Coherent mixed states and a generalised P representation

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Abstract. The pure Glauber (harmonic oscillator) coherent states provide a very useful basis for many purposes. They are complete in the sense that an arbitrary state in the Hilbert space may be expanded in terms of them. Furthermore, the well known P representation provides a diagonal expansion of an arbitrary operator in the Hilbert space in terms of the projection operators onto the coherent states. We study here the extensions of these results to the analogous mixed states which describe comparable harmonic oscillator systems in thermodynamic equilibrium at non-zero temperatures. Our results are given for the general density operator which describes the mixed squeezed coherent states of the displaced and squeezed harmonic oscillator. We show how these squeezed coherent mixed states similarly provide a very convenient complete description of a Hilbert space. In particular we show how the usual P and Q representations of operators in terms of pure states may be extended to finite temperatures with the corresponding mixed states, and various relations between them are demonstrated. The question of the existence of the generalised P representation for an arbitrary operator is further examined and some pertinent theorems are proven. We also show how our results relate to the Glauber-Lachs formalism in quantum optics for mixtures of coherent and incoherent radiation. Particular attention is focused both on the interplay between the quantum mechanical and thermodynamical uncertainties and on the entropy associated with such mixed states.

1. Introduction

The important role played by coherent states in a wide variety of problems in physics is by now widely appreciated (e.g. Klauder and Skagerstam 1985). Apart from the original Glauber and atomic coherent states, many other 'generalised coherent states' have subsequently beeen constructed in connection with specific applications and developments. In this particular work we are concerned with a generalisation suited to finite-temperature phenomena or to problems concerned with a superposition of coherent and chaotic states.

The usual coherent states discussed in the literature are pure states. By contrast, a quantum mechanical system in thermal equilibrium at finite temperatures is not in a pure state, but may be represented by a mixed state described by a density matrix (Fano 1957, Zubarev 1974, Harriman 1978, 1986). Consequently, our motivation is to study the coherence properties of mixed states. As a first step we consider here the mixed states described by the density matrices appropriate to systems of displaced and squeezed harmonic oscillators. These provide a much studied model system, and one of very considerable importance in various applications in the fields of quantum optics and quantum electronics. In our own case, they will serve as a particularly

illustrative example of our main aim which is to study and use these mixed states in the role of a 'basis' in the system Hilbert space. Clearly the term 'basis' is used here in a generalised sense, because in its strict sense it is a collection of pure states. A mixed state is, loosely, a set of pure states with a probability attached to each of them. It is usually described in terms of a density operator. One of our main conclusions is that the mixed coherent states of the harmonic oscillator provide a set of building blocks in the Hilbert space, comparable to the usual (pure state) basis vectors. In particular, we show how all bounded operators can be completely represented in terms of them.

In order to set the scene and to introduce the key concepts central to our motivation, let us first briefly consider various bases of pure states in a Hilbert space. We recall some of their properties, and express them in a language suitable for our later generalisations. In particular let us first compare and contrast conventional states and coherent states as bases. By conventional states, we refer here to the usual orthonormal eigenvectors that can be associated with any self-adjoint operator in the Hilbert space. A prime example is the orthonormal basis $\{|n\rangle\}$ associated with the number operator a^+a, where a and a^+ are the usual annihilation and creation operators of the linear harmonic oscillator, and which satisfy the basic commutation relation $[a, a^+] = I$. In this case it is instructive to compare the states $|n\rangle$ with the corresponding standard (or Glauber) coherent states $|A\rangle$. In terms of the eigenstates $|n\rangle$,

$$a^+a|n\rangle = n|n\rangle \qquad |n\rangle = (n!)^{-1/2}(a^+)^n|0\rangle \tag{1}$$

where $|0\rangle$ is the vacuum, $a|0\rangle = 0$ and the normalised standard coherent state $|A\rangle$ has the form

$$|A\rangle = \exp(-\tfrac{1}{2}|A|^2) \sum_{n=0}^{\infty} (n!)^{-1/2}A^n|n\rangle \tag{2}$$

for all complex numbers A. It is easily seen to be an eigenstate of the destruction operator

$$a|A\rangle = A|A\rangle. \tag{3}$$

It is of considerable interest for later purposes to compare the complete set of basis states $|n\rangle$ with the overcomplete set $|A\rangle$, and with later generalisations in mind we make this comparison via the corresponding density operators $I_n \equiv |n\rangle\langle n|$ and $I_A \equiv |A\rangle\langle A|$. In the case of the orthogonal basis, the operators I_n have three particularly important properties. In the first place, orthogonality immediately gives

$$I_n I_m = \delta_{mn} I_n \tag{4}$$

which clearly includes the weaker condition, $I_n^2 = I_n$, namely that the operators I_n are projection operators. Furthermore, the normalisation condition, $\mathrm{Tr}\, I_n = 1$, readily yields the result

$$\mathrm{Tr}(I_n I_m) = \delta_{mn}. \tag{5}$$

The second important relation is that the entropy associated with the density operator I_n is zero:

$$S_n \equiv -\mathrm{Tr}(I_n \ln I_n) = 0. \tag{6}$$

The third property concerns the completeness of the basis. This is conventionally expressed by the relation

$$\sum_n I_n = I. \tag{7}$$

In turn, (7) implies that an arbitrary ket $|k\rangle$ in the Hilbert space has the expansion

$$|k\rangle = \sum_n I_n |k\rangle = \sum_n \langle n|k\rangle |n\rangle \tag{8}$$

while an arbitrary (linear) operator Θ in the space has the expansion

$$\Theta = \sum_{m,n} \langle m|\Theta|n\rangle |m\rangle\langle n|. \tag{9}$$

In this latter regard it is particularly important to realise that *only* the very special class of operators $\{\Theta_c\}$ that commute with the number operator have the diagonal expansion

$$\Theta_c = \sum_n \langle n|\Theta_c|n\rangle |n\rangle\langle n| = \sum_n I_n \, \mathrm{Tr} \, (\Theta_c I_n). \tag{10}$$

Let us now compare these three properties of the density operators I_n in the number-state basis with their counterparts for the density operators I_A in the coherent-state basis. In the first place the operators I_A are not orthogonal, as in (4), although they are still projection operators,

$$I_A^2 = I_A. \tag{11}$$

It is easy to prove from (2) that the counterpart of (5) is now

$$\mathrm{Tr}(I_A I_B) = \exp(-|A - B|^2) \qquad \mathrm{Tr} \, I_A^2 = 1. \tag{12}$$

In the second place, the entropy associated with I_A is again identically zero

$$S_A \equiv -\mathrm{Tr}(I_A \ln I_A) = 0 \tag{13}$$

just as in (6). Finally, the completeness relation for the coherent states

$$\int \frac{\mathrm{d}^2 A}{\pi} I_A = I \qquad \mathrm{d}^2 A \equiv \mathrm{d}(\mathrm{Re} \, A) \, \mathrm{d}(\mathrm{Im} \, A) \tag{14}$$

is the counterpart of (7). Once again this implies that an arbitrary ket $|k\rangle$ may be expressed as

$$|k\rangle = \int \frac{\mathrm{d}^2 A}{\pi} I_A |k\rangle = \int \frac{\mathrm{d}^2 A}{\pi} \langle A|k\rangle |A\rangle. \tag{15}$$

A very important difference that now arises from the overcompleteness of the basis is that an *arbitrary* (trace class) operator Θ may be given the diagonal expansion

$$\Theta = \int \frac{\mathrm{d}^2 A}{\pi} I_A P(A; \Theta) \tag{16}$$

which is the so-called P representation of the operator Θ. Questions concerning the existence and uniqueness of the P representation have been much discussed in the literature, both with respect to the harmonic oscillator coherent states considered here and with respect to other or more general coherent states. For example, in the case of finite-dimensional Hilbert spaces, such as those associated with the atomic or

spin-coherent states, the corresponding P representative may not be uniquely defined (Lieb 1973, Gilmore 1976). In practice, certain rather general smoothness conditions, such as being infinitely differentiable, often dictate the form in such cases. In the present case of the oscillator coherent states, with infinite-dimensional Hilbert spaces, the P representations exist most generally in the sense of generalised functions or Schwartz distributions. Mehta (1967) has shown that one form for the P representative is

$$P(A; \Theta) = \exp(|A|^2) \int \frac{\mathrm{d}^2 B}{\pi} \langle -B|\Theta|B \rangle \exp(|B|^2) \exp(AB^* - A^*B) \qquad (17)$$

with an obvious implicit assumption that the integral exists. For many operators Θ the integrand will be such that $P(A; \Theta)$ is a smooth function. However in many other cases, the derivation of (17) and its interpretation must be within the framework of generalised function theory. We discuss such details more fully in § 3.

Now the above pure-state density matrices or projection operators play an extremely important role both in quantum mechanics (von Neumann 1955) and, more generally in mathematics, in the spectral theory of operators (Halmos 1958, Friedrichs 1973). Our main objective in the present work is to extend the treatment to the consideration of mixed states as a generalised basis in Hilbert space. The associated density operators ρ are clearly no longer projection operators and the counterparts of the relations expressed in (5), (6), (12) and (13) will no longer hold. Indeed the corresponding relations $\mathrm{Tr}(\rho^2) < 1$ and $S \equiv -\mathrm{Tr}(\rho \ln \rho) > 0$ provide a measure of how much these states are mixed. Amongst other things we will give a generalised P representation appropriate to mixed states or to finite temperatures. We also show how our density operators are related to those considered in a rather different language in the particular field of quantum optics in order to study superpositions of coherent and chaotic states (Glauber 1963, Lachs 1965, Mollow and Glauber 1967). The very wide use of the Glauber-Lachs formalism for various applications in quantum optics also leads us to believe that our own results should find wide applicability.

It is by now well known that the original Glauber coherent states have been generalised to the so-called squeezed (or two-photon or paired) coherent states (Yuen 1976, Caves 1981, 1982, Walls 1983, Bishop and Vourdas 1986). The standard Glauber states are just the limiting case of zero squeezing. For the sake of generality, and with future applications in mind, we present all of our results for the density operators and mixed states associated with the general displaced and squeezed harmonic oscillator. In § 2 we review the necessary properties of the displaced and squeezed harmonic oscillators and their coherent states, and in § 3 consider the mixed states and density matrices appropriate to such systems in thermal equilibrium at a finite temperature. We conclude in § 4 with a discussion of our results and some remarks concerning possible extensions.

2. The squeezed and displaced harmonic oscillator

We consider the Hamiltonian H_0 of the harmonic oscillator (in units mass $m = 1$ and $\hbar = 1$),

$$\begin{aligned}
H_0 &= \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega^2\hat{q}^2 = \omega(a_0^\dagger a_0 + \tfrac{1}{2}) \\
a_0 &\equiv (2\omega)^{-1/2}(\omega\hat{q} + \mathrm{i}\hat{p}) \qquad a_0^\dagger \equiv (2\omega)^{-1/2}(\omega\hat{q} - \mathrm{i}\hat{p}).
\end{aligned} \qquad (18)$$

We denote $|0\rangle_0$ as the vacuum state and $|n\rangle_0$ as the eigenstates of the number operator $\hat{n}_0 \equiv a_0^+ a_0$:

$$a_0 |0\rangle_0 = 0 \qquad a_0^+ a_0 |n\rangle_0 = n|n\rangle_0. \tag{19}$$

Introducing the unitary 'displacement operator', $U_1(A)$,

$$U_1(A) \equiv \exp(A a_0^+ - A^* a_0) \tag{20}$$

where A is an arbitrary complex parameter, it is simple to prove that its mode of action on the basic operators is given by

$$\begin{aligned} a_A &\equiv U_1(A) a_0 U_1^+(A) = a_0 - A \\ a_A^+ &\equiv U_1(A) a_0^+ U_1^+(A) = a_0^+ - A^*. \end{aligned} \tag{21}$$

Similarly, the unitarity of the operator $U_1(A)$ trivially yields that for any operator $f(a_0, a_0^+)$, we have

$$U_1(A) f(a_0, a_0^+) U_1^+(A) = f(a_A, a_A^+). \tag{22}$$

The operator $U_1(A)$ thus defines a unitary isomorphism of the Hilbert space onto itself, for any choice of parameter A. To each state $|s\rangle_0$ corresponds the mapped state $|s\rangle_A \equiv U_1(A)|s\rangle_0$, and to each operator $\Theta_0 = f(a_0, a_0^+)$ corresponds the mapped operator $\Theta_A \equiv U_1(A)\Theta_0 U_1^+(A) = f(a_A, a_A^+)$. It is trivial to prove that, if $|s\rangle_0$ is an eigenstate of Θ_0, then $|s\rangle_A$ is the corresponding eigenstate of Θ_A.

