Coherence resonance near the Hopf bifurcation in coupled chaotic oscillators

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We uncover a coherence resonance near the Hopf bifurcation from *chaos* in coupled chaotic oscillators. At the bifurcation, a nearly periodic rotating wave becomes stable as the state of synchronous chaos is destabilized. We find that noise can induce the bifurcation and, more strikingly, it can enhance the temporal regularity of the wave pattern in the coupled system. This resonant phenomenon is expected to be robust and physically observable.

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Coherence resonance has been a topic of recent interest. The phenomenon generally refers to resonant enhancement of the temporal regularity of a dynamical system by noise, which was first noticed by Sigeti and Horsthemke [1]. The work by Pikovsky and Kurths [2] laid a solid foundation for the phenomenon and since then, a plethora of theoretical works [3,4] and experimental works has appeared [5]. The common setting for investigating the coherence resonance is excitable systems [2], where the time trace of dynamical variables of physical interest consists of an infinite sequence of bursts occurring at random time intervals. At both small and large noise levels, the time series appear random in the sense that their Fourier spectra are broadband and apparently exhibit no pronounced peaks. At some intermediate noise levels, the bursting time series appears more regular, as characterized by the appearence of a finite set of peaks at certain frequencies. If one defines a measure, say the ratio of the height of the most pronounced peak in the Fourier spectrum to its width, to quantify the temporal regularity of the bursting time series, one finds that the measure tends to increase as the noise is strengthened and reach a maximum value at some optimal noise level. More recently, this resonant phenomenon has been studied for coupled chaotic systems [4].

In this paper, we report coherence resonance near the Hopf bifurcation from *chaos* in coupled chaotic systems which, to the best of our knowledge, has not been noticed before. In particular, it is known that when identical or nearly identical chaotic oscillators are coupled together in an asymmetric fashion, a rotating wave of relatively high frequency can appear [6], as the consequence of a Hopf bifurcation from the state of synchronous chaos [7–9]. To be concrete, consider the following system of N, nearest-neighbor coupled, identical chaotic oscillators:

$$\frac{d\mathbf{x}_i}{dt}=\mathbf{f}(\mathbf{x}_i)+(\varepsilon+r)\mathbf{C}\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)+(\varepsilon-r)\mathbf{C}\cdot(\mathbf{x}_{i-1}-\mathbf{x}_i), \tag{1}$$

where $i=1,\ldots,N$ is the index specifying the location of the ith oscillator in the space, $\mathbf{f}(\mathbf{x})$ is the vector field of an individual oscillator that generates a chaotic attractor, $\mathbf{C}$ is an $N\times N$ coupling matrix, ε is the nominal value of the coupling strength, r is the parameter characterizing the degree of asymmetry in the coupling, and the boundary condition is assumed to be periodic. An elementary Fourier analysis [8,10,11] indicates that Eq. (1) possesses N spatial eigenmodes with the following wave vectors: $2\pi k/N$, where $k=0,1,\ldots,N-1$, is the wave number, $k=0$ governs the dynamics in the synchronization manifold defined by $\mathbf{x}_1=\mathbf{x}_2=\cdots=\mathbf{x}_N$, and $k\neq 0$ correspond to the motions in the subspaces transverse to the synchronization manifold, which are modes in the space with the wavelength N/k. Assume that for $r=0$, the parameter ε is large enough so that the synchronous chaos is stable. When r is increased through a critical value so that the synchronization state becomes unstable, two conjugate spatial modes dominate the dynamics, triggering a rotating wave in the system [7,8]. The frequencies of the wave can be related to the generalized winding numbers associated with the corresponding transverse modes, which appear to be well defined for many known chaotic oscillators [7]. For $r\gg r_c$, the rotating wave is periodic and has been observed experimentally in coupled electronic circuits [6]. Subsequent works [7,8] indicate that for $r\gtrsim r_c$, an approximately periodic wave can still be observed with respect to the state of synchronous chaos. The onset of the periodic wave is the result of a Hopf bifurcation directly from synchronous chaos, where a pair of complex conjugate eigenvalues associated with the particular transverse spatial mode crosses the unit circle in opposite directions [7,8,12]. This bifurcation *from chaos* is an interesting high-dimensional phenomenon that is different from the commonly known phenomenon of Hopf bifurcation where a periodic motion is born from a steady state.

