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Coherent and focusing multidimensional nonlinear geometric optics

COHERENT AND FOCUSING MULTIDIMENSIONAL NONLINEAR GEOMETRIC OPTICS

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ABSTRACT. – Multiphase oscillating solutions of nonlinear hyperbolic systems, in any number of space variables are studied. The amplitudes and wavelengths correspond to the regime called weakly nonlinear geometric optics in the applied litterature.

Without strong assumptions on the set of phases, the problem is ill-posed. For instance, the Cauchy problem with oscillating data may have no solution on a domain independent of the frequencies. The underlying mechanism is focusing of oscillations. We give several examples of hidden or delayed focusing created by nonlinearities. Focusing, at least when it is too strong, must be avoided.

A special case in which focusing is avoided, occurs when one deals with a coherent set of phases. For coherent phases, the problem is well posed, and exact solutions with high frequency oscillations, admit a good asymptotic description which is nonlinear in the principal term. We do not make any finiteness assumption on the set of phases.

Introduction

We study asymptotic expansions for multiphase short wavelength solutions of nonlinear hyperbolic systems, in any number of space variables. The amplitude a and wavelength ε satisfy $a \sim \varepsilon^p$ where the critical exponent p is chosen so that the principal term in the asymptotics is nonlinear while for larger p it would be linear. This is the regime called weakly nonlinear geometric optics in the applied literature. An important first step is the correct formal study of this problem by [MR], [HMR].

As in the one dimensional case, we obtain rigorous results under hypotheses less strong than those imposed for the formal development. We refer the reader to [JMR1] for an introduction to the problem of proving rigorous asymptotic expansions for exact high frequency oscillating solutions to nonlinear hyperbolic systems. Similar expansions have been justified before including: single phase expansions in any number of space variables ([JR1] and [G1]); multiphase expansions in one space dimension ([JMR1] and *see* the references there); multiphase multi-d expansions assuming small divisors do not occur ([D], [JMR5]). Finally, [DPM], [ST], [S2], [C] study related asymptotics for weak

solutions, including the possiblity of shocks, in one space dimension. The results in the latter case are very far from complete for oscillatory solutions.

In this paper, there are three main points:

1) In general, *i. e.* without very strong assumptions on the set of phases, the oscillatory initial value problem is strongly ill-posed. For instance, the Cauchy problem with oscillating data has no solution on a domain whose size is independent of the frequencies. This is a typical multidimensional phenomenon, which is totally absent of the one dimensional case [JMR1]. The underlying problem is focusing of oscillations. What makes the problem hard, is that focusing can be created not only by the phases in the Cauchy data as in the linear case, but also by nonlinear interaction. We construct several examples of such hidden or delayed focusing created by nonlinearities.

The conclusion is that focusing, at least when it is too strong, must be avoided. Unfortunately, it seems impossible to make explicit the precise conditions that avoid this phenomenon.

2) An important case in which focusing is excluded, occurs when one deals with a coherent set of phases. The coherence assumption was introduced by [MR], [HMR] in their formal approach of the problem. They were led to that condition by different considerations, and in any event it arises in a very natural way. The first part of this paper will be devoted to coherent phases. In that case the Cauchy problem is solvable on a domain independent of the wavelength, and the exact solutions have an elegant asymptotic description.

3) In contrast to [HMR], we do not make any finiteness assumption on the set of phases. Such an assumption is very restrictive in multidimensional problems. We recover the phenomenon described in an example in [JR2]. Nonlinear interaction can excite oscillations propagating in an infinite number of directions, even if the Cauchy data have only a finite number of wave trains. The new oscillations appear in the principal part of the asymptotic expansion. Our study indicates that this phenomenon is generic. In [JMR6] we show that the asymptotic expansions proved here imply that this phenomenon occurs for the Euler equations of inviscid compressible flow.

In section 8, we prove a natural propagation theorem for the spectrum of oscillations. In particular, if one makes a finiteness assumption on the set of phases as in [HMR], then the principal term of the asymptotics oscillates with only this finite number of phases, and is given by the integrodifferential system of equations derived in [HMR]. This provides a rigorous justification to the formal nonlinear geometric optics elaborated in [HMR].

This paper is organized in two independent parts. In the first part, we study in detail the coherent situation. The main results are stated in section 2. Section 3 provides examples of rigorously valid geometric optics with finite or infinite number of phases. The proofs are given in sections 4 to 8.

The second part of the paper is devoted to the study of focusing in nonlinear problems, and to its disastrous consequences. Several focusing mechanisms are described in section 9. Examples showing that these phenomena occur are constructed in section 10 and 11. The results in section 11 show that our restriction to profiles which are smooth and quasiperiodic is natural.

Part of the results presented here were described in [JMR3]. Further developments and interesting applications of the constant coefficient case can also be found in [S1].

Part I. The Cauchy problem with coherent phases

2. NOTATIONS AND MAIN RESULTS.

2.1. *The equations.* – We consider oscillating solutions of semilinear systems

$$(2.1.1) \qquad \partial_t u + \sum_{j=1}^{d} A_j(t, x) \, \partial_j u = F(t, x, u)$$

as well as small oscillating perturbations $v = u_o + \varepsilon \, u^\varepsilon$ of a smooth solution u_o for quasilinear systems

$$(2.1.2) \qquad \partial_t v + \sum_{j=1}^{d} B_j(t, x, v) \, \partial_j v = G(t, x, v)$$

In order to put both cases in the same setting, we consider equations of the form

$$(2.1.3) \qquad \partial_t u + \sum_{j=1}^{d} A_j(t, x, \varepsilon u) \, \partial_j u = F(t, x, \varepsilon u, u)$$

In the semilinear case, the matrices $A_j(t, x, u)$ are independent of u. In the quasilinear case,

$$A_j(t, x, \varepsilon u) := B_j(t, x, u_0 + \varepsilon u),$$

$$F(t, x, \varepsilon u, u) := [G(t, x, u_0 + \varepsilon u) - G(t, x, u_0)]/\varepsilon$$

$$- \sum_{j=1}^{d} [B_j(t, x, u_0 + \varepsilon u) - B_j(t, x, u_0)] \, \partial_j u_0/\varepsilon$$

and $F(t, x, 0, u)$ is linear in $(u, \bar{u})$.

Because we want to use complex exponential functions, we allow complex valued functions u.

ASSUMPTION 2.1.1. – *The matrices $A_j(t, x, v)$ are hermitian symmetric, and are C^∞ functions of $(t, x, v, \bar{v})$. F is a C^∞ function of the variables $(t, x, v, \bar{v}, u, \bar{u})$.*

Remark 2.1.2. – 1) Semilinear symmetric hyperbolic systems beginning with a term $A_0 \, \partial_t u$ can also be put in this form.

2) We could consider quasilinear symmetric hyperbolic beginning with a term $A_0(t, x, u) \, \partial_t u$. This would complicate some of the analysis to follow. For the sake of brevity we suppose $A_0(t, x, u) = I$.

Introduce the unperturbed principal part

$$(2.1.4) \qquad L\left(t, x; \partial_t, \partial_x\right) := \partial_t + \sum_{j=1}^{d} A_j\left(t, x, 0\right)\partial_j$$

We assume the following hypothesis of *constant multiplicity*.

ASSUMPTION 2.1.2. – *The eigenvalues of* $A\left(t, x, 0; \xi\right) := \sum \xi_j A_j\left(t, x, 0\right)$ *have constant multiplicity for* $(t, x, \xi) \in \mathbb{R} \times \mathbb{R}^d \times (\mathbb{R}^d \backslash \{0\})$. *We denote them by* $\lambda_k\left(t, x; \xi\right)$.

2.2. *Asymptotic expansions. Coherent phases.* – We study families of solutions of (2.1.3) with oscillations of wavelength proportional to $\varepsilon \to 0$. In the similinear case (2.1.1) they have asymptotic expansions of the form

$$(2.2.1) \qquad u^\varepsilon\left(t, x\right) = \mathcal{U}\left(t, x, \vec{\varphi}\left(t, x\right)/\varepsilon\right) + o\left(1\right)$$

In the quasilinear case (2.1.2) the form is

$$(2.2.2) \qquad v^\varepsilon\left(t, x\right) = u_o\left(t, x\right) + \varepsilon \mathcal{U}\left(t, x, \vec{\varphi}\left(t, x\right)/\varepsilon\right) + o\left(\varepsilon\right).$$

Here $\vec{\varphi} = \left(\varphi_1, \cdots, \varphi_m\right)$ is a family of phases and the profile $\mathcal{U}\left(t, x, \theta\right)$ is a smooth function, periodic or almost-periodic in θ. In the applied literature, such expansions go by the name of nonlinear geometric optics and weakly nonlinear expansions.

The spectrum of $\mathcal{U}\left(t, x, \cdot\right)$ is assumed to be independent of (t, x). Resuming the Fourier series $\sum a_\alpha\left(t, x\right)e^{i\alpha \cdot \theta}$ of $\mathcal{U}$, summing first along rays through the origin, yields

$$(2.2.3) \qquad \mathcal{U}\left(t, x, \theta\right) = \sum_l \mathcal{U}_l\left(t, x, \alpha_l \cdot \theta\right)$$

for a countable collection of $\alpha_l \in \mathbb{R}^m$. In the applied litterature one has always taken the form (2.2.3) with a finite number of summands. This is a very strong condition on u requiring that its spectrum is contained in a finite union of lines. It is closely related to the finiteness assumption of [HMR] and one interesting aspect of our work is that no such assumption is needed.

The first condition on the phases is that they must be solutions of the eikonal equation associated to the unperturbed operator L:

$$(2.2.4) \qquad \det L\left(t, x; d\varphi\left(t, x\right)\right) = 0.$$

Clearly, one main object which appears in (2.2.1) is the space Φ, the linear span of the φ_j's. Our main assumption is that this space satisfies a *coherence property*, similar to the one introduced in [HMR].

DEFINITION 2.2.1. – *Suppose* $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ *is open and* $\Phi \subset C^\infty\left(\Omega; \mathbb{R}\right)$ *is a real vector space.* Φ *is L-coherent when for all* $\varphi \in \Phi \backslash \{0\}$, *one of the following two conditions holds*

(i) $\det L\left(t, x; d\varphi\left(t, x\right)\right) \equiv 0$ *and* $d\varphi\left(t, x\right) \neq 0$ *at every point* $\left(t, x\right) \in \Omega$,

(ii) $\det L\left(t, x; d\varphi\left(t, x\right)\right) \neq 0$ *at every point* $\left(t, x\right) \in \Omega$.

A typical example of coherence occurs when L has constant coefficients, and when Φ is a space of linear functions. This was the situation considered in [JMR2]. Further examples will be given in section 3.

Remark 2.2.2. – If Φ is coherent, then any $\varphi \in \Phi \backslash \{ 0 \}$ has nowhere vanishing differential. Therefore Φ has dimension, over $\mathbb{R}$, less than or equal to $d + 1$.

Remark 2.2.3. – Choose a basis $\psi_0, \ldots, \psi_\mu$ of Φ, and introduce the symbol

$$(2.2.5) \quad M\left(t, x; \gamma\right) := \sum_{j=0}^{\mu} \gamma_j\, L\left(t, x; d\psi_j\left(t, x\right)\right) = \gamma \cdot \partial_t\, \psi\left(t, x\right) + A\left(t, x, 0; \gamma \cdot \partial_x\, \psi\left(t, x\right)\right)$$

defined for $\left(t, x, \gamma\right) \in \Omega \times \mathbb{R}^{\mu+1}$. Here, $\gamma \cdot \partial_t\, \psi$ denotes $\sum \gamma_j\, \partial_t\, \psi_j$ and $\gamma \cdot \partial \psi := \sum \gamma_j\, \partial \psi_j \in \mathbb{R}^d$. The coherence assumption implies that the real roots $\gamma \neq 0$ of

$$(2.2.6) \qquad\qquad\qquad \det M\left(t, x, \gamma\right) = 0$$

are independent of $\left(t, x\right)$. Because the roots $\left(\tau, \xi\right) \in \mathbb{R}^{1+d} \backslash 0$ of $\det L\left(t, x; \tau, \xi\right) = 0$ have $\xi \neq 0$, we see that the roots $\gamma \neq 0$ of (2.2.6) satisfy $\gamma \cdot \partial_x\, \psi\left(t, x\right) \neq 0$, so that they are solution of exactly one among the equations:

$$(2.2.7) \qquad m_k\left(t, x; \gamma\right) := \gamma \cdot \partial_t\, \psi\left(t, x\right) + \lambda_k\left(t, x; \gamma \cdot \partial_x\, \psi\left(t, x\right)\right) = 0$$

The λ_k's were introduced in Assumption 2.1.2.

If Ω is connected, continuity implies that, if $\gamma \neq 0$ satisfies (2.2.7) at a point $(\underline{t}, \underline{x})$ then for all $\left(t, x\right) \in \Omega$, $\left(t, x, \gamma\right)$ is not only a solution of (2.2.6), but is a solution of (2.2.7) with the same index k.

2.3. *The Cauchy problem.* – We consider the Cauchy problem for equation (2.1.3). Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be a closed truncated cone of the form

$$(2.3.1) \qquad\qquad \Omega = \{ \left(t, x\right) | 0 \leq t \leq t_0 \text{ and } |x| + t/\delta \leq \rho \}$$

Let $\omega = \Omega \cap \{ t = 0 \} = \{ x|\ |x| \leq \rho \}$. We assume that δ is so small that for all $\left(t, x\right)$ and all v, $|v| \leq 1$, the boundary matrix

$$(2.3.2) \qquad\qquad \delta^{-1}\,\mathrm{Id} + \sum_{j=1}^{d} \frac{x_j}{|x|}\, A_j\left(t, x, v\right)$$

is definite positive. The boundaries of Ω are spacelike for solutions in Ω with $|v| \leq 1$ and Ω is contained in the domain of determinacy of ω for the operator L in (2.1.4).

Our goal is to construct oscillatory solutions of the form (2.2.1). We suppose that the space of phases $\Phi \subset C^\infty(\Omega)$ is given. Our main assumption is that Φ is L-coherent. Moreover, because we consider a Cauchy problem, we are led to assume that Φ contains a timelike function, *see* Remark 2.3.3 below.

ASSUMPTION 2.3.1. – Φ *is L-coherent and there is* $\varphi_0 \in \Phi$ *such that* $\varphi_{0|t=0} = 0$ *and* $\partial_t \varphi_{0|t=0}(x) \neq 0$ *at every point* $x \in \omega$.

The space of phases for the Cauchy data is

$$(2.3.3) \qquad \Phi^\circ := \{\, \varphi_{|t=0} \,|\, \varphi \in \Phi \,\} \subset C^\infty(\omega)$$

Coherence implies the following facts.

LEMMA 2.3.2. – (i) *The kernel of the mapping* $\psi \to \psi_{|t=0}$ *from* Φ *to* Φ° *is* $\mathbb{R}\,\varphi_0$. *Therefore* $\dim \Phi = 1 + \dim \Phi^\circ$.

(ii) *For any* $\varphi^\circ \in \Phi^\circ \backslash \{\, 0 \,\}$, $d\varphi^\circ$ *never vanishes and for all k the solution of the eikonal equation*

$$(2.3.4) \qquad \partial_t \varphi_k + \lambda_k(t, x; \partial_x \varphi_k) = 0, \qquad \varphi_{k|t=0} = \varphi^\circ$$

belongs to Φ.

Proof. – Let $\varphi \in \Phi$ be such that $\partial_x \varphi$ vanishes at one point $(0, \underline{x})$. Let $\beta = \partial_t \varphi(0, \underline{x})/\partial_t \varphi_0(0, \underline{x})$. Then the differential of $\varphi - \beta\varphi_0 \in \Phi$ vanishes at $(0, \underline{x})$ and therefore, by coherence and Remark 2.2.2, $\varphi - \beta\,\varphi_0 \equiv 0$. This implies part (i).

For $\varphi^\circ \in \Phi^\circ \backslash \{\, 0 \,\}$ choose $\varphi \in \Phi$ such that $\varphi(0, x) = \varphi^\circ(x)$. Set $\beta_k = \{\,\partial_t \varphi(0, 0) + \lambda_k(0, 0; d_x \varphi^\circ(0))\,\}/\partial_t \varphi_0(0, 0)$ and $\varphi_k := \varphi - \beta_k \varphi_0$. Then $\varphi_{k|t=0} = \varphi^\circ$. In particular $\varphi_k \in \Phi \backslash \{\, 0 \,\}$. Moreover, φ_k satisfies the eikonal equation (2.3.4) at $(0, 0)$ and therefore, by coherence and remark 2.3.2, on Ω, which proves part (ii).

Remark 2.3.3. – One could proceed the other way. Start with a space Φ° such that for any $\varphi^\circ \in \Phi^\circ \backslash \{\, 0 \,\}$, $d\varphi^\circ$ never vanishes. Take a basis $(\psi_1^\circ, \ldots, \psi_\mu^\circ)$ of Φ° and consider the solutions $\psi_{j,k}$ of the eikonal equations (2.3.4) with data ψ_j°. Define Φ as the linear span of the $\psi_{j,k}$ and *assume* that Φ is coherent. Then Φ satisfies assumption 2.3.1, provided that there are at least two distinct eigenvalues λ_1 and λ_2. Indeed, choose $\varphi_0 = \psi_{1,1} - \psi_{1,2}$.

Consider (2.1.3) together with the Cauchy data

$$(2.3.5) \qquad u^\varepsilon_{|t=0} = h^\varepsilon$$

We assume that h^ε has the form

$$(2.3.6) \qquad h^\varepsilon(x) = H^\varepsilon(x, \vec{\psi}^\circ(x)/\varepsilon)$$

where $\vec{\psi}^\circ := (\psi_1^\circ, \ldots, \psi_\mu^\circ)$ is a basis (over $\mathbb{R}$) of Φ°. $H^\varepsilon(x, X)$ is assumed to be smooth and *quasi-periodic* with respect to the variables X. This means that its spectrum is

contained in a finitely generated $\mathbb{Z}$-module. Chosing a basis of this module yields a finite family $\vec{\Phi}^o = (\varphi_1^o, \ldots, \varphi_m^o)$ of functions of $\Phi^o \backslash \{0\}$. We assume that h^ε has the form

$$(2.3.7) \qquad h^\varepsilon(x) = \mathcal{H}^\varepsilon(x, \vec{\varphi}^o(x)/\varepsilon)$$

where $\mathcal{H}^\varepsilon$ is a bounded family of smooth functions of $(x, \theta) \in \mathbb{R}^d \times \mathbb{R}^m$, periodic in θ. Note that *we do not assume* the φ_j^o to be $\mathbb{R}$-linearly independent. Of course, they can always be taken $\mathbb{Q}$-linearly independent.

We can choose linear coordinates in $\mathbb{R}^m$ such that the periods are equal to 1, *i.e.* the $\mathcal{H}^\varepsilon$ are functions of $\theta \in \mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$. More precisely, we assume

ASSUMPTION 2.3.4. – $\mathcal{H}^\varepsilon$, *for* $0 \leq \varepsilon \leq 1$, *is a bounded family in the Sobolev space* $H^s(\omega \times \mathbb{T}^m)$ *for some integer* $s > 1 + (d+m)/2$. *Moreover,* $\mathcal{H}^\varepsilon \to \mathcal{H}^o$ *as* $\varepsilon \to 0$ *in* $H^\sigma(\omega \times \mathbb{T}^m)$ *for all* $\sigma < s$.

The initial data oscillate with phases in Φ^o. The solutions are expected to have oscillations with respect to phases which are solutions to the eikonal equations (2.3.4), thus of the form $\mathcal{U}(t, x, \varphi_0(t, x)/\varepsilon, \ldots, \varphi_1(t, x)/\varepsilon, \ldots, \varphi_m(t, x)/\varepsilon)$ where, for $1 \leq j \leq m$, $\varphi_j \in \Phi$ is such that $\varphi_j(0, x) = \varphi_j^o(x)$. In particular, the profiles $\mathcal{U}$ depend on another fast variable $\tau = \varphi_0/\varepsilon$. Even though the initial profiles are periodic in θ, the profiles $\mathcal{U}$ in the following theorem will not in general be quasiperiodic in (τ, θ). They remain periodic in θ but are only almost periodic in (τ, θ). This is illustrated by the linear d'Alembert wave equation. Given initial phases $\omega_k \cdot x$, the profile will have spectrum containing the points

$$(\tau, \theta) = (|\sum n_k \omega_k|, \sum n_k \omega_k)$$

which in general do not belong to a finitely generated $\mathbb{Z}$-module.

THEOREM 2.3.5. – *With assumptions as above, one has the following:*

(i) *There are* $\varepsilon_1 > 0$ *and* $t_1 > 0$ *such that for all* $\varepsilon \in]0, \varepsilon_1]$ *the Cauchy problem* (2.1.3), (2.3.5) *has a unique solution* $u^\varepsilon \in C^1(\Omega_{t_1})$ *where* $\Omega_{t_1} = \Omega \cap \{0 \leq t \leq t_1\}$.

(ii) *For* $1 \leq j \leq m$, *let* $\varphi_j \in \Phi$ *be chosen so that* $\varphi_j(0, x) = \varphi_j^o(x)$ *and let* $\varphi_o \in \Phi$ *be a function as in assumption* 2.3.1. *Then there exists* $\mathcal{U}(t, x, \tau, \theta) \in C^1(\Omega_{t_1} \times \mathbb{R} \times \mathbb{T}^m)$, *almost periodic in* $\eta := (\tau, \theta) \in \mathbb{R} \times \mathbb{R}^m$ *such that*

$$(2.3.8) \qquad u^\varepsilon(t, x) - \mathcal{U}(t, x, \varphi_0(t, x)/\varepsilon, \varphi_1(t, x)/\varepsilon, \ldots, \varphi_m(t, x)/\varepsilon) \to 0$$

in $L^\infty(\bar{\Omega}_{t_o})$ *as* $\varepsilon \to 0$.

(iii) *With notations defined in the next paragraph,* $\mathcal{U}$ *is determined as the solution of*

$$(2.3.9) \qquad \begin{cases} \mathcal{U} = \mathbb{E}\mathcal{U} \\ \mathbb{E}\{L(t, x, \partial_t, \partial_x)\mathcal{U} + \underline{B}(t, x, \mathcal{U})\partial_\eta \mathcal{U}\} = \mathbb{E}\{\underline{F}(t, x, \mathcal{U})\} \\ \mathcal{U}_{|t=0, \tau=0}(x, \theta) = \mathcal{H}^o(x, \theta) \end{cases}$$

In (2.3.9) the following notations are used

$$\underline{F}(t, x, u) := F(t, x, 0, u), \qquad \underline{B}(t, x, u)\partial_\eta := \sum_{j=0}^m \underline{B}_j(t, x, u)\partial_{\eta j},$$

$$\underline{B}_j(t, x, u) := \sum_{l=1}^{d} \partial_l \varphi_j(t, x) \left\{ \frac{\partial A_l}{\partial u}(t, x, 0) \cdot u + \frac{\partial A_l}{\partial \bar{u}}(t, x, 0) \cdot \bar{u} \right\},$$

where the labelling of coordinates of $\eta = \{\tau, \theta)$ is such that $\eta_0 = \tau$ and $\eta_j = \theta_j$ for $1 \le j \le m$.

To define the averaging operator $\mathbb{E}$, let

(2.3.10) $$P(t, x, \partial_\eta) := \sum_{j=0}^{m} L(t, x; d\varphi_j(t, x)) \partial_{\eta j}.$$

P is related to the operator M that was introduced in (2.2.5). If $\psi_1^o, \ldots, \psi_\mu^o$ is a basis of Φ^o, and $\psi_1, \ldots, \psi_\mu$ are functions in Φ such that $\psi_{j|_{t=0}} = \psi_j^o$, then, by Lemma 2.3.2, $(\psi_0 := \varphi_0, \psi_1, \ldots, \psi_\mu)$ is a basis of Φ, and there is a constant coefficient $(m+1) \times (\mu+1)$ matrix R such that

(2.3.11) $${}^t(\varphi_0, \ldots, \varphi_m) = R\,{}^t(\psi_0, \ldots, \psi_\mu)$$

Then the symbol of M is

(2.3.12) $$M(t, x, \gamma) = \sum_{j=0}^{\mu} \gamma_j L(t, x; d\psi_j(t, x)) \quad \text{for} \quad (t, x, \gamma) \in \Omega \times \mathbb{R}^{\mu+1}$$

and P is related to M by the formula

(2.3.13) $$P(t, x, \sigma) = M(t, x, {}^t R \sigma) \quad \text{for} \quad (t, x, \sigma) \in \Omega \times \mathbb{R}^{m+1}$$

According to Remark (2.2.3), we call $\operatorname{Char} M$ [resp. $\operatorname{Char} P$] the set of the $\gamma \in \mathbb{R}^{\mu+1}$ [resp. $\alpha \in \mathbb{R}^{m+1}$] such that $\det\{M(t, x, \gamma)\} = 0$ [resp. $\det\{P(t, x, \sigma)\} = 0$]. As noted in Remark 2.2.3, these sets do not depend on the point (t, x). Furthermore

(2.3.14) $$\operatorname{Char} P = {}^t R^{-1}(\operatorname{Char} M)$$

Finally the operator $\mathbb{E}$ which occurs in (2.3.9) is the extension to a suitable space of almost periodic functions in θ, of the following action on exponential functions

(2.3.15) $$\mathbb{E}\{U(t, x) e^{i\sigma\eta}\} = \begin{cases} \Pi(t, x, \sigma) U(t, x) e^{i\sigma\eta} & \text{if } \sigma \in \operatorname{Char} P \\ 0 & \text{if } \sigma \notin \operatorname{Char} P \end{cases}$$

where, for $\sigma \in \operatorname{Char} P$, $\Pi(t, x, \sigma)$ is the orthogonal projector on the kernel of $P(t, x, \sigma)$. A precise definition will be given at section 6 (Definition 6.2.2).

Remark 2.3.6. – In subsection 2.5 we show that the profile equations (2.3.9) reduce to the integrodifferential system found in [HMR] when only a finite number of characteristic

phases are involved. A derivation of (2.3.9), using BKW methods and formal trigonometric series in θ, can be found in [JMR5]. For the sake of brevity we do not repeat the argument.

2.4. *Propagation of the oscillating spectrum.* – Once the general statement (2.3.8) is obtained, it remains to study the equations (2.3.9) of the profiles, to get further information.

We note that the condition $\mathcal{U} = \mathbb{E}\mathcal{U}$ is (formally) equivalent to

$$(2.4.1) \qquad\qquad P(t, x, \partial_\eta)\mathcal{U} = 0$$

and therefore, the Cauchy condition in (2.3.9)

$$(2.4.2) \qquad\qquad \mathcal{U}_{|t=0,\tau=0} = \mathcal{H}^\circ$$

amounts to a usual Cauchy condition

$$(2.4.3) \qquad\qquad \mathcal{U}_{|t=0} = \mathcal{V}^\circ, \qquad \text{with} \quad \mathbb{E}\,\mathcal{V}^\circ = 0$$

where $\mathcal{V}^\circ(t, x, \theta)$ is for each x the solution of the constant coefficient hyperbolic Cauchy problem in (τ, η)

$$(2.4.4) \qquad\qquad P(t, x, \partial_\tau, \partial_\theta)\mathcal{V}^\circ = 0; \qquad \mathcal{V}^\circ_{|\tau=0} = \mathcal{H}^\circ$$

A precise existence theorem for the Cauchy problem (2.3.9) will be given in section 5. We want to discuss here, qualitative properties of the solutions $\mathcal{U}$. They have the form

$$(2.4.5) \qquad\qquad \mathcal{U}(t, x, \eta) = \sum_{\sigma \in \mathcal{C}} U_\sigma(t, x)\, e^{i\sigma\eta},$$

where $U_\sigma(t, x) = \Pi(t, x, \sigma)\, U_\sigma(t, x)$, $\mathcal{C} := \text{Char}\,P \cap (\mathbb{R} \times \mathbb{Z}^m)$. Moreover, the series in (2.4.5) are absolutely convergent.

