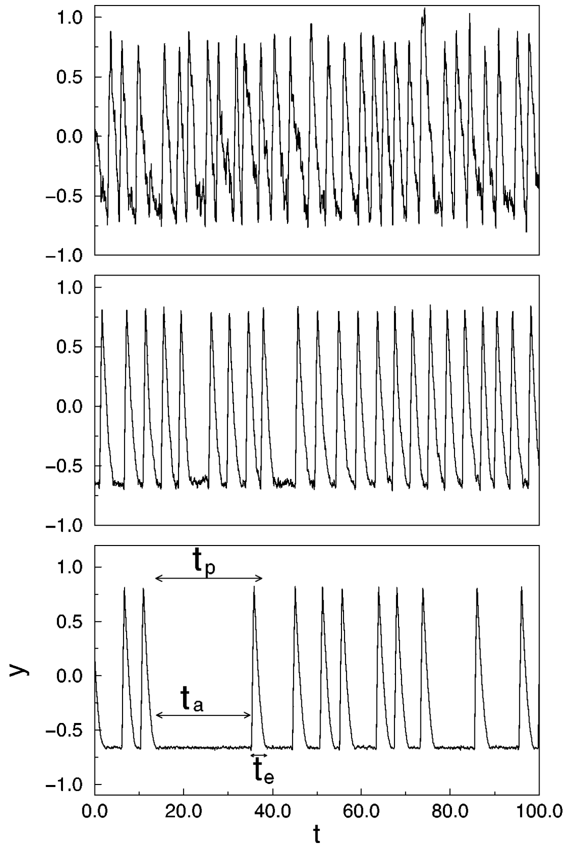

FIG. 1. The dynamics of the Fitz Hugh–Nagumo system [Eqs. (1), (2)] for $a = 1.05$, $\varepsilon = 0.01$, and different noise amplitudes: From bottom to top $D = 0.02$, $D = 0.07$, and $D = 0.25$. The mean durations of pulses are 7, 4, and 3.5, respectively. The activation and the excursion times for one pulse are depicted.

describe this effect with a single quantity, we calculate the characteristic correlation time as follows [20]:

$$\tau_c = \int_0^\infty C^2(t)\, dt\,. \tag{4}$$

The dependence of this quantity on the noise amplitude is presented in Fig. 3; it has a clear maximum at the noise amplitude $D_{\text{res}} \approx 0.06$. While the correlation time can be readily obtained numerically, for the convenience of the theoretical consideration we introduce another quantity (which can be interpreted, in the context of stochastic resonance terminology, as noise-to-signal ratio). Because the process Fig. 1 can be viewed as a sequence of pulses having durations t_p, we look at the normalized fluctuations of pulse durations

$$R_p = \frac{\sqrt{\text{Var}\,(t_p)}}{\langle t_p \rangle}\,. \tag{5}$$

This quantity, reported in Fig. 3, shows a minimum at D_{res}. Below we develop a theoretical approach to calculating R_p.

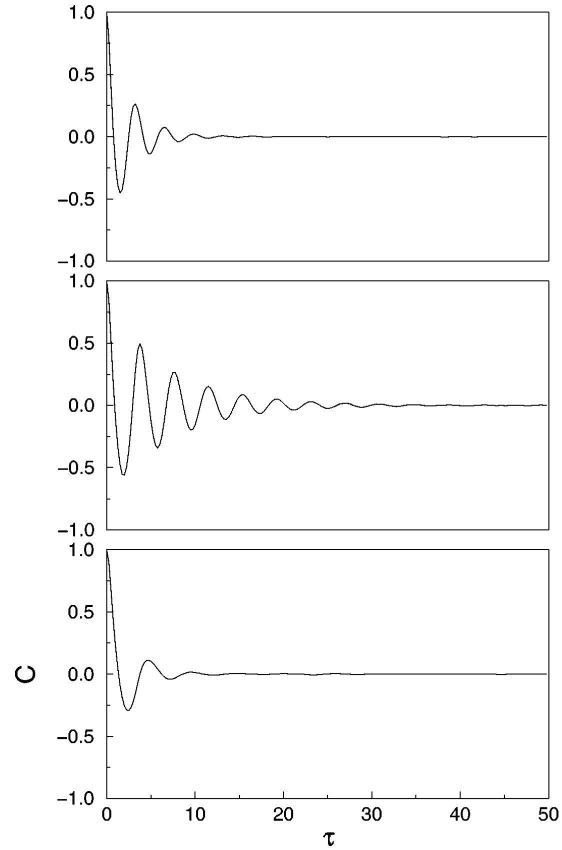

FIG. 2. The autocorrelation function of the regimes presented in Fig. 1.