These general ideas may now be applied to the displaced harmonic oscillator. The original (undisplaced) Hamiltonian of (18) is mapped onto the Hamiltonian

$$H_A \equiv U_1(A) H_0 U_1^+(A) = \omega(a_A^+ a_A + \tfrac{1}{2}) = \omega(a_0^+ a_0 - A^* a_0 - A a_0^+ + |A|^2 + \tfrac{1}{2}) \tag{23}$$

which describes a displaced oscillator with respect to the original operators a_0, a_0^+. Its ground state, $|0\rangle_A$, is given by

$$\begin{aligned} |0\rangle_A &= U_1(A)|0\rangle_0 \equiv |A\rangle_0 \\ a_A |0\rangle_A &= 0 \qquad a_0 |A\rangle_0 = A|A\rangle_0. \end{aligned} \tag{24}$$

This state is clearly a Glauber coherent state with respect to the original operators a_0, a_0^+, as (24) shows. We also define the displaced number operators, $\hat{n}_A$, as

$$\hat{n}_A \equiv U_1(A)\hat{n}_0 U_1^+(A) = a_A^+ a_A \tag{25}$$

whose eigenstates are clearly given by

$$\hat{n}_A |n\rangle_A = n|n\rangle_A \qquad |n\rangle_A = U_1(A)|n\rangle_0 \tag{26}$$

in terms of the states $|n\rangle_0$ defined in (19).

We now consider the unitary 'squeezing operators', $U_2(\rho, \theta, \lambda)$ (Yuen 1976, Caves 1981, 1982, Walls 1983, Bishop and Vourdas 1986) defined as

$$U_2(\rho, \theta, \lambda) \equiv \exp[-\tfrac{1}{4}\rho \exp(-i\theta) a_0^{+2} + \tfrac{1}{4}\rho \exp(i\theta) a_0^2] \exp(i\lambda a_0^+ a_0) \tag{27}$$

where ρ, θ, λ are real parameters, and $\rho > 0$. Their mode of action on the basic (displaced) operators is to generate the Bogoliubov transformations (Bishop and Vourdas 1986)

$$\begin{aligned} a_{A;\rho\theta\lambda} &\equiv U_2(\rho, \theta, \lambda) a_A U_2^+(\rho, \theta, \lambda) = \mu a_0 + \nu a_0^+ - A \\ a_{A;\rho\theta\lambda}^+ &\equiv U_2(\rho, \theta, \lambda) a_A^+ U_2^+(\rho, \theta, \lambda) = \nu^* a_0 + \mu^* a_0^+ - A^* \end{aligned} \tag{28}$$

$$\mu \equiv \exp(-i\lambda) \cosh(\tfrac{1}{2}\rho) \qquad \nu \equiv \exp[-i(\lambda + \theta)] \sinh(\tfrac{1}{2}\rho)$$

which leave the canonical commutation relations unchanged

$$[a_{A;\rho\theta\lambda}, a^+_{A;\rho\theta\lambda}] = [a_A, a^+_A] = I. \tag{29}$$

In order to simplify the notation, we shall henceforth omit the indices ρ, Θ, λ in the case $\rho = 0$. The unitarity of the operators U_2 again immediately gives the counterpart of (22) as

$$U_2(\rho, \theta, \lambda) f(a_A, a^+_A) U^+_2(\rho, \theta, \lambda) = f(a_{A;\rho\theta\lambda}, a^+_{A;\rho\theta\lambda}). \tag{30}$$

As before for U_1, each operator U_2 defines a unitary isomorphism of the Hilbert space onto itself, with similar mappings of each state $|s\rangle_A \to |s\rangle_{A;\rho\theta\lambda} \equiv U_2(\rho, \theta, \lambda)|s\rangle_A$, and of each operator $\Theta_A \to \Theta_{A;\rho\theta\lambda} \equiv U_2(\rho, \theta, \lambda)\Theta_A U^+_2(\rho, \theta, \lambda)$.

In particular, the Hamiltonian for the squeezed and displaced harmonic oscillator becomes

$$H_{A;\rho\theta\lambda} = U_2(\rho, \theta, \lambda) H_A U^+_2(\rho, \theta, \lambda) = \omega(a^+_{A;\rho\theta\lambda} a_{A;\rho\theta\lambda} + \tfrac{1}{2})$$

$$= \omega(f_0 a^+_0 a_0 + f_1 a^2_0 + f^*_1 a^{+2}_0 + f_2 a_0 + f^*_2 a^+_0 + f_3) \tag{31}$$

where the f coefficients are defined as

$$f_0 = \cosh \rho \qquad f_1 = \tfrac{1}{2} \exp(i\theta) \sinh \rho$$

$$f_2 = -A^* \exp(-i\lambda) \cosh(\tfrac{1}{2}\rho) - A \exp[i(\lambda + \theta)] \sinh(\tfrac{1}{2}\rho) \qquad f_3 = |A|^2 + \tfrac{1}{2} \cosh \rho. \tag{32}$$

Its ground state is the 'squeezed and displaced vacuum'

$$|0\rangle_{A;\rho\theta\lambda} \equiv U_2(\rho, \theta, \lambda)|0\rangle_A = U_2(\rho, \theta, \lambda) U_1(A)|0\rangle_0$$

$$= U_2(\rho, \theta, \lambda)|A\rangle_0 \equiv |A; \rho\theta\lambda\rangle_0 \tag{33}$$

$$a_{A;\rho\theta\lambda}|0\rangle_{A;\rho\theta\lambda} = 0.$$

The state $|A; \rho\theta\lambda\rangle_0$ is a squeezed coherent state with respect to the original operators a_0, a^+_0. Finally, we define the squeezed and displaced number operator, $\hat{n}_{A;\rho\theta\lambda}$,

$$\hat{n}_{A;\rho\theta\lambda} \equiv U_2(\rho, \theta, \lambda) \hat{n}_A U^+_2(\rho, \theta, \lambda) \tag{34}$$

whose eigenstates are given as

$$\hat{n}_{A;\rho\theta\lambda}|n\rangle_{A;\rho\theta\lambda} = n|n\rangle_{A;\rho\theta\lambda}$$

$$|n\rangle_{A;\rho\theta\lambda} \equiv |n;\rho\theta\lambda\rangle_A = U_2(\rho, \theta, \lambda)|n\rangle_A = U_2(\rho, \theta, \lambda) U_1(A)|n\rangle_0. \tag{35}$$

With this background established, we turn our attention in the next section to the corresponding system at a finite temperature.

3. Coherent density matrices

We now consider the canonical equilibrium states of the squeezed and displaced oscillator at some non-zero temperature T. As usual, they are the *mixed states* described by the thermodynamic density operator $\rho_{A;\rho\theta\lambda}(T)$, defined as

$$\rho_{A;\rho\theta\lambda}(T) \equiv Z^{-1} \exp(-\beta H_{A;\rho\theta\lambda}) \qquad Z \equiv \mathrm{Tr} \exp(-\beta H_{A;\rho\theta\lambda})$$

$$\beta \equiv (k_B T)^{-1} \qquad \mathrm{Tr}\, \rho_{A;\rho\theta\lambda}(T) = 1 \tag{36}$$

where k_B is Boltzmann's constant and $H_{A;\rho\theta\lambda}$ is the Hamiltonian of (31). The thermo-dynamic average, $\langle \Theta \rangle$, of an arbitrary operator Θ is defined in terms of the density operator as

$$\langle \Theta \rangle = \langle \Theta \rangle_{A;\rho\theta\lambda} = \text{Tr}[\rho_{A;\rho\theta\lambda}(T)\Theta]. \tag{37}$$

Using the results of § 2, we may also write

$$\rho_{A;\rho\theta\lambda}(T) = U_2(\rho, \theta, \lambda)\rho_A(T)U_2^{\dagger}(\rho, \theta, \lambda)$$
$$= U_2(\rho, \theta, \lambda)U_1(A)\rho_0(T)U_1^{\dagger}(A)U_2^{\dagger}(\rho, \theta, \lambda) \tag{38}$$

where the density operators $\rho_A(T)$ and $\rho_0(T)$ are defined as in (36) but in terms of the respective Hamiltonians H_A of (23) and H_0 of (18). The density operators $\rho_0(T)$, $\rho_A(T)$ and $\rho_{A;\rho\theta\lambda}(T)$ thus correspond respectively to the original oscillator, the displaced oscillator and the displaced squeezed oscillator. It is clear that each of these density operators is Hermitian and positive definite. Our intention is now to prove various properties of the density operators $\rho_{A;\rho\theta\lambda}$, which in particular will demonstrate how they may be viewed as a generalisation of the (pure) squeezed coherent states $|A;\rho\theta\lambda\rangle_0$, and how they may be used as a 'basis' in the corresponding Hilbert space.

3.1. Expansions of the density operator

The simplest expansion of the density operator $\rho_{A;\rho\theta\lambda}$ is clearly in terms of the projection operators $I_n(A; \rho\theta\lambda)$ formed from its corresponding eigenstates $|n\rangle_{A;\rho\theta\lambda} \equiv |n; \rho\theta\lambda\rangle_A$ of (35)

$$I_n(A; \rho\theta\lambda) \equiv |n; \rho\theta\lambda\rangle_{A\ A}\langle n; \rho\theta\lambda|$$
$$I_m(A; \rho\theta\lambda)I_n(A; \rho\theta\lambda) = I_n(A; \rho\theta\lambda)\delta_{mn} \qquad \sum_n I_n(A; \rho\theta\lambda) = I. \tag{39}$$

We thus have the diagonal representation

$$\rho_{A;\rho\theta\lambda}(T) = \sum_n p_n(T)I_n(A; \rho\theta\lambda)$$
$$p_n(T) = [1 - \exp(-\beta\omega)]\exp(-\beta\omega n) \qquad 0 \leq p_n < 1 \qquad \sum_n p_n = 1. \tag{40}$$

Clearly the quantities $p_n(T)$ represent the probabilities of finding the system in the state $|n\rangle_{A;\rho\theta\lambda}$. In the case of zero temperature, $T \to 0$, we see $p_n \to \delta_{n,0}$ and (40) reduces to the pure squeezed and displaced ground state

$$\rho_{A;\rho\theta\lambda}(T) \xrightarrow[T \to 0]{} |A; \rho\theta\lambda\rangle_0\ _0\langle A; \rho\theta\lambda| \tag{41}$$

in the notation of (33).