In order to recover the usual form (2.2.3) for the expansions, we group the terms in (2.4.5) with the same direction for α. Given a line $l \subset \mathbb{R}^{m+1}$, we define

$$(2.4.6) \qquad\qquad \mathcal{U}_l^*(t, x, \theta) = \sum_{\sigma \in \mathcal{C} \cap l \setminus \{0\}} U_\sigma(t, x)\, e^{i\sigma\eta}.$$

Introduce the average

$$(2.4.7) \qquad\qquad \bar{\mathcal{U}}(t, x) = U_0(t, x).$$

Then $\mathcal{U}$ can be written as the sum of the absolutely convergent series

$$(2.4.8) \qquad\qquad \mathcal{U} = \bar{\mathcal{U}} + \sum_l \mathcal{U}_l^*.$$

Note that the projector $\Pi\,(t,\,x,\,\sigma)$ is the same for all the σ belonging to the same line l. Call it $\Pi_l\,(t,\,x)$. Then the $\mathcal{U}_l^*$ have the following polarization

$$(2.4.9) \qquad\qquad \mathcal{U}_l^*\,(t,\,x) = \Pi_l\,(t,\,x)\mathcal{U}_l^*\,(t,\,x).$$

Moreover, if $\mathcal{C}_l := \mathcal{C} \cap l \neq \{\,0\,\}$, then it is a discrete one dimensional $\mathbb{Z}$-module. If σ_l is a basis of $\mathcal{C}_l$, then (2.4.6) shows that $\mathcal{U}_l^*$ can be viewed as a periodic function of $\sigma_l \cdot \theta$

$$(2.4.10) \qquad\qquad \mathcal{U}_l^*\,(t,\,x,\,\theta) = \tilde{\mathcal{U}}_l^*\,(t,\,x,\,\sigma_l \cdot \theta).$$

Introduce the phases

$$(2.4.11) \qquad\qquad \tilde{\varphi}_l\,(t,\,x) = \sigma_l \cdot \vec{\varphi}(t,\,x).$$

Then (2.3.8) takes the form

$$(2.4.12) \qquad u^\varepsilon\,(t,\,x) = \bar{\mathcal{U}}\,(t,\,x) + \sum_l \tilde{\mathcal{U}}_l\,(t,\,x,\,\tilde{\varphi}_l\,(t,\,x)/\varepsilon) + o\,(1).$$

Again, the series in (2.4.12) is absolutely convergent.

The intuitive rule for non linear interaction (resonance) is the following: take directions l_k in Char P which are in the spectrum of $\mathcal{U}$. Then nonlinearities produce spectrum in the linear span of the l_k. If this linear span intersect Char P in a new direction l', then oscillations will be created with spectrum in l'. To make this idea precise, we introduce the following definition.

DEFINITION 2.4.1. – *Let $\mathcal{L}$ be a set of lines l contained in* Char P, *such that $\mathcal{C}_l := \mathcal{C} \cap l \neq \{\,0\,\}$. $\mathcal{L}$ is said to be stable for interaction [resp for interaction of order $\leqq k$], if for any finite subset $l_1, \ldots, l_p$ [resp. any subset of $p \leqq k$ elements] of $\mathcal{L}$, one has:*

$$(2.4.13) \qquad (\mathcal{C}_{l_1} + \cdots + \mathcal{C}_{l_p}) \cap \mathrm{Char}\,P \subset \mathcal{C}_{\mathcal{L}} := \bigcup_{l \in \mathcal{L}} \mathcal{C}_l.$$

With this definition, one has the following result on the propagation of spectrum for solutions of (2.3.9)

THEOREM 2.4.2. – *Let $\mathcal{L}$ be a set of lines in* Char P. *Assume that $\mathcal{L}$ is stable for interaction. Let $\mathcal{U}$ be solution of (2.3.9). Assume that the spectrum of $\mathcal{U}_{|t=0}$ is contained in $\mathcal{C}_{\mathcal{L}}$, i.e. $U_\alpha\,(0,\,x) \equiv 0$ for all $\alpha \notin \mathcal{C}_{\mathcal{L}}$. Then for all $t \in [0,\,t_0]$ the spectrum of $\mathcal{U}$ is contained in $\mathcal{C}_{\mathcal{L}}$.*

Moreover, if $k \geqq 2$ and if $\underline{F}$ is polynomial (in u, $\bar{u}$) of degree $\leqq k$, the conclusion is still true if one only assumes that $\mathcal{L}$ is stable for interactions of order $\leqq k$.

Remark 2.4.3. – If our equation comes from a quasilinear problem (2.1.2), then $\underline{F}$ is linear in $(u,\,\bar{u})$. Therefore in that case, only stability for quadratic interactions has to be considered.

Similarly, for semilinear equations (2.1.1) with quadratic F, only stability for quadratic interactions is relevant.

Remark 2.4.4. – In particular, when the stable set $\mathfrak{L}$ is finite we recover the usual expansion with a finite number of phases

$$(2.4.14) \qquad u^{\varepsilon}(t, x) = \bar{\mathcal{U}}(t, x) + \sum_{l \in \mathfrak{L}} \tilde{\mathcal{U}}(t, x, \tilde{\varphi}_l(t, x)/\varepsilon) + o(1).$$

We will give in section 3, examples of finite sets $\mathfrak{L}$ which are stable for interaction of order $\leqq 2$. On the other hand, if the spectrum of the Cauchy data is contained in a finite but non stable set $\mathfrak{L}$, in general, the spectrum of the solution will not be contained in a finite union of lines. This is exactly what happens in the example of [JR2]. We refer to section 3 for another example of this type.

2.5. *The integro-differential form of the equations of profiles.* – In this section, we sketch the computations which show that (2.3.9) is the general form, for an infinite number of phases, of the integro-differential equations for profiles found in [HMR].

For $l = \mathbb{R}\sigma \subset \text{Char} P$, $\Pi_l(t, x)$ is the orthogonal projector on the kernel of $L(t, x, \sigma \cdot d\varphi(t, x))$ which is a spectral projector of $A(t, x, 0; \sigma \cdot \partial_x \varphi'(t, x))$. Then the constant multiplicity Assumption 2.1.2 implies that

$$(2.5.1) \qquad \Pi_l(t, x) L(t, x, \partial_t, \partial_x) \Pi_l(t, x) = \Pi_l(t, x) \mathbb{X}_l(t, x, \partial_t, \partial_x) \Pi_l(t, x)$$

where $\mathbb{X}_l$ is a first order operator whose principal part is scalar.

Introduce the projections $\mathbb{E}_l$

$$(2.5.2) \qquad \mathbb{E}_l\left\{ \sum_{\sigma} U_{\sigma}(t, x) e^{i\sigma\eta} \right\} = \sum_{\sigma \in \mathcal{C} \cap l \setminus \{0\}} \Pi_l(t, x) U_{\sigma}(t, x) e^{i\sigma\eta},$$

and

$$(2.5.3) \qquad \mathbb{E}_0\left\{ \sum_{\sigma} U_{\sigma}(t, x) e^{i\sigma\eta} \right\} = U_0(t, x).$$

Note that $\mathbb{E}_0$ is the averaging operator

$$(2.5.4) \qquad \mathbb{E}_0 \mathcal{F}(t, x) = \lim_{\rho \to \infty} \rho^{-1} \int_0^{\rho} \int_{\mathbb{T}^m} \mathcal{F}(t, x, \tau, \theta) \, d\tau \, d\theta$$

On the other hand, the polarization (2.4.9) and the diagonalization (2.5.1) imply that

$$(2.5.5) \qquad \mathbb{E}_l \mathbb{E}\left\{ L(t, x, \partial_t, \partial_x)\mathcal{U} \right\} = \Pi_l \mathbb{X}_l \mathcal{U}_l^*.$$

Projecting (2.3.9), we see that it is equivalent to

$$(2.5.6) \qquad L(\partial_t, \partial_x)\bar{\mathcal{U}} + \mathbb{E}_0\left\{ \underline{B}(\mathcal{U}) \partial_{\eta}\mathcal{U} \right\} = \mathbb{E}_0\left\{ \underline{F}(\mathcal{U}) \right\},$$

(2.5.7) $\Pi_l \, \mathbb{X}_l \, \mathcal{U}_l^* + \mathbb{E}_l \, \{ \, \underline{B} \, (\mathcal{U}) \, \partial_\theta \, \mathcal{U} \, \} = \mathbb{E}_l \, \{ \, \underline{F} \, (\mathcal{U}) \, \}, \quad \text{for all } l.$

Moreover, $\mathbb{E}_l = \Pi_l \, \mathbb{M}_l = \mathbb{M}_l \, \Pi_l$ where $\mathbb{M}_l$ is the mean value operator defined by

(2.5.8) $\mathbb{M}_l \, \mathcal{F} \, (t, \, x, \, \eta) = \lim_{\rho \to \infty} \rho^{-m} \int_{\rho Q} \mathcal{F}^* \, (t, \, x, \, \eta + \eta' \, \}) \, d\eta',$

where $\mathcal{F}^* = \mathcal{F} - \mathbb{E}_0 \, \mathcal{F}$ and Q is a cube of m dimensional measure 1 in $l^\perp$, the orthogonal space to l in $\mathbb{R}^{m+1}$.

Assuming strict hyperbolicity, $\Pi_l \, (t, \, x)$ is a projector on a one dimensional subspace of $\mathbb{R}^N$, generated y an eigenvector $r_l \, (t, \, x)$. The polarization (2.4.9) and (2.4.10) show that

(2.5.9) $\mathcal{U}_l \, (t, \, x, \, \theta) = p_l \, (t, \, x, \, \sigma_l \cdot \theta) \, r_l \, (t, \, x),$

for a scalar function p_l. Then, (2.5.7) can be written in terms of the p_l, and we leave to the reader to check that, when the set $\mathfrak{L}$ of directions l is finite, one gets exactly the integrodifferential equations obtained in [HMR].

3. Examples.

3.1. *Coherent spaces.*

Example 3.1.1. – A typical example of coherence is obtained when the unperturbed operator L of (2.1.4) has constant coefficients, and when Φ is a space of linear functions. This was the situation considered in [JMR2].

When the space dimension is $d = 1$, this situation was considered previously by L. A. Kaliakin [Ka1].

It is important to remark that there are other examples of coherences. To begin with, recall the following example from [HMR].

Example 3.1.2. – Consider a 3×3 system whose characteristic polynomial is

(3.1.1) $\tau \, (\tau^2 - |\xi|^2).$

For instance, linearize Euler's system of isentropic gas dynamics, around a state with zero velocity, constant density, and normalize the sound speed to one. Let $\varphi \, (x)$ and $\psi \, (x)$ be solutions of

(3.1.2) $|\nabla \, \varphi \, (x)|^2 = 1, \qquad |\nabla \, \psi \, (x)|^2 = 1$

with $\nabla \, \varphi \neq \nabla \, \psi$. Then the space Φ generated by $\varphi \, (x) - t$ and $\psi \, (x) - t$ is coherent.

Another example is found by adding variables in a constant coefficient operator

Example 3.1.3. – Assume that the variables $(t, \, x)$ are split into two groups, $y' = (y_0, \, \ldots, \, y_\mu)$ and $y'' = (y_{\mu+1}, \, \ldots, \, y_d)$. Write the principal part

(3.1.3) $L = \sum_{k=0}^{d} A_j^\sharp \, (y) \, \partial_{y_j}$

and assume that

(3.1.4)   for $j \leq \mu$, the matrices $A_j^\sharp$ are constant.

Then, the space of linear functions of y',

(3.1.5) $\Phi = \{ \varphi(y) = \alpha' \cdot y', \ \alpha' \in \mathbb{R}^{\mu+1} \}$

is L-coherent.

There are other possibilities. Let Φ be coherent with L. As already noted in remark 2.2.2, the dimension of Φ is $\leq d + 1$. Take a basis $\psi_0, \ldots, \psi_\mu$ of Φ. The differential $d\psi_j$ are linearly independent, and, locally, one can perform a change of variables such that Φ is the space of linear functions (3.1.3). In the new variables the principal part of L takes the form (3.1.3.). Call

(3.1.6) $$L'(y, \partial_{y'}) = \sum_{k=0}^{\mu} A_j^\sharp(y) \partial_{y_j}$$

then, coherence just means that the set

(3.1.7) $\{ \eta' \in \mathbb{R}^{\mu+1} | \det L'(y, \eta') = 0 \}$

does not depend on y, that is, the real characteristic variety of L' is independent of y. This does not imply that L' has constant coefficients, for several reasons. First, $\det L'(y, \eta') = 0$ may have complex roots and no assumption is made on these complex roots. This idea leads to example 3.1.4 below. Second, even if all the the roots are real and simple, this only implies that the polynomial $\det L'(y, \eta')$ has the form $a(y) p(\eta')$ with $a(y)$ a non vanishing factor and p a constant coefficient polynomial.

This raises the question of whether (strictly) hyperbolic operators whose characteristic polynomials are independent of (t, x) automatically have coefficients independent of (t, x). The answer is of course no since if $L(D)$ is a constant coefficient operator then the change of dependent variables $u = V(t, x) \tilde{u}$ procedures a variable coefficient operator with principal part $V^{-1} L V$ which has the same characteristic polynomial. A more refined question is whether these changes of dependent variables are the only possibilities, that is, is it true that (strictly) hyperbolic systems with constant coefficient characteristic polynomials can be conjugated to constant coefficient systems. When $\mu = 2$ in (3.1.6) the answer is yes: indeed, it suffices to diagonalize $(A_o^\sharp)^{-1} A_1^\sharp$. In particular, after a change of dependant and independant variables, example 3.1.3 is a particular case of exmple 3.1.2. Interestingly, when $\mu \geq 2$, the answer to the question above can be no: an example is given at example 3.1.5.

Example 3.1.4. – Consider the 4×4 system

$$L := \partial_t + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix} \partial_x + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \end{bmatrix} \partial_y$$

with $\alpha = \alpha(t, x, y)$ and $\beta = \beta(t, x, y)$. Its principal symbol is

$$(\tau^2 - \xi^2 - \eta^2)(\tau^2 - \alpha^2\xi^2 - \beta^2\eta^2)$$

If $\alpha > 2$, the real roots with $\tau = 2\xi$, are roots of $(\tau^2 - \xi^2 - \eta^2) = 0$. Therefore, the set Φ of linear phases generated by $2t + x$ and y is L-coherent.

Note that this is an example where Φ does not contain any timelike function.

Example 3.1.5. – Consider the strictly hyperbolic, hermitian symmetric, 3×3 system, in two space variables

$$L := \partial_t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \partial_x + \begin{bmatrix} 0 & 1/\sqrt{2}\cos\theta & 1/\sqrt{2}\cos\theta \\ 1/\sqrt{2}\cos\theta & 0 & i\sin\theta \\ 1/\sqrt{2}\cos\theta & -i\sin\theta & 0 \end{bmatrix} \partial_y$$

with $\theta = \theta(t, x, y)$. Its principal symbol is $\tau(\tau^2 - \xi^2 - \eta^2)$. Therefore, the linear phases form a L-coherent space. However, it is not hard to check that no change of dependent variables can conjugate L to a constant coefficient operator, unless $\theta(t, x, y)$ is constant.

Example 3.1.6. – The system in 3.1.5 can be transformed to the real 6×6 symmetric hyperbolic system, with constant multiplicity

$$L = \partial_t + A\,\partial_x + B(t, x, y)\,\partial_y$$

where A and B have the block structure

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix}; \qquad B = \begin{bmatrix} 0 & \alpha I & \alpha I \\ \alpha I & 0 & \beta\,^tJ \\ \alpha I & \beta J & 0 \end{bmatrix}$$

with

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \alpha = 1/\sqrt{2}\cos\theta, \qquad \beta = \sin\theta$$

and $\theta = \theta(t, x, y)$. The principal symbol of L is $\tau^2(\tau^2 - \xi^2 - \eta^2)^2$. Therefore, the linear phases form a coherent space. Again L is not conjugated to a constant coefficient operator, unless $\theta(t, x, y)$ is constant.

3.2. *Examples where* $\mathfrak{L}$ *is finite.* – We adopt notations (2.3.10)...(2.3.13), as well as the notations of section 2.4. In particular $\mathcal{C} = \mathrm{Char}\,P \cap (\mathbb{R} \times \mathbb{Z}^m)$. Call $\mathcal{D} = {}^tR\mathcal{C} = \mathrm{Char}\,M \cap {}^tR(\mathbb{R} \times \mathbb{Z}^m)$. First we remark

LEMMA 3.2.1. – (i) ${}^tR$ *is one to one from* $\mathcal{C}$ *onto* $\mathcal{D}$.

(ii) *If* $\mathcal{C}(l) := \mathcal{C} \cap l \neq \{0\}$, *then* $l' := {}^tR l \neq \{0\}$ *is a line in* $\mathbb{R}^{\mu+1}$.

(iii) *The interaction condition* $\{\mathcal{C}(l_1) + \cdots + \mathcal{C}(l_p)\} \cap \mathrm{Char}\,P \neq \{0\}$ *is equivalent to* $\{\mathcal{D}(l'_1) + \cdots + \mathcal{D}(l'_p)\} \cap \mathrm{Char}\,M \neq \{0\}$, *where* $l'_j := {}^tR l_j$ *and* $\mathcal{D}(l'_j) := \mathcal{D} \cap l'_j = {}^tR\mathcal{C}(l_j)$.

Proof. – The first point is a consequence of the independence over $\mathbb{Q}$ of the phases $\varphi_1, \ldots, \varphi_m$. Points (ii) and (iii) are immediate corollaries.

This lemma is usefull, because it is easier to work with the operator M than the operator P whose characteristic variety is larger.

Example 3.2.2. – Consider a constant coefficient two speed system [RR]. Then the principal symbol is equal to

$$q(\tau, \xi)^k$$

for some strictly hyperbolic quadratic form q and some integer k. Consider the space Φ of linear phases. A basis is $\psi_o = t$, $\psi_j = x_j$. Then, with the general notation (2.3.12)

$$(3.2.1) \qquad \operatorname{Char} M = \{\gamma \in \mathbb{R}^{d+1} | q(\gamma) = 0\}.$$

If l'_1 and l'_2 are two different lines in $\operatorname{Char} M$, then the linear span $l'_1 + l'_2$ intersects $\operatorname{Char} M$ only on $l'_1 \cup l'_2$. Therefore, quadratic interaction never produces new directions of oscillations. Thus for linear phases any set $\mathfrak{L}$ of directions $l \subset \operatorname{Char} P \cap (\mathbb{R} \times \mathbb{Z}^m)$ is stable for quadratic interactions.

With this example, we recover and generalize the second case studied by J. M. Delort [D], because we drop here all the arithmetic assumptions of [D].

Example 3.2.3. – Let the principal symbol be

$$(3.2.2) \qquad \{\tau^2 - |\xi|^2\}\{\tau^2 - c^2|\xi|^2\}.$$

Again we take Φ to be the space of linear functions on $\mathbb{R}^{d+1}$. Then $\operatorname{Char} M$ is the union of two cones; $\Sigma_1 := \{\tau^2 = |\xi|^2\}$ and $\Sigma_2 := \{\tau^2 = c^2|\xi|^2\}$. When $c > 1$, Σ_2 is inside Σ_1. Then, if $\mathfrak{L}$ is a set of directions $l_j \subset \Sigma_1$, $j = 1, \ldots, m$, such that none of the planes $l_j + l_k$, $j \neq k$, intersects $\Sigma_2 \setminus \{0\}$, then $\mathfrak{L}$ is stable for quadratic interaction.

Example 3.2.4. – Again we assume the principal symbol is given by (3.2.2), but we now take $c < 1$ and $\mathfrak{L} = \{l_j, 1 \leq j \leq m\}$ a family of lines contained in the interior sheet Σ_1. Then the plane $l_j + l_k$ does intersect the exterior sheet $\Sigma_2 \setminus \{0\}$. But it may happen for arithmetical reasons, that the lattice $C(l_j) + C(l_k)$ does not intersect $\Sigma_2 \setminus \{0\}$.

For instance, consider the following example in space dimension $d = 2$. With the indentification $\mathbb{R}^2 \simeq \mathbb{C}$, consider the three characteristic directions $(1, 1)$, $(1, j)$ and $(1, j^2)$ located on $\Sigma_1 := \{\tau^2 = |\xi|^2\}$. Here $j \neq 1$ is a cubic root of 1. A linear combination $\alpha(1, 1) + \beta(1, j)$, with $(\alpha, \beta) \in \mathbb{Z}^2$, belongs to $\Sigma_2 := \{\tau^2 = c^2|\xi|^2\}$ if and only if $(\alpha + \beta)^2 = c^2|\alpha + j\beta|^2 = c^2(\alpha^2 + \alpha\beta + \beta^2)$. When

$$(3.2.3) \qquad c^2 \notin \mathbb{Q},$$

this only occurs when $\alpha = \beta = 0$. The same reasonning holds by symmetry for all pairs $l_i \neq l_k$. Therefore, if (3.2.3) holds, the set of lines $l_k := \mathbb{R}(1, j^k)$, $k = 0, 1, 2$, is stable for quadratic interaction for any system whose principal symbol is (3.2.2).

Example 3.2.5. – The following example from [HMR] has linear phases and a 3×3 system in two space dimension, whose symbol is

$$(3.2.4) \qquad \tau \xi_2^2 + (\xi_1^2 - \tau^2)(\xi_1 + 3\tau).$$

The intersection of $\operatorname{Char} M$ with $\{\tau = 1\}$ is a cubic. [HMR] show that there is a link between the construction of sets $\mathfrak{L}$ that are stable for quadratic interaction, and the construction of subgroups for natural group structures on that cubic.

3.3 *Dense oscillations.* — We give here a variation of the example of [JR2] which shows that oscillations with a finite number of phases in the Cauchy data, can produce a family of waves moving in an infinity of distinct directions.

Consider in space dimension $d = 2$ the following problem

$$(3.3.1) \qquad \begin{cases} \Box u^{\varepsilon} = (\partial_t u^{\varepsilon})^3, \\ u^{\varepsilon}_{|t=0}(x) = \varepsilon\, H\,(x/\varepsilon), \qquad \partial_t u^{\varepsilon}_{|t=0}(x) = H_0\,(x/\varepsilon), \end{cases}$$

$$(3.3.2) \qquad H(X) = \sum_{k=1}^{3} A\,(\alpha_k\, X), \qquad H_0(X) = \sum_{k=1}^{3} A'\,(\alpha_k\, X).$$

where $A(\theta)$ is a 2π-periodic C^{∞} function on $\mathbb{R}$. For $k = 1, 2, 3$, $\alpha_k \in \mathbb{R}^2$, $|\alpha_k| = 1$, and the α_k are not located on a line $\subset \mathbb{R}^2$. Moreover, we assume that all the Fourier coefficients of A are nonzero.

In particular the Cauchy data oscillate with the three phases $\varphi_k^0(x) = \alpha_k \cdot x$.

(3.3.1) can be written as a first order system

$$(3.3.3) \qquad \begin{cases} L\,\vec{u}^{\varepsilon} = F\,(\vec{u}^{\varepsilon}), \\ \vec{u}^{\varepsilon}_{|t=0}(x) = \vec{H}\,(x/\varepsilon). \end{cases}$$

with $\vec{u} = (u_0, u_1, u_2) = (\partial_t u, \partial_{x_1} u, \partial_{x_2} u)$, $F(\vec{u}) = ((u_0)^3, 0, 0)$ and $\vec{H} = (H_0, \partial_{x_1} H, \partial_{x_2} H)$. The principal symbol of L is $\tau\,(\tau^2 - |\xi|^2)$.

Theorem 2.3.5 asserts that $\vec{u}^{\varepsilon}(t, x) = \mathcal{U}(t, x, t/\varepsilon, x/\varepsilon) + o(1)$ with $\mathcal{U}$ solution to the system (2.3.9). It is clear that $\mathcal{U}$ does not depend on the variable x and satisfies

$$(3.3.4) \qquad \mathcal{U} = \mathbb{E}\,\mathcal{U}; \qquad \partial_t \mathcal{U} = \mathbb{E}\,\underline{F}\,(\mathcal{U}); \qquad \mathcal{U}_{|t=0,\,T=0}(X) = \vec{H}\,(X).$$

The general formula (2.5.6) implies that

$$(3.3.5) \qquad \mathcal{U}(t, T, X) = \sum_{(\lambda,\,\alpha)} U_{\lambda,\,\alpha}(t)\, e^{i\,(\lambda T + \alpha \cdot X)}$$

the absolutely convergent summation being over the indices (λ, α) such that $\alpha \in \mathfrak{Z} :=$ $\mathbb{Z}\,\alpha_1 + \mathbb{Z}\,\alpha_2 + \mathbb{Z}\,\alpha_3$ and $\lambda = 0$ or $\lambda = \pm\,|\alpha|$. Moreover, the coefficients $U_{\lambda,\,\alpha}$ have the polarization indicated at (2.5.6).

Our choice (3.3.2) has been made so that, at time $t = 0$, the coefficients $U_{\lambda,\,\alpha}$ have a particular polarization. More pecisely, $U_{\lambda,\,\alpha}(0) = 0$ when $\lambda = 0$ or $\lambda = -|\alpha|$ and

$$(3.3.6) \qquad \mathcal{U}(0, T, X) = \sum_{k=1}^{3} A'\,(\alpha_k\, X + T)\, r_k$$

where r_k is the eigenvector $(1,\ \alpha_k)$.

The next remark is that if $\alpha \in 3\backslash\{\,0\,\}$ and $\vec{v} = (v_o,\ 0,\ 0)$, then $\Pi\,(0,\ \alpha)\,\vec{v} = 0$. Thus the Fourier coefficients of $\mathbb{E}\,\underline{F}\,(\mathcal{U})$, and therefore those of $\partial_t \mathcal{U}$, of index $(0,\ \alpha)$ vanish. Therefore, all the $U_{0,\,\alpha}\,(t)$ vanish identically. Moreover, using the explicit form of the projectors $\Pi\,(\pm\,|\alpha|,\ \alpha)$, one can show that the first component $\mathcal{U}_o$ of the vector valued function $\mathcal{U}$ has the form

$$(3.3.7) \qquad \mathcal{U}_o\,(t,\ T,\ X) = \sum_{(\lambda,\,\alpha)\in\mathcal{C}} \sigma_{\lambda,\,\alpha}\,(t)\,e^{i\,(\lambda T + \alpha\cdot X)}$$

where $\mathcal{C} = \{\,\pm\,|\alpha|,\ \alpha),\ \alpha \in 3\}$ and satisfies

$$(3.3.8) \qquad \partial_t \mathcal{U}_o = \mathbb{P}\,\{\,(\mathcal{U}_o)^3\,\}; \qquad \mathcal{U}_{o|_{t=0}}\,(T,\ X) = \sum_{k=1}^{3} A'(\alpha_k\,X + T).$$

Here $\mathbb{P}$ is the operator which maps $\sum a_{\lambda,\,\alpha}\,(t)\,e^{i\,(\lambda T + \alpha\cdot X)}$ to

$$(3.3.9) \qquad a_{0,\,0}\,(t) + \sum_{(\lambda,\,\alpha)\in\mathcal{C}\backslash\{\,0\,\}} \frac{1}{2}\,a_{\lambda,\,\alpha}\,(t)\,e^{i\,(\lambda T + \alpha\cdot X)}.$$

In particular, (3.3.8) implies that $\mathcal{V}_1 := \partial_1 \mathcal{U}_{0|_{t=0}}$ is given by

$$(3.3.10) \qquad \mathcal{V}_1 = \mathbb{P}\,\{\,(\mathcal{U}_{0|_{t=0}})^3\,\}$$

we now expand A in its rapidly convergent Fourier series, $A = \sum a_m\,e^{im\theta}$, getting an expression for $(\mathcal{U}_{0|_{t=0}})$. With (3.3.9) and (3.3.10) we find that $\mathcal{V}_1$ has the form

$$(3.3.11) \qquad \mathcal{V}_1\,(T,\ X) = \sum b_\mu\,e^{i\varphi_\mu}$$

with phases $\varphi_\mu\,(T,\ X) = \sum \mu_k\,(\alpha_k\cdot X + T)$. In (3.3.11), the summation is taken over the set $\mathcal{M}$ of multi-indices $\mu = (\mu_1,\ \mu_2,\ \mu_3) \in \mathbb{Z}^3$ such that $d\varphi_\mu = (\sum \mu_k,\ \sum \mu_k\,\alpha_k) \in \mathcal{C}$. Call $\mathcal{M}_\sharp$ the subset of the $\mu \in \mathcal{M}$ such that the three components μ_k are $\neq 0$. When $\mu \in \mathcal{M}_\sharp$, a direct computation also gives that

$$(3.3.12) \qquad b_\mu = 3\,\mu_1\,\mu_2\,\mu_3\,a_{\mu_1}\,a_{\mu_2}\,a_{\mu_3} \neq 0$$

Because the vectors $(1,\ \alpha_k)$ are linearly independent, $d\varphi_\mu \neq d\varphi_{\mu'}$, when $\mu \neq \mu'$, and $d\varphi_\mu$ and $d\varphi_{\mu'}$ are parallel if and only if μ and μ' are proportional. Now, (3.3.12) shows that the first component of $\partial_t \mathcal{U}_{l|_{t=0}}$ and therefore of $\mathcal{U}$, has a nonvanishing oscillation in any direction l generated by the $d\varphi_{\mu'}$ $\mu \in \mathcal{U}_\sharp$. Call $\mathfrak{L}$ the collection of these directions.