Physically, the appearance of coherence resonance is deeply related to the excitable nature of the Fitz Hugh–Nagumo system. The system has two characteristic times: the activation time t_a and the excursion time t_e. The

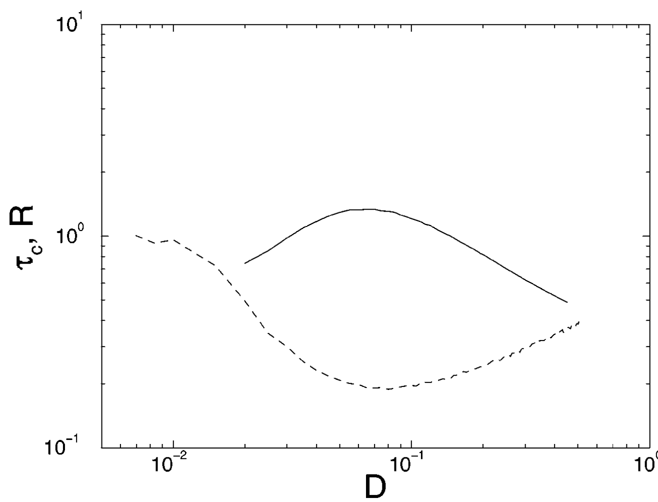

FIG. 3. Correlation time τ_c (solid line) and the noise-to-signal ratio R [Eq. (5), dashed line] vs noise amplitude for the Fitz Hugh–Nagumo system with $a = 1.05$, $\varepsilon = 0.01$.

activation time is the time needed to excite the system from the stable fixed point $x = -a,\ y = a^3/3 - a$; while the excursion time is the time needed to return from the excited state to the fixed point. The pulse duration t_p is the sum of these times $t_p = t_a + t_e$. The crucial point is that these times and their fluctuations have a different dependence on the noise amplitude. The activation time decreases rapidly with the noise amplitude according to the Kramers formula $\langle t_a \rangle \sim \exp(\text{const} \times D^{-2})$ [21,22]. It can be also shown that for small noise $\mathrm{Var}(t_a) \approx \langle t_a \rangle^2$ [23]. Thus for small noise, where $t_a \gg t_e$ and the period is dominated by the activation time $t_p \approx t_a$, the fluctuations of the pulse durations are relatively large: $R_p \approx R_a \approx 1$. For large noise the contribution of the activation time t_a to the period is negligible, here the excursion time dominates $t_p \approx t_e$. If the motion in the excited state is nearly uniform, $\langle t_e \rangle$ weakly depends on the noise amplitude, but its variance can be estimated as $\mathrm{Var}(t_e) \sim D^2 \langle t_e \rangle$ [24], so the fluctuations grow with the noise amplitude. In this regime $R_p \approx R_e \sim D\langle t_e \rangle^{-1/2}$. The coherence resonance, i.e., a minimum in the dependence $R(D)$, appears if the threshold of excitation is small and the excursion time is large. In this case the minimum corresponds to a sufficiently large noise amplitude so that $t_a \ll t_e$, but not very large so that fluctuations of the excursion time are small $R_e(D_{\mathrm{res}}) < 1$.

To make these arguments quantitative, we suggest a simple analytical model of the coherence resonance. Note first that due to the smallness of the parameter ε in the Fitz Hugh–Nagumo model the motion is restricted to the "nearly limit" cycle in the phase space, consisting of two lines of slow motion and two straight lines of fast motion. On each line of slow motion the variable x is a function of y. Thus along the lines of slow motion the dynamics can be represented with the one-dimensional Langevin equation