More generally, for $T \neq 0$, the density operator may be represented as a superposition of the projection operators of (41) as already discussed in § 1. This leads us to a second very important expansion for the density operator, namely its P representation

$$\rho_B(T) = \int \frac{d^2A}{\pi} P(A; \rho_B(T))|A\rangle_0\ _0\langle A| \tag{42a}$$

$$\rho_{B;\rho\theta\lambda}(T) = \int \frac{d^2A}{\pi} P(A; \rho_B(T))|A; \rho\theta\lambda\rangle_0\ _0\langle A; \rho\theta\lambda| \tag{42b}$$

where $(42b)$ follows from $(42a)$ by using (33) and (38), and where the function $P(A; \rho_B(T))$ is given from (17) as

$$P(A; \rho_B(T)) = \int \frac{d^2\Gamma}{\pi} \, _0\langle -\Gamma | \rho_B(T) | \Gamma \rangle_0 \exp(|A|^2 + |\Gamma|^2 + A\Gamma^* - A^*\Gamma). \quad (43)$$

In order to evaluate (43), we first need to calculate the matrix elements of the density operator between Glauber coherent states. Using (24) and (38), we may write

$$_0\langle A | \rho_B(T) | \Gamma \rangle_0 = \, _0\langle 0 | U_1^\dagger(A) U_1(B) \rho_0(T) U_1^\dagger(B) U_1(\Gamma) | 0 \rangle_0. \quad (44)$$

Using (20) it is simple to prove the group relation:

$$U_1^\dagger(B) U_1(\Gamma) = \exp[\tfrac{1}{2}(\Gamma B^* - \Gamma^* B)] U_1(\Gamma - B). \quad (45)$$

Also, (1) and (2) easily yield the relation

$$\exp(\phi a_0^\dagger a_0)|A\rangle_0 = \exp[\tfrac{1}{2}|A|^2(|e^\phi|^2 - 1)]|A \, e^\phi\rangle_0. \quad (46)$$

Use of (45) and (46) in (44) then readily gives

$$_0\langle A | \rho_B(T) | \Gamma \rangle_0 = (1 - e^{-\beta\omega}) \exp[(1 - e^{-\beta\omega})(B^*\Gamma + A^*B - |B|^2) - \tfrac{1}{2}|A|^2 - \tfrac{1}{2}|\Gamma|^2 + A^*\Gamma \, e^{-\beta\omega}] \quad (47)$$

where we have used the overlap relation between coherent states

$$_0\langle B | A \rangle_0 = \exp(-\tfrac{1}{2}|A|^2 - \tfrac{1}{2}|B|^2 + AB^*) \quad (48)$$

which follows from (2). After some straightforward integration, (43) and (47) yield the final result

$$P(A; \rho_B(T)) = (e^{\beta\omega} - 1) \exp[-(e^{\beta\omega} - 1)|A - B|^2]. \quad (49)$$

The zero-temperature limit of (49) is clearly singular and it is easy to see that

$$P(A; \rho_B(T)) \xrightarrow[T \to 0]{} \pi\delta^{(2)}(A - B) \quad (50)$$

as required.

It is of interest to enquire whether the more general squeezed density operator $\rho_{B;\rho\theta\lambda}(T)$ also has a P representation in terms of the (unsqueezed) Glauber coherent state:

$$\rho_{B;\rho\theta\lambda}(T) = \int \frac{d^2A}{\pi} P(A; \rho_{B;\rho\theta\lambda}(T))|A\rangle_0 \, _0\langle A| \quad (51)$$

as well as in (42) in terms of the squeezed coherent states. It is not difficult to show that in this case, however, the expression of Mehta (1967) given by (17) now diverges. We stress that this does *not* imply that the representation of (51) does not exist, but rather that it cannot be found by this method. The general question of the existence of the P representation is complicated (Klauder and Skagerstam 1985). Sudarshan (1963) seems to have been the first to investigate the universality of the diagonal representation for operators in the case of the standard coherent states. A rather general and particularly illuminating discussion has been given more recently by Yaffe (1982).

The situation can be summarised as follows. The existence proofs, which enable us to assert that the diagonal representations of (16) can always be found, do not guarantee that the weight factor $P(A; \Theta)$ is a well behaved function. In general this is not the case and such P representatives must be interpreted in the sense of Schwartz distributions.

The construction of Mehta (1967) is, in the first instance, justified in its derivation only when the factor $\langle -B|\Theta|B\rangle \exp(|B|^2)$ is square integrable, and hence so also is $P(A;\Theta) \exp(-|A|^2)$. More generally, in the case that $\langle -B|\Theta|B\rangle \exp(|B|^2)$ grows as a (finite-order) polynomial at infinity, the Fourier integral can still be understood as a Schwartz tempered distribution—loosely speaking, combinations of Dirac delta functions and their (finite-order) derivatives.

However, there are still other cases where the integrand in (17) grows exponentially as one or both of $|\mathrm{Re}\, B|$ and $|\mathrm{Im}\, B|$ tend to infinity. The squeezed density operator $\rho_{B;\rho\theta\lambda}(T)$ is just such a case for all $\rho \neq 0$. In such cases the P representation now still exists as a (Schwartz) distribution, which is, however, no longer a tempered distribution. Klauder *et al* (1965) applied the techniques of distribution theory to this case. They showed that every operator in the trace class (namely, those whose trace exists in the sense of being finite and basis independent) is the weak limit of a sequence of operators, each of which admits a diagonal representation of the form of (16), with square-integrable P representatives. From a practical point of view, such limiting sequences are often not easy to use, and it is largely to take into account these difficulties that alternative generalised (off-diagonal) pure-state P representatives have been considered (e.g. Drummond and Gardiner 1980). From the formal mathematical point of view, however, we can summarise by saying that the P representative exists for an arbitrary bounded operator (of the trace class) as a Fourier transform of a distribution. Such distributions, whether tempered or not, may be defined as sequences of integrable functions or by linear maps. Both viewpoints have proven useful to the mathematician, but little practical use seems to have come from either of them in the present context.

Since such P representatives for $\rho_{B;\rho\theta\lambda}(T)$ in the case of squeezing are not central to our present purposes we make no attempt to pursue any alternative construction of the P representatives for an appropriate sequence of operators for this case and do not discuss this topic further here. We shall however return to such considerations in § 3.4 where we generalise the concept of the P representation to mixed states.

Instead, we turn to a third useful expansion for the density operator, in terms of the original number eigenstates $|n\rangle_0$

$$\rho_B(T) = \sum_{m,n} \rho_{B;nm}(T)|n\rangle_0 \, {}_0\langle m| \qquad \rho_{B;nm}(T) \equiv {}_0\langle n|\rho_B(T)|m\rangle_0$$

$$\rho_{B;\rho\theta\lambda}(T) = \sum_{m,n} \rho_{B;nm}(T)|n;\rho\theta\lambda\rangle_0 \, {}_0\langle m;\rho\theta\lambda| \tag{52}$$

which, as we have already remarked, cannot now be a diagonal expansion. Again, in order to avoid unnecessary complication we restrict ourselves to finding the matrix elements $\rho_{B;nm}(T)$ for the unsqueezed density operator. These matrix elements can be evaluated directly from the previous representation of (42) and (49), although the resulting integrals are somewhat cumbersome. A simple method is to take matrix elements of the expansion of (52) between Glauber coherent states

$$_0\langle A|\rho_B(T)|\Gamma\rangle_0 = \sum_{m,n} \rho_{B;nm}(T) \exp[-\tfrac{1}{2}|A|^2 - \tfrac{1}{2}|\Gamma|^2](m!\,n!)^{-1/2} A^{*n}\Gamma^m \tag{53}$$

where we have made use of (2). This result may now be compared directly with (47) to give

$$\sum_{m,n} \rho_{B;nm}(T)(m!\,n!)^{-1/2} A^{*n}\Gamma^m$$

$$= (1 - e^{-\beta\omega}) \exp[-(1 - e^{-\beta\omega})|B|^2]$$

$$\times \exp[(1 - e^{-\beta\omega})(B^*\Gamma + A^*B) + e^{-\beta\omega} A^*\Gamma]. \tag{54}$$

The necessary expansion of the last term in (54) may now be made by using one of the generating functions for the associated Laguerre polynomials (Gradshteyn and Ryzhik 1965)

$$e^{-xz}(1+z)^m = \sum_{n=0}^{\infty} L_n^{m-n}(x)z^n \qquad |z| < 1 \tag{55}$$

to prove rather easily the relation

$$\exp(\alpha_1\alpha_2 + \alpha_1 r + \alpha_2 r) = \sum_{m,n} \frac{1}{m!}(\alpha_1 r)^m \left(\frac{\alpha_2}{r}\right)^n L_n^{m-n}(-r^2). \tag{56}$$

Upon making the following substitutions into (56):

$$\alpha_1 = \exp(-\tfrac{1}{2}\beta\omega)\frac{B^*\Gamma}{|B|} \qquad \alpha_2 = \exp(-\tfrac{1}{2}\beta\omega)\frac{A^*B}{|B|} \qquad r = 2|B|\sinh(\tfrac{1}{2}\beta\omega) \tag{57}$$

and inserting the resulting expression in (52), the desired result is obtained:

$$\rho_{B;nm}(T) = \left(\frac{n!}{m!}\right)^{1/2}(B^*)^{m-n}\exp[-\tfrac{1}{2}\beta\omega(m+n+1)][2\sinh(\tfrac{1}{2}\beta\omega)]^{m-n+1}$$

$$\times \exp[-(1-e^{-\beta\omega})|B|^2]L_n^{m-n}[-4|B|^2\sinh^2(\tfrac{1}{2}\beta\omega)]. \tag{58}$$

The Hermiticity of the density operator $\rho_B(T)$, which implies that $\rho_{B;nm}(T) = \rho_{B;mn}^*(T)$, is not immediately apparent from (58) but may readily be checked using the relation

$$\frac{x^m}{m!}L_n^{m-n}(-x) = \frac{x^n}{n!}L_m^{n-m}(-x). \tag{59}$$

Using the limiting relation $L_n^0(0) = 1$, it is simple to show that in the special case $B = 0$, (58) reduces to

$$\rho_{0;nm}(T) = \delta_{mn}(1-e^{-\beta\omega})e^{-n\beta\omega} \tag{60}$$

which is just the Planck distribution. Similarly by using the asymptotic form

$$x^{-n}L_n^{m-n}(-x) \xrightarrow[x\to\infty]{} \frac{1}{n!} \tag{61}$$

we can easily show that the zero-temperature limit of (58) is given by the correct form:

$$\rho_{B;nm}(T) \xrightarrow[T\to 0]{} (n!\,m!)^{-1/2}B^{*m}B^n\exp(-|B|^2) = {}_0\langle n|B\rangle_0\,{}_0\langle B|m\rangle_0. \tag{62}$$

Before proceeding we pause here to point out the connection between our density operators $\rho_B(T)$ and the Glauber–Lachs formalism in quantum optics for mixtures of incoherent (or, loosely, thermal) and coherent radiation. In this connection, it has been shown by Glauber (1963) how the density operator for an electromagnetic field can profitably be expanded in the P representation. Furthermore, Glauber showed that the P representation for a field, which is the superposition of two separate fields with individual P representations $P_1(A)$ and $P_2(A)$, is given by their convolution:

$$P(A) = \int \frac{d^2\Gamma}{\pi} P_1(\Gamma)P_2(A-\Gamma). \tag{63}$$

In the case of a single ω mode, the P representation for an oscillator of corresponding frequency in a coherent state $|B\rangle_0$ is clearly

$$P_1(A) = \pi\delta^{(2)}(A-B). \tag{64}$$

Glauber (1963) has also shown that the P representation for the corresponding mode of a random incoherent field, with average number of photons $\bar{n}$, is given by

$$P_2(A) = (\bar{n})^{-1} \exp(-|A|^2/\bar{n}). \tag{65}$$

Substitution of (64) and (65) into (63) shows that the P representation for the mixed incoherent and coherent field is

$$P(A) = (\bar{n})^{-1} \exp(-|A - B|^2/\bar{n}) \tag{66}$$

which has been used in this form by Lachs (1965).

In our case of thermodynamic equilibrium for a system with a single mode ω, we have $\bar{n} \to \langle n_0 \rangle_0$ in the notation of (37), which by the use of (40) reduces to the usual Planck distribution

$$n_T \equiv \langle n_0 \rangle_0 = \sum_n n p_n(T) = (e^{\beta\omega} - 1)^{-1}. \tag{67}$$

A comparison of our (49) with (66) and (67) then shows immediately how our $\rho_B(T)$ can be used in the Glauber–Lachs formalism to represent, for a single-frequency (ω) mode, a superposition of a coherent field in state $|B\rangle_0$ with a corresponding thermal (incoherent) black-body radiation field at temperature T. We also note that within the context of the amplification of electromagnetic fields, Mollow and Glauber (1967) were also led, during a discussion of a simple theoretical model of the parametric amplifier, to a relation similarly analogous to our representation of (58).