To conclude, it remains to show that $\mathfrak{L}$ can be infinite. As in [JR2] choose

$$(3.3.13) \qquad \alpha_1 = (1,\ 0), \qquad \alpha_2 = (-1/2,\ \sqrt{3}/2), \qquad \alpha_3 = (-1/2,\ -\sqrt{3}/2).$$

It is proved in [JR2] that then not only $\mathfrak{L}$ is infinite, but the union of lines in $\mathfrak{L}$ is dense in the light cone $\{\,\tau^2 = |\xi|^2\,\}$.

4. SINGULAR SYSTEMS. – In the proof of theorem 2.3.5 the first and main step is to solve singular systems. In this section we present a study of such systems, which may be of independent interest.

4.1. *Statement of the result.* – Consider a problem

$$(4.1.1) \quad \partial_t u^\varepsilon + \sum_{j=1}^d A_j\left(t, x, \varepsilon u^\varepsilon\right) \partial_{x_j} u^\varepsilon + \varepsilon^{-1} \sum_{l=1}^m B_k\left(t, x, \varepsilon u^\varepsilon\right) \partial_{\theta_l} u^\varepsilon = F\left(t, x, u^\varepsilon\right),$$

where we assume that all the matrices A_j and B_k are hermitian symmetric and smooth functions of their arguments. We work on a closed domain $\Omega \subset \mathbb{R}^{1+d}$ of the form (2.3.1) and assume (2.3.2). The unknowns u^ε are functions of $(t, x, \theta) \in \Omega \times \mathbb{T}^m$.

The singular term of (4.4.1) is ε^{-1} times

$$(4.1.2) \quad Q\left(t, x, \partial_\theta\right) u^\varepsilon := \sum_{l=1}^m B_k\left(t, x, 0\right) \partial_{\theta_l} u^\varepsilon$$

and we need assumptions on Q in order to be able to solve (4.4.1). The main difficulty is to commute Q with the derivatives ∂_x. An easy case is when Q has constant coefficients. In our case, coherence only implies that the characteristic set

$$(4.1.3) \quad \operatorname{Char} Q = \left\{ \sigma \in \mathbb{R}^m \,|\, \det Q\left(t, x, \sigma\right) = 0 \right\}$$

is independent of (t, x). The discussion in section 3.1 shows that this does not imply that Q can be reduced to a constant coefficient operator by a conjugation, *i.e.* a change of dependent variables. Nevertheless, the idea is that this reduction can be performed by a pseudo-differential conjugation $u = V\left(t, x, \partial_\theta\right) v$.

Before giving the precise condition we need on Q, we introduce the following notation for multiplication operators on Fourier series depending smoothly on (t, x).

DEFINITION 4.1.1. – $\Sigma^0(\Omega \times \mathbb{R}^m)$, *or simply* $\Sigma^0\left(\Omega\right)$ *when the dimension m is clear from the context, denotes the space of matrices $E\left(t, x, \sigma\right)$, defined for $(t, x, \sigma) \in \Omega \times \mathbb{R}^m$, homogeneous of degree 0 in σ, such that the set of functions $\{ E\left(\cdot, \cdot, \sigma\right), \sigma \in \mathbb{R}^m \}$ is bounded in $C^\infty\left(\Omega\right)$. A symbol $E \in \Sigma^0\left(\Omega\right)$ is elliptic when $E\left(t, x, \sigma\right)$ is invertible for all (t, x, σ) and $E^{-1} \in \Sigma^0\left(\Omega\right)$.*

Note that such symbols are assumed to be defined for all $\sigma \in \mathbb{R}^m$. On the other hand, we do not require any smoothness in σ. With this definition we can now state the condition we impose on Q.

ASSUMPTION 4.1.2. – *There is an elliptic symbol $V \in \Sigma^0\left(\Omega\right)$ such that*

$$(4.1.4) \quad \forall\left(t, x, \sigma\right) \in \Omega \times \mathbb{R}^m : V^{-1}\left(t, x, \sigma\right) Q\left(t, x, \sigma\right) = Q^\sharp\left(\sigma\right),$$

where $Q^\sharp$ is a matrix independent of $(t, x) \in \Omega$, homogeneous of degree one in $\sigma \in \mathbb{R}^m$.

We now introduce some function spaces. Denote by ω_t the section at time t of Ω : ω_t is the ball in $\mathbb{R}^d$ of radius $\rho - t/\delta$, centered at the origin. We call $E^0\left(t_1\right)$ the space of those functions $u \in L^2\left(\Omega_{t_1} \times \mathbb{T}^m\right)$ such that their extension by 0 outside Ω_{t_1} is continuous in

time $t \in [0, t_1]$ with values in $L^2 (\mathbb{R}^d \times \mathbb{T}^m)$. For an integer $s \geq 0$, we call $E^s (t_1)$ the space of the $u \in E^0 (t_1)$ whose (x, θ)-derivatives $\partial^{\alpha}_{x, \theta} u$ belong to $E^0 (t_1)$ for $|\alpha| \leq s$. For $u \in E^s (t_1)$ we denote by $\|u(t)\|_s$ the $H^s (\omega_t \times \mathbb{T}^m)$-norm of $u(t, \cdot, \cdot)$:

$$(4.1.5) \qquad \|u(t)\|_s := \|u(t, \cdot, \cdot)\|_{H^s (\omega_t \times \mathbb{T}^m)}$$

and

$$(4.1.6) \qquad \|u\|_{E^s(t_1)} = \sup_{0 \leq t \leq t_1} \|u(t)\|_s.$$

The main result of this section is .

THEOREM 4.1.3. – *Let Ω be a closed set as in (2.3.1). Assume the symmetry of the matrices A_j and B_k, (2.3.2) and that Q satisfies Assumption 4.1.2. Consider the Cauchy problem for (4.1.1) with initial data*

$$(4.1.7) \qquad u^{\varepsilon}_{|t=0} = h^{\varepsilon},$$

where h^{ε} is a bounded family in $H^s (\omega \times \mathbb{T}^m)$ with $s > 1 + (d + m)/2$. Then there are $\varepsilon_1 > 0$ and $t_1 > 0$ such that for all $\varepsilon \in]0, \varepsilon_1]$ the problem (4.1.1) (4.1.7) has a unique solution $u^{\varepsilon} \in E^s (t_1)$. Moreover, the family u^{ε} is bounded in $E^s (t_1)$.

4.2. L^2 *estimates.* – The first step is to study the linearized equations

$$(4.2.1) \quad L^{\varepsilon} u^{\varepsilon} := \partial_t u^{\varepsilon} + \sum_{j=1}^{d} A_j (t, x, \varepsilon v^{\varepsilon}) \partial_{x_j} u^{\varepsilon} + \varepsilon^{-1} \sum_{l=1}^{m} B_k (t, x, \varepsilon v^{\varepsilon}) \partial_{\theta_l} u^{\varepsilon} = f^{\varepsilon}.$$

Recall that Ω is defined in (2.3.1), and we fix $t_0 < \delta\rho$. To begin with, we assume that $v^{\varepsilon} \in C^1 (\Omega_{t_1} \times \mathbb{T}^m)$ for some $t_1 \leq t_0$ and that

$$(4.2.2) \qquad \varepsilon \|v^{\varepsilon}\|_{L^{\infty}} \leq 1,$$

so that Ω_{t_1} is contained in the domain of determinacy of ω for L^{ε}.

PROPOSITION 4.2.1. – *Let $f^{\varepsilon} \in L^2 (\Omega_{t_1} \times \mathbb{T}^m)$ and let $u^{\varepsilon} \in E^0 (t_1)$ be solution to (4.2.1). Then the following uniform estimate holds*

$$(4.2.3) \qquad \|u^{\varepsilon} (t)\|_0 \leq e^{Kt} \|u^{\varepsilon} (0)\|_0 + \int_0^t e^{K (t-t')} \|f^{\varepsilon} (t')\|_0 \, dt'.$$

where K depends only on the norm of $\nabla_{x, \theta} v^{\varepsilon}$ in $L^{\infty} (\Omega_{t_1} \times \mathbb{T}^m)$

Proof. – The usual integration by parts gives

$$2 \operatorname{Re} \int_{\Omega_T} \langle u^{\varepsilon}, L^{\varepsilon} u^{\varepsilon} \rangle \, dt \, dx = \|u^{\varepsilon} (T)\|^2_0 - \|u^{\varepsilon} (0)\|^2_0 + \mathcal{B}_1 + \mathcal{B}_2.$$

The first term $\mathcal{B}_1$ is equal to

$$\int_{0 \leq t = \delta\,(\rho - |x|) \leq T} \left\langle\, u^\varepsilon + \delta \sum_{j=1}^{d} \frac{x_j}{|x|}\, A_j\,(t,\, x,\, \varepsilon\, v^\varepsilon)\, u^\varepsilon,\, u^\varepsilon \,\right\rangle dx.$$

It is non negative by (2.3.2) and (4.2.2). Because of the symmetry of the matrices, the second term $\mathcal{B}_2$ is the integral over Ω_T of

$$\left\langle\, \left\{ \sum_{j=1}^{d} \partial_{x_j}\, A_j\,(t,\, x,\, \varepsilon\, v^\varepsilon) + \varepsilon^{-1} \sum_{l=1}^{m} \partial_{\theta_k}\, B_k\,(t,\, x,\, \varepsilon\, v^\varepsilon) \right\}\, u^\varepsilon,\, u^\varepsilon \,\right\rangle.$$

Because $B_k\,(t,\, x,\, \varepsilon\, v\,(t,\, x,\, \theta))$ depends on θ only through $\varepsilon\, v$, this term is $= O\,(K|u^\varepsilon|^2)$ so that

$$2\,\mathrm{Re} \int_0^T \|u^\varepsilon\,(t)\|_0\, \|f^\varepsilon\,(t)\|_0 dt \geq \|u^\varepsilon\,(T)\|_0^2\, -\, \|u^\varepsilon\,(0)\|_0^2 - K \int_0^T \|u^\varepsilon\,(t)\|_0^2\, dt$$

and (4.2.3) follows. Note that assumption 4.1.2 is not needed here.

4.3. *Estimate of the derivatives.* – Our goal in this section is to get Sobolev estimates by differentiating (4.2.1). Obviously, the $(t,\, x)$-dependance of the B_k's is a serious obstacle to a straightforward computation. The role of assumption 4.1.2 is to reduce to the case where $B\,(t,\, x,\, 0)$ is independent of $(t,\, x)$. Before the proof, we need some preparation. First, to a symbol $E \in \Sigma^0$, we associate the operator $E\,(t,\, x,\, \partial_\theta)$ which is defined as follows: for

$$(4.3.1) \qquad\qquad u\,(t,\, x,\, \theta) = \sum_{\sigma \in \mathbb{Z}^m} u_\sigma\,(t,\, x)\, e^{i\sigma \cdot \theta},$$

the image is

$$(4.3.2) \qquad \{ E\,(t,\, x,\, \partial_\theta)\, u \}\,(t,\, x,\, \theta) = \sum_{\sigma \in \mathbb{Z}^m} E(t,\, x,\, \sigma)\, u_\sigma\,(t,\, x)\, e^{i\sigma \cdot \theta}.$$

The definition certaintly makes sense for smooth u. As usual, when no confusion is possible, we will often denote by the same letter E the symbol $E\,(t,\, x,\, \sigma)$ and the operator $E\,(t,\, x,\, \partial_\theta)$. The following proposition summarizes the main properties of this very elementary calculus of Fourier multipliers with $(t,\, x)$ as parameters.

PROPOSITION 4.3.1. – *Let* $E \in \Sigma^0\,(\Omega)$. *Then:*

(i) $E\,(t,\, x,\, \partial_\theta)$ *extends continuously from* $E^s\,(t_1)$ *into itself. There is a constant* C *such that for all* $t_1 \leq t_o$, *all* $u \in E^s\,(t_1)$ *and all* $t \leq t_1$ *one has*

$$(4.3.3) \qquad\qquad \|(Eu)\,(t)\|_s \leq C\|u\,(t)\|_s.$$

(ii) E *commutes with* ∂_θ *and the commutator of* ∂_t *or* ∂_{x_j} *with* $E\,(t,\, x,\, \partial_\theta)$ *is* $(\partial_t\, E)$ $(t,\, x,\, \partial_\theta)$ *or* $(\partial_{x_j}\, E)\,(t,\, x,\, \partial_\theta)$.

(iii) *If $E^\sharp$ is another symbol in $\Sigma^0(\Omega)$ then $E(t, x, \partial_\theta) \circ E^\sharp(t, x, \partial_\theta) = (EE^\sharp)(t, x, \partial_\theta)$.*

Proof. – Definition (4.3.2) and the fact that the norm in $H^s(\omega_t \times \mathbb{T}^m)$ is equivalent to

$$(4.3.4) \qquad \Big\{ \sum_{\alpha \in \mathbb{Z}^m} \sum_{|\beta| \leq s} (1 + |\alpha|)^{2s - 2|\beta|} \| \partial_x^\beta u_\alpha(t, x) \|^2_{L^2(\omega_t)} \Big\}^{1/2}$$

easily imply the results.

Next consider (4.2.1). For simplicity, we call A_j the matrix $A_j(t, x, \theta) = A_j(t, x, \varepsilon v^\varepsilon(t, x, \theta))$ as well as the operator of multiplication by this matrix. From the singular part of L^ε write

$$(4.3.5) \qquad B_k(t, x, \varepsilon v^\varepsilon) = B_k(t, x, 0) + \varepsilon C_k(t, x, \theta).$$

The classical properties of Sobolev spaces on balls of radius bounded from above (by ρ) and from below (by $\rho - t_0 \delta > 0$), show that

LEMMA 4.3.2. – *Let $s > (d + m)/2$. There is a function $M(K)$ such that for all $t_1 \leq t_o$, and all $v^\varepsilon \in E^s(t_1)$ with norm $\leq K$, the coefficients A_j and C_k belong to $E^s(t_1)$ with norm $\leq M(K)$.*

We now take advantage of Assumption 4.1.2. Setting

$$(4.3.6) \qquad u^\varepsilon = V(t, x, \partial_\theta)\, \hat{u}^\varepsilon, \qquad \hat{u}^\varepsilon = V^{-1}(t, x, \partial_\theta)\, u^\varepsilon$$

and multiplying (4.2.1) by $V^{-1}(t, x, \partial_\theta)$, we get for $\hat{u}^\varepsilon$ the following equation

$$(4.3.7) \qquad \hat{L}^\varepsilon \hat{u}^\varepsilon = g^\varepsilon := V^{-1}(t, x, \partial_\theta)\, f^\varepsilon,$$

with

$$(4.3.8) \qquad \hat{L}^\varepsilon = \partial_t + \sum_{j=1}^d V^{-1} A_j V \partial_{x_j} + \sum_{l=1}^m V^{-1} C_k V \partial_{\theta_l} + \varepsilon^{-1} Q^\sharp(\partial_\theta) + E$$

and

$$(4.3.9) \qquad E = V^{-1}(\partial_t V) + \sum_{j=1}^d V^{-1} A_j (\partial_{x_j} V).$$

We can now state the commutation estimates.

PROPOSITION 4.3.3. – *Assume the coefficients A_j and C_k of $\hat{L}^\varepsilon$ belong to $E^s(t_1)$, with $s > 1 + (d + m)/2$. Let $\hat{u}^\varepsilon \in E^s(t_1)$ be solution to (4.3.7) with $g^\varepsilon \in E^s(t_1)$. Let α be a multi-index in $\mathbb{N}^d \times \mathbb{N}^m$ of length $|\alpha| \leq s$, and let $\hat{u}_\alpha^\varepsilon = \partial_{x, \theta}^\alpha \hat{u}^\varepsilon$. Then*

$$(4.3.10) \qquad \hat{L}^\varepsilon \hat{u}_\alpha^\varepsilon = g_\alpha^\varepsilon$$

satisfies

$$(4.3.11) \qquad \| g_\alpha^\varepsilon(t) \|_0 \leq C \| \text{Coef}(t) \|_s \| \hat{u}_\alpha^\varepsilon(t) \|_s + \| g^\varepsilon(t) \|_s$$

where Coef *denotes the collection of coefficients A_j and C_k.*

Proof. – The fundamental remark is that $\partial^\alpha := \partial^\alpha_{x,\theta}$ commutes exactly both with ∂_t and with the singular part $\varepsilon^{-1} Q^\sharp (\partial_\theta)$. Therefore, the bracket of $\hat{L}^\varepsilon$ with ∂^α is a sum of terms

$$(4.3.12) \qquad (\partial^{\alpha'} V^{-1}) (\partial^\beta \mathrm{Coef}) (\partial^{\alpha''} V) \partial^\gamma \hat{u}^\varepsilon$$

with $|\alpha'| + |\alpha''| + |\beta| + |\gamma| \leq |\alpha| + 1$ and $|\gamma| \leq |\alpha|$. Because of proposition 4.3.1, the $L^2 (\omega_t \times \mathbb{T}^m)$ norm of (4.3.12) is dominated by

$$(4.3.13) \qquad C\|(\partial^\beta \mathrm{Coef}) (t) w (t)\|_0$$

where $w = (\partial^{\alpha''} V) \partial^\gamma \hat{u}^\varepsilon$. Proposition 4.3.1 also implies that

$$(4.3.14) \qquad \|w (t)\|_{H^\sigma (\omega_t \times \mathbb{T}^m)} \leq C\|\hat{u}^\varepsilon_\alpha (t)\|_s, \qquad \sigma = s - |\gamma| \geq 0.$$

On the other hand

$$(4.3.15) \qquad \|\partial^\beta \mathrm{Coef} (t)\|_{H^{\sigma'} (\omega_t \times \mathbb{T}^m)} \leq C\|\mathrm{Coef} (t)\|_s, \qquad \sigma' = s - |\beta| \geq 0.$$

Finally, the conditions $\sigma \geq 0$, $\sigma' \geq 0$ and $\sigma + \sigma' > (d+m)/2$ imply that the multiplication is bounded from $H^{\sigma'} (\omega_t \times \mathbb{T}^m) \times H^\sigma (\omega_t \times \mathbb{T}^m)$ into $L^2 (\omega_t \times \mathbb{T}^m)$, with a norm uniformly bounded, provided that the radius of ω_t does not shrink to 0. Thus

$$(4.3.16) \qquad \|(\partial^\beta \mathrm{Coef}) (t) w (t)\|_0 \leq C\|\partial^\beta \mathrm{Coef} (t)\|_{\sigma'} \|w (t)\|_\sigma$$

With (4.3.14) and (4.3.15), the proposition follows.

PROPOSITION 4.3.4. – *Under Assumption* 4.1.2, *let* $v^\varepsilon \in E^s (t_1)$ *satisfying* (4.2.2), $f^\varepsilon \in E^\varepsilon (t_1)$ *and* $h \in H^s (\omega \times \mathbb{T}^m)$. *Then the solution* μ^ε *of the Cauchy problem* (4.2.1) (4.1.7) *belongs to* $E^s (t_1)$ *and*

$$(4.3.19) \qquad \|u^\varepsilon (t)\|_s \leq Ce^{Kt}\|h^\varepsilon\|_s + C \int_0^t e^{K (t-t')}\|f^\varepsilon (t)\|_s \, dt'$$

where K *depends only on the norm of* v^ε *in* $E^s (t_1)$, *and* C *is independent of* ε, v^ε *and* f^ε.

Proof. – Proposition 4.3.1 shows that the L^2 energy estimate (4.2.3) for L^ε can be transported to $\hat{L}^\varepsilon$. This yields an estimate of the form

$$(4.3.20) \qquad \|\hat{u}^\varepsilon (t)\|_0 \leq C e^{Kt}\|\hat{u}^\varepsilon(0)\|_0 + C \int_0^t e^{K (t-t')}\|\hat{L}^\varepsilon \hat{u}^\varepsilon (t')\|_0 \, dt'$$

where C depends only on V, not on v^ε. Here we use that the L^∞-norm of $\nabla_{x,\theta} v^\varepsilon$ is dominated by the $E^s (t_1)$ norm of v^ε since $s > 1 + (d + m)/2$. Using proposition 4.3.3,

lemma 4.3.2 and applying estimate (4.3.20) to the derivatives $\hat{u}_\alpha^\varepsilon$, we get the following estimate

$$(4.3.21) \qquad \|\hat{u}^\varepsilon(t)\|_s \leqq C\,e^{Kt}\|\hat{u}^\varepsilon(0)\|_s + C\int_0^t e^{K(t-t')}\|\hat{L}^\varepsilon\,\hat{u}^\varepsilon(t')\|_s\,dt'$$

$$+ CK\int_0^t e^{K(t-t')}\|\hat{u}^\varepsilon(t')\|_s\,dt'.$$

Applying Gronwall's lemma and going back to L^ε, once more using proposition 4.3.1, leads to estimate (4.3.19).

4.4. *Proof of Theorem* 4.1.3. — Consider the iterative scheme

$$(4.4.1) \quad \partial_t\,u_{\nu+1}^\varepsilon + \sum_{j=1}^d A_j\,(t,\,x,\,\varepsilon\,u_\nu^\varepsilon)\,\partial_{x_j}\,u_{\nu+1}^\varepsilon + \varepsilon^{-1}\sum_{l=1}^m B_k\,(t,\,x,\,\varepsilon\,u_\nu^\varepsilon)\,\partial_{\theta_l}\,u_{\nu+1}^\varepsilon$$

$$= F\,(t,\,x,\,u_\nu^\varepsilon).$$

For $\nu = 0$, start with a function $u_0^\varepsilon\,(t,\,x,\,\theta)$, such that $u_0^\varepsilon\,(0,\,x,\,\theta) = h^\varepsilon\,(x,\,\theta)$, the family u_0^ε being bounded in $E^s\,(t_o)$.

PROPOSITION 4.4.1. – *Let h^ε be a bounded family in $H^s\,(\omega \times \mathbb{T}^m)$ with $s > 1+(d+m)/2$. Then there is $\varepsilon_1 > 0$ and $t_1 > 0$ such that*

(i) *for all $\varepsilon \leqq \varepsilon_1$, the Cauchy problem* (4.4.1) (4.1.7) *has a (unique) solution $u_{\nu+1}^\varepsilon$ in $E^s\,(t_1)$ and the family u_ν^ε is bounded in $E^s\,(t_1)$.*

(ii) *The sequence u_ν^ε converges to the solution u^ε of* (4.1.1) (4.1.7) *in $E^{s'}\,(t_1)$ for all $s' < s$, uniformly with respect to ε.*

(iii) *$u^\varepsilon \in E^s\,(t_1)$ and the family u^ε is bounded in $E^s\,(t_1)$.*

Proof. – a) Let R be such that $\|u_0^\varepsilon\,(t)\|_s \leqq R$ for all ε and all $t \in [0,\,t_o]$. In particular $\|h^\varepsilon\|_s \leqq R$. Assume that for all $t \in [0,\,t_1]$:

$$(4.4.2) \qquad\qquad \|u_\nu^\varepsilon\,(t)\|_s \leqq C\,(R+1)$$

where C is the constant that appears in (4.3.19). Then, from the Sobolev imbedding theorem, we see that there is $\varepsilon_1 > 0$ such that for $\varepsilon \leqq \varepsilon_1$, (4.4.2) implies that:

$$(4.4.3) \qquad\qquad \varepsilon\|u_\nu^\varepsilon\|_{L^\infty} \leqq 1.$$

Then proposition 4.3.4 applies and with lemma 4.3.2 we conclude that there is $K = K\,(R)$ and $M\,(R)$ such that

$$(4.4.4) \qquad \|u_{\nu+1}^\varepsilon\,(t)\|_s \leqq Ce^{Kt}\{R + t\,M\,(C\,(R+1))\} \leqq C\,(R+1)$$

the last inequality beeing valid provided that $t \leqq t_1$ is small enough.

We can always assume that $C \geq 1$, so that (4.4.2) certainly holds for $\nu = 0$. Therefore, we conclude that (4.4.2) holds for all $\varepsilon \leq \varepsilon_1$, $t \leq t_1$ and $\nu \in \mathbb{N}$.

b) Using the uniform bound (4.4.2), writing the equation for $u_{\nu+1}^\varepsilon - u_\nu^\varepsilon$ and using the L^2 estimate (4.2.3), one gets the following

$$(4.4.5) \qquad \| \{ u_{\nu+1}^\varepsilon - u_\nu^\varepsilon \} (t) \|_0 \leq \int_0^t \| \{ u_\nu^\varepsilon - u_{\nu-1}^\varepsilon \} (t') \|_0 \, dt'$$

from which it follows that u_ν^ε converges in $E^0(t_1)$, uniformly in ε. Because of the bound (4.4.2), the convergence also holds in $E^{s'}(t_1)$, for all $s' < s$.

c) The uniform bound (4.4.2) shows that for all $t \in [0, t_1]$, $u^\varepsilon(t)$ belongs to $H^s(\omega_t \times \mathbb{T}^m)$, with a uniform estimate of its norm in this space. On the other hand, for a fixed $\varepsilon > 0$, equation (4.1.1) is a classical symmetric hyperbolic system, and the classical results show that u^ε is indeed continuous in time with values in H^s.

5. EXISTENCE OF SOLUTIONS. – The main purpose of this section is to prove part (i) of theorem 2.3.5.

5.1. *Reduction to a singular problem.* – We suppose from now on that the assumptions of theorem 2.3.5 are fullfilled. In particular, there is $\varphi_0 \in \Phi$ such that $\varphi_{0|_{t=0}} = 0$ and $\partial_t \varphi_{0|_{t=0}} \neq 0$. One can perform a change of variables, in such a way that $\varphi_0 \equiv t$. This does not affect the form of the equations nor the assumptions. On the other hand, Ω is slightly perturbed, but one can always decrease t_0 and δ so that theorem 2.3.5 in the old variables can be deduced from the same statement in the new variables. So, from now on we assume that

$$(5.1.1) \qquad \text{the function } \varphi_0(t) \equiv t \text{ belongs to } \Phi.$$

Let $\Phi^\circ := \{ \varphi_{|_{t=0}}, \varphi \in \Phi \}$ as in (2.3.3). Lemma 2.3.2 implies

LEMMA 5.1.1. – *Let* $\psi_0 := \varphi_0, \psi_1, \ldots, \psi_\mu$ *be a basis of* Φ. *For* $k = 1, \ldots, \mu$ *let* $\psi_k^\circ = \psi_{k|_{t=0}}$. *Then* $(\psi_1^\circ, \ldots, \psi_\mu^\circ)$ *is a basis of* Φ°.

Fix a basis $\psi_0 := \varphi_0, \psi_1, \ldots, \psi_\mu$ of Φ, and the corresponding basis $\psi_1^\circ, \ldots, \psi_\mu^\circ$ of Φ°. According to formulas (2.3.6) (2.3.7), we are given a family $(\varphi_1^\circ, \ldots, \varphi_m^\circ)$ of initial phases. They belong to $\Phi^\circ \backslash \{ 0 \}$ and are $\mathbb{Q}$-linearly independent. Denote by R° the $m \times \mu$ constant matrix such that

$$(5.1.2) \qquad {}^t(\varphi_1^\circ, \ldots, \varphi_m^\circ) = R^\circ \, {}^t(\psi_1^\circ, \ldots, \psi_\mu^\circ).$$

Then

$$(5.1.3) \qquad {}^t(\varphi_1, \ldots, \varphi_m) = R^\circ \, {}^t(\psi_1, \ldots, \psi_m) \in \Phi^m$$

and $\varphi_j(0, x) = \varphi_j^\circ(x)$. Set $\vec{\varphi}^{\,\circ} := {}^t(\varphi_1^\circ, \ldots, \varphi_m^\circ)$ and $\vec{\varphi} := {}^t(\varphi_1, \ldots, \varphi_m)$.