The general question to be addressed in this paper is: what is the effect of noise on the Hopf bifurcation from chaos? Our findings are the following. (1) In parameter regimes below the Hopf bifurcation where synchronous chaos is stable, noise can destabilize the synchronous chaotic motion, induce the bifurcation and, consequently, *stabilize* a rotating wave in the coupled chaotic system. The stable wave pattern is observable even for weak noise. More strikingly, the temporal regularity of the wave can be greatly enhanced as the noise level is increased, in a resonant manner. (2) In the parameter regime slightly above the Hopf bifurcation point where a chaotic rotating wave is already stable, noise can again enhance the regularity of the wave. These phenomena are robust in the sense that they persist under small perturbations, such as parameter mismatches among coupled oscillators. We expect our findings to be important in addressing role of noise in high-dimensional dynamical systems [13].

As a numerical example, we consider the following ring of coupled Lorenz oscillators [7–9]:

$$\frac{dx_i}{dt}=10(y_i-x_i)+D\xi_i(t),$$

$$\frac{dy_i}{dt}=\beta_i x_i-y_i-x_i z_i+(\varepsilon+r)(x_{i+1}-x_i)+(\varepsilon-r)(x_{i-1}-x_i),$$

$$\frac{dz_i}{dt}=x_i y_i-z_i \quad (i=1,2,\dots,N), \tag{2}$$

where $\beta_i=28$, D is the noise amplitude, $\xi_i(t)$'s ($i=1,\dots,N$) are independent Gaussian random variables of zero mean and unit variance, and periodic boundary condition is utilized. Each Lorenz oscillator for the set of standard parameter values as in Eqs. (2), when uncoupled, exhibits a chaotic attractor. For illustrative purpose, we fix $N=6$ and $\varepsilon=15.1$, and choose r to be the bifurcation parameter, as in Refs. [7,8]. For small values of r, the state of synchronous chaos is transversely stable. The Hopf bifurcation from chaos occurs at $r_c\approx 5.87$, at which the largest transverse Lyapunov exponent crosses zero and becomes positive for $r>r_c$. For the coupled system Eqs. (2), the wave number associated with the largest transverse Lyapunov exponent [7,8] is $k=1$. Thus, for $r\gtrsim r_c$, the synchronous chaotic state is transversely unstable and a rotating wave of wavelength $N=6$ ($k=1$) emerges [7].

The remarkable phenomenon is that noise can induce a similar rotating wave even for $r<r_c$ where the state of synchronous chaos would be transversely stable without noise. For instance, for $r=5.8\lesssim r_c$, a rotating wave is observed even when the noise amplitude is small. To visualize the wave pattern, it is necessary to focus on quantities that do not involve the chaotic dynamics in the synchronization manifold. The following set of derivation vectors is thus convenient [9]: $\Delta x_i(t)=x_i(t)-(1/N)\Sigma_{j=1}^{N}x_j(t)$, for $i=1,\dots,N$, on which our subsequent analysis will be based. Figure 1 shows, for $D=1.0$, the evolution of the wave pattern in this variable-difference space, where the abscissa denotes the spatial location of the oscillator, the ordinate is the time axis, and the gray scale is determined by the values of $\Delta x_i(t)$. Apparently, there is an approximate periodicity in time for all the oscillators, and the phase differences among the neighboring oscillators are roughly T/N: $\Delta x_{i+1}(t)\approx\Delta x_i(t-[1/N]T)$ ($i=1,\dots,N$), where $T\approx 0.5$ is approximately the period of $\Delta x(t)$. Simulations suggest that the wave pattern in Fig. 1 is not a transient phenomenon. Notice that if the state of synchronous chaos is stable, then there is no wave because $\Delta x_i(t)=0$ for all i. The wave pattern observed in Fig. 1 is thus induced by noise. The frequency of the wave, which is $f\approx 2.0$, is nothing but the generalized winding number associated with the $k=1$ transverse mode, which is given by [7]