3.2. Applications to statistical correlations

The expansions of both (42) and (52) lead to useful expressions for the average values $\langle \Theta \rangle_B \equiv \mathrm{Tr}[\rho_B(T)\Theta]$ of an arbitrary operator in a mixed state described by the density operator $\rho_B(T)$. We find, respectively,

$$\langle \Theta \rangle_B = \int \frac{d^2A}{\pi} P(A; \rho_B(T)) {}_0\langle A|\Theta|A\rangle_0 \tag{68a}$$

$$\langle \Theta \rangle_B = \sum_{m,n} \rho_{B;nm}(T) {}_0\langle m|\Theta|n\rangle_0. \tag{68b}$$

Particularly interesting applications of (68) arise either when Θ is known in normal-ordered form or when it is a function of the number operator $a_0^\dagger a_0$ only. In the former case, if Θ has the normal-ordered expansion

$$\Theta = \sum_{m,n} c_{mn}(a_0^\dagger)^m (a_0)^n \tag{69}$$

use of (68a) immediately gives

$$\langle \Theta \rangle_B = \sum_{m,n} c_{mn} \int \frac{d^2A}{\pi} P(A; \rho_B(T))(A^*)^m A^n. \tag{70}$$

The remaining integrals in (70) are best performed in a polar coordinate basis, with the result

$$\int \frac{d^2A}{\pi} P(A; \rho_B(T))(A^*)^m A^n = n!(e^{\beta\omega} - 1)^{-n}(B^*)^{m-n} L_n^{m-n}[-(e^{\beta\omega} - 1)|B|^2] \tag{71}$$

which can be inserted into (70) to get the final result. One can again confirm that (70) is consistent with the exact result $\langle \Theta \rangle_B = \langle \Theta^\dagger \rangle_B^*$ by using (59). We also note that (68b)

is less immediately applicable to operators of normal-ordered form than $(68a)$. Alternatively, if Θ is a function of the number operator $\hat{n}_0 = a_0^+ a_0$, of the form

$$\Theta = \sum_k d_k (a_0^+ a_0)^k \tag{72}$$

$(68b)$ immediately yields the result

$$\langle \Theta \rangle_B = \sum_k d_k \sum_n n^k \rho_{B;nn}(T) \tag{73}$$

which has been used by Lachs (1965).

Even in this case, $(68a)$ can be profitably used and we conclude this section by demonstrating some of the statistical relations of great practical usefulness in such fields as quantum optics (Saleh 1978). We now make use of the relation

$$_0\langle A|(a_0^+ a_0)^k|A\rangle_0 = \exp(-|A|^2) \sum_n (n!)^{-1} |A|^{2n} n^k \tag{74}$$

which follows from inserting complete sets of number states on both sides of the operator $(a_0^+ a_0)^k$ in the matrix element, and making use of (2). We see that the matrix elements in (74) are simply the kth moments of a Poisson distribution with mean $|A|^2$, which are just polynomials of order k in $|A|^2$. A combination of $(68a)$, (71) and (74) then leads to an easy evaluation of $\langle \Theta \rangle_B$ for operators Θ of the form of (72). For example, it is then trivial to prove the relation

$$\langle \hat{n}_0(\hat{n}_0 - 1) \ldots (\hat{n}_0 - k + 1) \rangle_B = k!(e^{\beta\omega} - 1)^{-k} L_k^0[-(e^{\beta\omega} - 1)|B|^2] \tag{75}$$

from which the moments $\langle (\hat{n}_0)^k \rangle_B$ may readily be found by iteration.

For many purposes it is of greater interest to deal directly with the so-called cumulants (or semi-invariants), rather than with the moments. These are well known in probability theory where they are defined as follows. Taking the number operator $\hat{n}_0$ as an example, just as the operator $\exp(\hat{n}_0 t)$ generates the moments, as

$$\langle e^{\hat{n}_0 t} \rangle_B \equiv \text{Tr}[\rho_B(T) \exp(\hat{n}_0 t)] = \sum_{n=0}^{\infty} \frac{t^n}{n!} m_n(B; T) \qquad m_k(B; T) \equiv \langle (\hat{n}_0)^k \rangle_B \tag{76}$$

so the logarithm of this expression is used as the generating function which defines the cumulants, M_n,

$$\ln \text{Tr}[\rho_B(T) \exp(\hat{n}_0 t)] = \sum_{n=1}^{\infty} \frac{t^n}{n!} M_n(B; T). \tag{77}$$

A comparison of (76) and (77) shows that the first few cumulants are given as

$$M_1 = m_1 \qquad M_2 = m_2 - m_1^2 \qquad M_3 = m_3 - 3m_2 m_1 + 2m_1^3$$
$$M_4 = m_4 - 4m_3 m_1 - 3m_2^2 + 12m_2 m_1^2 - 6m_1^4. \tag{78}$$

The cumulants are particularly easy to evaluate either in the case of zero displacement $(B = 0)$ or zero temperature $(T = 0)$.

In the first place, when $B = 0$, it is trivial to see from (77) that the cumulants are given by

$$M_n(0; T) = \frac{d^n}{dt^n} \left[\ln\left(\frac{1 - e^{-\beta\omega}}{1 - e^{-\beta\omega + t}} \right) \right] \Bigg|_{t=0}$$

$$= -\frac{d^n}{dt^n} [\ln(1 + n_T - n_T e^t)]|_{t=0}. \tag{79}$$

In terms of the pure thermal ($B = 0$) mean n_T of (67), we thus find the usual cumulants of the Planck distribution, the first few of which are given by

$$M_1(0; T) = n_T \qquad M_2(0; T) = n_T(n_T + 1) \qquad M_3(0; T) = n_T(n_T + 1)(2n_T + 1)$$
$$M_4(0; T) = n_T(n_T + 1)(6n_T^2 + 6n_T + 1). \tag{80}$$

Similarly, when $T = 0$, we find

$$\ln \mathrm{Tr}[\rho_B(0)\exp(\hat{n}_0 t)] = \ln[_0\langle B|\exp(\hat{n}_0 t)|B\rangle_0] = |B|^2(e^t - 1) \tag{81}$$

where the last equality in (81) follows rather easily by making use of (74). A comparison of (77) and (81) then shows that in this case the cumulants are all identical:

$$M_n(B; 0) = |B|^2 \equiv n_B \tag{82}$$

which is the result expected for a Poisson distribution with mean number n_B of coherent photons.

The general case of (77) can also be evaluated by making use of (68b) to give

$$\mathrm{Tr}(\rho_B(T)\exp(\hat{n}_0 t)) = \sum_m \rho_{B;mm}(T)\,e^{mt}. \tag{83}$$

Use of (58), together with another of the generating functions for the Laguerre polynomials (Gradshteyn and Ryzhik 1965)

$$(1-z)^{-1}\exp\!\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^0(x)z^n \qquad |z| < 1 \tag{84}$$

readily yields the result

$$\ln \mathrm{Tr}(\rho_B(T)\exp(\hat{n}_0 t)) = -\ln(1 + n_T - n_T\,e^t) + \frac{n_B}{(1 + n_T)}\left(\frac{e^t}{1 + n_T - n_T\,e^t} - 1\right). \tag{85}$$

By comparing (77) and (85), we can now readily evaluate the cumulants $M_n(B; T)$. A further comparison with (79) then gives the important result

$$M_n(B; T) = M_n(0; T) + \frac{n_B}{n_T(1 + n_T)}M_{n+1}(0; T). \tag{86}$$

We note also that the limiting forms of (79) and (82) follow directly from (86) by making use of the relation

$$M_{n+1}(0; T) = n_T(1 + n_T)P_{n-1}(n_T) \qquad n \geq 1$$
$$P_{n-1}(0) = 1 \tag{87}$$

where $P_n(x)$ is a polynomial in x of order n and where (87) follows simply from (79). The cumulants $M_n(B; T)$ are then easily obtained from (79), (80) and (86) and the moments $m_n(B; T)$ may then also be obtained by inverting the relations of (78). We quote the results for the first few cumulants:

$$M_1(B; T) = n_B + n_T \qquad M_2(B; T) = n_B + n_T(n_T + 1) + 2n_B n_T$$
$$M_3(B; T) = n_B + M_3(0; T) + 6n_B n_T(n_T + 1) \tag{88}$$
$$M_4(B; T) = n_B + M_4(0; T) + n_B n_T(24n_T^2 + 36n_T + 14).$$

3.3. Momentum-position uncertainty relations

By inverting (18), (21) and (28) it is easy to express the position and momentum operators in the form

$$\hat{q} = (\tfrac{1}{2}\omega^{-1})^{1/2}[(\mu - \nu)a^{\dagger}_{A;\rho\theta\lambda} + (\mu^* - \nu^*)a_{A;\rho\theta\lambda} + 2\,\mathrm{Re}\,A]$$

$$\hat{p} = (\tfrac{1}{2}\omega)^{1/2}[\mathrm{i}(\mu + \nu)a^{\dagger}_{A;\rho\theta\lambda} - \mathrm{i}(\mu^* + \nu^*)a_{A;\rho\theta\lambda} + 2\,\mathrm{Im}\,A].$$

(89)

Making use of this representation, together with (37), (39) and (40), one easily shows that the density matrix $\rho_{A;\rho\theta\lambda}$ describes a mixed state localised around the phase space point:

$$\langle q \rangle \equiv \langle q \rangle_{A;\rho\theta\lambda} = (2/\omega)^{1/2}\,\mathrm{Re}\,A \qquad \langle p \rangle \equiv \langle p \rangle_{A;\rho\theta\lambda} = (2\omega)^{1/2}\,\mathrm{Im}\,A$$

(90)

and with the following uncertainties:

$$(\Delta q)^2 \equiv \sigma_{qq} \equiv \langle (q - \langle q \rangle)^2 \rangle_{A;\rho\theta\lambda} = \tfrac{1}{2}\omega^{-1}\coth(\tfrac{1}{2}\beta\omega)(\cosh\rho - \cos\theta\sinh\rho)$$

$$(\Delta p)^2 \equiv \sigma_{pp} \equiv \langle (p - \langle p \rangle)^2 \rangle_{A;\rho\theta\lambda} = \tfrac{1}{2}\omega\coth(\tfrac{1}{2}\beta\omega)(\cosh\rho + \cos\theta\sinh\rho)$$

(91)

$$\sigma_{pq} \equiv \langle \tfrac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) - \langle p \rangle\langle q \rangle \rangle_{A;\rho\theta\lambda} = \tfrac{1}{2}\coth(\tfrac{1}{2}\beta\omega)\sin\theta\sinh\rho.$$

Of particular interest is the 'uncertainty determinant' or 'variance determinant', $\det(\sigma)$, defined as below and easily evaluated from (91)

$$\det(\sigma) \equiv \sigma_{pp}\sigma_{qq} - \sigma_{pq}^2 = [\tfrac{1}{2}\coth(\tfrac{1}{2}\beta\omega)]^2 = (\tfrac{1}{2} + n_T)^2$$

(92)

in terms of the Planck distribution of (67). Equation (92) expresses a generalised uncertainty relation for the mixed squeezed coherent states. We note in particular that the quantity $\det(\sigma)$ is independent of both the displacement parameter A and the squeezing parameters ρ, θ, λ, although the individual variances and means do depend on these parameters. This invariance for $\det(\sigma)$ is of course just a reflection of the underlying unitary nature of the displacement and squeezing operators, $U_1(A)$ and $U_2(\rho, \theta, \lambda)$ respectively. In the particular case when $\theta = 0$, π, then $\sigma_{pq} = 0$, and

$$\Delta p\,\Delta q = \tfrac{1}{2} + n_T \qquad \Delta p/\Delta q = \omega\exp(\pm\rho).$$

(93)

We recognise in (93) the 'squeezing factor' $\exp(\pm\rho)$ in the ratio of uncertainties $\Delta p/\Delta q$ which gives the squeezed states their name and note that it is independent of temperature. We also see that, at zero temperature, the squeezed coherent states with $\theta = 0$ or π are minimum uncertainty states. At finite temperatures, we recognise in both (92) and (93) the combination of the 'quantum mechanical uncertainty', $\tfrac{1}{2}$, and the 'thermodynamic uncertainty', $n_T = (\exp\beta\omega - 1)^{-1}$.