We now look for u^ε in the form

$$(5.1.4) \qquad\qquad u^\varepsilon(t, x) = U^\varepsilon(t, x, \vec{\varphi}(t, x)/\varepsilon).$$

For u^ε to satisfy (2.1.3) (2.3.5) it is *sufficient* that $U^\varepsilon(t, x, \theta)$ solves

$$(5.1.5) \qquad \partial_t U^\varepsilon + \sum_{j=1}^d A_j(t, x, \varepsilon U^\varepsilon) \partial_{x_j} U^\varepsilon + \varepsilon^{-1} \sum_{k=1}^m B_k(t, x, \varepsilon U^\varepsilon) \partial_{\theta_k} U^\varepsilon$$
$$= F(t, x, \varepsilon U^\varepsilon, U^\varepsilon)$$

$$(5.1.6) \qquad\qquad U^\varepsilon(0, x, \theta) = \mathcal{H}^\varepsilon(x, \theta)$$

with

$$(5.1.7) \qquad B_k(t, x, v) = \partial_t \varphi_k \, \mathrm{Id} + \sum_{j=1}^d \partial_{x_j} \varphi_k(t, x) A_j(t, x, v).$$

This is an operator of form (4.1.1) with Q given by

$$(5.1.8) \qquad Q(t, x, \partial_\theta) := \sum_{k=1}^m B_k(t, x, 0) \partial_{\theta_l} = \sum_{k=1}^m L(t, x; d\varphi_k(t, x)) \partial_{\theta_k}.$$

PROPOSITION 5.1.2. – *Q satisfies Assumption 4.1.2.*

The proof is given in the next subsection. Thus Proposition 4.4.1 can be applied. The corresponding scheme (4.4.1) is

$$(5.1.9) \ \partial_t U^\varepsilon_{\nu+1} + \sum_{j=1}^d A_j(t, x, \varepsilon U^\varepsilon_\nu) \partial_{x_j} U^\varepsilon_{\nu+1} + \varepsilon^{-1} \sum_{l=1}^m B_k(t, x, \varepsilon U^\varepsilon_\nu) \partial_{\theta_l} U^\varepsilon_{\nu+1}$$
$$= F(t, x, \varepsilon U^\varepsilon_\nu, U^\varepsilon_\nu)$$

$$(5.1.10) \qquad\qquad U^\varepsilon_{\nu+1}(0, x, \theta) = \mathcal{H}^\varepsilon(x, \theta).$$

The choice of U^ε_0 does not really matter. For simplicity, choose $U^\varepsilon_0(t, x, \theta) := \mathcal{H}^\varepsilon(x, \theta)$.

PROPOSITION 5.1.3. – *Assume that the hypothesis of theorem* 2.3.5 *and* (5.1.1) *are satisfied. Then there is* $\varepsilon_1 > 0$ *and* $t_1 > 0$ *such that*

(i) *for all* $\varepsilon \leq \varepsilon_1$, *the Cauchy problem* (5.1.9) (5.1.10) *has a (unique) solution* $U^\varepsilon_{\nu+1}$ *in* $E^s(t_1)$ *and the family* U^ε_ν *is bounded in* $E^s(t_1)$.

(ii) *The sequence* U^ε_ν *converges to the solution* U^ε *of* (5.1.5) (5.1.6) *in* $E^{s'}(t_1)$ *for all* $s' < s$, *uniformly with respect to* ε.

(iii) $U^\varepsilon \in E^s(t_1)$ *and the family* U^ε *is bounded in* $E^s(t_1)$.

(iv) *The family u^ε given by* (5.1.4) *is a family of (smooth) solutions of* (2.1.3) (2.3.5) *on the same domain* $\Omega_{t_1} = \Omega \cap \{\, 0 \leq t \leq t_1 \,\}$.

5.2. *Proof of proposition* 5.1.2. – *a*) Introduce

$$(5.2.1) \qquad\qquad A(t, x; \xi) = \sum_{j=1}^{d} \xi_j A_j(t, x, 0)$$

so that

$$(5.2.2) \qquad\qquad Q(t, x, \sigma) = \sigma \cdot \partial_t \vec{\varphi} + A(t, x, \sigma \cdot \partial_x \vec{\varphi})$$

with the obvious notation $\sigma \cdot \partial \vec{\varphi} = \sum \sigma_k \partial \varphi_k$. We also make use of the basis ψ_0, $\psi_1, \ldots, \psi_\mu$ of Φ and introduce for $(t, x, \gamma') \in \Omega \times \mathbb{R}^\mu$, $\gamma' := (\gamma_1, \ldots, \gamma_\mu)$

$$(5.2.3) \quad N(t, x, \gamma') := L(t, x; \gamma' \cdot d\vec{\psi}'(t, x)) = \gamma' \cdot \partial_t \vec{\psi}' \,\mathrm{Id} + (A(t, x; \gamma' \cdot \partial_x \vec{\psi}'(t, x))$$

where $\gamma' \cdot \partial \vec{\psi} = (\gamma_1 \partial \psi_1 + \cdots + \gamma_\mu \partial \psi_\mu)$. Because $\psi_0 \equiv t$, the matrix M introduced in (2.2.5) is

$$(5.2.4) \qquad M(t, x, \gamma) = \gamma_0 \,\mathrm{Id} + N(t, x, \gamma') \qquad \text{if} \quad \gamma = (\gamma_0, \gamma') \in \mathbb{R} \times \mathbb{R}^\mu.$$

With the matrix R° as in (5.1.3), we have:

$$(5.2.5) \qquad\qquad Q(t, x, \sigma) = N(t, x, {}^t R^\circ \sigma)$$

b) Let $\gamma' \in \mathbb{R}^\mu \backslash \{\, 0 \,\}$ and let $\psi := \gamma_1 \psi_1 + \cdots + \gamma_\mu \psi_\mu$. Then $\partial_x \psi$ never vanishes on Ω. If one had $\partial_x \psi(\underline{t}, \underline{x}) = 0$, then $\chi := \psi - \partial_t \psi(\underline{t}, \underline{x}) \psi_0$ would be an element of Φ such that $d_{t,x} \chi(\underline{t}, \underline{x}) = 0$. The coherence assumption would imply that $\chi \equiv 0$, which is impossible since $\psi_0, \ldots, \psi_\mu$ are linearly independent and $\gamma' \neq 0$.

c) Part *b*) and Assumption 2.1.2 imply that $N(t, x, \gamma')$ is a smooth family of symmetric matrices with eigenvalues of constant multiplicity for $\gamma' \neq 0$. Thus, for any $\underline{\gamma}' \neq 0$, there is a conical neighorhood Γ of $\underline{\gamma}'$ and an orthogonal matrix $W(t, x, \gamma')$ such that

$$(5.2.6) \qquad W^{-1}(t, x, \gamma') N(t, x, \gamma') W(t, x, \gamma') = \Lambda(t, x, \gamma')$$

where Λ diagonal with entries the eigenvalues of N repeated accordingly to their multiplicity. Here, W is a smooth function of (t, x, γ') for $\gamma' \in \Gamma$, homogeneous of degree zero in γ'. In particular, the families $\{\, W(\cdot, \cdot, \gamma') \,\}_{\gamma' \in \Gamma}$, $\{\, W^{-1}(\cdot, \cdot, \gamma') \,\}_{\gamma' \in \Gamma}$ are bounded in $C^\infty(\Omega)$.

Let Γ_l be a finite covering of $\mathbb{R}^m \backslash \{\, 0 \,\}$ by such open cones and let W_l, Λ_l be the associated matrices. There is a partition of $\mathbb{R}^m \backslash \{\, 0 \,\}$ by disjoints conical sets $G_l \subset \Gamma_l$. Define $W(t, x, \gamma') = W_l(t, x, \gamma')$, $\Lambda(t, x, \gamma') = \Lambda_l(t, x, \gamma')$ when $\gamma' \in G_l$. Then

(5.2.6) holds for all points $(t, x, \gamma') \in \Omega \times \mathbb{R}^m \backslash \{ 0 \}$. We extend the definitions to $\gamma' = 0$ by setting $W(t, x, 0) = \mathrm{Id}$, $\Lambda(t, x, 0) = 0$. Note that $N(t, x, 0) = 0$ so that (5.2.6) also holds when $\gamma' = 0$. Thus the whole family $\{ W(\cdot, \cdot, \gamma'), W^{-1}(\cdot, \cdot, \gamma'), \gamma' \in \mathbb{R}^\mu \}$ is bounded in $C^\infty(\Omega)$.

d) $\Lambda(t, x, \gamma')$ is diagonal with entries equal to the eigenvalues of $N(t, x, \gamma')$. Because of (5.2.4) these eigenvalues are the solutions λ of $\det M(t, x, (-\lambda, \gamma')) = 0$. Remark 2.2.3 and the coherence assumption imply that the roots of $\det M(t, x, (\alpha_0, \alpha)) = 0$ do not depend on (t, x). Therefore $\Lambda(t, x, \gamma') = \Lambda(0, 0, \gamma')$ does not depend on $(t, x) \in \Omega$.

e) Define

$$(5.2.7) \qquad V(t, x, \sigma) = W(t, x, {}^t\!R^\circ \sigma).$$

Then the family $\{ V(\cdot, \cdot, \sigma), V^{-1}(\cdot, \cdot, \gamma') \}_{\sigma \in \mathbb{R}^m}$ is bounded in $C^\infty(\Omega)$. Moreover (5.2.5) (5.2.6) show that

$$(5.2.8) \qquad Q^\sharp(t, x, \sigma) := V^{-1}(t, x, \sigma) Q(t, x, \sigma) V(t, x, \sigma)$$

is the diagonal matrix $\Lambda(t, x, {}^t\!R^\circ \sigma)$ which by *d)* does not depend on (t, x). The proof of Proposition 5.1.2 is complete.

6. EXISTENCE OF PROFILES. – The aim of this section is to study the system (2.3.9) for the profiles. First, we describe function spaces and the averaging operator $\mathbb{E}$. Then we consider the existence theorem of solutions to (2.3.9).

Assume that the hypotheses of theorem 2.3.5 are satisfiedd, including the strong form (5.1.1). In addition to φ_o, we are given phases $(\varphi_1, \ldots, \varphi_m)$ which satisfy (5.1.2) (5.1.3). Profiles are functions of $(t, x) \in \Omega$ and of the fast variables $(\tau, \theta) \in \mathbb{R} \times \mathbb{T}^m$.

6.1. *Spaces of profiles.* – Recall that Ω is given by (2.3.1), with parameters $0 < t_0 < \rho\delta$; ω_t denotes the ball of radius $\rho - t/\delta$ centered at the origin, in $\mathbb{R}^d$. For $t_1 \leq t_0$, $\mathcal{E}^0(t_1)$ is the space of functions $\mathcal{U}(t, x, \tau, \theta)$ on $\Omega_{t_1} \times \mathbb{R} \times \mathbb{T}^m$ such that their extension by 0 outside Ω_{t_1} is continuous and bounded in $(t, \tau) \in [0, t_1] \times \mathbb{R}$ with values in $L^2(\mathbb{R}^d \times \mathbb{T}^m)$. For an integer $s \geq 0$, $\mathcal{E}^s(t_1)$ is the space of the $\mathcal{U} \in E^0(t_1)$ whose (x, θ)-derivatives $\partial_{x,\theta}^\alpha \mathcal{U}$ belong to $\mathcal{E}^0(t_1)$ for $|\alpha| \leq s$. For $\mathcal{U} \in \mathcal{E}^s(t_1)$ we denote by $\|\mathcal{U}(t, \tau)\|_s$ the $H^s(\omega_t \times \mathbb{T}^m)$-norm of $\mathcal{U}(t, \cdot, \tau, \cdot)$:

$$(6.1.1) \qquad \|\mathcal{U}(t, \tau)\|_s := \|\mathcal{U}(t, \cdot, \tau, \cdot)\|_{H^s(\omega_t \times \mathbb{T}^m)}$$

and

$$(6.1.2) \qquad \|\mathcal{U}\|_{\mathcal{E}^s(t_1)} = \sup_{0 \leq t \leq t_1, \tau \in \mathbb{R}} \|\mathcal{U}(t, \tau)\|_s.$$

DEFINITION 6.1.1. – (i) *By trigonometric polynomial we mean any finite sum*

$$(6.1.3) \qquad \mathcal{U}(t, x, \tau, \theta) = \sum_{\lambda, \alpha} U_{\lambda, \alpha}(t, x) e^{i(\lambda\tau + \alpha \cdot \theta)},$$

where the summation runs over a finite subset of $\mathbb{R} \times \mathbb{Z}^m$, *and where the coefficients* $U_{\lambda, \alpha}$ *are* C^∞ *functions with compact support in* $[0, t_0] \mathbb{R}^n$.

(ii) *For* $s \in \mathbb{N}$, $\mathcal{P}^s(t_1)$ *denotes the closure in* $\mathcal{E}^s(t_1)$ *of the space of trigonometric polynomials.*

LEMMA 6.1.2. – *For* $s > (d+m)/2$, $\mathcal{E}^s(t_1)$ *and* $\mathcal{P}^s(t_1)$ *are Banach algebras. Moreover, if* F *is a* C^∞ *function, there is a function of one real variable* $F^*(K)$, *such that for all* $t_1 \leq t_0$, *the mapping* $u \to F(t, x, \mathcal{U}(t, x, \tau, \theta))$ *is a continuous map of the ball of radius* K *in* $\mathcal{E}^s(t_1)$ *[resp.* $\mathcal{P}^s(t_1)$*] into the ball of radius* $F^*(K)$ *in* $\mathcal{E}^s(t_1)$ *[resp.* $\mathcal{P}^s(t_1)$*].*

Proof. – The result for $\mathcal{E}^s(t_1)$ is a immediate consequence of the Schauder lemma in $H^s(\omega_t \times \mathbb{T}^m)$ with (t, τ) acting as parameters.

It remains to show that $F(t, x, \mathcal{U}(t, x, \tau, \theta))$, in short $F(\mathcal{U})$, belongs to $\mathcal{P}^s(t_1)$ if u does. First, we remark that if $F - G$ is small in some C^N_{loc} norm, then for $\mathcal{U} \in \mathcal{E}^s(t_1)$, $F(\mathcal{U}) - G(\mathcal{U})$ is small in $\mathcal{E}^s(t_1)$. Next, by continuity, if $\mathcal{U} - \mathcal{V}$ is small in $\mathcal{E}^s(t_1)$, $G(\mathcal{U}) - G(\mathcal{V})$ is also small. Thus, approximating F by polynomials G and $\mathcal{U}$ by trigonometric polynomials $\mathcal{V}$, provides approximations in $\mathcal{E}^s(t_1)$, of $F(\mathcal{U})$ by trigonometric polynomials $G(\mathcal{V})$. The lemma follows.

6.2. *Averaging operators.* – Next consider the averaging operator $\mathbb{E}$. For later purposes, we need to generalize it. Recall that the operator P is defined in (2.3.10) and that $\mathcal{C}$ denotes the set

(6.2.1) $$\mathcal{C} := \operatorname{Char} P \cap (\mathbb{R} \times \mathbb{Z}^m).$$

More generally, if Λ is a subset of $\mathbb{R} \times \mathbb{R}^m$, we call $\mathcal{C}_\Lambda := \mathcal{C} \cap \Lambda$, and the projector $\mathbb{E}_\Lambda$ is defined on trigonometric polynomials (6.1.3) by the formula

(6.2.2) $$\mathbb{E}_\Lambda \mathcal{U}(t, x, \tau, \theta) = \sum_{(\lambda, \alpha) \in \mathcal{C}_\Lambda} \{\Pi(t, x, \lambda, \alpha) U_{\lambda, \alpha}(t, x)\} e^{i(\lambda\tau + \alpha \cdot \theta)},$$

where, for $(\lambda, \alpha) \in \operatorname{Char} P$, $\Pi(t, x, \lambda, \alpha)$ is the orthogonal projector in $\mathbb{C}^N$ on the kernel of $P(t, x, \lambda, \alpha)$. With these notations, one has

PROPOSITION 6.2.1. – $\mathbb{E}_\Lambda$ *has a unique continuous extension to an operator from* $\mathcal{P}^0(t_1)$ *to* $\mathcal{P}^0(t_1)$ *which by restriction maps* $\mathcal{P}^s(t_1)$ *to* $\mathcal{P}^s(t_1)$. *Furthermore, the norm of these operators is bounded uniformly with respect to* t_1, *for* $0 \leq t_1 \leq t_0$.

Proof. – First, observe that for all $\alpha \in \mathbb{Z}^m$, the number of λ such that $(\lambda, \alpha) \in \mathcal{C}$ is not larger than N. Second, the projectors $\Pi(t, x, \lambda, \alpha)$ have norm ≤ 1 in $\mathbb{C}^N$.

For a trigonometric polynomial $\mathcal{U}$, introduce the following notation

(6.2.3) $$\mathcal{U}_\alpha(t, x, \tau) = \sum_\lambda U_{\lambda, \alpha}(t, x) e^{i\lambda\tau}$$

and similarly

(6.2.4) $$(\mathbb{E}_\Lambda \mathcal{U})_\alpha(t, x, \tau) = \sum_{\{\lambda | (\lambda, \alpha) \in \mathcal{C}_\Lambda\}} \{\Pi(t, x, \lambda, \alpha) U_{\lambda, \alpha}(t, x)\} e^{i\lambda\tau}.$$

Then

$$(6.2.5) \qquad |(\mathbb{E}_\Lambda \mathcal{U}_\alpha)(t,\, x,\, \tau)|^2 \leqq N \sum_{\{\,\lambda\,|\,(\lambda,\,\alpha)\,\in\,\mathcal{C}_\Lambda\,\}} |U_{\lambda,\,\alpha}(t,\, x)|^2.$$

On the other hand, (6.2.3) implies that

$$(6.2.6) \qquad U_{\alpha,\,\lambda}(t,\, x) = \lim_{R\to\infty} R^{-1} \int_0^R \mathcal{U}_\alpha(t,\, x,\, T)\, e^{-i\lambda T}\, dT.$$

Thus

$$(6.2.7) \qquad |(\mathbb{E}_\Lambda \mathcal{U})_\alpha(t,\, x,\, \tau)|^2 \leqq N \lim_{R\to\infty} R^{-1} \int_0^R |\mathcal{U}_\alpha(t,\, x,\, \tau')|^2 d\tau'.$$

Fatou's lemma implies

$$(6.2.8) \quad |(\mathbb{E}_\Lambda \mathcal{U})_\alpha(t,\, \cdot,\, \tau)\|^2_{L^2(\omega_t)} \leqq N \lim_{R\to\infty} R^{-1} \int_0^R \sum \|\mathcal{U}_\alpha(t,\, \cdot,\, \tau')\|^2_{L^2(\omega_t)} d\tau' =$$

$$(6.2.9) \qquad = N \lim_{R\to\infty} R^{-1} \int_0^R \|\mathcal{U}(t,\, \cdot,\, \tau',\, \cdot)\|^2_{L^2(\omega_t \times \mathbb{T}^n)} d\tau'$$

$$\leqq N \sup_{\tau'} \|\mathcal{U}(t,\, \cdot,\, \tau',\, \cdot)\|^2_{L^2(\omega_t \times \mathbb{T}^n)}$$

Hence

$$(6.2.10) \qquad \|\mathbb{E}_\Lambda \mathcal{U}\|_{\mathcal{E}^0(t_1)} \leqq \sqrt{N} \|\mathcal{U}\|_{\mathcal{E}^0(t_1)}$$

and therefore $\mathbb{E}_\Lambda$ extends from $\mathcal{P}^0(t_1)$ into itself.

The proof of the $\mathcal{E}^s$ estimates is similar. $\mathbb{E}_\Lambda$ commutes with derivations ∂_θ. To estimate the commutator with ∂_x-derivatives, note that condition (5.1.1) implies P has the form

$$(6.2.11) \qquad P(t,\, x,\, \partial_\tau,\, \partial_\theta) = \partial_\tau + Q(t,\, x,\, \partial_\theta),$$

with Q as in (5.1.8). Hence, for $(\lambda,\, \alpha\} \in \mathcal{C}$, the projector $\Pi(t,\, x,\, \lambda,\, \alpha)$ on the kernel of $P(t,\, x,\, \lambda,\, \alpha)$ is a spectral projector of $Q(t,\, x,\, \alpha)$. The diagonalization (5.2.8) shows that the family $\{\,\Pi(\cdot,\, \cdot,\, \lambda,\, \alpha),\, (\lambda,\, \alpha) \in \mathcal{C}\,\}$ is bounded in $C^\infty(\Omega)$. Therefore the commutators of $\mathbb{E}$ with ∂_x-derivatives have the same form as (6.2.4). Thus one gets that, for some constant C independent of t_1, and for any trigonometric polynomial $\mathcal{U}$, one has

$$(6.2.12) \qquad \|\mathbb{E}_\Lambda \mathcal{U}\|_{\mathcal{E}^s(t_1)} \leqq C \|\mathcal{U}\|_{\mathcal{E}^s(t_1)}$$

so that $\mathbb{E}_\Lambda$ maps $\mathcal{P}^s(t_1)$ into itself.

DEFINITION. 6.2.2 .– *Let* $\mathbb{E} := \mathbb{E}_\Lambda$ *with* $\Lambda = \mathbb{R} \times \mathbb{Z}^n$. *Let* $\mathcal{N}^s(t_1)$ *be the range* $\mathbb{E}\,\mathcal{P}^s(t_1)$.

Since $\mathbb{E}$ is a bounded projector, $\mathcal{N}^s(t_1)$ is also the kernel in $\mathcal{P}^s(t_1)$ of Id-$\mathbb{E}$, and therefore, $\mathcal{N}^s(t_1)$ is a closed subspace of $\mathcal{P}(t_1)$.

LEMMA 6.2.3. – $\mathcal{N}^s(t_1)$ *is the kernel in* $\mathcal{P}^s(t_1)$ *of the operator* $P(t, x, \partial_\tau, \partial_\theta)$.

Proof. – Let $\mathcal{U} \in \mathcal{P}^0(t_1)$ be such that $P\mathcal{U} = 0$. Then one can expand $\mathcal{U}$ into its Fourier series in θ

$$(6.2.13) \qquad \mathcal{U}(t, x, \tau, \theta) = \sum_{\alpha \in \mathbb{Z}^m} \mathcal{U}_\alpha(t, x, \tau)\, e^{i\alpha \cdot \theta}.$$

Integrating by parts $\displaystyle\int P\mathcal{U}\, e^{-i\alpha\theta}\, d\theta$, the condition $P\mathcal{U} = 0$ and (5.2.11) imply that for all $\alpha \in \mathbb{Z}^m$

$$(6.2.14) \qquad \partial_\tau \mathcal{U}_\alpha + i\, Q(t, x, \alpha)\mathcal{U}_\alpha = 0$$

and hence

$$(6.2.15) \qquad \mathcal{U}_\alpha(t, x, \tau) = \sum_{\lambda \in \mathcal{C}(\alpha)} \Pi(t, x, \lambda, \alpha)\, \mathcal{U}_\alpha(t, x, 0)\, e^{i\lambda\tau},$$

with $\mathcal{C}(\alpha) = \{\lambda \in \mathbb{R} | (\lambda, \alpha) \in \mathcal{C}\}$. This implies that $\mathbb{E}\mathcal{U} = \mathcal{U}$.

Conversely, if $\mathcal{U} = \mathbb{E}\mathcal{U}$ is a limit of trigonometric polynomials $\mathcal{U}^\nu$, then by continuity of $\mathbb{E}$, $\mathcal{U}$ is also the limit of the trigonometric polynomials $\mathbb{E}\mathcal{U}^\nu$ which satisfy $P(\mathbb{E}\mathcal{U}^\nu) = 0$. Passing to the limit yields $P\mathcal{U} = 0$.

We use the following scalar product for t fixed. Let

$$(6.2.16) \quad \langle \mathcal{U}, \mathcal{V} \rangle(t) := \lim_{\rho \to \infty} \rho^{-1} \int_0^\rho \left\{ \int_{\omega_t \times \mathbb{T}^m} \langle \mathcal{U}(t, x, \tau, \theta), \mathcal{V}(t, x, \tau, \theta) \rangle\, dx\, d\theta \right\} d\tau.$$

The scalar product in the integral is taken in $\mathbb{C}^N$. For trigonometric polynomials (6.1.3), one immediately checks that the limit exists and satisfies

$$(6.2.17) \qquad \langle \mathcal{U}, \mathcal{V} \rangle(t) = (2\pi)^n \sum_{(\lambda, \alpha)} \langle U_{\lambda, \alpha}, V_{\lambda, \alpha} \rangle_{L^2(\omega_t)},$$

$$(6.2.18) \qquad |\langle \mathcal{U}, \mathcal{V} \rangle(t)| \leq \|\mathcal{U}\|_{\mathcal{E}^0(t_1)} \|\mathcal{V}\|_{\mathcal{E}^0(t_1)}.$$

This shows that in (6.1.16), the limit exists for all $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{P}^0(t_1)$ and satisfies (6.2.17) (6.2.18), the Fourier series (6.2.13) having its usual sense.

LEMMA 6.2.4. – *For* $\mathcal{U} \in \mathcal{N}^0(t_1)$ *and for* $t \in [0, t_1]$ *and* $\tau \in \mathbb{R}$ *one has*

$$(6.2.19) \qquad \|\mathcal{U}(t, \tau)\|_0 = \|\mathcal{U}(t, 0)\|_0 = \{\langle \mathcal{U}, \mathcal{U} \rangle(t)\}^{1/2}.$$

Proof. – If $\mathcal{U} \in \mathcal{N}^0(t_1)$, then $\mathcal{U}$ has the form (6.2.13) (6.2.15). For every fixed $\alpha \in \mathbb{Z}^m$, the projectors $\Pi(t, x, \lambda, \alpha)$, $\lambda \in \mathcal{C}(\alpha)$, are mutually orthogonal and $\sum_{\lambda \in c(\alpha)} \Pi(t, x, \lambda, \alpha) = \mathrm{Id}_{\mathbb{C}^N}$. Thus (6.2.15) implies that for all α, t, x and τ

$$(6.2.20) \qquad |\mathcal{U}_\alpha(t, x, \tau)|^2 = \sum_{\lambda \in \mathcal{C}(\alpha)} |\Pi(t, x, \lambda, \alpha)\,\mathcal{U}_\alpha(t, x, 0)|^2 = |\mathcal{U}_\alpha(t, x, 0)|^2.$$

Integrating in x gives (6.2.19).

Remark 6.2.5. – For any $\Lambda \subset \mathbb{R} \times \mathbb{R}^m$, $\mathbb{E}_\Lambda$ is a projector in $\mathcal{P}^0(t_1)$ whose range is contained in $\mathcal{N}^0(t_1)$. Because the $\Pi(t, x, \lambda, \alpha)$ are orthogonal projector in $\mathbb{C}^N$, it can be checked on trigonometric polynomials, and hence on $\mathcal{P}^0(t_1)$, that $\mathbb{E}$ is orthogonal for the scalar product

$$(6.2.21) \qquad \langle \mathbb{E}\,\mathcal{U}, \mathcal{V} \rangle(t) = \langle \mathcal{U}, \mathbb{E}\,\mathcal{V} \rangle(t) = \langle \mathbb{E}\,\mathcal{U}, \mathbb{E}\,\mathcal{V} \rangle(t).$$

6.3. *The linearized equations.* – We solve (2.3.9) by an iterative method. First consider the linear systems

$$(6.3.1) \qquad \begin{cases} \mathcal{U} \in \mathcal{N}^s(t_1), \\ \mathcal{L}\mathcal{U} := \mathbb{E}\{L(t, x, \partial_t, \partial_x)\mathcal{U} + \underline{B}(t, x, \mathcal{V})\partial_\theta \mathcal{U}\} = \mathbb{E}\,\mathcal{F}, \\ \mathcal{U} =_{|t=0, \tau=0}(x, \theta) = \mathcal{H}(x, \theta). \end{cases}$$

with

$$(6.3.2) \qquad \underline{B}(t, x, u)\,\partial_\theta := \sum_{j=1}^{m} \underline{B}_j(t, x, u)\,\partial_{\theta_j}$$

and

$$\underline{B}_j(t, x, u) := \sum_{l=1}^{d} \partial_l\,\varphi_j(t, x)\left\{\frac{\partial A_l}{\partial u}(t, x, 0)\cdot u + \frac{\partial A_l}{\partial \bar{u}}(t, x, 0)\cdot \bar{u}\right\}.$$

$\underline{B}$ is the same as in (2.3.9), but it is important to remark that condition (5.1.1) implies that no ∂_τ derivative appears.

In (6.3.1), $\mathcal{F}$ and $\mathcal{V}$ are given members of $\mathcal{P}^s(t_1)$.

The main idea is that $\mathcal{L}$ is symmetric hyperbolic in $\mathcal{N}^0$. In particular, one has the following L^2 estimate.