$$\frac{dy}{dt} = -\frac{dU}{dy} + D\xi(t) \tag{6}$$

with noisy term ξ and a nonlinear potential $U(y)$ having a single minimum (a stable fixed point). The fast motion can be modeled in this approach as a jump (reinjection) of the variable y, if the excitation threshold is arrived [25]. Thus we can consider Eq. (6) as defined on the half line $-\infty < y < 0$, with reinjection of y from the threshold $y = 0$ to the point $y = y_0$. The sequence of pulses is in this interpretation a sequence of walks from y_0 to 0, with reinjections. Because each walk is described by the Langevin equation (6), we can apply the method of the Fokker–Planck equation to find statistical characteristics of pulse durations t_p. These durations are nothing else but first passage times for the random process (1) starting at $y = y_0$, with the absorbing boundary $y = 0$. The equations for the moments of these times are well known [22,26]. The solutions for the first two moments have the form:

$$\langle t_p(y_0) \rangle = 2D^{-2} \int_{y_0}^{0} dv \int_{-\infty}^{v} du\, \exp\!\left(2\,\frac{U(v) - U(u)}{D^2}\right),$$

$$\langle t_p^2(y_0) \rangle = 4D^{-2} \int_{y_0}^{0} dv \int_{-\infty}^{v} du\, \langle t_p(u) \rangle \times \exp\!\left(2\,\frac{U(v) - U(u)}{D^2}\right).$$

Except for extremely simplified models, the resulting formulas are very tedious. We were able to get closed analytical results for a simple model of the phase motion with a piece-wise linear potential $U(y) = -Ay$ if $y < -1$ and $U(y) = A + B + By$ if $0 > y > -1$ (the minimum of the potential at $y = -1$ determines the position of the stable fixed point), although these formulas are still too cumbersome to be presented in this short Letter. From these analytic expressions we calculate the ratio R which characterizes the coherence of the oscillations, and plot it in Fig. 4. Two asymptotics in accordance with the qualitative arguments above are clearly seen: For small noise $R \approx 1$ what corresponds to the Poissonian statistics of the activation times for small noise; for large noise $R \sim D$. The sharpness of the coherence resonance depends on the model parameters A, B, y_0. In agreement with the qualitative consideration above, the minimum is deeper for larger excursion times (large values of $|y_0|$). We emphasize that the phase dynamics equation (6) provides a general description of the coherence resonance (with details of a particular system coming through the potential U and boundary conditions), provided the excited state is regular (nonchaotic); otherwise the one-dimensional description is not sufficient.

In conclusion, we have demonstrated that the dynamical regimes appearing in noise-driven excitable systems can be rather nontrivial. The coherence of noise-excited

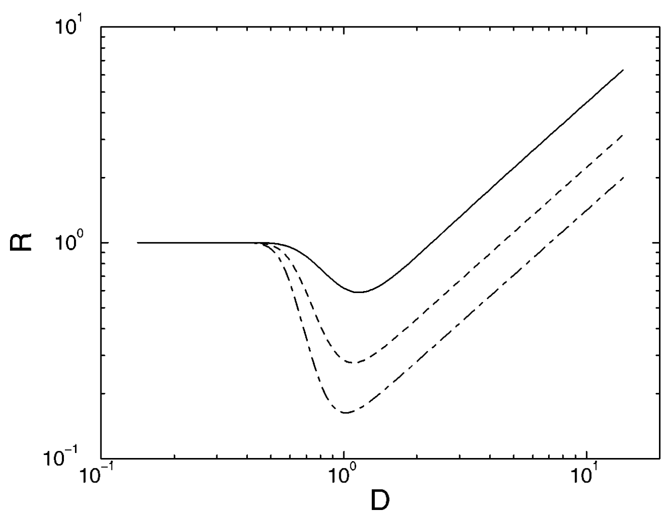

FIG. 4. The relative first passage time fluctuations vs noise amplitude in the one-dimensional phase dynamics model (described in text) for $A = B = 1$ and $y_0 = -5$ (solid line), $y_0 = -20$ (dashed line) and $y_0 = -50$ (dot-dashed line).

oscillations is maximal for a certain noise amplitude. This maximum is explained by the interplay between different statistical properties of the activation and excursion times. We have also proposed a simplified description of this phenomenon, based on the one-dimensional Langevin phase dynamics. The phenomenon described may be of particular importance in neurophysiology, where large ensembles of neurons may become ordered due to interaction with a noisy environment. We hope that experimental observation of this effect is possible with standard equipment used in studies of stochastic resonance [8].

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