3.4. Completeness relations and a generalised P representation

We now turn our attention to the central feature of this work, namely the investigation of the completeness properties of the squeezed coherent density matrices. In the first place it is easy to prove the relation

$$\int \frac{\mathrm{d}^2 B}{\pi}\rho_{B;\rho\theta\lambda}(T) = I$$

(94a)

which is clearly a generalisation to the mixed states appropriate to non-zero temperatures of the zero-temperature completeness relation for the pure squeezed coherent states

$$\int \frac{\mathrm{d}^2 B}{\pi}\rho_{B;\rho\theta\lambda}(0) \equiv \int \frac{\mathrm{d}^2 B}{\pi}|B;\rho\theta\lambda\rangle_0\,{}_0\langle B;\rho\theta\lambda| = I$$

(94b)

which itself follows from the well known resolution of the identity of (14) for the Glauber coherent states, by pre- and post-multiplication with the operators $U_2(\rho, \theta, \lambda)$ and $U_2^\dagger(\rho, \theta, \lambda)$ respectively. There are various ways of proving (94a). The most direct is perhaps to insert into the left-hand side of (94a) the P representation of (42). Making use of the explicit relation of (49), an interchange of the order of integration immediately relates the finite-temperature result of (94a) to its zero-temperature limit in (94b). If one wishes to avoid the rigorous justification of the interchange of the order of integration inherent to the above proof, one may also prove (94a) by taking matrix elements of both sides with respect to arbitrary number states $_0\langle n|$ and $|m\rangle_0$. Use of the explicit result of (58) for the resulting matrix elements, and an evaluation of the remaining integral by going to a plane polar coordinate representation, then shows that the only non-zero matrix elements are the diagonal ones by first performing the angular integration. Finally, one shows that the above matrix elements are precisely equal to δ_{mn} by using the relation

$$\int_0^x dx\, e^{-x} L_n^0(\lambda x) = (1 - \lambda)^n. \tag{95}$$

The relation (94a) is important because it shows that the mixed coherent basis $\rho_{A;\rho\theta\lambda}(T)$ for arbitrary fixed values of the temperature T and the squeezing parameters ρ, θ, λ and for all $A \in \mathbb{C}$, is complete in the sense that an arbitrary ket $|k\rangle$ may be written as

$$|k\rangle = \int \frac{d^2 A}{\pi} \rho_{A;\rho\theta\lambda}(T)|k\rangle$$

$$= \int \frac{d^2 A}{\pi} \sum_n p_n(T)\, _A\langle n;\rho\theta\lambda|k\rangle|n;\rho\theta\lambda\rangle_A \tag{96}$$

where we have made use of (39) and (40).

We also wish to investigate to what extent an arbitrary operator Θ may be expanded in terms of the coherent density matrices as a generalisation to finite temperatures of the P representation. If such an expansion exists, we write it as

$$\Theta = \int \frac{d^2 A}{\pi} \rho_{A;\rho\theta\lambda}(T) P_T(A, \rho\theta\lambda; \Theta) \tag{97a}$$

in terms of the mixed squeezed coherent states, or as

$$\Theta = \int \frac{d^2 A}{\pi} \rho_A(T) P_T(A; \Theta) \tag{97b}$$

in terms of the mixed Glauber coherent states. In the limit as $T \to 0$, (97b) becomes the usual P representation

$$\Theta = \int \frac{d^2 A}{\pi} P(A; \Theta)|A\rangle_0\, _0\langle A| \tag{97c}$$

which we have discussed previously.

In order to proceed, it is very useful to prove a theorem concerning the corresponding generalised Q representation. The usual (zero-temperature) Q representation of an arbitrary operator is just its diagonal matrix elements between Glauber coherent states, $Q(A; \Theta) \equiv\, _0\langle A|\Theta|A\rangle_0$. Their finite-temperature analogues are obviously just the thermodynamic averages of (37), which we now write in the equivalent form:

$$Q_T(A, \rho\theta\lambda; \Theta) \equiv \text{Tr}[\rho_{A;\rho\theta\lambda}(T)\Theta] \qquad Q_T(A; \Theta) \equiv \text{Tr}[\rho_A(T)\Theta]. \tag{98}$$

The theorem that we now prove is that the set S_1 of operators which are uniquely specified by their generalised Q representation of (98), considered as a function of A (for all $A \in \mathbb{C}$) and for fixed and arbitrary temperature and squeezing parameters, is identical to the set S_2 of operators admitting the generalised P representation of (97a, b). The proof is very similar to the analogous zero-temperature case discussed by Klauder and Skagerstam (1985), and which is itself based on earlier work of Simon (1980). We therefore only sketch the proof below, making no pretence at full rigour. Purely for ease of notation we consider only the case of zero squeezing, although the proof is identical when squeezing is also present.

Let us first suppose the existence of some set S_1' of non-zero operators Φ for which $Q_T(A; \Phi) = 0$ for all $A \in \mathbb{C}$. Clearly $\Phi \notin S_1$. Hence we trivially have for all functions $P_T(A; \Theta)$

$$\int \frac{\mathrm{d}^2 A}{\pi} P_T(A; \Theta) Q_T(A; \Phi) = 0. \tag{99}$$

But, by writing an operator $\Theta_1 \in S_2$ in the form of (97b), and by using (98), we see that

$$\mathrm{Tr}(\Theta_1 \Theta_2) = \int \frac{\mathrm{d}^2 A}{\pi} P_T(A; \Theta_1) Q_T(A; \Theta_2). \tag{100}$$

A comparison of (99) and (100) shows that $\mathrm{Tr}(\Theta_1^\dagger \Phi) = 0$ for all $\Theta_1^\dagger \in S_2$, and hence for all $\Theta_1 \in S_2$, when $\Phi \in S_1'$. Now we say that two operators Θ_1 and Θ_2 are orthogonal if $\mathrm{Tr}(\Theta_1^\dagger \Theta_2) = 0$. Hence the set S_2 has a non-trivial orthogonal complement $S_2^\perp$ when a non-zero Φ exists for which $Q_T(A; \Phi) = 0$. In other words, if $\Phi \in S_1'$ then $\Phi \in S_2^\perp$. Conversely, if $\Phi \in S_2^\perp$, then $\mathrm{Tr}(\Theta_1^\dagger \Phi) = 0$ for all $\Theta_1 \in S_2$. But, since for all integrable functions $f(A)$ we have a trace-class operator $\Theta_1^\dagger \equiv \int \mathrm{d}^2 A f(A) \rho_A(T)$ belonging to S_2, then (100) shows that this implies that the integral $\int \mathrm{d}^2 A f(A) Q_T(A; \Phi)$ vanishes for all integrable $f(A)$. In turn, this implies that $Q_T(A; \Phi) \equiv 0$; and hence $\Phi \in S_2^\perp$ implies that $\Phi \in S_1'$. We have thus shown the equivalence $S_1' \equiv S_2^\perp$. For the case of a d-dimensional Hilbert space, with d finite, the desired result now follows immediately, since as a linear vector space the space of all $d \times d$ matrices is just $V = S_2 \oplus S_2^\perp$ for any $S_2 \subset V$. Our result above then shows that an arbitrary operator θ, i.e. an arbitrary $d \times d$ matrix, has a unique decomposition $\Theta = \Theta_P + \Theta_{Q'}$ where $\Theta_P \in S_2$ and $\Theta_{Q'} \in S_1'$, which proves our theorem. Finally, as discussed by Klauder and Skagerstam (1985), infinite-dimensional Hilbert spaces may be similarly treated provided attention is restricted to the space S_2 of Hilbert–Schmidt operators admitting a generalised P representation, namely those satisfying (97b) and which further satisfy the Hilbert–Schmidt criterion

$$\mathrm{Tr}(\Theta^\dagger \Theta) = \int \frac{\mathrm{d}^2 A}{\pi} \int \frac{\mathrm{d}^2 B}{\pi} [P_T(A; \Theta)]^* P_T(B; \Theta) \, \mathrm{Tr}[\rho_A(T)\rho_B(T)] < \infty. \tag{101}$$

An immediate corollary of the above theorem is that a general Hilbert–Schmidt operator may be represented by a generalised P representation whenever the set S_1' is empty, namely whenever their generalised Q representatives suffice to determine all operators. Furthermore, since a completely general operator can always be constructed as the limit of a sequence of Hilbert–Schmidt operators, we can in this case completely generalise to finite temperatures the results of Klauder et al (1965) to which we have previously alluded. As a second theorem we now prove by an extension of the method used by Klauder (1964) that, just as in the zero-temperature case, an operator Θ is indeed uniquely determined by its generalised Q representation for fixed and arbitrary T (and ρ, Θ, λ), considered as a function of A, for all $A \in \mathbb{C}$. In other words we prove

now that S'_1 is empty. Again, for the sake of clarity we present the proof only for the case of zero squeezing.

Both as a preliminary for the proof and for later purposes it is useful to introduce the Weyl representation of an arbitrary operator. This follows most easily from the fundamental relation between the displacement or Weyl operators, $U_1(A)$, of (20)

$$\int \frac{d^2 A}{\pi} {}_0\langle B | U_1(A) | C\rangle_0 {}_0\langle D | U_1^\dagger(A) | E\rangle_0 = {}_0\langle B | E\rangle_0 {}_0\langle D | C\rangle_0 \qquad (102a)$$

which was given by Moyal (1949), and which is rather easily proven using the group relation (45) and the overlap relation (48) between coherent states. Making use of the completeness relation (14) for coherent states, we may write $(102a)$ in the form

$$\int \frac{d^2 A}{\pi} U_1(A) {}_0\langle D | U_1^\dagger(A) | E\rangle_0 = | E\rangle_0 {}_0\langle D | \qquad (102b)$$

from which immediately follows the Weyl representation of an arbitrary operator

$$\Theta = \int \frac{d^2 A}{\pi} \operatorname{Tr}[U_1^\dagger(A)\Theta] U_1(A) \qquad (103a)$$

by writing Θ as a non-diagonal decomposition in terms of coherent states, which by completeness may always be done. It is convenient for later purposes to make a trivial change of variables to write $(103a)$ in the form

$$\Theta = \int \frac{d^2 K}{(2\pi)^2} \tilde{W}(K; \Theta) U_1(-\tfrac{1}{2}iK) \qquad \tilde{W}(K; \Theta) \equiv \pi \operatorname{Tr}[U_1^\dagger(-\tfrac{1}{2}iK)\Theta]. \qquad (103b)$$

To return to the proof of our second theorem, let us now assume that Θ_1 and Θ_2 are two operators with the same generalised Q representation. Their difference $\Phi \equiv \Theta_1 - \Theta_2$ then belongs to S'_1, and hence $Q_T(A; \Phi) = 0$ for all $A \in \mathbb{C}$. By writing Φ in the Weyl form of $(103b)$, we have from the definition of (98),

$$Q_T(A; \Phi) = \int \frac{d^2 K}{(2\pi)^2} \tilde{W}(K; \Phi) \operatorname{Tr}[\rho_A(T) U_1(-\tfrac{1}{2}iK)]. \qquad (104)$$

We now make use of the zero-temperature version of (100) to write

$$\operatorname{Tr}[\rho_A(T) U_1(-\tfrac{1}{2}iK)] = \int \frac{d^2 B}{\pi} P(B; \rho_A(T)) {}_0\langle B | U_1(-\tfrac{1}{2}iK) | B\rangle_0. \qquad (105)$$

The Q representatives of the Weyl operators are easily evaluated using (22) and (48) as

$${}_0\langle B | U_1(-\tfrac{1}{2}iK) | B\rangle_0 = \exp(-iB \cdot K - |K|^2/8) \qquad (106)$$

where we have introduced the convenient scalar product notation

$$B \cdot K \equiv \tfrac{1}{2}(BK^* + B^*K) = \operatorname{Re} B \cdot \operatorname{Re} K + \operatorname{Im} B \cdot \operatorname{Im} K. \qquad (107)$$

Use of (49) and (106) in (105) and (104) then readily gives

$$Q_T(A; \Phi) = \int \frac{d^2 K}{(2\pi)^2} \tilde{W}(K; \Phi) \exp(-\tfrac{1}{8}|K|^2 \coth(\tfrac{1}{2}\beta\omega)) \exp(-iB \cdot K) \qquad (108)$$

valid for arbitrary Φ. If we now restrict ourselves to those $\Phi \in S'_1$ as above for which $Q_T(A; \Phi) = 0$ for all $A \in \mathbb{C}$, we then have by the standard theorem of Fourier analysis that $\tilde{W}(K; \Phi) = 0$ almost everywhere, since the remaining exponential term never vanishes. Hence $\Phi = 0$, the null operator, and the proof is completed.