LEMMA 6.3.1. – *Let* $\mathcal{V} \in \mathcal{P}^s(t_1)$, $s > 1 + (d+m)/2$, $\mathcal{U} \in \mathcal{N}^1(t_1)$ *and* $\mathcal{F} \in \mathcal{P}^0(t_1)$ *satisfy* (6.3.1). *Then for all* $t \in [0, t_1]$

$$(6.3.3) \qquad \langle \mathcal{U}, \mathcal{U} \rangle(t) \leq e^{Ct}\|\mathcal{H}\|_0^2 + C\int_0^t e^{C(t-t')}\langle \mathcal{F}, \mathcal{F} \rangle(t')\,dt',$$

where C only depends on the $L^\infty\left(\Omega_{t_1} \times \mathbb{R} \times \mathbb{T}^m\right)$ norm of $\partial_\theta V$.

Proof. – Assume $\mathcal{U} \in \mathcal{N}^1\left(t_1\right)$. Because $\mathcal{U} = \mathbb{E}\mathcal{U}$, remark 6.2.5 shows

$$\langle \mathcal{L}\mathcal{U}, \mathcal{U} \rangle\left(t\right) = \langle L\left(t, x, \partial_t, \partial_x\right)\mathcal{U} + \underline{B}\left(t, x, V\right)\partial_\theta \mathcal{U}, \mathcal{U} \rangle\left(t\right).$$

Definition (6.2.16) together with an integration by parts yields

$$(6.3.4)\quad 2\operatorname{Re}\langle \mathcal{L}\mathcal{U}, \mathcal{U} \rangle\left(t\right) = \frac{d}{dt}\langle \mathcal{U}, \mathcal{U} \rangle\left(t\right)$$

$$+ \operatorname{Re}\langle \partial_x \underline{A}\mathcal{U}, \mathcal{U} \rangle\left(t\right) + \operatorname{Re}\langle \underline{B}\left(t, x, \partial_\theta V\right)\mathcal{U}, \mathcal{U} \rangle\left(t\right)$$

with $\partial_x \underline{A} := \sum \partial_{x_j} A_j\left(t, x, 0\right)$. To derive (6.3.4) we use that $\underline{B}$ is linear in V.

Introduce the following notation. For $\mathcal{F} \in \mathcal{P}^s\left(t_1\right)$ and for $t \in \left[0, t_1\right]$, let

$$(6.3.5)\qquad \|\mathcal{F}\left(t\right)\|_{\mathcal{E}^s} := \sup_{\tau \in \mathbb{R}} \|\mathcal{F}\left(t, \tau\right)\|_s.$$

One clearly has for $\mathcal{U} \in \mathcal{P}^0\left(t_1\right)$ $\mathcal{F} \in \mathcal{P}^s\left(t_1\right)$, $s > \left(d + m\right)/2$, and $t \leqq t_1$

$$(6.3.6)\qquad \|\left(\mathcal{F}\mathcal{U}\right)\left(t\right)\|_{\mathcal{E}^0} \leqq \|\mathcal{F}\left(t\right)\|_{L^\infty}\|\mathcal{U}\left(t\right)\|_{\mathcal{E}^0}.$$

Because $\langle \cdot, \cdot \rangle$ is a prehilbertian scalar product, for all $\mathcal{F} \in \mathcal{P}^0\left(t_1\right)$ and $\mathcal{G} \in \mathcal{P}^0\left(t_1\right)$ and for all $t \in \left[0, t_1\right]$ one also has

$$(6.3.7)\qquad |\langle \mathcal{F}, \mathcal{G} \rangle\left(t\right)|^2 \leqq \langle \mathcal{F}, \mathcal{F} \rangle\left(t\right)\langle \mathcal{G}, \mathcal{G} \rangle\left(t\right).$$

Thus, both terms $\operatorname{Re}\langle \partial_x \underline{A}\mathcal{U}, \mathcal{U} \rangle\left(t\right) + \operatorname{Re}\langle \underline{B}\left(t, x, \partial_\theta v\right)\mathcal{U}, \mathcal{U} \rangle\left(t\right)$ are $O\left(C\langle \mathcal{U}, \mathcal{U} \rangle\left(t\right)\right)$ while $\langle \mathcal{L}\mathcal{U}, \mathcal{U} \rangle\left(t\right)$ is $O\left(\langle \mathcal{L}\mathcal{U}, \mathcal{L}\mathcal{U} \rangle^{1/2}\left(t\right)\langle \mathcal{U}, \mathcal{U} \rangle^{1/2}\left(t\right)\right)$.

Finally, note that the initial condition in (6.3.1) and lemma 6.3.2 imply that

$$(6.3.8)\qquad \langle \mathcal{U}, \mathcal{U} \rangle\left(0\right) = \|\mathcal{H}\|_0^2.$$

Thus, with lemma 6.2.4, estimate (6.3.3) follows from (6.3.4) along the usual lines.

PROPOSITION 6.3.2. – *Let $V \in \mathcal{P}^s\left(t_1\right)$, $s > 1 + \left(d + m\right)/2$, $\mathcal{U} \in \mathcal{N}^{s+1}\left(t_1\right)$ and $\mathcal{F} \in \mathcal{P}^s\left(t_1\right)$ satisfy (6.3.1). Then for all $t \in \left[0, t_1\right]$*

$$(6.3.9)\qquad \|\mathcal{U}\left(t\right)\|_{\mathcal{E}^s} \leqq e^{Ct}\|\mathcal{H}\|_0 + C\int_0^t e^{C\left(t - t'\right)}\|\mathcal{F}\left(t'\right)\|_{\mathcal{E}^s}dt',$$

where C only depends on the $\mathcal{E}^s\left(t_1\right)$ norm of V.

Proof. – For $|\alpha| \leqq s$, let $\mathcal{U}_\alpha = \partial_{x,\theta}^\alpha \mathcal{U}$. According to proposition 6.2.1, $\mathbb{E}$ is continuous on the $\mathcal{P}^s$-spaces, as well as the commutators $\left[\partial_{x,\theta}, \mathbb{E}\right]$. Using that $H^{s-1}\left(\omega_t \times \mathbb{R}^m\right)$ is an algebra for $s - 1 > \left(d + m\right)/2$, yields

$$\|\left[\partial_{x,\theta}^\alpha, \mathcal{L}\right]\mathcal{U}\left(t\right)\|_{\mathcal{E}^0} \leqq C_0\|V\left(t\right)\|_{\mathcal{E}^s}\|\mathcal{U}\left(t\right)\|_{\mathcal{E}^s}.$$

Thus estimate (6.3.9) follows from lemma 6.3.1 applied to the $\mathcal{U}_\alpha$'s.

THEOREM 6.3.3. – *Let* $\mathcal{V} \in \mathcal{P}^s(t_1)$ *and* $\mathcal{F} \in \mathcal{P}^s(t_1)$, $s > 1 + (d+m)/2$. *Then* (6.3.1) *has a unique solution* $\mathcal{U} \in \mathcal{N}^s(t_1)$ *which satisfies* (6.3.9).

This result can be derived from the a priori estimates (6.3.9) repeating classical arguments from the theory of symmetric hyperbolic systems. Equation (6.3.1) can be viewed as a symmetric evolution equation in $\mathcal{N}^s(t)$. However, $\mathbb{E}$ introduces technical difficulties. In particular, (6.3.1) is an equation for $\mathcal{U}(t, \cdot)$ in a time dependent space. We sketch a different proof which uses another classical tool, the approximation of ∂_θ by symmetric finite differences.

Proof. – Uniqueness follows from Lemma 6.3.1. To prove existence, for $h > 0$ and $j = 1, \ldots, m$, introduce the finite difference operator

$$(6.3.10) \qquad \delta_j^h \mathcal{U}(t, x, \tau, \theta) := \{ \mathcal{U}(t, x, \tau, \theta + he_j) - \mathcal{U}(t, x, \tau, \theta - he_j) \}/2\,h,$$

where e_j denotes the j-th vector of the canonical basis in $\mathbb{R}^m$. The strategy is to solve the regularized system

$$(6.3.11) \qquad \begin{cases} \mathcal{U}^h \in \mathcal{N}^s(t_1), \\ \mathbb{E}\{ L(t, x, \partial_t, \partial_x)\mathcal{U}^h + \underline{B}(t, x, \mathcal{V})\delta_\theta^h \mathcal{U}^h \} = \mathbb{E}\,\mathcal{F}, \\ \mathcal{U}^h_{|t=0, \tau=0}(x, \theta) = \mathcal{H}(x, \theta). \end{cases}$$

and to pass to the limit as $h \to 0$.

   a) Consider the equation

$$(6.3.12) \qquad \begin{cases} \mathcal{U} \in \mathcal{N}^s(t_1), \\ \mathbb{E}\{ L(t, x, \partial_t, \partial_x)\mathcal{U} \} = \mathbb{E}\,\mathcal{F}, \\ \mathcal{U}_{|t=0, \tau=0}(x, \theta) = \mathcal{H}(x, \theta). \end{cases}$$

Suppose first that $\mathcal{F}(t, x, \tau, \theta) := \sum F_{\lambda,\alpha}(t, x) e^{i(\lambda\tau+\alpha\theta)}$ and $H(x, \theta) := \sum \mathcal{H}_\alpha(t, x) e^{i\alpha\theta}$ are trigonometric polynomials. A trigonometric polynomial $\mathcal{U}(t, x, \tau, \theta) := \sum U_{\lambda,\alpha}(t, x) e^{i(\lambda\tau+\alpha\theta)}$ satisfies (6.3.10) if and only if

$$(6.3.13) \qquad U_{\lambda,\alpha} = 0 \quad \text{when } (\lambda, \alpha) \notin \mathcal{C}$$

$$(6.3.14) \qquad \left\{ \begin{array}{ll} \Pi_{\lambda,\alpha} U_{\lambda,\alpha} = U_{\lambda,\alpha}, & \Pi_{\lambda,\alpha} LU_{\lambda,\alpha} = \Pi_{\lambda,\alpha} F_{\lambda,\alpha} \\ \\ U_{\lambda,\alpha}(0, x) = \Pi_{\lambda,\alpha}(0, x) H_\alpha(x) \end{array} \right\} \quad \text{when } (\lambda, \alpha) \in \mathcal{C}.$$

Here, $\Pi_{\lambda,\alpha}$ denotes multiplication by the projector $\Pi(\cdot, \cdot, (\lambda, \alpha))$.

Assumption 2.1.2 implies that $\Pi_{\lambda,\alpha} L \Pi_{\lambda,\alpha}$ is a first order system, whose principal part is diagonal. Hence the equation in (6.3.14) takes the form

$$(6.3.15) \qquad X_{\lambda,\alpha} U_{\lambda,\alpha} + \Pi_{\lambda,\alpha} E_{\lambda,\alpha} U_{\lambda,\alpha} = \Pi_{\lambda,\alpha} F_{\lambda,\alpha}$$

where $X_{\lambda,\alpha} := \partial_t + \sum a_{\lambda,\alpha,j}\,(t,\,x)\partial_{x_j}$ is a scalar real vector field. More precisely, $X_{\lambda,\alpha}$ is the projection on the $(t,\,x)$-space of the Hamiltonian vector field of L on the characteristic Lagrangian manifold $(t,\,x,\,d\,(\lambda\,t+\alpha\cdot\varphi\,(t,\,x)\,))$. Hence Ω is contained in the domaine of determinacy of ω for $X_{\lambda,\alpha}$ and for smooth data $F_{\lambda,\alpha}$ and H_α, (6.3.14) has a unique solution on Ω. Thus (6.3.12) has a unique solution. Adding the estimates for the different Fourier components, we see that there is a constant K such that

$$(6.3.15) \qquad \|\mathcal{U}\,(t)\|_{\mathcal{E}^s} \leq e^{Kt}\|\mathcal{H}\|_s + K\int_0^t e^{K\,(t-t')}\|\mathcal{F}\,(t')\|_{\mathcal{E}^s}dt'.$$

Thus the existence result and (6.3.15) extend to all $\mathcal{F}\in\mathcal{P}^s\,(t_1)$ and $\mathcal{H}\in H^s\,(\omega\times\mathbb{T}^m)$.

b) Suppose $\mathcal{V}\in\mathcal{P}^s\,(t_1)$, $\mathcal{F}\in\mathcal{P}^s\,(t_1)$ and $\mathcal{H}\in H^s\,(\omega\times\mathbb{T}^m)$. For each fixed $h>0$, δ_θ^h and $\underline{B}\,(t,\,x,\,\mathcal{V})\,\delta_\theta^h$ are bounded operators in $H^s\,(\omega_t\times\mathbb{T}^m)$. Therefore (6.3.11) is solved by standard Picard's iterations and part a).

c) Next one proves uniform estimates for the solutions $\mathcal{U}^h\in\mathcal{N}^s\,(t_1)$. Since the δ_j^h are anti-adjoint and the commutators $[\underline{B}\,(t,\,x,\,\mathcal{V}),\,\delta_\theta^h]$ are uniformly bounded in L^2 with norm $O\,(\|\partial_\theta\mathcal{V}\|_{L^\infty})$, repeating the proof of Lemma 6.3.1 yields the uniform estimates

$$(6.3.16) \qquad \|\mathcal{U}^h\,(t)\|_{\mathcal{E}^0} \leq e^{Ct}\|\mathcal{H}\|_0 + C\int_0^t e^{C\,(t-t')}\|\mathcal{F}\,(t')\|_{\mathcal{E}^0}dt'$$

where $C = O\,(\|\partial_\theta\mathcal{V}\|_{L^\infty}) \leq O\,(\|\mathcal{V}\|_{\mathcal{E}^s})$.

Similarly, using that $H^{s-1}\,(\omega_t\times\mathbb{R}^m)$ is an algebra for $s-1>(d+m)/2$, and that the δ_j^h are uniformly bounded from $H^s\,(\omega_t\times T^m)$ into $H^{s-1}\,(\omega_t\times\mathbb{T}^m)$ yields

$$\|[\partial_{x,\theta}^\alpha,\,\underline{B}\,(t,\,x,\,\mathcal{V})\,\delta_\theta^h]\mathcal{U}^h\,(t)\|_{\mathcal{E}^0} \leq C_0\|\mathcal{V}\,(t)\|_{\mathcal{E}^s}\|\mathcal{U}^h\,(t)\|_{\mathcal{E}^s}$$

for all $|\alpha|\leq s$. As in Proposition 6.3.2, this implies that for all $t\in[0,\,t_1]$

$$(6.3.17) \qquad \|\mathcal{U}^h\,(t)\|_{\mathcal{E}^s} \leq e^{Ct}\|\mathcal{H}\|_s + C\int_0^t e^{C\,(t-t')}\|\mathcal{F}\,(t')\|_{\mathcal{E}^s}dt',$$

where C depends only on the $\mathcal{E}^s\,(t_1)$ norm of $\mathcal{V}$. This proves that the family $\mathcal{U}^h$ is uniformly bounded in $\mathcal{N}^s\,(t_1)$.

d) The equation (6.3.11), together with the analysis of $\mathbb{E}\,L\,\mathbb{E}$ made in part a), show that $\partial_t\mathcal{U}^h$ is uniformly bounded in $\mathcal{E}^{s-1}(t_1)$. Therefore one can extract a subsequence $\mathcal{U}^{h_n}$ which converges in $\mathcal{E}^{s-1}\,(t_1)$ to $\mathcal{U}\in\mathcal{E}^s\,(t_1)$, which satisfies (6.3.1). Passing to the limit in (6.3.17) yields (6.3.9).

6.4. *The nonlinear equations.* – Consider the iterative scheme

$$(6.4.1) \qquad \begin{cases} \mathcal{U}_{\nu+1}\in\mathcal{N}^s\,(t_o), \\ \mathbb{E}\,\{\,L\,(t,\,x,\,\partial_t,\,\partial_x)\mathcal{U}_{\nu+1} + \underline{B}\,(t,\,x,\,\mathcal{U}_\nu)\,\partial_\theta\mathcal{U}_{\nu+1}\,\} = \mathbb{E}\,\{\,\underline{F}\,(t,\,x,\,\mathcal{U}_\nu)\,\}, \\ \mathcal{U}_{\nu+1|_{t=0,\,\tau=0}}\,(x,\,\theta) = \mathcal{H}\,(x,\,\theta). \end{cases}$$

For $\nu = 0$, we choose $\mathcal{U}_0\,(t,\,x,\,\tau,\,\theta) := \mathcal{H}\,(x,\,\theta)$.

THEOREM 6.4.1. – *Given $\mathcal{H} \in H^s (\omega \times \mathbb{T}^m)$, $s > 1 + (d + m)/2$, there is $t_1 \in]0, t_o]$, and a bounded sequence in $\mathcal{N}^s (t_1)$ solution of (6.4.1). Moreover, this sequence converges in $\mathcal{N}^\sigma (t_1)$ for all integer $\sigma < s$, and the limit $\mathcal{U}$ is the unique solution in $\mathcal{N}^s (t_1)$ of (2.3.9).*

The proof of existence is classical, applying inductively Theorem 6.3.3. Uniqueness follows from the L^2 estimates of Lemma 6.3.1.

7. ASYMPTOTIC BEHAVIOUR OF EXACT SOLUTIONS. – The exact solutions u^ε have been constructed in section 5, and the profiles $\mathcal{U}$ in section 6. Our goal is now to prove the asymptotic formula (2.3.8). This will finish the proof of theorem 2.3.5.

7.1. *Reduction to a linear problem.* – Recall that the u^ε have the form (5.1.4), with U^ε solution to (5.1.5) (5.1.6). Let t_1 be the smaller of the times found in proposition 5.1.5 and theorem 6.4.1. Because of the Sobolev imbedding, $H^s(\omega_t \times \mathbb{T}^m) \subset L^\infty$ when $s > (d + m)/2$, the estimate (2.3.8) is a consequence of the following more precise result.

PROPOSITION 7.1.1. – *Introduce $U_a^\varepsilon(t, x, \theta) := \mathcal{U}(t, x, t/\varepsilon, \theta)$. Then U_a^ε is a bounded family in $E^s (t_1)$ and $U^\varepsilon - U_a^\varepsilon \to 0$ in $E^\sigma (t_1)$ for all $\sigma < s$, as $\varepsilon \to 0$.*

Because the families U_a^ε and U^ε are bounded in $E^s(t_1)$ it is sufficient to prove that $U^\varepsilon - U_a^\varepsilon \to 0$ in $E^{s-1} (t_1)$.

On the other hand, we know from proposition 5.1.5 that U^ε is the limit of U_ν^ε, uniformly in ε. Moreover $\mathcal{U}$ is the limit of $\mathcal{U}_\nu$. Introduce $U_{a,\nu}^\varepsilon (t, x, \theta) := \mathcal{U}_\nu (t, x, t/\varepsilon, \theta)$. Proposition 7.1.1 follows from

PROPOSITION 7.1.2. – *For each ν, $U_\nu^\varepsilon - U_{a,\nu}^\varepsilon \to 0$ in $E^{s-1}(t_1)$ as $\varepsilon \to 0$.*

The proof of proposition 7.1.2 is by induction on ν. For $\nu = 0$, $U_0^\varepsilon (t, x, \theta) := \mathcal{H}^\varepsilon (x, \theta)$, while $\mathcal{U}_0 (t, x, \tau, \theta) := \mathcal{H}^0 (x, \theta)$ so that $U_{0,a}^\varepsilon (t, x, \theta) = \mathcal{H}^0(x, \theta)$. Hence, by assumption 2.3.4, the property is satisfied for $\nu = 0$. The induction argument is a consequence of the following linear result for singular systems. Consider the linearized problem

$$(7.1.1) \quad \partial_t U^\varepsilon + \sum_{j=1}^d A_j (t, x, \varepsilon V^\varepsilon) \partial_{x_j} U^\varepsilon + \varepsilon^{-1} \sum_{l=1}^m B_k (t, x, \varepsilon V^\varepsilon) \partial_{\theta_l} U^\varepsilon))$$
$$= F (t, x, \varepsilon V^\varepsilon, V^\varepsilon),$$

$$(7.1.2) \quad U^\varepsilon(0, x, \theta) = \mathcal{H}^\varepsilon (x, \theta).$$

Assume that the family V^ε is bounded in $E^s (t_1)$, and there is $\mathcal{V} \in \mathcal{P}^s (t_1)$ such that

$$(7.1.3) \quad V^\varepsilon - V_a^\varepsilon \to 0 \quad \text{in } E^{s-1} (t_1), \quad \text{with} \quad V_a^\varepsilon (t, x, \theta) := \mathcal{V}(t, x, t/\varepsilon, \theta).$$

Parallel to (7.1.1), we consider equation (6.3.1) for profiles

$$(7.1.4) \quad \begin{cases} \mathcal{U} \in \mathcal{N}^s (t_1), \\ \mathcal{L}\mathcal{U} := \mathbb{E} \{ L (t, x, \partial_t, \partial_x)\mathcal{U} + \underline{B} (\mathcal{V}) \partial_\theta \mathcal{U} \} = \mathbb{E} \mathcal{F}, \\ \mathcal{U}_{|t=0, \tau=0} (x, \theta) = \mathcal{H} (x, \theta). \end{cases}$$

with $\mathcal{F}(t,\, x,\, \tau,\, \theta) \;:=\; \underline{F}(t,\, x,\, \mathcal{V}(t,\, x,\, \tau,\, \theta))$.

THEOREM 7.1.3. – *With the above assumptions, let* $U^\varepsilon \in E^s(t_1)$ *and* $\mathcal{U} \in \mathcal{N}^s(t_1)$ *be the solutions of* (7.1.1) (7.1.2), *and* (7.1.4) *respectively. Then, with* $U_a^\varepsilon(t,\, x,\, \theta) :=$ $\mathcal{U}(t,\, x,\, t/\varepsilon,\, \theta)$, *one has* $\mathcal{U}^\varepsilon - \mathcal{U}_a^\varepsilon \to$ *in* $E^{s-1}(t_1)$.

The main difficulty comes from the fact that the solution $\mathcal{U}$ to (7.1.4) does not immediately provide a good approximate solution $U_a^\varepsilon(t,\, x,\, \theta) := \mathcal{U}(t,\, x,\, t/\varepsilon,\, \theta)$ to the equation (7.1.1). More precisely, if $\mathfrak{L}$ denotes the operator in the left hand side of (7.1.1), we have that $\mathfrak{L}(U^\varepsilon - U_a^\varepsilon)$ is $O(1)$ and converges only *weakly* to 0. The first try is to produce a corrected approximate solution U_{ca}^ε of the form $U_a^\varepsilon + \varepsilon\, U_c^\varepsilon$ such that $\mathfrak{L}(U^\varepsilon - U_{ca}^\varepsilon)$ is $o(1)$. But, for the problem considered here our assumptions are not sufficient to ensure the existence of such a simple correction [JMR2] and we have to modify the argument. The strategy is as follows. Given a $\delta > 0$, we exhibit a "corrected asymptotic approximate solution" $U_{0,\,a}^\varepsilon + \varepsilon\, U_{1,\,a}^\varepsilon$ such that

$$(7.1.5) \qquad \| \mathfrak{L}_0 \,(U^\varepsilon - \{\, U_{0,\,a}^\varepsilon + \varepsilon\, U_{1,\,a}^\varepsilon\})\|_{E^{s-1}(t_1)} \leqq K\,\delta + c\,(\varepsilon,\, \delta),$$

$$(7.1.6) \qquad \| \,(U^\varepsilon - \{\, U_{0,\,a}^\varepsilon + \varepsilon\, U_{1,\,a}^\varepsilon\})_{|t=0}\|_{H^{s-1}} \leqq K\,\delta + c\,(\varepsilon,\, \delta),$$

where $c\,(\varepsilon,\, \delta) \to 0$ as $\varepsilon \to 0$, for each $\delta > 0$. In (7.1.5), $\mathfrak{L}_0$ denotes an approximation of the operator $\mathfrak{L}$ appearing in the left hand side of (7.1.1). Moreover, $U_{i,\,a}^\varepsilon(t,\, x,\, \theta) := \mathcal{U}_i(t,\, x,\, t/\varepsilon,\, \theta)$ for $i = 0,\, 1$, and

$$(7.1.7) \qquad\qquad \| \mathcal{U} - \mathcal{U}_o\|_{\mathcal{E}^s(t_1)} \leqq \delta.$$

Energy estimates for $\mathfrak{L}_0$ imply that

$$(7.1.8) \qquad \|U^\varepsilon - (U_{0,\,a}^\varepsilon + \varepsilon\, U_{1,\,a}^\varepsilon)\|_{E^{s-1}(t_1)} \leqq K\,\delta + c(\varepsilon,\, \delta)$$

and with (7.1.7)

$$(7.1.9) \qquad\qquad \|U^\varepsilon - U_a^\varepsilon\|_{E^{s-1}(t_1)} \leqq K\,\delta + c\,(\varepsilon,\, \delta).$$

Theorem 7.1.3 follows, fixing first δ and next letting ε tend to 0.

7.2. *Proof of theorem* 7.1.3. – First note the following elementary lemma.

LEMMA 7.2.1. – *If* (7.1.3) *holds, then* $F^\varepsilon(t,\, x,\, \theta) := F(t,\, x,\, \varepsilon\, V^\varepsilon,\, V^\varepsilon) \in E^s(t_1)$ *satisfies*

$$(7.2.1) \qquad F^\varepsilon - F_a^\varepsilon \to 0 \quad \text{in } E^{s-1}(t_1), \qquad \text{with} \quad F_a^\varepsilon(t,\, x,\, \theta) := \mathcal{F}(t,\, x,\, t/\varepsilon,\, \theta),$$

where $\mathcal{F}(t,\, x,\, \tau,\, \theta) := \underline{F}(t,\, x,\, \mathcal{V}(t,\, x,\, \tau,\, \theta)) \in \mathcal{P}^s(t_1)$, *as above.*

Recall that $F(t,\, x,\, v,\, w)$ is linear in w.

LEMMA 7.2.2. – *For any $\delta > 0$, there are trigonometric polynomials $\mathcal{U}_0$, $\mathcal{V}_0$ and $\mathcal{F}_0$ such that $\mathbb{E}\mathcal{U}_0 = \mathcal{U}_0$ and*

$$(7.2.2) \qquad \|\mathcal{U} - \mathcal{U}_0\|_{\mathcal{E}^s(t_1)} \leq \delta \qquad \text{and} \qquad \|\partial_t \mathcal{U} - \partial_t \mathcal{U}_0\|_{\mathcal{E}^{s-1}(t_1)} \leq \delta,$$

$$(7.2.3) \qquad \|\mathcal{V} - \mathcal{V}_0\|_{\mathcal{E}^s(t_1)} \, \delta; \qquad \|\mathcal{F} - \mathcal{F}_0\|_{\mathcal{E}^s(t_1)} \leq \delta.$$

Proof. – The approximation in $\mathcal{E}^s(t_1)$ of $\mathcal{V}$ and $\mathcal{F}$ by trigonometric polynomials, are consequences of the definition of $\mathcal{P}^s(t_1)$.

The problem for $\mathcal{U}$ is slightly more difficult, because we also want an approximation of the time derivative. Expand $\mathcal{U}$ in a Fourier series in θ

$$(7.2.4) \qquad \mathcal{U}(t,\,x,\,\tau,\,\theta) = \sum_{\alpha \in \mathbb{Z}^m} \mathcal{U}_\alpha(t,\,x,\,\tau)\, e^{i\alpha \cdot \theta}.$$

Then

$$(7.2.5) \qquad \|\mathcal{U}\|_{\mathcal{E}^s(t_1)} \approx \sup_{0 \leq t \leq t_1,\, \tau \in \mathbb{R}} \Big\{ \sum_{\alpha \in \mathbb{Z}^m} \sum_{|\beta| \leq s} (1 + |\alpha|)^{2s - 2|\beta|} \|\partial_x^\beta \mathcal{U}_\alpha(t,\,\cdot,\,\tau)\|_{L^2(\omega_t)}^2 \Big\}^{1/2}.$$

Because $\mathcal{U} \in \mathcal{N}^s(t_1)$, each $\mathcal{U}_\alpha$ is a trigonometric polynomial in τ

$$(7.2.6) \qquad \mathcal{U}_\alpha(t,\,x,\,\tau) = \sum_{\lambda \in \mathcal{C}(\alpha)} U_{\lambda,\alpha}(t,\,x)\, e^{i\lambda \tau},$$

where $\mathcal{C}(\alpha) := \{\,\lambda \in \mathbb{R}\,|\,(\lambda,\,\alpha) \in \mathcal{C}\,\}$ has at most N elements.