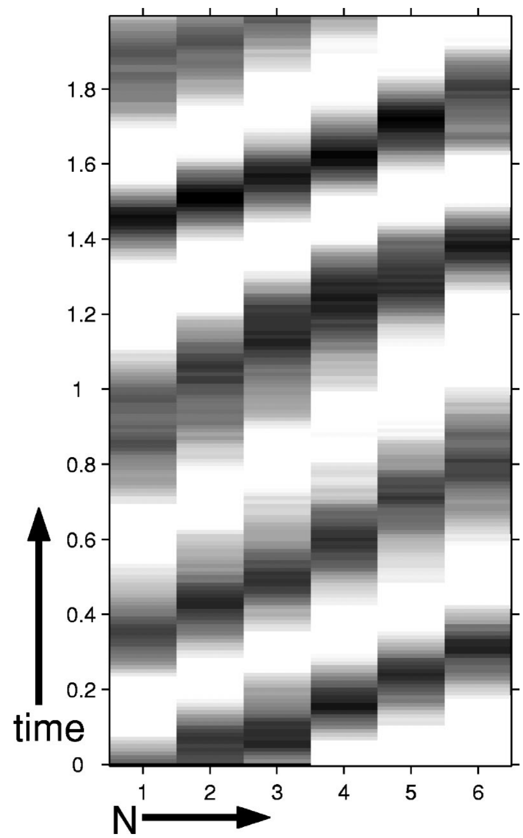

FIG. 1. Space-time plot of $\Delta x_i(t)$, $i=1,\dots,6$ for the rotating wave induced by noise for $r=5.8<r_c$. The noise amplitude is $D=1.0$. The period of the wave is approximately 0.5 (the frequency is $f\approx 2.0$).

$$\langle\omega\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T\dot{\theta}(t)dt, \tag{3}$$

where $\theta(t)$ is defined via $\tan\theta=\Delta y/\Delta x$, Δx and Δy are two (arbitrary) components of the infinitesimal tangent vector associated with the $k=1$ transverse Lyapunov exponent. For r in the vicinity of r_c, this frequency remains approximately constant, which is the reason why the wave induced by noise for $r<r_c$ appears similar in characteristics to that for $r>r_c$ in the absence of noise.

To better assess the influence of noise on the wave pattern, we show in Figs. 2(a–c) the power spectra of the temporal oscillations of the wave at the location of a specific oscillator, say $\Delta x_1(t)$, for $D=0.01$, $D=1.0$, and $D=50.0$, respectively. For small noise, the wave is weak in the sense that the peak at $f_p\approx 2.0$ is low, as in Fig. 2(a). The wave becomes increasingly strong as the noise level is raised, as exemplified by Fig. 2(b). If the noise amplitude is too large, the wave pattern is smeared out and the power spectrum becomes broadband, as in Fig. 2(c). These observations point to a resonant behavior: the approximately periodic wave be-