To recapitulate, we have thus proven from a combination of our two theorems the fundamental result that all operators in the trace class either directly have the generalised P representation of (97) or may be regarded as the limit of a sequence of such operators. Since our theorems involved existence proofs only, we now proceed to consider the construction of the generalised P representatives and various relations involving them. As a first step, we consider generalising the Mehta (1967) construction of (17) to our finite-temperature case. By taking the matrix elements of (97a) between the states ${}_0\langle -B; \rho\theta\lambda|$ and $|B; \rho\theta\lambda\rangle_0$ of (33), we find

$$
{}_0\langle -B; \rho\theta\lambda|\Theta|B; \rho\theta\lambda\rangle_0 = \int \frac{\mathrm{d}^2A}{\pi} P_T(A, \rho\theta\lambda; \Theta) \,{}_0\langle -B|\rho_A(T)|B\rangle_0 \tag{109}
$$

where we have made use of (33) and (38) to eliminate the squeezing parameters in the matrix element in the right-hand side of (109). We may now use (47) to show that (109) may be written as

$$
{}_0\langle -B; \rho\theta\lambda|\Theta|B; \rho\theta\lambda\rangle_0 \exp[(1+\mathrm{e}^{-\beta\omega})|B|^2]
$$
$$
= \int \frac{\mathrm{d}^2A}{\pi} P_T(A, \rho\theta\lambda; \Theta)(1-\mathrm{e}^{-\beta\omega})
$$
$$
\times \exp[(1-\mathrm{e}^{-\beta\omega})(-|A|^2+A^*B-AB^*)]. \tag{110}
$$

Since the factor $(A^*B - AB^*)$ in the exponential term in the integrand of (110) is pure imaginary, we see that the integral is just a double Fourier transform. If the left-hand side of the equation is square integrable (which is certainly not always the case), we may perform the usual Fourier inversion. In such cases the inversion is readily performed explicitly to give

$$
P_T(A, \rho\theta\lambda; \Theta) = (1-\mathrm{e}^{-\beta\omega}) \exp[(1-\mathrm{e}^{-\beta\omega})|A|^2] \int \frac{\mathrm{d}^2B}{\pi} \,{}_0\langle -B; \rho\theta\lambda|\Theta|B; \rho\theta\lambda\rangle_0
$$
$$
\times \exp[(1+\mathrm{e}^{-\beta\omega})|B|^2] \exp[(1-\mathrm{e}^{-\beta\omega})(AB^* - A^*B)]. \tag{111}
$$

More generally we may also interpret the generalised P representative as a tempered distribution in those cases where the left-hand side of (110) is limited to polynomial growth at infinity in both Re B and Im B. Most generally, there will also be cases where, for example, the growth is exponential and is hence not limited by any polynomial. The Fourier inversion can still be dealt with in the framework of distribution theory. As we have already mentioned, one commonly used technique would, in our case, now enable us to express the generalised P representative as the limit of a suitable sequence of square-integrable functions, analogously to the work of Klauder *et al* (1965) in the zero-temperature or pure-state case. Equation (111) clearly reduces in the case of zero squeezing and in the limit of zero temperature to the expression (17) of Mehta (1967).

3.5. Relations between the generalised P and Q representatives

We turn our attention now to various relations that exist between the generalised P and Q representatives. They are extensions of other similar such relations for the zero-temperature case (Hillery *et al* 1984, Balazs and Jennings 1984). Firstly, using (97a) and (98), we see that

$$
Q_{T'}(B, \rho\theta\lambda; \Theta) = \int \frac{\mathrm{d}^2A}{\pi} P_T(A, \rho\theta\lambda; \Theta) \,\mathrm{Tr}[\rho_A(T)\rho_B(T')] \tag{112}
$$

where have used (38) and the unitarity of the squeezing operators $U_2(\rho, \theta, \lambda)$ to eliminate the squeezing parameters from the trace term in the integrand. This trace may now be evaluated by making use of (68a) to write

$$\mathrm{Tr}[\rho_A(T)\rho_B(T')] = \int \frac{\mathrm{d}^2\Gamma}{\pi} P(\Gamma; \rho_A(T)) Q(\Gamma; \rho_B(T')). \tag{113}$$

By making use of (49) and the relation

$$Q(A; \rho_B(T)) = (1 - e^{-\beta\omega}) \exp[-(1 - e^{-\beta\omega})|A - B|^2] \tag{114}$$

which follows immediately from (47), the integration in (113) may be performed explicitly to give

$$\mathrm{Tr}[\rho_A(T)\rho_B(T')] = \frac{2 \sinh(\tfrac{1}{2}\beta\omega) \sinh(\tfrac{1}{2}\beta'\omega)}{\sinh[\tfrac{1}{2}(\beta+\beta')\omega]} \exp\left(-\frac{2 \sinh(\tfrac{1}{2}\beta\omega) \sinh(\tfrac{1}{2}\beta'\omega)}{\sinh[\tfrac{1}{2}(\beta+\beta')\omega]}|A - B|^2 \right). \tag{115}$$

Insertion of (115) into (112) then gives a rather general relation between the generalised P and Q representatives at different temperatures for a given operator.

Before proceeding further we wish to make a few comments concerning (115). In the first place, the zero-temperature limit $T = 0 = T'$ correctly reduces to the form

$$\mathrm{Tr}[|A\rangle_0 \, {}_0\langle A|B\rangle_0 \, {}_0\langle B|] \equiv |{}_0\langle A|B\rangle_0|^2 = \exp(-|A - B|^2) \tag{116}$$

which expresses the overlap between coherent states as given in (48). Secondly, using (39) and (40) we may write

$$\mathrm{Tr}[\rho_A(T)\rho_B(T')] = \sum_{m,n} p_m(T)p_n(T')|{}_A\langle m|n\rangle_B|^2. \tag{117}$$

Of the terms in (117) we have that $p_m(T)$ is the probability to find the system described by the density matrix $\rho_A(T)$ in the state $|m\rangle_A$, and secondly that the term $\sum_n p_n(T')|{}_A\langle m|n\rangle_B|^2$ is the probability to find the system described by the density matrix $\rho_B(T')$ in the state $|m\rangle_A$. Hence the trace $\mathrm{Tr}[\rho_A(T)\rho_B(T')]$ is simply the total joint probability to find the two separate systems described by density operators $\rho_A(T)$ and $\rho_B(T')$ in the same (but otherwise arbitrary) pure state. It is easy to see from the explicit form of (115) that this joint probability never exceeds one and only equals unity in the case $A = B$ at zero temperature, in which case the two systems are clearly definitely in the same pure state. In the case $A = B$, the right-hand side of (117) reduces to

$$\mathrm{Tr}[\rho_A(T)\rho_A(T')] = \sum_n p_n(T)p_n(T') = \frac{(1 - e^{-\beta\omega})(1 - e^{-\beta'\omega})}{(1 - e^{-(\beta+\beta')\omega})} \leqslant 1 \tag{118}$$

where the sum is trivially performed using the explicit form of $p_n(T)$ from (40), in total agreement with (115). We note in passing that the same argument as above can be readily generalised to two arbitrary mixed states, where it leads to the interpretation of $\mathrm{Tr}(\rho_1\rho_2)$ as the total joint probability to find the two systems described separately by density operators ρ_1 and ρ_2, in the same (arbitrary) pure state.

Equations (112) and (115) may be written in two other equivalent forms. In the first place we may exploit the Gaussian form of (115) to make use of the general relation

$$\int \frac{\mathrm{d}^2A}{\pi} f(A^*, A)k \exp(-k|A - B|^2) = \exp\left(\frac{1}{k} \frac{\partial^2}{\partial B^* \partial B} \right) f(B^*, B) \qquad k > 0 \tag{119}$$

which holds for an arbitrary function $f(A^*, A) \equiv f(A)$ of the complex variable A, which permits a Taylor expansion. Equation (119) is readily proven by first shifting variables to $A = B + k^{-1/2} z$; then Taylor expanding the function f about the complex point B, and finally using the trivial integral,

$$\int \frac{d^2 z}{\pi} \exp(-|z|^2) z^n (z^*)^m = n! \, \delta_{mn} \tag{120}$$

which is readily proven in polar coordinate representation. In this way we find

$$Q_T(B, \rho\theta\lambda; \Theta) = \exp\left(\frac{\sinh[\frac{1}{2}(\beta + \beta')\omega]}{2 \sinh(\frac{1}{2}\beta\omega) \sinh(\frac{1}{2}\beta'\omega)} \frac{\partial^2}{\partial B^* \, \partial B} \right) P_T(B, \rho\theta\lambda; \Theta). \tag{121}$$

In the second place, we recognise that the integral in (112) with the form of the trace from (115) is a simple convolution. Again, when $P_T(A, \rho\theta\lambda; \Theta)$ is a tempered distribution both sides can be Fourier transformed to give

$$\tilde{Q}_T(K, \rho\theta\lambda; \Theta) = \tilde{P}_T(K, \rho\theta\lambda; \Theta) \exp\left(-\frac{|K|^2}{8} \frac{\sinh[\frac{1}{2}(\beta + \beta')\omega]}{\sinh(\frac{1}{2}\beta\omega) \sinh(\frac{1}{2}\beta'\omega)} \right) \tag{122}$$

where our Fourier transform $\tilde{f}(K)$ of an arbitrary function $f(B)$ of complex parameter B is defined as

$$\tilde{f}(K) \equiv \int d^2 B \exp(i\boldsymbol{B} \cdot \boldsymbol{K}) f(B) \leftrightarrow f(B) = \int \frac{d^2 K}{(2\pi)^2} \exp(-i\boldsymbol{B} \cdot \boldsymbol{K}) \tilde{f}(K). \tag{123}$$

Three particularly important cases of (112) and (115) are: (i) $T' = T$, (ii) $T' = 0$ and (iii) $T = 0$. They lead respectively to the results below, where we also give the corresponding form of (122):

$$Q_T(B, \rho\theta\lambda; \Theta) = \int \frac{d^2 A}{\pi} P_T(A, \rho\theta\lambda; \Theta) \tanh(\frac{1}{2}\beta\omega) \exp[-\tanh(\frac{1}{2}\beta\omega)|A - B|^2]$$

$$\tilde{Q}_T(K, \rho\theta\lambda; \Theta) = \tilde{P}_T(K, \rho\theta\lambda; \Theta) \exp[-\frac{1}{4}|K|^2 \coth(\frac{1}{2}\beta\omega)] \tag{124a}$$

$$Q(B, \rho\theta\lambda; \Theta) = \int \frac{d^2 A}{\pi} P_T(A, \rho\theta\lambda; \Theta)(1 - e^{-\beta\omega}) \exp[-(1 - e^{-\beta\omega})|A - B|^2]$$

$$\tilde{Q}(K, \rho\theta\lambda; \Theta) = \tilde{P}_T(K, \rho\theta\lambda; \Theta) \exp[-\frac{1}{4}|K|^2 (1 - e^{-\beta\omega})^{-1}] \tag{124b}$$

$$Q_T(B, \rho\theta\lambda; \Theta) = \int \frac{d^2 A}{\pi} P(A, \rho\theta\lambda; \Theta)(1 - e^{-\beta\omega}) \exp[-(1 - e^{-\beta\omega})|A - B|^2]$$

$$\tilde{Q}_T(K, \rho\theta\lambda; \Theta) = \tilde{P}(K, \rho\theta\lambda; \Theta) \exp[-\frac{1}{4}|K|^2 (1 - e^{-\beta\omega})^{-1}] \tag{124c}$$

where the various P and Q functions with no temperature subscript indicate just their usual zero-temperature counterparts.