On the other hand, equation (7.1.4) implies that

$$(7.2.7) \qquad \partial_t \mathcal{U} = -\mathbb{E}\,\{\,A(t,\,x,\,\partial_x)\mathcal{U} + \underline{B}(\mathcal{V})\,\partial_\theta \mathcal{U}\,\} + \mathbb{E}\,\mathcal{F} + [\partial_t,\,\mathbb{E}]\mathcal{U} \in \mathcal{P}^{s-1}(t_1).$$

Thus

$$(7.2.8) \qquad \sup_{0 \leq t \leq t_1,\, \tau \in \mathbb{R}} \Big\{ \sum_{\alpha \in \mathbb{Z}^m} \sum_{|\beta| \leq -1} (1 + |\alpha|)^{2s - 2|\beta| - 2} \|\partial_t\, \partial_x^\beta\, \mathcal{U}_\alpha(t,\,\cdot,\,\tau)\|_{L^2(\omega_t)}^2 \Big\}^{1/2}$$

$$\approx \|\partial_t \mathcal{U}\|_{\mathcal{E}^{s-1}(t_1)} < +\infty.$$

Thanks to (7.2.5) (7.2.8), approximating $\mathcal{U}$ by a finite sum in (7.2.4), yields a $\mathcal{U}_o$ which satisfies (7.2.2). It remains to smooth the corresponding coefficients $U_{\lambda,\alpha}$, to get a trigonometric polynomial in the sense of definiton 6.1.1. Replacing the smoothed $\mathcal{U}_0$ by $\mathbb{E}\mathcal{U}_0$ we can also assume that $\mathbb{E}\mathcal{U}_0 = \mathcal{U}_0$.

As a corollary, let

$$(7.2.9) \qquad \mathcal{G}_0 := \{ L(t, x, \partial_t, \partial_x) \mathcal{U}_0 + \underline{B}(t, x; \mathcal{V}_0) \partial_\theta \mathcal{U}_0 \}.$$

$\mathcal{G}_0$ is a trigonometric polynomial. From lemma 6.1.2, proposition 6.2.1, and equation (7.1.4) we deduce

$$(7.2.10) \qquad \| \mathbb{E}\,\mathcal{G}_0 - \mathbb{E}\,\mathcal{F}_0 \|_{\mathcal{E}^{s-1}(t_1)} = O(\delta).$$

Moreover, the norm of $\mathcal{G}_0$ in $\mathcal{E}^{s-1}(t_1)$, is bounded uniformly in δ.

Next, introduce the operator

$$(7.2.11) \qquad \mathfrak{L}_0 := L(t, x, \partial_t, \partial_x) + Q(t, x, \partial_\theta) + \underline{B}(t, x, V_{o,a}) \partial_\theta,$$

with Q as in (5.1.8). The notation $W_a(t, x, \theta) := \mathcal{W}(t, x, t/\varepsilon, \theta)$, will be used systematically in the sequel.

LEMMA 7.2.3. – *There are trigonometric polynomials $\mathcal{U}_1$ such that*

$$(7.2.12) \qquad \| \mathfrak{L}_0 \{ U^\varepsilon_{0,a} + \varepsilon\, U^\varepsilon_{1,a} \} - F^\varepsilon_{0,a} \|_{E^{s-1}(t_1)} \leqq K\,\delta + c(\varepsilon, \delta),$$

where K is a constant independent of δ and ε, and $c(\varepsilon, \delta) \to 0$ as $\varepsilon \to 0$, for each $\delta > 0$.

Proof. – We use the WKB method, following [L2]. Using $\mathbb{E}\,\mathcal{U}_0 = \mathcal{U}_0$, $P\mathcal{U}_0 = (\partial_\tau + Q)\mathcal{U}_0 = 0$ and (7.2.9), we see that

$$\mathfrak{L}_0\, U^\varepsilon_{0,a} = G^\varepsilon_{0,a}.$$

Let $\mathcal{F}_1 := (\mathrm{Id} - \mathbb{E}) \{ \mathcal{F}_0 - \mathcal{G}_0 \}$. Then $\mathcal{F}_1$ is a trigonometric polynomial such that $\mathbb{E}\,\mathcal{F}_1 = 0$. Hence

$$(7.2.14) \qquad \mathcal{F}_1(t, x, \tau, \theta) = \sum_{(\lambda, \alpha) \in \Lambda} F_{\lambda, \alpha}(t, x)\, e^{i\,(\lambda\tau + \alpha\cdot\theta)},$$

where Λ is finite and

$$(7.2.15) \qquad \Pi(t, x, \lambda, \alpha)\, F_{\lambda, \alpha}(t, x) = 0 \quad \text{whenever } (\lambda, \alpha) \in \mathcal{C}.$$

By definition of $\mathcal{C}$ and coherence, for $(\lambda, \alpha) \notin \mathcal{C}$, $P(t, x, \lambda, \alpha)$ is an invertible matrix and there is a unique smooth function $U_{1, \lambda, \alpha} \in C^\infty(\Omega_{t_1})$ such that

$$(7.2.16) \qquad i^{-1} P(t, x, \lambda, \alpha)\, U_{1, \lambda, \alpha} = F_{\lambda, \alpha}.$$

On the other hand, when $(\lambda, \alpha) \in \mathcal{C}$, $P(t, x, \lambda, \alpha)$ is symmetric and $\Pi(t, x, \lambda, \alpha)$ is the orthogonal projector on its kernel. Therefore condition (7.2.15) implies that there is a unique smooth function $U_{1, \lambda, \alpha} \in C^\infty(\Omega_{t_1})$ such that

$$(7.2.16') \qquad i^{-1} P(t, x, \lambda, \alpha)\, U_{1, \lambda, \alpha} = F_{\lambda, \alpha}, \qquad \Pi(t, x, \lambda, \alpha)\, U_{1, \lambda, \alpha} = 0.$$

Finally, we get a trigonometric polynomial

$$(7.2.17) \qquad \mathcal{U}_1\left(t, x, \tau, \theta\right) := \sum_{(\lambda, \alpha) \in \Lambda} U_{1, \lambda, \alpha}\left(t, x\right) e^{i\left(\lambda \tau + \alpha \cdot \theta\right)},$$

such that $P\left(t, x, \partial_\tau, \partial_\theta\right)\mathcal{U}_1 = \mathcal{F}_1$. Hence $\varepsilon U_{1, a}^\varepsilon$ satisfies

$$(7.2.18) \qquad \left\|\mathcal{L}_0\left(\varepsilon U_{1, a}^\varepsilon\right) - F_{1, a}^\varepsilon\right\|_{E^{s-1}(t_1)} \leqq C\left(\delta\right)\varepsilon,$$

for some constant $C\left(\delta\right)$ independent of ε. (7.2.13) implies that

$$\mathcal{L}_0\left(U_{0, a}^\varepsilon + \varepsilon U_{1, a}^\varepsilon\right) - F_{0, a}^\varepsilon = \left(\mathbb{E}\left\{\mathcal{G}_0 - \mathcal{F}_0\right\}\right)_a^\varepsilon + \mathcal{L}_0\left(\varepsilon U_{1, a}^\varepsilon\right) - F_{1, a}^\varepsilon$$

and the lemma follows from estimates (7.2.10) and (7.2.18).

LEMMA 7.2.4. – *One has*

$$(7.2.19) \qquad \left\|\mathcal{L}_0\left\{U^\varepsilon - U_{0, a}^\varepsilon - \varepsilon U_{1, a}^\varepsilon\right\}\right\|_{E^{s-1}(t_1)} \leqq K\delta + c(\varepsilon, \delta),$$

$$(7.2.20) \qquad \left\|\left(U^\varepsilon - U_{0, a}^\varepsilon - \varepsilon U_{1, a}^\varepsilon\right)_{|t=0}\right\|_{H^{s-1}} \leqq K\delta + c\left(\varepsilon, \delta\right),$$

where K is a constant independent of δ and ε, and $c\left(\varepsilon, \delta\right) \to 0$, for each $\delta > 0$.

Proof. – In equation (7.1.1), we neglect terms in $O\left(\varepsilon\right)$ and we replace V and F first by V_a and F_a and next by $V_{0, a}$ and $F_{0, a}$. Since the family U^ε is bounded in $E^\varepsilon(t_1)$ and because of (7.1.3) (7.2.1) and (7.2.3), we get that

$$(7.2.21) \qquad \left\|\mathcal{L}_0 U^\varepsilon - F_{0, a}^\varepsilon\right\|_{E^{s-1}(t_0)} \leqq K\delta + c\left(\varepsilon\right).$$

with K still independent of δ and ε, and $c\left(\varepsilon\right) \to 0$ as $\varepsilon \to 0$. Comparing with (7.2.12) yields (7.2.19).

Finally,

$$(7.2.22) \qquad \left(U^\varepsilon - U_{0, a}^\varepsilon - \varepsilon U_{1, a}^\varepsilon\right)_{|t=0} = \mathcal{H}^\varepsilon - \mathcal{H}^0 + \left(\mathcal{U} - \mathcal{U}_0\right)_{|t=0, \tau=0} - \varepsilon \mathcal{U}_{1_{|t=0, \tau=0}}$$

and estimate (7.2.20) follows.

The coefficients of $\mathcal{L}_0$ have $E^s\left(t_1\right)$ norms which are bounded independent of δ and ε. Hence, a slight modification of proposition 4.3.4, shows that

$$(7.2.23) \qquad \left\|U - U_{0, a}^\varepsilon - \varepsilon U_{1, a}^\varepsilon\right\|_{E^{s-1}(t_1)} \leqq K\delta + c\left(\varepsilon, \delta\right).$$

Now,

$$(7.2.24) \qquad \left\|\varepsilon U_{1, a}^\varepsilon\right\|_{E^{s-1}(t_1)} \leqq c\left(\varepsilon, \delta\right)$$

and by (7.2.2)

$$(7.2.25) \qquad\qquad \|U^\varepsilon_{0,\,a} - U^\varepsilon_a\|_{E^{s-1}\,(t_1)} \leqq K\,\delta.$$

Therefore

$$\|U^\varepsilon - U^\varepsilon_a\|_{E^{s-1}\,(t_1)} \leqq K\,\delta + c\,(\varepsilon,\,\delta)$$

and theorem 7.1.3 follows.

8. PROPAGATION OF THE SPECTRUM. – This section contains the proof of theorem 2.4.2 concerning the propagation of the oscillating spectrum.

Assumptions are as in section 6. $\mathcal{U} \in \mathcal{N}^s\,(t_1)$, $s > (d + m)/2$, is a given solution of (2.3.9). We consider a family $\mathfrak{L}$ of lines $l \subset \operatorname{Char} P \subset \mathbb{R} \times \mathbb{R}^m$, and we introduce

$$(8.1.1) \qquad\qquad \Lambda := \bigcup_{l \in \mathfrak{L}} l, \qquad M = \operatorname{Char} P \backslash \Lambda.$$

The projectors $\mathbb{E}_\Lambda$ and $\mathbb{E}_M$ have been defined in section 6.2. They act from $\mathcal{P}^s\,(t_1)$ into $\mathcal{N}^s\,(t_1)$ and

$$(8.1.2) \qquad\qquad \mathbb{E} = \mathbb{E}_\Lambda + \mathbb{E}_M.$$

In particular,

$$(8.1.3) \qquad\qquad \mathcal{U} = \mathbb{E}_\Lambda\mathcal{U} + \mathbb{E}_M\mathcal{U}.$$

Theorem 2.4.2 can be stated as follows

THEOREM 8.1.1. – *Assume that either $\mathfrak{L}$ is stable for interaction or $\underline{F}$ is polynomial of degree $\leqq k$ and $\mathcal{L}$ is stable for interaction of order $\leqq k$, for some $k \geqq 2$. If $\mathbb{E}_M\mathcal{U}_{|t=0} = 0$, then $\mathbb{E}_M\mathcal{U} = 0$ on $\Omega_{t_1} \times \mathbb{R} \times \mathbb{T}^m$.*

The proof is based on the following two results

PROPOSITION 8.1.2. – *Assume that either $\mathfrak{L}$ is stable for interaction or $\underline{F}$ is polynomial of degree $\leqq k$ and $\mathfrak{L}$ is stable for interaction of order $\leqq k$. Then for all $\mathcal{U} \in \mathcal{P}^s\,(t_1)$, $s > (d + m)/2$,*

$$(8.1.4) \qquad\qquad \mathbb{E}_M\left\{ \underline{F}\,(t,\,x,\,\mathbb{E}_\Lambda\mathcal{U}) \right\} = 0.$$

PROPOSITION 8.1.3. – *Assume that $\mathfrak{L}$ is stable for interaction of order $\leqq 2$. Then for all $\mathcal{U} \in \mathcal{P}^s\,(t_1)$, $s > 1 + (d + m)/2$.*

$$(8.1.5) \qquad\qquad \mathbb{E}_M\left\{ \underline{B}\,(t,\,x,\,\mathbb{E}_\Lambda\mathcal{U})\,\partial_\theta\,\mathbb{E}_\Lambda\mathcal{U} \right\} = 0.$$

Proof. – a) Suppose first that $\underline{F}$ is polynomial in $(u,\,\bar{u})$ of degree k. If $\mathcal{U}$ is a trigonometric polynomial, so are $\mathbb{E}_\Lambda\mathcal{U}$ and $\underline{F}\,(t,\,x,\,\mathbb{E}_\Lambda\mathcal{U})$, and the spectrum of the latter is contained

in the set of $\mathbb{Z}$-linear combinations with at most k terms, of frequencies in the spectrum of the former. If $\mathfrak{L}$ is stable for interaction of order $\leq k$, we conclude that the spectrum of $\underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U})$ is contained in Λ, hence (8.1.4) is satisfied.

For a general $\mathcal{U} \in \mathcal{P}^s(t_1)$, let $\mathcal{U}_\nu$ be a sequence of trigonometric polynomials such that $\mathcal{U}_\nu \to \mathcal{U}$ in $\mathcal{P}^s(t_1)$. By lemma 6.1.2, $\underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U}_\nu) \to \underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U})$ in $\mathcal{P}^s(t_1)$ and $\mathbb{E}_M\{\underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U})\} = \lim \mathbb{E}_M\{\underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U}_\nu)\} = 0$.

b) Let $\underline{F}$ be a general C^∞ function, and let $\mathcal{U} \in \mathcal{P}^s(t_1)$. Because $s > (d+m)/2$, $\mathcal{U}$ and $\mathbb{E}_\Lambda \mathcal{U}$ beelong $L^\infty(\Omega_{t_1} \times \mathbb{R} \times \mathbb{T}^m)$. Therefore, one can approximate $\underline{F}$ by polynomials in $(u, \bar{u})$, F_ν, so that $F_\nu(t, x, \mathbb{E}_\Lambda \mathcal{U}) \to \underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U})$ uniformly and in $\mathcal{P}^0(t_1)$. If $\mathfrak{L}$ is stable for interaction at any order, then step a) implies that $\mathbb{E}_M\{F_\nu(t, x, \mathbb{E}_\Lambda \mathcal{U})\} = 0$. Thanks to proposition 6.2.1, we can pass to the limit to prove (8.1.4).

c) Let $\mathcal{U} \in \mathcal{P}^s(t_1)$, $s > 1 + (d+m)/2$. Then $\mathbb{E}_\Lambda \mathcal{U}$ and $\partial_\theta \mathbb{E}_\Lambda \mathcal{U}$ have their spectrum contained in Λ, and so does $\underline{B}(t, x, \mathbb{E}_\Lambda \mathcal{U}) \partial_\theta \mathbb{E}_\Lambda \mathcal{U}$ as soon as $\mathfrak{L}$ is stable for quadratic interaction. Recall that $\underline{B}(t, x, u)$ is linear in $(u, \bar{u})$. Then (8.1.5) holds.

Proof of theorem 8.1.1. – Assume either that $\mathfrak{L}$ is stable for interaction or that $\underline{F}$ is polynomial of degree $\leq k$ and that $\mathfrak{L}$ is stable for interaction of order $\leq k$, where $k \geq 2$.

Apply $\mathbb{E}_M$ to equation (2.3.9). The first term is linear in $\mathcal{U}$ and does not increase the spectrum, so that

$$\mathbb{E}_M \mathbb{E}\{L(t, x, \partial_t, \partial_x)\mathbb{E}_\Lambda \mathcal{U} = 0$$

and

(8.1.6) $\qquad \mathbb{E}_M \mathbb{E}\{L(t, x, \partial_t, \partial_x)\mathcal{U}\} = \mathbb{E}_M\{L(t, x, \partial_t, \partial_x)\mathbb{E}_M\mathcal{U}.$

With proposition 8.1.3, we see that

(8.1.7) $\qquad \mathbb{E}_M \mathbb{E}\{\underline{B}(t, x, \mathcal{U})\partial_\theta \mathcal{U}\} = \mathbb{E}_M\{\underline{B}(t, x, \mathcal{U})\partial_\theta \mathbb{E}_M \mathcal{U}\}$

$$+ \mathbb{E}_M\{\underline{B}(t, x, \mathbb{E}_M \mathcal{U})\partial_\theta \mathbb{E}_\Lambda \mathcal{U}\}.$$

We remark that the last term is linear in $\mathbb{E}_M \mathcal{U}$ and it can be written

(8.1.8) $\qquad \mathbb{E}_M\{\underline{B}(t, x, \mathbb{E}_M \mathcal{U})\partial_\theta \mathbb{E}_\Lambda \mathcal{U}\} = \mathbb{E}_M\{G_o(t, x, \partial_\theta \mathbb{E}_\Lambda \mathcal{U})\mathbb{E}_M \mathcal{U}\}.$

Moreover, one has

(8.1.9) $\qquad \underline{F}(t, x, \mathcal{U}) = \underline{F}(t, x, \mathbb{E}_\Lambda \mathcal{U}) + G(t, x, \mathbb{E}_\Lambda \mathcal{U}, \mathbb{E}_M \mathcal{U})\mathbb{E}_M \mathcal{U}$

for some matrix valued function $G(t, x, v, w)$ which depends smoothly on the variables $(t, x, v, w) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N$. Hence, proposition 8.1.2 implies

(8.1.10) $\qquad \mathbb{E}_M \mathbb{E}\underline{F}(t, x, \mathcal{U}) = \mathbb{E}_M\{G(t, x, \mathbb{E}_\Lambda \mathcal{U}, \mathbb{E}_M \mathcal{U})\mathbb{E}_M \mathcal{U})\mathbb{E}_M \mathcal{U}\}.$

Thus $\mathcal{V} := \mathbb{E}_M \mathcal{U}$ satisfies

(8.1.11)
$$\begin{cases} \mathcal{V} = \mathbb{E}_M \, \mathcal{V}, \\ \mathbb{E}_M \left\{ L\left(t, \, x, \, \partial_t, \, \partial_x\right) \mathcal{V} + \underline{B}\left(t, \, x, \, \mathcal{U}\right) \partial_\theta \, \mathcal{V} \right\} = \mathbb{E}_M \left\{ \mathcal{G} \mathcal{V} \right\}. \end{cases}$$

where $\mathcal{G}$ is a matrix in $\mathcal{P}^{s-1}\left(t_1\right)$, which depends smoothly on t, x, $\mathbb{E}_\Lambda \mathcal{U}$, $\mathbb{E}_M \mathcal{U}$ and $\partial_\theta \mathbb{E}_\Lambda \mathcal{U}$.

We can now repeat the proof of lemma 6.3.1, and see that we have the following L^2 energy-estimate

(8.1.12)
$$\|\mathcal{V}(t)\|_0 \leqq e^{Ct} \|\mathcal{V}(0)\|_0,$$

where C depends only on an L^∞ bound for $\partial_\theta \mathcal{U}$ and $\mathcal{G}$.

In particular, if $\mathcal{V}(0) = 0$, then $\mathcal{V} \equiv 0$, and the proof of theorem 8.1.1 is complete.

Part II. Focusing of nonlinear hyperbolic waves

In the first part we showed how a strong coherence assumption leads to a satisfactory multidimensional nonlinear geometric optics. When the space dimension is equal to one such a strong assumption is not needed [JMR1]. In the second part we show that these one dimensional results do not carry over to the multidimensional situations. The culprit is *hidden focussing*.

9. DESCRIPTION OF SEVERAL MECHANISMS. – For linear hyperbolic problems, focusing is characterized by two striking consequences which are closely related. First the evolution operators are not bounded in L^∞ and second the evolution operators loose derivatives when measured in the classical C^k spaces. For nonlinear problems the consequences are much more grave. Solutions may fail to exist beyond focusing and there is focusing that is *hidden*, that is not revealed by straightforward analysis of the incoming phases. In the subsections of section 9 we describe a variety of different mechanisms by which focusing can cause explosions. All involve in one way or another a phase which degenerates. In section 10 we construct examples showing that these possibilities actually occur. In section 11, we give examples of focusing from the linear superposition of waves with nondegenerate phases. The conclusion is that in space dimension greater than one, smooth almost periodic profiles and bounded periodic profiles are not good enough. Smooth quasiperiodic profiles is the choice carried out in part I.

9.1. *Linear of direct focusing.* – The simplest examples of linear focusing are the smooth rotationally symmetric solutions of $\square u = 0$ in $\mathbb{R}^3$. They have the form

(9.1.1)
$$u\left(t, \, r\right) = \frac{\left[f\left(t+r\right) - f\left(t-r\right)\right]}{r}, \qquad \text{for} \quad r \equiv |x| \neq 0,$$

(9.1.2)
$$u\left(t, \, r\right) = 2 \, f'\left(t\right) \qquad \text{for} \quad r = 0.$$

Suppose that $f \in C_0^\infty(\mathbb{R})$ has support in $[T, \infty[$ with $T > 0$, then for $t < T$, one has an incoming wave $f(t + r)/r$. As $t \to T$, the inner edge of the wave approaches the origin and the $1/r$ factor causes the wave to grow. The wave is *reflected* from the origin emerging with the opposite sign. When the wave focuses at the origin its size in sup norm may be much larger than its initial size. To see this take a uniformly bounded sequence f whose first derivatives are not uniformly bounded. Thus the solution operator which takes the initial data $\nabla_{t,x} u(0, \cdot)$ to data $\nabla_{t,x} u(t, \cdot)$ at time $t \neq 0$, is not bounded on $L^\infty(\mathbb{R}^d)$. Another aspect of the same phenomenon is that taking $f \in C^k$ yields solutions in C^{k-1}. There is a loss of one derivative with respect to the regularity of the Cauchy data. The L^∞ unboundedness of the solution operator is seen for the oscillatory initial data

$$(9.1.3) \qquad u^\varepsilon(0, t) = 0, \qquad \partial_t u^\varepsilon(0, r) = h(r) e^{i\varphi(r)/\varepsilon}.$$

Here we suppose that h and φ are smooth even functions of r, and that φ is real valued. Formula (9.1.1) applies with f^ε even given by

$$(9.1.4) \qquad \partial_r f^\varepsilon(r) = (r/2) h(r) e^{i\varphi(r)/\varepsilon}.$$

Then

$$(9.1.5) \qquad \partial_t u^\varepsilon(t, 0) = 2(f^\varepsilon)''(t) = \varepsilon^{-1} i \varphi'(t) t h(t) e^{i\varphi(t)/\varepsilon} + O(1)$$

So that for $t > 0$, $\nabla u^\varepsilon(t)$ is of the order ε^{-1} times as large as its initial data provided that $\varphi'(t) h(t) \neq 0$. For the equation, $\Box u = F(\nabla u)$ these large amplitudes are fed back into the equation. Here there is an important distinction between the cases of linear and nonlinear F. For linear problems one has a family of global solution for which $\nabla u^\varepsilon(t, \cdot)$ is unbounded. For the nonlinear problems one has existence on an interval $[0, T^\varepsilon[$ and the large amplitudes from focusing at time T may cause $\limsup(T^\varepsilon) \leqq T$. In this case we say that there is explosive nonexistence at time T. In his classic analysis [L2], Lax showed that solutions of the first order $N \times N$ linear strtictly hyperbolic oscillatory initial value problem

$$(9.1.6) \qquad L u^\varepsilon = 0, \qquad u^\varepsilon(0, x) = h(x) e^{i\varphi^0(x)/\varepsilon}$$

have asymptotic expansions as superpositions of N asymptotic solutions

$$(9.1.7) \qquad u^\varepsilon(t, x) = \sum_{k=0}^{N} u_k^\varepsilon(t, x),$$

$$(9.1.8) \qquad u_k^\varepsilon(t, x) = \sum_{j=0}^{\infty} \varepsilon^j a_{kj}(t, x) e^{i\varphi_k(t, x)/\varepsilon}.$$

Here we have supposed that $h \in C_0^\infty(\mathbb{R}^d)$ and φ is a smooth real valued function with $d\varphi \neq 0$ on $\mathrm{supp}(h)$. The phases φ_k are the N smooth solutions of the eikonal equation

$$(9.1.9) \qquad p(t, x, d\varphi_k) = 0, \qquad \varphi_k(0, x) = \varphi^0(x)$$

where

(9.1.10) $p\left(t,\ x,\ \tau,\ \xi\right) = \det L_1\left(t,\ x,\ \tau,\ \xi\right)$

is the determinant of the principal symbol. If $\lambda_k\left(t,\ x,\ \xi\right)$ denote the N smooth real roots of $p\left(t,\ x,\ \lambda,\ \xi\right) = 0$ the φ_k are solutions of the reduced eikonal equations

(9.1.11) $\partial_t\,\varphi_k = \lambda_k\left(t,\ x,\ \partial_x\,\varphi_k\right), \qquad \varphi_k\left(0,\ x\right) = \varphi^0\left(x\right).$

The graph of $\partial_{t,\,x}\,\varphi$ is determined at $t = 0$ from (9.1.11) and for $t \neq 0$ from the fact that it is invariant under the flow of the time dependent Hamilton field H_p. The projections on t, x space of the integral curves of H_p which foliate the graph of $d\varphi$ are called the rays associated to φ. The rays are the integral curves of the vector field

(9.1.12) $\partial_\tau\,p\left(t,\ x,\ d\varphi_k\left(t,\ x\right)\right)\partial_t + \sum \partial_{\xi_n}\left(t,\ x,\ d\varphi_k\left(t,\ x\right)\right)\partial_{x_n}.$

Let ψ_t denote the corresponding flow from $\{\,0\,\}\times\mathbb{R}^d$ to $\{\,t\,\}\times\mathbb{R}^d$. The phenomenon of linear focusing is linked to the behavior of φ_k, ψ and of the principal amplitude $a_{k,\,0}\left(t,\ x\right)$ along the rays beginning in $\mathrm{supp}\left(h\right)$. At the time t_0 of first focusing for the k-waves, four phenomena occur.

(i) The eikonal equation (9.1.11) is not solvable past $t = t_0$ along one of the rays beginning in $\mathrm{supp}\left(h\right)$.

(ii) $\mathrm{Inf}\left\{\,\det D\psi_t\left(x\right);\ x\in\mathrm{supp}\left(h\right)\,\right\}\to 0$ as $t\to t_0$.

(iii) The rays fail to define a regular foliation on a neighborhood of $t = t_0$.

(iv) $\mathrm{Sup}\left\{\,|D_x^2\,\varphi_k|;\ x\in\mathrm{supp}\left(h\right)\,\right\}\to+\infty$ as $t\to t_0$.

If $a_{k,\,0}\left(0,\ x\right)\neq 0$ at the starting point of a ray along which focusing occurs, then as $t\to t_0$, $a_{k,\,0}\to\infty$ along the ray. Such increasing amplitudes when plugged into nonlinear equations can result in lack of solvability beyond time t_0. We say that there is direct focusing at time t when the four equivalent conditions (i)-(iv) hold for the solution φ of the eikonal equation. A simple example of direct focusing occurs for the wave equation in $\mathbb{R}^{1+d}$ with $d > 1$. Consider the solutions u to (the epsilon dependence has been suppressed for ease of reading)

(9.1.13) $\Box\,u = 0, \qquad u\left(0,\ \cdot\right) = 0, \qquad \partial_t\,u\left(0,\ x\right) = h\left(x\right)e^{ir^2/\varepsilon}.$

The initial phase is r^2 and in order to satisfy the hypotheses of this subsection we suppose that h is a smooth function vanishing on a neighborhood of $r = 0$. The phases for this problem are the solution of

(9.1.14) $\partial_t\,\varphi_\pm = \pm\,|\partial_x\,\varphi_\pm|, \qquad \varphi_\pm\left(0,\ x\right) = |x|^2 \qquad \text{for} \quad |x|\geqq T.$

The solutions are

(9.1.15) $\varphi\left(t,\ x\right) = \left(|x|\ \pm t\right)^2$

and $d\varphi$ is constant along the rays $x_0 \pm tx_0/|x_0|$. For $t > 0$ we see that there is focusing at $t = T$ where $T = \inf\{|x| : x \in \text{supp}(h)\}$. The focusing occurs along the ray(s) approaching the origin from the point(s) in $\text{supp}(h)$ closest the origin. In section 10.1 we show that this focusing leads to blowup and nonexistence beyond $t = T$ for some semilinear problems of the form

$$(9.1.16) \qquad \Box u = f(u, \nabla u), \qquad u(0, \cdot) \quad \text{and} \quad \partial_t u(0, \cdot) \quad \text{as in } (9.1.13).$$

9.2. *Degenerate direct focusing.* -- Degenerate direct focusing occurs when the phases present in the initial waves are stationary $(d\varphi = 0)$ on a small set. The geometric optics expansion (9.1.8) is no longer applicable for the initial value problem (9.1.6) at points of $\text{supp}(h)$ where $d\varphi = 0$. If the set of such stationary points is small the solutions of the initial value problem may be unbounded. An example is (9.1.13) at $x = 0$, provided that $h(0) \neq 0$. Considering this as the limit of the previous cases when $T \to 0$ suggests that unboundedness may occur instantaneously at $T = 0$. One can have instantaneous focusing.