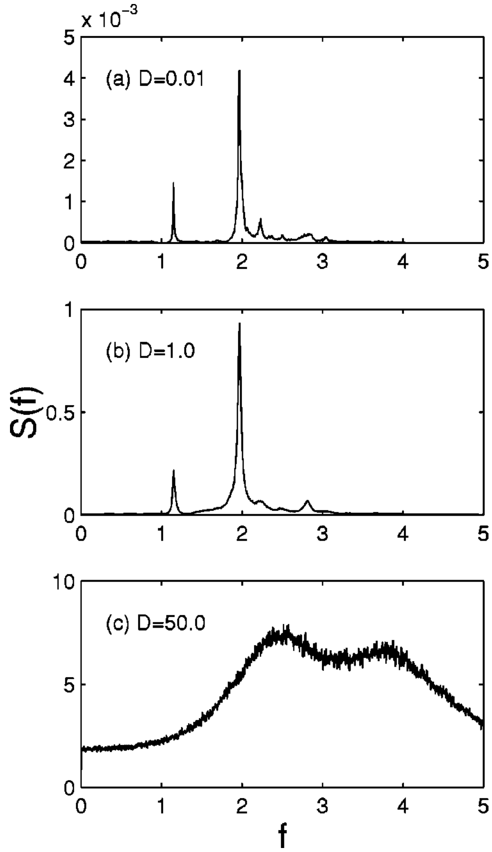

FIG. 2. For $r=5.8<r_c$, power spectra of $\Delta x_1(t)$ from the noise-induced wave: (a) $D=0.01$, (b) $D=1.0$, and (c) $D=50.0$.

comes more regular in temporal evolution as the noise amplitude is increased initially, but the regularity becomes deteriorated when the noise is large and strengthened further.

The temporal regularity of the noise-induced wave can be quantified by the following measure [14], defined with respect to the dominant spectral peak: $\beta_s = H f_p/\Delta f$, where H is the height of the spectral peak at f_p, and Δf is the half-width of the peak. The higher the peak and/or the narrower the peak, the more temporally regular the wave pattern. The

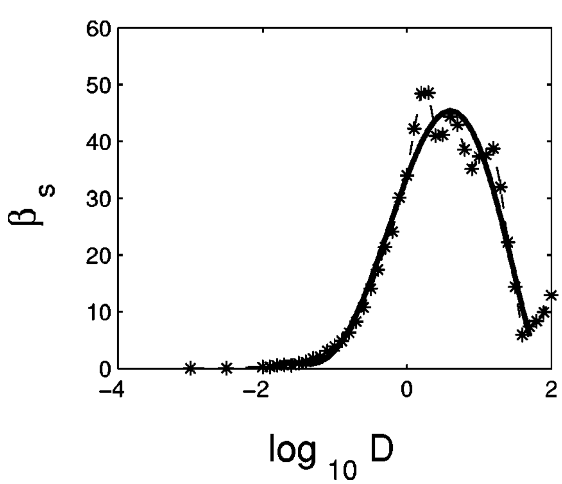

FIG. 3. The measure β_s versus the noise amplitude. The resonant behavior (coherence resonance) is apparent.

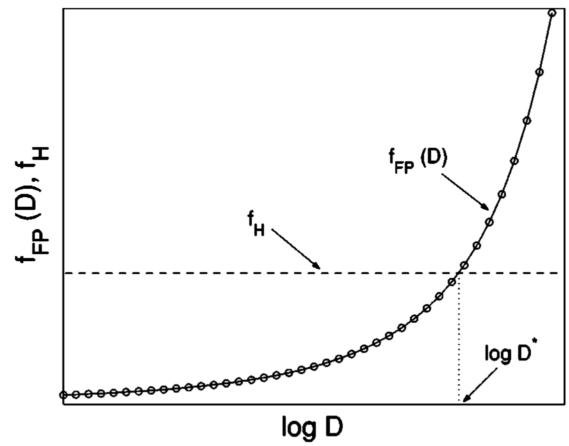

FIG. 4. Schematic illustration of the mechanism of coherence resonance of wave in coupled chaotic oscillators: the dashed horizontal line denotes the deterministic wave frequency that hardly changes with noise, and the solid curve indicates the general behavior of the first-passage frequency of the underlying stochastic process. Coherence resonance occurs because there can be a match between the two independent time scales at some optimal noise level D^*.

variation of the spectral peak at f_p with the noise amplitude in Fig. 2(a–c) suggests a resonant behavior in β_s: it is small for weak noise, increases with noise and reaches a maximum at some optimal noise level, and then decreases as the noise amplitude is increased further, as shown in Fig. 3 for $r=5.8$, where the measure β_s achieves its maximum at $\log_{10} D^* \approx 0.6$.