We may also derive a relation between the finite- and zero-temperature Q representatives as follows. Using (42) in the definition (98), we find immediately the relation

$$Q_T(B, \rho\theta\lambda; \Theta) = \int \frac{d^2 A}{\pi} Q(A, \rho\theta\lambda; \Theta)(e^{\beta\omega} - 1) \exp[-(e^{\beta\omega} - 1)|A - B|^2] \tag{125a}$$

where we have used (49). This may again be solved by convolution to yield

$$\tilde{Q}_T(K, \rho\theta\lambda; \Theta) = \tilde{Q}(K, \rho\theta\lambda; \Theta) \exp[-\frac{1}{4}|K|^2 (e^{\beta\omega} - 1)^{-1}]. \tag{125b}$$

More generally, we may use (125b) twice to prove

$$\tilde{Q}_{T'}(K, \rho\theta\lambda; \Theta) = \tilde{Q}_T(K, \rho\theta\lambda; \Theta) \exp\left(-\frac{|K|^2}{8} \frac{\sinh[\frac{1}{2}(\beta - \beta')\omega]}{\sinh(\frac{1}{2}\beta\omega)\sinh(\frac{1}{2}\beta'\omega)} \right). \tag{126}$$

It is perhaps also worth noting that (126) also follows by eliminating $\tilde{P}_T(K, \rho\theta\lambda; \Theta)$ between (122) and another version of itself with $T' = T''$. In a very similar fashion we may also eliminate $\tilde{Q}_{T'}(K, \rho\theta\lambda; \Theta)$ from (122) to derive the comparable relation,

$$\tilde{P}_{T'}(K, \rho\theta\lambda; \Theta) = \tilde{P}_T(K, \rho\theta\lambda; \Theta) \exp\left(\frac{|K|^2}{8} \frac{\sinh[\frac{1}{2}(\beta - \beta')\omega]}{\sinh(\frac{1}{2}\beta\omega)\sinh(\frac{1}{2}\beta'\omega)} \right) \tag{127}$$

which has the special case, when $T' = 0$,

$$\tilde{P}(K, \rho\theta\lambda; \Theta) = \tilde{P}_T(K, \rho\theta\lambda; \Theta) \exp[-\tfrac{1}{4}|K|^2(e^{\beta\omega} - 1)^{-1}] \tag{128a}$$

or, equivalently, upon Fourier inversion

$$P(B, \rho\theta\lambda; \Theta) = \int \frac{d^2A}{\pi} P_T(A, \rho\theta\lambda; \Theta)(e^{\beta\omega} - 1) \exp[-(e^{\beta\omega} - 1)|A - B|^2]. \tag{128b}$$

It is useful to summarise the various relations between the generalised P and Q representatives by making contact again with the Weyl representation introduced previously in (103b). In fact, we may also slightly generalise the Weyl representation to include squeezing. By obvious insertions of the unitary squeezing operator $U_2(\rho, \theta, \lambda)$, we may rewrite (102$b$) in the analogous form

$$\int \frac{d^2A}{\pi} U_1(A_{\rho\theta\lambda}) {}_0\langle D; \rho\theta\lambda| U_1^{\dagger}(A_{\rho\theta\lambda})|E; \rho\theta\lambda\rangle_0 = |E; \rho\theta\lambda\rangle_0 {}_0\langle D; \rho\theta\lambda| \tag{129}$$

where we have made use of (20), (28) and (30) to show that

$$\begin{aligned} U_2(\rho, \Theta, \lambda) U_1(A) U_2^{\dagger}(\rho, \Theta, \lambda) &= U_1(A_{\rho\theta\lambda}) \\ A_{\rho\theta\lambda} &\equiv A e^{i\lambda} \cosh(\tfrac{1}{2}\rho) - A^* \exp[-i(\lambda + \Theta)] \sinh(\tfrac{1}{2}\rho). \end{aligned} \tag{130}$$

In the same way that (103a) follows from (102b), we may now show from (129) that an arbitrary operator Θ may be written as

$$\Theta = \int \frac{d^2A}{\pi} \mathrm{Tr}[U_1^{\dagger}(A_{\rho\theta\lambda})\Theta]U_1(A_{\rho\theta\lambda}) \tag{131a}$$

which we again write in the equivalent form

$$\begin{aligned} \Theta &= \int \frac{d^2K}{(2\pi)^2} \tilde{W}(K, \rho\theta\lambda; \Theta) U_2(\rho, \theta, \lambda) U_1(-\tfrac{1}{2}iK) U_2^{\dagger}(\rho, \theta, \lambda) \\ \tilde{W}(K, \rho\theta\lambda; \Theta) &\equiv \tilde{W}(K_{\rho\theta\lambda}; \Theta) \equiv \pi \, \mathrm{Tr}[U_1(-\tfrac{1}{2}iK_{\rho\theta\lambda})\Theta]. \end{aligned} \tag{131b}$$

We may now easily take the generalised Q representative of (98) to get

$$Q_T(A, \rho\theta\lambda; \Theta) = \int \frac{d^2K}{(2\pi)^2} \tilde{W}(K, \rho\theta\lambda; \Theta) \, \mathrm{Tr}[\rho_A(T)U_1(-\tfrac{1}{2}iK)] \tag{132}$$

where we have made use of (38) and the unitarity of the squeezing operators $U_2(\rho, \Theta, \lambda)$. Further evaluation of (132) then proceeds exactly as in the case of zero squeezing

which led from (104) to (108). A further comparison with (124a) then gives us the two relations

$$Q_T(A, \rho\theta\lambda; \Theta) = \int \frac{\mathrm{d}^2 K}{(2\pi)^2} \, \tilde{W}(K, \rho\theta\lambda; \Theta) \exp[-\tfrac{1}{8}|K|^2 \coth(\tfrac{1}{2}\beta\omega)] \exp(-\mathrm{i}\boldsymbol{B}\cdot\boldsymbol{K}) \quad (133a)$$

$$P_T(A, \rho\theta\lambda; \Theta) = \int \frac{\mathrm{d}^2 K}{(2\pi)^2} \, \tilde{W}(K, \rho\theta\lambda; \Theta) \exp[\tfrac{1}{8}|K|^2 \coth(\tfrac{1}{2}\beta\omega)] \exp(-\mathrm{i}\boldsymbol{B}\cdot\boldsymbol{K}) \quad (133b)$$

which may be given also in the equivalent form:

$$Q_T(A, \rho\theta\lambda; \Theta) = \exp\left(\tfrac{1}{2}\coth(\tfrac{1}{2}\beta\omega)\frac{\partial^2}{\partial A^*\partial A}\right) W(A, \rho\theta\lambda; \Theta) \quad (134a)$$

$$P_T(A, \rho\theta\lambda; \Theta) = \exp\left(-\tfrac{1}{2}\coth(\tfrac{1}{2}\beta\omega)\frac{\partial^2}{\partial A^*\partial A}\right) W(A, \rho\theta\lambda; \Theta). \quad (134b)$$

We note that the generalised P representative may just be regarded as the analytic continuation to negative temperatures of the corresponding generalised Q representative, namely we have formally that the former is obtained by the replacement $T \to -T$ in the latter. We emphasise again, however, that this is only valid when the resulting generalised P function is integrable and square integrable, so that the above derivations are valid, and so that the resulting operators represented by the right-hand side of (97a, b) are bona fide operators.

A simple example of the above is provided by the operator $\rho_B(T')$ which has a generalised Q representative, $Q_T(A; \rho_B(T'))$ given by (115). One immediately sees that the corresponding generalised P representative $P_T(A; \rho_B(T'))$, obtained by the replacement $T \to -T$ in (115), is valid only for $T \le T'$.

Finally we note that if Θ is put into normal-ordered form

$$\Theta = :f_n(a_0, a_0^\dagger): \quad (135a)$$

where the normal-ordering operator :( ): indicates that in every term all the destruction operators a_0 stand to the right of all the creation operators $a_0^\dagger$ then we have trivially that

$$Q(B; \Theta) \equiv {}_0\langle B|\Theta|B\rangle_0 = f_n(B, B^*) \quad (135b)$$

where we have used (24). Furthermore, the two relations (135a,b) are totally equivalent. (Note also that a similar analysis may be made for $Q(B, \rho\theta\lambda; \Theta)$ with respect to normal ordering in terms of the operators $a_{0;\rho\theta\lambda}$ and $a_{0;\rho\theta\lambda}^\dagger$.) In this case we have from (119) and (125a)

$$\exp\left(-(\mathrm{e}^{\beta\omega}-1)^{-1}\frac{\partial^2}{\partial B\,\partial B^*}\right)Q_T(B; \Theta) = f_n(B, B^*). \quad (136)$$

We note also that if we write Θ in the form of (135a), its generalised Q representative may be evaluated directly as

$$Q_T(B; \Theta) = \int \frac{\mathrm{d}^2 A}{\pi} \, {}_0\langle A|\rho_B(T):f_n(a_0, a_0^\dagger):|A\rangle_0$$

$$= \int \frac{\mathrm{d}^2 A}{\pi} \int \frac{\mathrm{d}^2\Gamma}{\pi} \, {}_0\langle A|\rho_B(T)|\Gamma\rangle_0 f_n(A, \Gamma^*) \, {}_0\langle\Gamma|A\rangle_0 \quad (137)$$

where we have inserted a complete set of coherent states. Using (47) and (48), further evaluation of (137) is possible and it may be verified that this leads precisely to the inverse of (136), as required.

Similarly, Θ may also be put into antinormal-ordered form

$$\Theta = {}''f_a(a_0, a_0^\dagger)'' \tag{138}$$

where the antinormal-ordering operator $''(\ )''$ indicates that in every term all the destruction operators stand to the left of all the creation operators. In this case, if we assume that $f_a(B, B^*)$ has the expansion

$$f_a(B, B^*) = \sum_{m,n} f_{mn} B^m (B^*)^n \tag{139}$$

we may evaluate the P representative $P(A; \Theta)$ by the Mehta construction of (17). Specifically, by inserting a complete set of coherent states, we have

$$\begin{aligned}
{}_0\langle -B|\Theta|B\rangle_0 &= \sum_{m,n} f_{mn} \int \frac{\mathrm{d}^2\Gamma}{\pi} \, {}_0\langle -B|\Gamma\rangle_0 \Gamma^m \, {}_0\langle \Gamma|B\rangle_0 (\Gamma^*)^n \\
&= \sum_{m,n} f_{mn} \int \frac{\mathrm{d}^2\Gamma}{\pi} \Gamma^m (\Gamma^*)^n \exp(-|B|^2 - |\Gamma|^2 + B\Gamma^* - B^*\Gamma) \\
&= \exp(-|B|^2) \sum_{m,n} f_{mn} \left(\frac{\partial}{\partial B}\right)^n \left(-\frac{\partial}{\partial B^*}\right)^m \int \frac{\mathrm{d}^2\Gamma}{\pi} \exp(-|\Gamma|^2 + B\Gamma^* - B^*\Gamma) \\
&= \exp(-|B|^2) \sum_{m,n} f_{mn} \left(\frac{\partial}{\partial B}\right)^n \left(-\frac{\partial}{\partial B^*}\right)^m \exp(-|B|^2).
\end{aligned} \tag{140}$$

Insertion of (140) into (17) gives

$$\begin{aligned}
P(A; \Theta) &= \exp(|A|^2) \sum_{m,n} f_{mn} \int \frac{\mathrm{d}^2B}{\pi} \exp(AB^* - A^*B) \left(\frac{\partial}{\partial B}\right)^n \left(-\frac{\partial}{\partial B^*}\right)^m \exp(-|B|^2) \\
&= \exp(|A|^2) \sum_{m,n} f_{mn} \int \frac{\mathrm{d}^2B}{\pi} \exp(-|B|^2) \left(-\frac{\partial}{\partial B}\right)^n \left(\frac{\partial}{\partial B^*}\right)^m \exp(AB^* - A^*B) \\
&= \exp(|A|^2) \sum_{m,n} f_{mn} A^m (A^*)^n \int \frac{\mathrm{d}^2B}{\pi} \exp(-|B|^2 + AB^* - A^*B) \\
&= f_a(A, A^*)
\end{aligned} \tag{141}$$

where in the second line of (141) we have integrated by parts. In this case we also have from (119) and (128b)

$$\exp\left((e^{\beta\omega} - 1)\frac{\partial^2}{\partial B\, \partial B^*}\right) P_T(B; \Theta) = f_a(B, B^*) \tag{142}$$

which may again be verified directly for Θ of the form of (138).