9.3. *Four types of linear interaction.* -- The nonlinear interaction of waves with phases $\varphi_1, \ldots, \varphi_m$ lead to the appearance of source terms waves with phases equal to real linear combinations $\psi = \sum \alpha_k \varphi_k = \alpha \cdot \varphi$ of the incoming phases. We consider four distinct possibilities.

(i) *Nondegerate resonance.* $d\psi \neq 0$ and ψ satisfies the eikonal equation $p(t, x, d, \psi) = 0$.

One might think that the phase ψ could focus at a time earlier than the incoming phases. However, since $d\psi$ is a linear combination of the $d\varphi_k$, it is clear from criterion (iv) above that this cannot happen. In summary, nondegenerate resonance does not create focusing apart from that already present in the incoming waves.

(ii) $p(t, x, d\psi) \neq 0$ *for all* t, x. In this case $\text{graph}(d(\alpha \cdot \varphi))$ is disjoint from the characteristic variety. Then $L^{-1}(be^{i\alpha \cdot \varphi/\varepsilon})$ is an elliptic inversion and one expects no propagation for the resulting terms.

(iii) *Weakly resonant nondegenerate focusing.* Here $d(\alpha \cdot \varphi) \neq 0$ and $\text{graph}(d(\alpha \cdot \varphi))$ intersects but does not lie inside $\text{Char}(L)\backslash 0$. This possibility is discussed in the next section.

(iv) *Degenerate interaction.* Here $d(\alpha \cdot \varphi)$ vanishes at at least one point. This is discussed in Section 9.5.

9.4. *Weak resonant nondegenerate focusing.* -- Graph $(d(\alpha \cdot \varphi))$ intersects but does not lie inside $\text{Char}(L)\backslash 0$. Note that in the $2d + 2$ dimensional contangent bundle $T^*(R^{1+d})$

$$(9.4.1) \qquad \dim(\text{Char}(L)) = 2d + 1 \qquad \text{and} \qquad \dim(\text{graph}(d(\alpha \cdot \varphi))) = d + 1.$$

The generic case of transverse intersection yields

$$(9.4.2) \qquad \dim[\text{Char}(L) \cap \text{graph}(d(\alpha \cdot \varphi)] = d,$$

and

$$(9.4.3) \qquad \dim \pi_x(\text{Char}(L) \cap \text{graph}(d(\alpha \cdot \varphi))) = d.$$

In this case, a stationary phase computation using the Fourier integral parametrix for L yields for points after interaction

(9.4.4) $$L^{-1}\left(ae^{i\alpha\cdot\varphi/\varepsilon}\right) \sim \sqrt{\varepsilon}\,be^{i\psi/\varepsilon}$$

with a new phase, ψ satisfying the eikonal equation and constructed according to the recipe

(9.4.5)   graph $(d\psi)$ is the H_p flow out of Char $(L) \cap$ graph $(d(\alpha \cdot \varphi))$

(9.4.6) $\psi = \alpha \cdot \varphi$ on Char $(L) \cap$ graph $(d(\alpha \cdot \varphi))$.

First note that the outgoing wave is small of order $\sqrt{\varepsilon}$. In formal nonlinear geometric optics such terms are traditionally ignored. In one space dimension it has been rigorously proven that dropping these terms is justified [JMR1]. In higher dimensions, the outgoing phase ψ may focus at a time earlier than the focusing of the incoming waves. This can amplify the small wave to one that is large, say of order $1/\sqrt{\varepsilon}$. Nonlinear terms can further amplify to ε^{-n} for any n, and one can have explosive nonexistence at a positive time $\underline{t}$ which is strictly smaller than the time at which the incoming phases φ focus. This is an example of hidden focusing. The phases which focus are not apparent in the initial phases but are created by nonlinear interaction. The mechanism just described is the focusing at $\underline{t} > 0$ of a wave created by weakly nonlinear resonance.

9.5. *Degenerate nonlinear focusing.* – This term refers to the following situation. There is a real linear combination $\alpha \cdot \varphi$ of incoming phases and a point $\underline{t}$, $\underline{x}$ at which $d(\alpha \cdot \varphi) = 0$. As in instantaneous direct focusing, this can lead to nonlinear explosive nonexistence at $t = \underline{t}$. As above, this is a type of hidden focusing. If $\underline{t} = 0$ we say that there is instantaneous degenerate nonlinear focusing.

Note that the interaction of two phases with independent differentials cannot yield degenerate nonlinear focusing while three phases with pairwise independent phases can.

9.6. *Second generation focusing.* – In the above examples initial phase φ were the ingredients for producing focusing phenomena. We showed in Section 9.3 that new phases created by nondegenerate resonant interaction did not lead to new troubles. In Section 9.4 we suggested that small waves with new phases ψ can be created by weakly resonant nonlinear interaction and that these new waves may focus earlier than the original waves. *Second generation* phenomena are interactions of the new waves with incoming waves.

The phases to consider in this second generation are real linear combinations $\alpha \cdot \varphi + b\,\psi$. Taking for example, α equal to the coefficients in (9.4.4) which lead to the creation of the ψ wave and taking $b = -1$ we see that $d(\alpha \cdot \varphi + b\,\psi) = 0$ on a codimension one subset of space-time. More generally if there is a real combination $\alpha \cdot \varphi + b\,\psi$ and a small set on which $d(\alpha \cdot \varphi + b\,\psi) = 0$ we may expect to find *second generation degenerate focusing*. If there are such points on $t = 0$, the focusing can be *instantaneous*.

If there is a real linear combination so that graph $(d(\alpha \cdot \varphi + b\,\psi))$ cuts Char (L) transversely one may have second generation weakly nonlinear interaction leading to

second generation weakly resonant focusing. Before focusing, these second generation waves will have small amplitudes. Interactions involving these second generations waves are called *higher generation interactions.* We will content ourselves with naming and constructing waves of the first and second generation.

10. EXAMPLES OF FOCUSING. – In this section we construct examples showing that the phenomena described in section 9 occur. § 10.1 is devoted to direct focusing while 10.2 gives examples of hidden focusing.

10.1. *Direct focusing.* – We begin with the simplest examples. The later examples are elaborations of these.

Example 10.1.1. – *Non degenerate direct focusing.*

Consider radial solutions $u(t, x)$ on $\mathbb{R}^{1+3}$ of a semilinear system whose principal part is the strictly hyperbolic system $(\Box, \partial_t)$

$$(10.1.1) \qquad \Box u^\varepsilon = 0, \qquad u^\varepsilon(0, x) = 0, \qquad \partial_t u^\varepsilon(0, x) = h(r) e^{ir^2/\varepsilon}$$

$$(10.1.2) \qquad \partial_t v^\varepsilon = |\partial_t u^\varepsilon|^2 (v^\varepsilon)^2, \qquad v^\varepsilon(0, x) = v_0(x).$$

Note that the state variables appearing in L^2 energy estimates are $\nabla_{t,x} u^\varepsilon$ and v^ε so that semilinear systems allow nonlinear functions of these variables. For any $v_0 \in C_0^\infty(\mathbb{R}^3)$, there is a maximal interval of existence of smooth solutions, $[0, T(\varepsilon)[$, characterized by

If $T(\varepsilon) < \infty$, then there is a unique solution $(v, u) \in C_0^\infty([0; T(\varepsilon)[\times \mathbb{R}^3)$ and $\sup(|v^\varepsilon(t, \cdot)|, |\Delta_{t,x} u^\varepsilon(t; \cdot)|) \to \infty$ as t increases to $T(\varepsilon)$.

PROPOSITION 10.1.2. – *Suppose that the initial data $(v_0, h) \in C_0^\infty(\mathbb{R}^3)$ satisfy $\mathrm{supp}(h) \subset [\underline{t}, \infty[, t > 0, h(t) > 0$ on $]\underline{t}, \underline{t} + \delta]$, and $v_0(0) > 0$. Then as ε tends to zero, the maximal interval of existence for (10.1.1) (10.1.2) satisfies $T(\varepsilon) \leqq \underline{t} + o(1)$.*

Proof. – For any $\eta \in]0, \delta]$ apply formula (9.15) to find for ε small

$$(10.1.3) \qquad |\partial_t u^\varepsilon(t, 0)| > c\varepsilon^{-1} \qquad \text{for} \quad \underline{t} + \eta/2 < t < \underline{t} + \eta.$$

To finish the proof apply the following lemma with $y(t) := v^\varepsilon(t, 0)$, $a(t) := |\partial_t u^\varepsilon(t, 0)|^2$. The lemma is proved by explicitly solving the initial value problem.

LEMMA 10.1.3. – *Suppose that $a(t)$ is a smooth nonnegative function on $[0, \infty[$ and $\int_0^\infty a(t)\,dt > 1/y_0$. Then, the maximal interval of existence, $[0, T[$, for the initial value problem*

$$y' = a(t)y, \qquad y(0) = y_0 > 0$$

in given by

$$(10.1.4) \qquad \int_0^T a(t)\,dt = 1/y_0.$$

Using the lower bound (10.1.2) yields

$$(10.1.5) \qquad \int_{\underline{t}}^{t+\delta} |\partial_t u^\varepsilon (t, 0)|^2 \, dt \geq c \, \delta \varepsilon^{-2}.$$

This together with the lemma proves the proposition.

Example 10.1.4. – *Degenerate direct focusing.* A variant of the previous example provides an example of degenerate direct focusing.

PROPOSITION 10.1.5. – *Suppose that the initial data* $(v_0, h) \in C_0^\infty (\mathbb{R}^3)$ *satisfy* $h(0) \neq 0$ *and* $v_0(0) > 0$. *Then, there is a constant* $c > 0$ *so that for* $0 < \varepsilon < 1$, *the maximal interval of existence for* (10.1.1) (10.1.2) *satisfies* $T(\varepsilon) < c\varepsilon^{2/5}$.

Proof. – Formula (9.1.5) yields

$$(10.1.6) \qquad |\partial_t u^\varepsilon (t, 0)| > c_1 t^2 / \varepsilon \qquad \text{for} \quad c_2 \varepsilon^{1/2} \leq t \leq c_3.$$

Applying Lemma 10.1.2 yields the desired result.

Remark. – The explosion in this example occurs at $x = 0$ for very small t. Thus the finite propagation speed shows that if ω is any neighborhood of $(0, 0)$, there are positive constants c and ε_0 such that if $\varepsilon \leq \varepsilon_0$ and there is a smooth solution on $\omega \cap \{ 0 \leq t \leq T \}$, then $T < c\varepsilon^{2/5}$.

Critiques. – The two examples above have the advantage of being completely explicit. They have several weaknesses.

1. They are examples for systems and one might ask whether the same phenomena occur for scalar equations $\Box u = F(Du)$ and $\Box u = F(u)$.

2. The effects of focusing of spherical solutions are stronger as the dimension grows. The above examples are in $\mathbb{R}^{1+3}$ and one might wonder whether the same effects are present in two dimensional space.

3. The focusing effects are studied on the $x = 0$ axis which is a very small set. One could imagine that they occur only there and that the right hand sides $|\partial_t u^\varepsilon|^2 (v^\varepsilon)^2$ are uniformly bounded in L^1_{loc} in which case there would exist global weak solutions uniformly bounded in $L^1_{\text{loc}} (\mathbb{R}^{1+3})$.

4. One could ask if focusing which is not as radical as for spherical waves, for example focusing along a nondegenerate caustic causes analogous difficulties.

5. There are problems where in spite of such focusing, weak solutions continue to exist and it is desireable to describe their behavior.

The next three examples remove some of the first three criticisms. They are presented for instantaneous degenerate focusing. Putting a factor $h(r)$ with $h = 0$ on $[0, T]$ in front of the oscillatory initial condition yields examples of nondegenerate direct focusing explosions at time $T > 0$ as in example 10.1.1. The details are left to the reader.

Example 10.1.6. – *A simple scalar 5-d example of degenerate focusing.*

This example takes advantage of the explicit linearization of Nirenberg's nonlinearity. Consider in $\mathbb{R}^{1+5}$ the solutions of

(10.1.7) $\Box u^\varepsilon = (\partial_t u^\varepsilon)^2 - |\nabla_x u^\varepsilon|^2, \qquad u^\varepsilon(0,x) = 0, \qquad \partial_t u^\varepsilon(0,x) = -\cos(|x|^2/\varepsilon).$

Nirenberg's linearization,

(10.1.8) $w^\varepsilon := 1 - \exp(u^\varepsilon)$

transforms the nonlinear problem to the linear problem

(10.1.9) $\Box w^\varepsilon = 0, \qquad w^\varepsilon(0,x) = 0, \qquad \partial_t w^\varepsilon(0,x) = \cos(|x|^2/\varepsilon).$

Recall that the solution $w(t,x)$ of the 5-d linear wave equation with initial value equal to zero and initial time derivative equal to $f(x)$ is given by the formula

(10.1.10) $w(t,0) = t\, A\, v(f, |x| = t) + (t^2/3)\, A\, v(\partial_r f, |x| = t),$

where $A\, v(\cdot, |x| = t)$ means the average over the sphere of radius t, and ∂_r denotes the radial derivative. Thus the explicit solution to our problem is equal to

(10.1.11) $w^\varepsilon(t,0) = -\varepsilon^{-1} 2t^3 \sin(t^2/\varepsilon) + t\cos(t^2/\varepsilon).$

PROPOSITION 10.1.7. – *If ω is a neighborhood of* $(0,0)$, *there are positive constants c and $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and there is a smooth solution of* (10.1.8) *on* $\omega \cap \{0 < t < T\}$ *then* $T < c\varepsilon^{1/3}$.

Proof. – If u^ε is such a solution then w^ε defined by (10.1.8) is a solution of (10.1.9) and $w^\varepsilon < 1$. The formula (10.1.11) shows that if c is large then $w^\varepsilon(t,0) < 1$ cannot hold for all $t \in [0, c\varepsilon^{1/3}]$.

For the examples that follow, we need to study focusing of solutions of the linear wave equation in dimension 2 and 4. It is just as easy to treat all dimensions, recovering the main term in (10.1.11) at the same time. The time derivative of the solution v^ε to

(10.1.12) $\Box v^\varepsilon = 0, \quad v^\varepsilon(0,x) = 0, \quad \partial_t v^\varepsilon(0,x) = e^{i|x|^2/2\varepsilon}, \qquad x \in \mathbb{R}^d,$

is given by the oscillatory integral

(10.1.13) $\partial_t v^\varepsilon(t,x) = (2\pi)^{-d} \int \int \cos(t|\xi|)\, e^{i\,(x\xi - y\xi + |y|^2/2\varepsilon)}\, dy\, d\xi,$

(10.1.14) $= (2\pi)^{-d} \{V_- + V_+\}/2,$

where

$$(10.1.15) \qquad V_{\pm}(t,\, x) := \int\int e^{i\,(\pm t|\xi|+x\xi-y\xi+|y|^2/2\varepsilon)}\, dy\, d\xi,$$

$$= \text{const.}\,\varepsilon^{d/2} \int e^{i\,(\pm t|\xi|+x\xi-\varepsilon|\xi|^2/2)}\, d\xi.$$

This integral is analysed by the method of stationary phase. Introduce the variables $\xi := \varepsilon\xi/t$ to find

$$(10.1.16) \qquad V_{\pm} = \text{const.}\,(t^2/\varepsilon)^{d/2} \int e^{i\,(\pm|\zeta|-|\xi|^2/2)\,t^2/\varepsilon}\, e^{i\zeta xt/\varepsilon}\, d\zeta.$$

At points near $(0,\, 0)$ where

$$(10.1.17) \qquad \lambda := t^2/\varepsilon \gg 1 \qquad \text{and} \qquad |xt/\varepsilon| \leqq 1,$$

on has $1 \gg t \gg |x|$, and $e^{i\zeta xt/\varepsilon}$ is not rapidly varying with respect to ζ. The latter factor is treated as an amplitude term.

For $|\zeta| \geqq 2$, the phases $|\zeta|^2/2 \pm |\zeta|$ are smooth and have gradient bounded below by const.$|\zeta|$. Thus if $\chi\,(|\zeta|)$ is smooth and vanishes for $|\zeta| > 3$ and is equal to 1 for $|\zeta| < 2$ then integration by parts shows that in the region $(10.1.17)$

$$(10.1.18) \qquad \int e^{i\,(\pm|\zeta|-|\xi|^2/2)\,t^2/\varepsilon}\, e^{i\zeta xt/\varepsilon}\, \{\, 1 - \chi\,(|\zeta|)\,\}\, d\zeta = O\,(\lambda^{-\infty}).$$

Therefore one can introduce the factor $\chi\,(\zeta)$ into $(10.1.16)$ without changing the assymptotic behavior.

Introduce polar coordinates

$$(10.1.19) \qquad \zeta = \rho\theta, \qquad \theta \in S^{d-1}, \qquad d\zeta = \rho^{d-1}\, d\rho\, d\theta,$$

to find

$$(10.1.20) \qquad V_{\pm} = \text{const.}\,\lambda^{d/2} \int_0^\infty e^{i\,(\pm\rho-\rho^2/2)\,\lambda}\, J\,(\rho\, xt/\varepsilon)\, \chi\,(\rho)\, \rho^{d-1}\, d\rho + O\,(\lambda^{-\infty}),$$

where the Bessel function J is given by

$$(10.1.21) \qquad J\,(y) := \int e^{iy\theta}\, d\theta.$$

The integral $(10.1.20)$ is analysed by standard stationary phase methods. For the minus sign there are no stationary points. With the plus sign there is exactly one nondegenerate

stationary point at $\rho = 1$. At that point, the phase is equal to $1/2$ and the second derivative is equal to 1.

The contribution of the nonstationary points is estimated by integration by parts. Thanks to the ρ^{d-1} term in (10.1.20), the boundary terms at $\rho = 0$ vanish for the $d - 2$ first integrations by parts. This shows that in the region (10.1.17), the stationary phase expansion gives the behavior of V_+. Thus,

$$(10.1.22) \qquad \partial_t v^\varepsilon (t, x) = \text{const.} \lambda^{(d-1)/2} e^{i\lambda/2} J (tx/\varepsilon) + O (\lambda^{(d-2)/2}).$$

For $d = 2$ and $0 < \alpha < 1/2$, (10.1.22) is used in the cylinder

$$(10.1.23) \qquad R(\varepsilon) := \{ (t, x) \in \mathbb{R}^3 : 2\varepsilon^\alpha \geqq t \geqq \varepsilon^\alpha \text{ and } |x| \leqq \delta\varepsilon^{(1-\alpha)} \},$$

where δ is chosen so that $J(y) > J(0)/2 > 0$ for $|y| \leqq 2\delta$. Then

$$(10.1.24) \qquad \exists c > 0, \qquad \forall (t, x) \in R(\varepsilon), \qquad |\partial_t v^\varepsilon (t, x)| \geqq c\varepsilon^{\alpha - 1/2}.$$

For $d \geqq 2$, an entirely parallel argument shows that in the region (10.1.17),

$$(10.1.25) \quad v^\varepsilon (t, x) = \text{const.} \, t \, \lambda^{(d-2)/2} \int |\zeta|^{-1} \sin (\lambda|\zeta|) e^{-i\lambda|\xi|^2/2} e^{i\xi xt/\varepsilon} d\zeta$$

$$= \text{const.} \lambda^{(d-2)/2} \int_0^\infty \sin (\lambda\rho) e^{-i\lambda\rho^2/2} J (\rho xt/\varepsilon)$$

$$\times \chi (\rho) \rho^{d-2} d\rho + O (\lambda^{-\infty})$$

$$= \text{const.} \, t \, \lambda^{(d-3)/2} e^{i\lambda/2} J (tx/\varepsilon) + O (t \lambda^{(d-4)/2}).$$

Example 10.1.8. – *2-d degenerate focusing.*

Consider the semilinear system oscillatory initial value problem in $\mathbb{R}^{1+2}$

$$(10.1.26) \qquad \Box v^\varepsilon = 0, \qquad v^\varepsilon (0, x) = 0, \qquad \partial_t v^\varepsilon (0, x) = e^{i|x|^2/2\varepsilon},$$

$$(10.1.27) \qquad \Box u^\varepsilon = |\partial_t v^\varepsilon|^\gamma |u^\varepsilon|^\beta, \qquad u^\varepsilon (0, x) = 1, \qquad \partial_t u^\varepsilon (0, x) = 0.$$

The monotonicity of $\Box^{-1}$ when $d = 2$, shows that $u^\varepsilon > 1$ for $t > 0$.

PROPOSITION 10.1.9. – *Fix* $d = 2$, $\beta > 1$, $\gamma > 4$, *and* ω *a neighborhood of* $(0, 0)$. *There are positive constants* c, μ *and* ε_0 *such that if* $\varepsilon < \varepsilon_0$ *and there is a smooth solution of* (10.1.26) (10.1.27) *on* $\omega \cap \{ 0 < t < T \}$ *then* $T < c\varepsilon^\mu$.

Proof. – The proof rests on two comparison theorems. The first is the following.

LEMMA 10.1.10. – *Suppose that* Γ *is the truncated light cone*

$$\Gamma := \{\, (t,\, x) \in \mathbb{R}^{1+2} : 0 \leqq t \leqq T \leqq R \text{ and } |x| \leqq R - t \,\},$$

that $a_k \in L^\infty(\Gamma)$ *satisfy* $a_1 \geqq a_2 > 0$ *and that* $0 \leqq u_k \in C(\Gamma)$ *satisfy*

$$(10.1.28) \qquad \Box u_k = a_k |u_k|^\beta, \qquad u_1 = u_2, \quad \text{and} \quad \partial_t u_1 = \partial_1 u_2$$

when $t = 0$ *and* $|x| < R$.
Then $u_1 \geqq u_2$ *on* Γ.

Proof. – Subtracting the equation for u_2 from that for u_1 yields the equation for $U := u_1 - u_2$

$$(10.1.29) \qquad \Box U = a_2 \left(|u_1|^\beta - |u_2|^\beta \right) + (a_1 - a_2)|u_1|^\beta := a_2 \left(|u_1|^\beta - |u_2|^\beta \right) + f$$

where $f := (a_1 - a_2)|u_1|^\beta$ is nonnegative. The first term on the right is equal to $b(t, x)\, U$ with $b \in C(\Gamma)$. Because u_1 and u_2 are $\geqq 0$, b is nonegative. The linear initial value problem

$$(10.1.30) \qquad \Box U = bU + f, \qquad U(0,\, x) = \partial_t U(0,\, x) = 0 \quad \text{for } |x| < R$$

is solved as the limit of Picard iterates defined by $U^0 = 0$ and

$$(10.1.31) \qquad \Box U^{\nu+1} = bU^\nu + f, \qquad U^{\nu+1}(0,\, x) = \partial_t U^{\nu+1}(0,\, x) = 0.$$

The positivity of $\Box^{-1}$ for $d = 2$ proves inductively that $U^\nu \geqq 0$. Passing to the limit $\nu \to \infty$ proves that $U \geqq 0$ on Γ, and the lemma is proved.

Let

$$(10.1.32) \qquad a(t,\, x) := \begin{cases} \{\, c\,\varepsilon^{(\alpha - 1/2)}\,\}^\gamma & \text{for } (t,\, x) \in R(\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

where $R(\varepsilon)$ is defined in (10.1.23). Let w^ε be the solution of the initial value problem

$$(10.1.33) \qquad \Box w^\varepsilon = a^\varepsilon |w^\varepsilon|^\beta, \qquad w^\varepsilon(0,\, x) = 1, \qquad \partial_t w^\varepsilon(0,\, x) = 0.$$

Lemma 10.1.10 together with (10.1.24) imply that $u^\varepsilon > w^\varepsilon$ on any truncated light cone on which both exist.

For $t \leqq \varepsilon^\alpha$, the right hand side of (10.1.33) vanishes so $\omega^\varepsilon = 1$ for $t \leqq \varepsilon^\alpha$. In the tiny tip of backward light cone

$$(10.1.34) \qquad K := \{\, (t,\, x) : \varepsilon^\alpha \leqq t \leqq \varepsilon^\alpha + \delta\varepsilon^{1-\alpha} \text{ and } |x| \leqq \delta^{1-\alpha} - (t - \varepsilon^\alpha) \,\},$$

the coefficient $a^\varepsilon(t, x)$ and the Cauchy data at $t = \varepsilon^\alpha$ of w^ε are independent of x. This implies, as in J. Keller's proof of a similar explosion [Ke], that in K, w^ε is independent of x and in fact $w^\varepsilon(t, x) = y(t, \varepsilon)$ where y is the solution of

$$(10.1.35) \qquad d^2 y/dt^2 = \{ c\, \varepsilon^{(\alpha-1/2)} \}^\gamma y^\beta, \qquad y(\varepsilon^\alpha, \varepsilon) = 1, \qquad y'(\varepsilon^\alpha, \varepsilon) = 0.$$

The solution y explodes at time $\varepsilon^\alpha + t(\varepsilon)$, with

$$t(\varepsilon) = \int_1^\infty \{ 2\,(c\,\varepsilon^{(\alpha-1/2)})^\gamma \,(\gamma^{\beta+1} - 1)/(\beta + 1) \}^{-1/2}\, dy = \text{const.}\,\varepsilon^{(1/2-\alpha)\,\gamma/2}.$$

The tip of the light cone has height $\delta\varepsilon^{1-\alpha}$, the same as the radius of the base. Thus the function y explodes before the tip precisely when α and γ satisfy $(1 - \alpha) < (1/2 - \alpha)\,\gamma/2$. This has solutions α with $0 < \alpha < 1/2$ precisely when $\gamma > 4$.

Thus, if u^ε exists in $K \cap \{ t < \varepsilon^\alpha + \delta\varepsilon^{1-\alpha} \}$ it must satisfy $u^\varepsilon \geqq y(t, \varepsilon)$ and therefore, if ε is small enough, u^ε explodes at or before $\varepsilon^\alpha + \delta\varepsilon^{1-\alpha}$.

Remark. – At the blowup time for y one has

$$(10.1.36) \qquad \infty = \int_{\varepsilon^\alpha}^{\varepsilon^\alpha + t(\varepsilon)} y''\, dt = \int_{\varepsilon^\alpha}^{\varepsilon^\alpha + t(\varepsilon)} \{ c\,\varepsilon^{(\alpha-1/2)} \}^\gamma y^\beta\, dt.$$

Since the construction is by inequality of exponents of ε, y blows up way before the time of the peak of the cone. In this way we see that, for ε small enough, $u^\varepsilon(\cdot, x)$ blows up in K for $|x| < \delta'\varepsilon^{1-\alpha}$ with any $\delta' < \delta$. Furthermore the second integrand in (10.1.36) is a lower bound for $|\partial_t v^\varepsilon|^\gamma |u^\varepsilon|^\beta$. Thus, by (10.1.36), it is not possible that

$$\int_K |\partial_t v^\varepsilon|^\gamma |u^\varepsilon|^\beta dt < +\infty.$$

This answers critique ♮3 above. A similar analysis of explosion for example 2 would yield a similar answer.

Example 10.1.11. – *4-d degenerate focusing without derivatives in the nonlinearity.*

PROPOSITION 10.1.12. – *Fix $d = 4$, $\gamma > 0$, $\beta > 1$ and ω a neighborhood of $(0, 0)$. There are positive constants c, μ and $\varepsilon_0 > 0$ such that if $\varepsilon \leqq \varepsilon_0$ and there is a smooth solution of (10.1.26) coupled to the ordinary differential equation*

$$(10.1.43) \qquad \partial_t^2 u = |v^\varepsilon|^\gamma |u^\varepsilon|^\beta, \qquad u^\varepsilon(0, x) = 1, \qquad \partial_t u^\varepsilon(0, x) = 0.$$

on $\omega \cap \{ 0 < t < T \}$ then $T < c\varepsilon^\mu$.