The characteristics of the wave versus the noise, as exemplified by Figs. 1–3, appear to be general for the model system Eqs. (2). For instance, a similar resonant behavior is observed for $r \gtrsim r_c$, where the nearly periodic wave is generated through the Hopf bifurcation in the absence of noise. In this case, the temporal periodicity of the wave can be enhanced by noise, although the wave itself is not induced by noise. We have also tested cases where the coupled Lorenz chaotic oscillators are nonidentical with small amounts of random parameter mismatch (for example, β_i's have been varied from 26 to 30). Although the notions of the synchronization manifold and transverse Lyapunov exponents no longer hold upon such a symmetry-breaking perturbation, the wave pattern and the resonant behavior under noise persist. Thus, coherence resonance with respect to the wave pattern appears to be a robust phenomenon in coupled chaotic oscillators.

We now present a physical theory for the observed resonance phenomenon. In order for a resonance to occur, it is necessary to have two independent and competing time scales. At least one time scale should depend on noise. Resonance occurs for a proper noise level when the two time scales match. In our problem of wave, one apparent scale is the average wave periodicity, which is a fundamental time scale of the coupled chaotic system determined by one of the generalized winding numbers. This time scale is thus deterministic and it hardly changes with noise. Let f_H be the frequency corresponding to this deterministic time scale. The second time scale is determined by the stochastic dynamics. In particular, when there is noise, if the system has an invariant subspace, such as the synchronization manifold, then noise can cause a trajectory initiated in the invariant subspace to wander away from it. But if the system is bounded, at a later time the trajectory will come back to the invariant subspace. On an average, this process defines a time scale, which is the stochastic first-passage time with respect to the invariant subspace. This time changes with the noise amplitude. The existence and behavior of this stochastic time scale can be understood more quantitatively by considering the following simple one-dimensional model with a steady state, under the influence of noise:

$$\frac{dv}{dt}=[-\lambda+h(t)]v+D\xi(t), \tag{4}$$

where λ determines the asymptotic stability of the steady state $v=0$, which mimics the largest transverse Lyapunov exponent of the invariant subspace, $h(t)$ is a zero-mean chaotic process that models the fluctuations of the finite-time Lyapunov exponent, and $D\xi(t)$ is the external noise. The probability distribution function $P(v,t)$ of the stochastic process $V(t)$ obeys the Fokker-Planck equation

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial v}\left[\left(-\lambda v+\frac{1}{2}\eta v\right)P\right]+\frac{1}{2}\frac{\partial^2}{\partial v^2}[(\eta v^2+D)P], \tag{5}$$

where η is the amplitude of $h(t)$. To compute the first-passage time, imagine there is an absorbing boundary at $v=a>0$. The boundedness of the system implies that there must be a reflecting boundary at $v=b<0$. With these boundary conditions, the Fokker-Planck equation can be solved to yield the following expression for the first-passage time [15]:

$$\langle T\rangle=2\int_{v_0}^{a}dy(\eta y^2+D)^{\lambda/\eta-1/2}\int_{b}^{y}(\eta z^2+D)^{-1/2-\lambda/\eta}dz, \tag{6}$$

where v_0 is the initial value of $v(t)$. Figure 4 shows a typical behavior of the first-passage frequency $f_{FP}(D)\equiv 1/\langle T\rangle$ versus the noise amplitude D, which is obtained utilizing an arbitrary value of b and an arbitrary initial condition v_0. The general feature is that the frequency increases with noise. Since the deterministic frequency f_H is approximately constant, generically the $f_{FP}(D)$ curve can intersect f_H at some optimal noise amplitude D^*, leading to the time-scale match required for coherence resonance, as schematically shown in Fig. 4. The optimal noise level D^* depends on the details of the system and cannot be predicted by our heuristic theory.

In summary, we have investigated the effect of noise on wave pattern associated with the Hopf bifurcation from synchronous chaos in coupled chaotic oscillators. A resonant behavior is identified and a heuristic theory is given. The general conclusion is that noise can induce and, more importantly, has the ability to enhance the temporal regularity of the wave pattern in such high-dimensional systems. Regular wave patterns are ubiquitous in nature, and our work may provide a hint to the observability and robustness of such waves, despite the fact that the underlying local dynamics may potentially be chaotic.

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