It is clear from our earlier discussion that our results should therefore be of considerable use in putting arbitrary operators into normal- or antinormal-ordered form. In particular, the generalised Q representative $Q_T(A; \Theta)$ (or $Q_T(A, \rho\theta\lambda; \Theta)$) may be evaluated both in the limit as $T \to 0^+$, and after the analytic continuation to negative temperatures in the limit as $T \to 0^-$, to give both the normal- and antinormal-ordered functions f_n and f_a with respect to the operators a_0 and $a_0^\dagger$ (or $a_{0;\rho\theta\lambda}$ and $a_{0;\rho\theta\lambda}^\dagger$).

3.6. Entropy relations

At zero temperature, our systems are described by pure states, $\rho_{A;\rho\theta\lambda}(0) = |A; \rho\theta\lambda\rangle_0 {}_0\langle A; \rho\theta\lambda|$, and as consequences of this fact we have that these zero-temperature density matrices, generically $\rho(0)$, satisfy the statements: (i) that they are projection operators, $\rho^2(0) = \rho(0)$; (ii) that the trace of their square equals unity, $\text{Tr }\rho^2(0) = 1$; and (iii) that the associated entropy vanishes, $S(0) \equiv -\text{Tr}[\rho(0) \ln \rho(0)] = 0$. It is not difficult to show that any one of these statements implies the other two. Conversely, at finite temperatures the density matrices, generically $\rho(T)$, describe real mixed states, and the corresponding statements become: (i) $\rho^2(T) \neq \rho(T)$; (ii) $0 < \text{Tr }\rho^2(T) < 1$; and (iii) $S(T) \equiv -\text{Tr}[\rho(T) \ln \rho(T)] > 0$. Both quantities $\text{Tr }\rho^2(T)$ and the entropy $S(T)$ are independent of the displacement and squeezing parameters $\{A; \rho\theta\lambda\}$, and depend solely on temperature. Both provide a quantitative measure of the state mixing described by $\rho(T)$. Thus, the smaller is $\text{Tr }\rho^2(T)$ or the greater is $S(T)$, the more mixed is the state or, correspondingly, the less 'information' do we have about the basis. Specifically we have from (39) and (40)

$$\text{Tr }\rho^2_{A;\rho\theta\lambda}(T) = \sum_n p_n^2(T) = \tanh(\tfrac{1}{2}\beta\omega)$$

$$S(T) \equiv S(T; \omega) \equiv -\text{Tr}[\rho_{A;\rho\theta\lambda}(T) \ln \rho_{A;\rho\theta\lambda}(T)] = -\sum_n p_n(T) \ln p_n(T)$$

$$= -\ln(1 - e^{-\beta\omega}) + \beta\omega(e^{\beta\omega} - 1)^{-1}. \tag{143}$$

For many purposes we are interested in systems with more than one type of oscillator (or boson or mode), and it is not difficult to extend our results accordingly. For example, suppose that we have two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with corresponding operators $(a_1, a_1^\dagger)$ and $(a_2, a_2^\dagger)$ which obey the usual canonical commutation relations:

$$[a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger] \qquad [a_i, a_j^\dagger] = \delta_{ij} \qquad i, j = 1, 2. \tag{144}$$

Let ρ_1 and ρ_2 be density matrices in the respective spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then, if $\rho_1 \otimes \rho_2$ represents the corresponding density matrix for the mixed state of the joint system in $\mathcal{H}_1 \otimes \mathcal{H}_2$, it is a simple matter to show (Fano 1957, Segal 1960, Wehrl 1978) the absolutely fundamental property of the additivity of the entropy

$$S\{\rho_1 \otimes \rho_2\} = S\{\rho_1\} + S\{\rho_2\}. \tag{145}$$

The relation (145) is often of considerable practical use. For example, it may be used together with the simply proven fact that unitary transformations do not change the entropy to handle systems with general quadratic Hamiltonians. As an illustration, we consider the Hamiltonian

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \kappa a_1^\dagger a_2^\dagger + \kappa^* a_1 a_2. \tag{146}$$

We have explained elsewhere (Bishop and Vourdas 1987) that, in the case where $\tfrac{1}{4}(\omega_1 + \omega_2)^2 - |\kappa|^2 > 0$, there is a unitary transformation generated by an operator V, where

$$V = \exp(-\tfrac{1}{2}\rho \, e^{-i\theta} \, a_1^\dagger a_2^\dagger + \tfrac{1}{2}\rho \, e^{i\theta} a_1 a_2)$$

$$\rho = \tanh^{-1}\left(\frac{2\kappa}{\omega_1 + \omega_2}\right) \qquad \theta = -\arg \kappa \tag{147}$$

which transforms (146) into the decoupled form:

$$H \to \tilde{H} \equiv V^\dagger H V = \Omega_1 a_1^\dagger a_1 + \Omega_2 a_2^\dagger a_2 + \text{constant}$$

$$\Omega_n = \text{sgn}(\omega_1 + \omega_2)[\tfrac{1}{4}(\omega_1 + \omega_2)^2 - |\kappa|^2]^{1/2} - (-1)^n \tfrac{1}{2}(\omega_1 - \omega_2) \qquad n = 1, 2. \tag{148}$$

Furthermore the spectrum of the transformed Hamiltonian $\tilde{H}$ is positive-definite if $\omega_1 + \omega_2 > 0$ and $\frac{1}{4}(\omega_1 + \omega_2)^2 - |\kappa|^2 > \frac{1}{4}(\omega_1 - \omega_2)^2$ or, equivalently, if $\omega_1 > 0$ and $\omega_1\omega_2 > |\kappa|^2$. In this case, the entropy of the system with Hamiltonian H at thermodynamic equilibrium at temperature T, and hence described by the density matrix $\rho = e^{-\beta H}/\text{Tr}(e^{-\beta H})$, is the same as for the corresponding system with Hamiltonian $\tilde{H}$. Using (143) and (145) we see immediately that this entropy is just $S(T; \Omega_1) + S(T; \Omega_2)$, where $S(T; \omega)$ is defined in (143). For completeness we also note that in the case $\frac{1}{4}(\omega_1 - \omega_2)^2 > \frac{1}{4}(\omega_1 + \omega_2)^2 - |\kappa|^2 > 0$, although the Hamiltonians H and $\tilde{H}$ are unitarily equivalent, the energy spectrum is not bounded from below, and hence the density matrix cannot be defined since the operator $e^{-\beta H}$ does not belong to the trace class. Finally, we have explained elsewhere (Bishop and Vourdas 1987) that in the case $\frac{1}{4}(\omega_1 + \omega_2)^2 < |\kappa|^2$, the Hamiltonian H is unitarily equivalent to another which may be employed to describe the damped harmonic oscillator. The spectrum now becomes complex (if one is prepared to admit non-normalisable eigenfunctions) and it is not possible to define an entropy, at least not in the conventional sense.

Finally we note that, even in the case of a single mode, it may sometimes be convenient to work with more than a single squeezed coherent state $|A; \rho\theta\lambda\rangle_0$ at a time, or correspondingly with more than a single associated Hamiltonian $H_{A;\rho\theta\lambda}$ of (31) or density matrix $\rho_{A;\rho\theta\lambda}(T)$ of (36). In such cases the following points may be useful. If we consider a subset $\rho_{A_i;\rho\theta\lambda}(T)$, $i = 1, \dots, N$, of the density matrices of (36), then the operator $\tilde{\rho}(T) \equiv \Sigma_i \lambda_i \rho_{A_i;\rho\theta\lambda}(T)$ is also a density matrix provided the $\{\lambda_i\}$ are a set of real positive integers that sum to unity, $\Sigma_i \lambda_i = 1$. It may then be shown that the entropy $\tilde{S} \equiv \tilde{S}\{\tilde{\rho}\}$ associated with this density matrix obeys the inequalities

$$S \leq \tilde{S} \leq S - \sum_i \lambda_i \ln \lambda_i \qquad (149)$$

where S is the entropy of (143). The left inequality in (149) is just the well known property of the concavity of the entropy (Segal 1960, Wehrl 1978) expressed in our particular framework, while the right inequality has also been studied by Lanford and Robinson (1968).

4. Conclusions

We have attempted to show in this paper to what extent the *mixed* squeezed coherent states appropriate to a displaced and squeezed harmonic oscillator in thermodynamic equilibrium at some finite temperature can be used as a generalised basis in Hilbert space. In particular we have demonstrated in the first place how the associated thermodynamic density operator may be used to expand arbitrary ket states by providing an appropriate decomposition of the identity operator. Secondly, and much more importantly, we have investigated the possibility of expanding an arbitrary operator in the Hilbert space in terms of these density operators. We have explicitly demonstrated how this latter aspect may be viewed as a rather natural generalisation to finite temperatures of the well known P representation of quantum mechanical operators as integrals over projection operators onto the pure Glauber coherent state vectors in the space.

We have explicitly investigated the question of the existence of the generalised P representation and have shown how the situation is quite similar to the usual pure-state

or zero-temperature case. Thus, we have shown how the generalised P representative for an arbitrary trace-class operator Θ in the space may be straightforwardly obtained in a variety of ways, but where each of these ultimately relies on the existence of the two-dimensional Fourier transform of some appropriate associated quantity. Certainly the class of operators for which such transforms exist as smooth functions is rather large. On the other hand, there exist other quite ordinary and respectable operators for which the respective Fourier integral only finds meaning within the framework of the theory of generalised functions or distributions. This typically arises due to the behaviour at infinity of the integrand. In some cases where the behaviour at infinity is limited to polynomial growth, the associated generalised P representative can be interpreted as a tempered distribution. In other cases, typically where there is exponential growth at infinity, the generalised P representative still exists as a distribution (e.g. in the sequential sense). In such cases, because the distribution is no longer tempered, its practical usefulness will be accordingly curtailed.

In these latter cases, one may either take the view that the generalised P representative does not exist or, as we have shown, even here one may define the generalised P representative as the limit of an infinite sequence of such representatives.

It is also worth pointing out that, while a given operator may have a generalised P representative only in the Schwartz sense in the case of mixed ordinary (unsqueezed) coherent states, its generalised P representative may be quite well behaved in terms of the mixed squeezed coherent states. These extra squeezing degrees of freedom may in such cases be of real practical use.

We have also shown how our results may be related to the Glauber–Lachs formalism in quantum optics for mixtures of coherent and incoherent radiation. In particular we have been able to give some rather general expressions for the cumulants of the associated photon distribution functions for such mixtures. These expressions are of practical use in evaluating the various photon correlation effects observed in experiments of the Hanbury Brown and Twiss (1956) type.

The pure coherent states are well known to be minimum-uncertainty states in that they realise the Heisenberg position–momentum uncertainty relation $\Delta p \Delta q \geqslant \frac{1}{2}$ as an equality. We have shown here how the corresponding mixed coherent states obey a finite-temperature generalisation of this relation which is formulated in terms of the uncertainty determinant or variance determinant of (92). The corresponding expression involves the corresponding quantum mechanical minimum uncertainty in the zero-temperature limit, but at finite temperatures also contains an extra thermodynamic uncertainty which is given in terms of the Planck distribution for the average number of thermal photons. It is of course just this additional term which is ultimately responsible for the thermodynamic entropy which we have also discussed.

In this latter context we have illustrated how the entropy may be evaluated in the particular case of two modes coupled through a quadratic Hamiltonian. More generally, although most of our results have been given for a single mode (i.e. a single one-dimensional harmonic oscillator), the generalisation to systems having more than one degree of freedom is straightforward in principle.

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