Proof. – Use (10.1.25) and proceed as in the proof of Proposition 10.1.9.

Critique. – The criticisms 1, 2, 3 have all been addressed to some extent. The above analysis leaves several open questions.

Question 1. – Produce blow up for $d = 2$ and a scalar equation.

Question 2. – The fact that $H^1 (\mathbb{R}^2)$ is contained in L^p for all $p < \infty$, allows one to show that for $\square\, u = f\, (u)$ with f polynomially bounded and initial data $u^\varepsilon\, (0,\, x) = \varepsilon\, a_0\, e^{i\varphi\, (x)/\varepsilon}$, $\partial_t\, u^\varepsilon\, (0,\, x) = a_1\; e^{i\varphi\, (x)/\varepsilon}$, the domain of smooth existence does not shrink with ϵ. Is the same true for arbitrary smooth f ?

Question 3. – Produce blow up for $d = 4$ and a scalar equation without derivative in the nonlinearity.

Question 4. – In dimension $d = 3$, investigate the analogous questions of blow up for scalar waves equations, with or without derivative in the nonlinearity. See the non explosive example of [D].

10.2. *Hidden focusing, focusing from nonlinear interaction.* – In this subsection we construct examples of instantaneous explosions which are caused by the focusing in phases produced by nonlinear interactions.

Example 10.2.1. – *Degenerate nonlinear focusing.*

The space dimension is 3. Cauchy data are prescribed which in the linear case would yield four high frequency wave trains one for each of the phases φ_1, φ_2, φ_3, φ_4. Polynomial interaction leads one to source terms with phases which are linear combinations of the φ_k. We choose the phases such that there is at least one combination of the first three phases which has an isolated stationary point on $\{\, t = 0\, \}$ and the resulting degenerate focusing causes instantaneous explosion.

Fix the notations $\partial_k = \dfrac{\partial}{\partial x_k}$ and $\square_c = \partial_t^2 - c^2\, \Delta_x$ where $c\, (t,\, x)$ is positive real valued function. The system of equations in $\mathbb{R}^{1+3}$ is

$$(10.2.1) \qquad \begin{cases} \square_1\, u^\varepsilon = 0, \\ \square_c\, v^\varepsilon + a\, (t,\, x)\, \partial_1\, v^\varepsilon = 0 \\ \square_3\, w^\varepsilon = b\, (t,\, x)^{-2}\, (\partial_t\, \bar{u}^\varepsilon + \partial_1\, \bar{u}^\varepsilon)^3\, (\partial_t\, u^\varepsilon - \partial_1\, u^\varepsilon)\, (\partial_t\, v^\varepsilon)^2, \\ \partial_t\, z^\varepsilon = |\partial_1\, w^\varepsilon|^2\, (z^\varepsilon)^2. \end{cases}$$

The initial conditions are

$$(10.2.2) \qquad \begin{cases} u^\varepsilon\, (0,\, x) = 0,\; \partial_t\, u^\varepsilon\, (0,\, x) = e^{ix_1/\varepsilon}, \\ v^\varepsilon\, (0,\, x) = \varepsilon\, e^{ix_1/\varepsilon},\; \partial_t\, v^\varepsilon\, (0,\, x) = b\, (0,\, x)\, e^{ix_1/\varepsilon}, \\ w^\varepsilon\, (0,\, x) = 0,\; \partial_t\, w^\varepsilon\, (0,\, x) = 0, \\ z^\varepsilon\, (0,\, x) = 0. \end{cases}$$

Here a, b, c are smooth functions chosen below and satisfying $c\, (0,\, 0) = 2$ so that the system is strictly hyperbolic on a neighborhood of the origin.

PROPOSITION 10.2.2. – *If ω is a neighborhood of* $(0, 0)$ *there are positive constants C and ε_0 such that if $\varepsilon < \varepsilon_0$ and there is a solution of* (10.2.1), (10.2.2) *on $\omega \cap \{\, t < T\, \}$, Then $T < C\, \varepsilon^{1/6}$.*

Proof. – It suffices to show that there are positive constants η, γ with the property that

$$(10.2.3) \qquad |\partial_t\, w^\varepsilon\, (t,\, 0)| \geqq \gamma \varepsilon^{-1/2}\, t^{5/2} \qquad \text{for} \quad t \geqq \eta \varepsilon^{1/3}.$$

The initial value problem for u is explicitly solvable.

$$(10.2.4) \qquad u^\varepsilon = \varepsilon \left[e^{i\varphi_1/\varepsilon} - e^{i\varphi_2/\varepsilon} \right]/2\,i, \qquad \varphi_1 = x - t, \quad \varphi_2 = x + t.$$

Since $c(0,\,0) = 2$, the eikonal equation for $\Box_c$ with initial value equal to x_1 has two solutions

$$\varphi_3 = x + 2\,t + O\left(|t|\,|t,\,x|\right), \qquad \varphi_4 = x - 2\,t + O\left(|t|\,|t,\,x|\right).$$

Then

$$\psi = -3\,\varphi_2 + \varphi_1 + 2\,\varphi_3 = O\left(|t|\,|t,\,x|\right).$$

The right hand side $F\left(\nabla u^\varepsilon,\, \nabla v^\varepsilon\right)$ of the equation for w^ε is chosen to single out this phase. Note that $d\psi\,(0,\,0) = 0$ and this degeneracy accounts for the focusing explosion.

Next choose the coefficients to simplify the calculations. Let

$$\varphi_3\,(t,\,x) = x_1 + 2\,t + t\left(|x|^2 - t^2\right), \qquad c\,(t,\,x) = |\nabla \varphi_3|^{-1} \partial_t \varphi_3$$

With these choices φ_3 satisfies the eikonal equation and the equation for v^ε has the explicit solution

$$v^\varepsilon = \varepsilon\, e^{i\varphi_3/\varepsilon}.$$

Choosing

$$a\,(t,\,x) = |\partial_1 \varphi_3|^{-1}\Box_c\,\varphi_3, \qquad b\,(t,\,x) = i\,\partial_t\,\varphi_3$$

the initial value problem for w^ε simplifies to

$$\Box_3 w^\varepsilon = e^{i\psi/\varepsilon}, \qquad \psi = 2\,t\left(|x|^2 - t^2\right), \qquad w^\varepsilon\,(0,\,x) = \partial_t w^\varepsilon\,(0,\,x) = 0.$$

Explicit solution of this three dimensional wave equation yields

$$w^\varepsilon\,(t,\,0) = \int_0^t (t-s)\,A\,v\left(e^{i\psi\,(s,\,\cdot)/\varepsilon},\, |x| = 3\,(t-s)\right) ds = \int_0^t (t-s)\,e^{2is\,[9\,(t-s)^2 - s^2]/\varepsilon}\,ds,$$

$$\partial_t w^\varepsilon\,(t,\,0) = \int_0^t \left(1 + 36\,i\,(t-s)^2\,s/\varepsilon\right) e^{2is\,[9\,(t-s)^2 - s^2]/\varepsilon}\,ds.$$

The change of variable $s = \tau\,t$ yields with $\lambda = t^3/\varepsilon$,

$$\partial_t w^\varepsilon\,(t,\,0) = t \int_0^1 \left(1 + 36\,i\,\lambda\,(1-s)^2\,s\right) e^{2i\lambda s\,[9\,(1-s)^2 - s^2]/\varepsilon}\,ds.$$

When $\lambda \to \infty$, this is a nondegenerate stationary phase integral with a unique nondegenerate stationary point $\tau = b \in\,]0,\,1[$. One finds

$$\partial_t w\,(t,\,0) = c_1\,t\left[\lambda^{1/2} + O\,(1)\right] \quad \text{as } \lambda \to \infty$$

uniformly for $\varepsilon^{1/3} \le t \le \delta$ which implies 10.2.3.

Example 10.2.3. – *Weakly resonant focusing.*

As noted in section 9.5, degenerate nonlinear focusing requires the interaction of three pairwise independent phases. Next we construct an example with two phases creating explosive nonexistence by the mechanism of weakly resonant focusing.

The example is in six dimensional space with spatial variable $x = (x_1, y) \in \mathbb{R} \times \mathbb{R}^5$. The incoming wave is

$$(10.2.5) \qquad u^\varepsilon(t; x) := \varepsilon \left\{ e^{i(x\omega^+ + t)/\varepsilon} - e^{i(x\omega^- + t)/\varepsilon} \right\}$$

with $\omega^\pm := (1/2, \pm 1/2, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^4$ so that u^ε is a solution of

$$(10.2.6) \qquad \Box_{\sqrt{2}} \, u^\varepsilon = 0$$

with two independent phases.

These two waves have a weakly resonant interaction to yield a wave v^ε solution to

$$(10.2.7) \qquad \begin{aligned} \Box_c \, v^\varepsilon &= \chi(t)\,(\partial_t u^\varepsilon - 2\,\omega^+ \cdot \nabla_x u^\varepsilon)\,(\partial_t u^\varepsilon - 2\,\omega^- \cdot \nabla_x u^\varepsilon) \\ &= \chi(t)\,e^{i(x_1 + 2t)/\varepsilon} =: f^\varepsilon, \end{aligned}$$

$$(10.2.8) \qquad v^\varepsilon = 0 \qquad \text{for} \quad t < 0.$$

Here $\chi \in C^\infty(\mathbb{R})$ vanishes for $t < 0$ and is strictly positive on $t > 0$. The speed c is chosen as

$$(10.2.9) \qquad c(t, x) := 2 + (|y|^2 + \delta^2 t^2)(|y|^2 - t^2).$$

The main contributions to v^ε come from those points (t, x) where the covector $(2, 1, 0, \ldots, 0) = d(x + 2t)$ belongs to

$$\mathrm{Char}(\Box_c) = \left\{ (t, x, \tau; \xi) : \tau^2 = c^2(t, x)|\xi|^2 \right\},$$

that is where $c = 4$. With our $c(t, x)$ this is exactly the cone $\{ (t, x_1, y) : |y| = t \}$.

The wave v^ε then interacts with u^ε to create an outgoing wave w^ε which has instantaneous focusing. This wave satisfies

$$(10.2.10) \qquad \Box_1 \, w^\varepsilon = \chi(t)\,\partial_t v^\varepsilon\,(\partial_t \bar{u}^\varepsilon - 2\,\omega^+ \cdot \nabla_x \bar{u}^\varepsilon)\,(\partial_t \bar{u}^\varepsilon - 2\,\omega^- \cdot \nabla_x \bar{u}^\varepsilon),$$

$$(10.2.11) \qquad w^\varepsilon = 0 \qquad \text{for} \quad t < 0.$$

The speeds 1, $\sqrt{2}$, and c being distinct, the system for $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ is strictly hyperbolic on a neighborhood of $(0, 0)$.

PROPOSITION 10.2.4. – *There is a $\delta_0 > 0$ so that for $0 < \delta < \delta_0$ there is a positive $t_0(\delta)$ and smooth functions $b_j(t)$ and $\theta_j(t)$ $j = 1, 2$ defined on $[0, t_0]$ and such that for all $0 < t < t_0$*

$$(10.2.12) \qquad \theta_1(t) \neq \theta_2(t)) \ \text{are real}, \qquad b_1(t) \neq 0, \qquad b_2(t) \neq 0,$$

and

$$(10.2.13) \qquad \partial_t w^\varepsilon(t, 0) = \varepsilon^{-1} \left\{ b_1(t) e^{i\theta_1(t)/\varepsilon} + b_2(t) e^{i\theta_2(t)/\varepsilon} \right\} + O\left(\varepsilon^{-1/2}\right)$$

uniformly for t in compact subsets of $]0, t_0]$.

Consequently, if one adjoints the equation

$$\partial_t z^\varepsilon = |\partial_t w^\varepsilon|^2 (z^\varepsilon)^2, \qquad z^\varepsilon(0, x) = z_0(x).$$

then the domain of smooth existence can include no ε independent neighborhood of the origin.

Remark. – The functions b_k are infinitely flat at $t = 0$.

Proof. – The function v^ε is given by Duhamel's integral

$$v^\varepsilon(t, x) = \int_0^t V^\varepsilon(t, s, x) \, ds$$

where V solves the initial value problem

$$\Box_c V^\varepsilon(\cdot, s, \cdot) = 0, \qquad V^\varepsilon = 0 \quad \text{and} \quad \partial_t V^\varepsilon = \chi(s) e^{i(x_1 + 2s)/\varepsilon} \quad \text{at } t = s.$$

Since c depends only on $(t, |y|)$, it follows that $V^\varepsilon e^{-i(x_1 + 2t)/\varepsilon}$ depends only on $(t, |y|)$. Linear geometric optics for V^ε yields

$$(10.2.14) \quad \partial_t v^\varepsilon(t, x) = \int_0^t \left\{ a^+((t, x, s, \varepsilon) e^{i\varphi^+(t, s, x)/\varepsilon} + a^-(t, x, s, \varepsilon) e^{i\varphi^-(t, s, x)/\varepsilon} \right\} ds$$

The phases are solutions of the eikonal equations

$$(10.2.15) \qquad \partial_t \varphi^+ = \pm c(t, x)|\nabla \varphi^\pm|, \qquad \varphi^\pm(s, x, s) = x_1 + 2s$$

so have the form

$$(10.2.16) \qquad \varphi^\pm(t, x, s) = x_1 + 2t + \psi^\pm(t, s, |y|).$$

The amplitudes have asymptotic expansions $\displaystyle\sum_{j \geq 0} \varepsilon^j a_j^\pm(t, |y|, s)$.

Near the origin in (t, s, x) space,

$$\varphi^-(t, x, s) = x_1 + 2s - 4t + O(|t - s|(|t| + |s| + |x|))$$

and the corresponding integral in (10.2.14) is oscillatory without stationary points. The integrand vanishes at the endpoint $t = 0$, so the principal contributions come from the endpoint $t = s$ so the integral has the form

$$(10.2.17) \qquad g^\varepsilon(t, x) = \varepsilon\, b(t, x, \varepsilon)\, e^{i(x_1 + 2t)/\varepsilon}$$

where b has an expansion of order zero like that of $a^\pm$ valid when $\varepsilon \to 0$.

Denote by $h^\varepsilon(t, x)$ the φ^+ contribution to the right hand side of (10.2.14). Then w^ε satisfies $\Box_1 w^\varepsilon = \bar{f}^\varepsilon(g^\varepsilon + h^\varepsilon)$. The oscillatory factors in $\bar{f}^\varepsilon g^\varepsilon$ cancel so that the contribution of this term to w^ε is $O(1)$ in C^∞. Thanks to (10.2.14) and (10.2.16), the second term $l^\varepsilon := \bar{f}^\varepsilon h^\varepsilon$ is a function only of t and $|y|$. Denoting the contribution of this term by z^ε on has

$$(10.2.18) \qquad \Box_1 z^\varepsilon = l^\varepsilon, \qquad z^\varepsilon = 0 \qquad \text{for} \quad t \leqq 0.$$

To analyse z^ε, note that equation (10.2.15) implies

$$(10.2.19) \qquad \psi := \psi^+(t, s, r) = C(t, r) - C(s, r) + O\left(|t - s|^3(|t| + |s| + |r|)^4\right)$$

$$(10.2.20) \qquad C(t, r) = r^4 t - (1 - \delta^2)\, r^2 t^2/3 - \delta^2 t^5/5.$$

The solution formula (10.1.10) for $\Box_1$ on $\mathbb{R} \times \mathbb{R}^5$ yields

$$(10.2.21) \qquad \partial_t z^\varepsilon(t, 0) = \varepsilon^{-2} \int_0^t \int_0^s b(t, \sigma, s, \varepsilon)\, e^{i\psi(s, \sigma, t-s)/\varepsilon}\, d\sigma\, ds$$

where b has an asymptotic expansion of degree zero with leading term

$$(10.2.22) \qquad b_0(t, \sigma, s) = \text{const.}\, \chi(s)\, (t - s)^2\, (\partial_r \psi(s, \sigma, t - s))^2\, a_0^+(s, \sigma, t - \sigma)$$

Note that a_0^+, the leading term in the expansion of a^+, is a function of $(t, s, |y|)$ which satisfies

$$(10.2.23) \qquad a_0^+(t, s, x) > 0 \quad \text{for} \quad t > s > 0, \quad \text{and} \quad a^+(t, s, x) = 0 \quad \text{when} \quad s = 0$$

It remains to analyse (10.2.21) by the method of stationary phase.

For t small the principal part of the phase $\psi(s, \sigma, t - s)$ is

$$\Theta(t, s, \sigma) := C(s, t - s) - C(t - s, t - s)$$

which is homogeneous of degree 5. Introducing the scaled variables $s' := s/t$, $\sigma' := \sigma/t$ yields

$$\psi(s, \sigma, t - s) = t^5 \left\{ \Theta(1, s', \sigma') + t^2 \eta(t, s', \sigma') \right\}$$

where η is a smooth function of (t, s', σ') the latter two variables running over $\Delta := \{ (s', \sigma')/0 \leq \sigma' \leq s' \leq 1 \}$. Then

$$(10.2.24) \quad \partial_t z^\varepsilon (t, 0) = \varepsilon^{-2} \int_0^1 \int_0^{s'} b (t, t \sigma', ts', \varepsilon) e^{i \{\Theta (1, s', \sigma') + t^2 \eta (t, s', \sigma')\} t^5/\varepsilon} \, d\sigma \, ds$$

This is a standard stationary phase integral with large parameter $\lambda := t^5/\varepsilon$, and phase function

$$\theta (t, s', \sigma') := \Theta (1, s', \sigma') + t^2 \eta (t, s', \sigma').$$

LEMMA 10.2.5. – *There is a positive δ_0 so that for all $0 < \delta < \delta_0$ there is a positive $t_0 (\delta)$ so that for all $t < t_0 (\delta)$ the phase $\theta (t, \cdot, \cdot)$ as a function of (s', σ') has exactly three stationary points in the triangle $\Delta := \{ (s', \sigma')/0 \leq \sigma' \leq s' \leq 1 \}$. Exactly one of the three lies on the boundary, in fact at the point $(1/2, 1/2)$. The stationary points are uniformly nondegenerate. The values of the phase at the two interior stationary points are unequal.*

Proof. – The phase $\theta (t, \cdot, \cdot)$ is perturbation of $\Theta (1, s', \sigma')$. For $\Theta (1, s', \sigma')$ one verifies by direct calculation that there is $\delta_0 > 0$ so that for $0 < \delta \leq \delta_0$ there are exactly three stationary points in Δ. Two are at interior points $(\alpha_k, 1 - \alpha_k)$ where $1/2 < \alpha_1 < \alpha_2 < 1$ and the third is at the boundary point $(1/2, 1/2)$. All are nondegenerate and the values of Θ at the interior stationary points are unequal.

The implicit function theorem implies that there is a neighborhood of Δ such that for t small, the phase $\theta (t, \cdot, \cdot)$ has exactly three uniformly nondegenerate stationary points, two in the interior and the third near the point $(1/2, 1/2)$. One checks directly that $(1/2, 1/2)$ is a stationary point for all t and the Lemma follows.

Continuing with the demonstration of Proposition 10.2.4, denote by $(s'_k, \sigma'_k) k = 1, 2$ the interior stationary points of θ near $(\alpha_k, 1 - \alpha_k)$. Using (10.2.23) one checks that the amplitude $b_0 (t, t \sigma', ts', \varepsilon)$ is nonzero at these points, though it is infinitely flat as $t \to 0$. Furthermore (10.2.22) implies that $b_0 = 0$ on the part $s' = \sigma'$ of $\partial \Delta$, in particular at the point $(1/2, 1/2)$. Denote by $\theta_j (t)$ the value of the phase at the interior stationary points.

It follows that for $\lambda = t^5/\varepsilon$ large, the integral in (10.2.21) has asymptotic expansion

$$(10.2.25) \qquad \lambda^{-1} [\underline{b}_1(t) e^{i\theta_1 (t) \lambda} + \underline{b}_2(t) e^{i\theta_2 (t) \lambda}] + O (\lambda^{-3/2}).$$

with $\underline{b}_k (t)$ nonzero for $0 < t$ small, and infinitely flat at $t = 0$. Plug (10.2.25) in (10.2.24) with $\lambda = t^5/\varepsilon$, then absorb a t^{-3} factor in the $\underline{b}_k$ and a t^5 in θ_k to prove Proposition 10.2.4.

11. PROFILE RESTRICTIONS FROM FOCUSING BY SUPERPOSITION. – Consider the linear initial value problem

$$(11.1) \qquad \Box u^\varepsilon = 0, \qquad u^\varepsilon (t, x) = 0, \qquad \partial_t u^\varepsilon (0, x) = H (x/\varepsilon)$$

where H is an almost periodic function

$$(11.2) \qquad H = \sum h_\alpha e^{i\alpha \cdot x}.$$

The solution is a sum

(11.3) $$\nabla u^{\varepsilon}(t,\, x) = \sum u_{\alpha}^{+}\, e^{i\,(\alpha \cdot x + |\alpha|t)/\varepsilon} + u_{\alpha}^{-}\, e^{i\,(\alpha \cdot x - |\alpha|t)/\varepsilon}$$

with coefficients comparable to those in (11.2),

(11.4) $$C^{-1}|h_{\alpha}| \leqq |u_{\alpha}^{+}| + |u_{\alpha}^{-}| \leqq C|h_{\alpha}|.$$

Question. – Under what conditions on the periodic or almost periodic function H is the family ∇u^{ε} uniformly bounded in $L^{\infty}_{\mathrm{loc}}$? More precisely, for what norms on periodic functions H is it true that

$$\|\nabla u^{\varepsilon}\|_{L^{\infty}\,(\mathbb{R}^d)} \leqq c\,(t)\|H\| \quad \text{for all } 0 < \varepsilon < 1 ?$$

If the ∇u^{ε} are not uniformly bounded then by adjoining $\partial_{t} z^{\varepsilon} = F\,(\nabla u^{\varepsilon},\, z^{\varepsilon})$ one will have explosive nonexistence for a semilinear problem of nonlinear geometric optics form. The examples also put strong restrictions on the spaces from which profiles may be chosen (*see* the discussion at the end of this section).

The phases in the solution (11.3) neither focus nor degenerate. When the ∇u^{ε} are unbounded it is because of *focusing by superposition*.

In dimension $d = 1$, the fact that almost periodic functions are bounded together with the fact that the map $\nabla u\,(0,\, \cdot) \to \nabla u\,(t,\, \cdot)$ is bounded on L^{∞} shows that any bounded function H, in particular any almost periodic H, yields bounded ∇u^{ε}. That is, the $L^{\infty}\,(\mathbb{R})$ norm has the property required in the question.

The simple Proposition 11.1 shows that periodicity plus smoothness is sufficient. In dimension $d > 1$ one can neither drop smoothness (Proposition 11.2) nor replace periodicity by almost periodicity (Proposition 11.3).

PROPOSITION 11.1. – *If H is almost periodic with absolutely summable Fourier coefficients then the ∇u^{ε} are bounded and*

(11.5) $$\|\nabla u^{\varepsilon}\|_{L^{\infty}\,([0,\,1]\,\times\,\mathbb{R}^d)} \leqq c \sum |h_{\alpha}|.$$

In particular, if H is periodic and is in $H^{s}_{\mathrm{loc}}\,(\mathbb{R}^d)$ for $s > d/2$, then ∇u^{ε} is uniformly bounded and one has

(11.6) $$\|\nabla u^{\varepsilon}\|_{L^{\infty}\,([0,\,1]\,\times\,\mathbb{R}^d)} \leqq c\,(R)\|H\|_{H^{s}\,(|x|\leqq R)}$$

as soon as the R-ball contains a period parallelopiped.

PROPOSITION 11.2. – *If $d > 1$, there is a bounded nonsmooth periodic H so that for any neigbhorhood $\mathcal{O}$ of the origin in $\mathbb{R}^{1+d}$ and any $\varepsilon > 0$, ∇u^{ε} is not in $L^{\infty}\,(\mathcal{O})$.*

PROPOSITION 11.3. – *If $d > 1$, there is a sequence of smooth periodic functions H_{k} of period p_{k} decreasing to zero and such that for every α, the family $\{\,\partial^{\alpha} H_{k}\,\}$ is bounded in*

$L^\infty(\mathbb{R}^d)$ and so that if u_k^ε denotes the solution of (11.1) with $H = H_k$, then $\nabla u_k^{1/k}(t, 0)$ is not bounded in $L^\infty([0, T])$ for any $T > 0$.

PROPOSITION 11.4. – Let u^ε denote the solutions of (11.1) and suppose that $d > 1$. There is a smooth almost periodic function H all of whose derivatives are bounded on $\mathbb{R}^d$ and such that $\nabla u^\varepsilon(t, 0)$ is not bounded in $L^\infty([0, T])$ for any $T > 0$.

Proof of Proposition 11.2. – Choose $g \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ so that ∇v is not bounded on a neighborhood of the origin where v is the compactly supported solution of

(11.7) $\Box v = 0, \qquad v(0, x) = 0, \qquad \partial_t v(0, x) = g.$

Let H be any bounded periodic function which is equal to g on a neighborhood of the origin. Finite speed shows that near the origin in space time one has $\nabla u^\varepsilon = \nabla v(t/\varepsilon, x/\varepsilon)$ and the proof is complete.

Proof of Proposition 11.3. – Let $a(r)$ be smooth radial function supported in $r < 1$ and with $a(0) \neq 0$. For $k > 0$ let

(11.8) $H_k(x) = a(|x|/k) e^{i|x|^2/k} \qquad \text{for} \quad \max|x_l| \leq k.$

Extend H_k to be periodic of period k. Finite speed shows that for $\varepsilon = 1/k$ and $|t| + |x| < 1$, $v_k := u_k^{1/k}$ satisfies the initial value problem

$$\Box v_k = 0, \qquad v_k(0, \cdot) = 0, \qquad \partial_t v_k(0, x) = a(|x|) e^{ik|x|^2}.$$

Formula (10.1.23) shows that for any $t > 0$, $\partial_t v_k(t, 0)$ grows like $k^{(d-1)/2}$.

Proof of Proposition 11.4. – Let $\mathcal{F}$ be the space of almost periodic functions on $\mathbb{R}^d$ each of whose derivates is bounded. $\mathcal{F}$ is a Frechet space with seminorms $\|\partial^\alpha H\|_{L^\infty(\mathbb{R}^d)}$, $\alpha \in \mathbb{N}^d$. Let $K_{m,n} : \mathcal{F} \to L^\infty\left(\left[0, \frac{1}{m}\right]\right)$ be the map which sends H to $\partial_t u^{1/n}_{|[0;\frac{1}{m}] \times \{0\}}$. Proposition 11.3 shows that for m fixed the family $\{K_{m,n}\}$ is not uniformly bounded. The uniform boundedness principle implies that

$$E_m = \left\{ H \in \mathcal{F} : \{K_{m,n}(H)\} \text{ is bounded in } L^\infty\left(\left[0, \frac{1}{m}\right]\right) \right\}$$

is of first category in $\mathcal{F}$. Thus the same is true of the union $\cup E_m$.

In particular the complement of the union is of second category and therefore nonempty. Elements of the complement satisfy the conditions of the Proposition.

Discussion. – (i) The positive results above suggest that one may choose profiles to be smooth and periodic or to be almost periodic with absolutely summable Fourier series. The latter space is an algebra but is not closed under smooth but not real analytic functions which restricts its applicability. It is invariant under real analytic functions and in that case it serves quite well [JMR4].

(ii) Periodicity is too strong an assumption. For example, in the constant coefficient case one would like to study the case of three linearly independent initial phases $\alpha_k \cdot x$,

$k = 1$, 2, 3. These are susceptible to a treatment with periodic profiles only if the α_k are $\mathbb{Z}$-linearly dependent. In Part I, H is assumed to be quasiperiodic

$$H\left(x,\ \theta\right) = \mathcal{H}\left(x,\ M\,\theta\right), \qquad M \in \mathrm{Hom}\left(\mathbb{R}^d,\ \mathbb{R}^m\right)$$

so they lift to smooth functions $\mathcal{H}$ on an m dimensional torus which allows an analysis like that in Proposition 11.2, and also allows one to treat the triple interation just mentioned. The number of derivatives needed for the lift must be greater than $m/2$ where m is the number of $\mathbb{Z}$-independent phases.

(iii) This yields another explanation why smooth almost periodic profiles present a problem. For such almost periodic profiles one would like to take $m = \infty$, which would require lifting to an infinite dimensional torus and infinitely many derivatives.

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