Coherent-mode decomposition of partially polarized, partially coherent sources

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It is shown that any partially polarized, partially coherent source can be expressed in terms of a suitable superposition of transverse coherent modes with orthogonal polarization states. Such modes are determined through the solution of a system of two coupled integral equations. An example, for which the modal decomposition is obtained in closed form in terms of fully linearly polarized Hermite Gaussian modes, is given.

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1. INTRODUCTION

The determination of the coherent modes[1] of a partially coherent beam and of their relative weights represents a key problem in characterizing real laser beams and sources, and several methods have been proposed to solve such a problem in the scalar case.[2–19]

In view of the current interest in light beams that are both partially polarized and partially coherent,[20–28] it is desirable to extend the coherent-mode representation to the case of vectorial electromagnetic beams. In the present paper, we show that such an extension can be carried out in a more or less straightforward way.

The similarity between the coherent-mode representation in the scalar and vectorial cases becomes particularly transparent when the structure of the theory is viewed from a standpoint slightly more formal and general than the position representation, which is more often used in optics. For this reason, we will briefly present this general point of view, showing how the coherent-mode representation of partially coherent scalar sources simply arises as a particular case, and then we will apply the coherent-mode decomposition to the case of partially polarized, partially coherent vectorial beams described by the beam-coherence-polarization (BCP) matrix.[25,26] Following such a formalism, we will neglect the longitudinal component of the field and consider the paraxial approximation to be valid. We will show that, in general, the determination of the coherent modes of a partially polarized, partially coherent source can be a rather difficult task, but it can be made easier if the pertinent BCP matrix can be reduced to a diagonal form. As an example, we will consider partially polarized sources characterized by BCP matrices having Gaussian Schell-model (GSM)[1] diagonal elements and specular mutual intensity (SMI)[29] antidiagonal ones.

The paper is organized as follows: In Section 2, the formalism based on the Hilbert space is recalled. After introduction of the notation to be used, such a formalism is applied to the scalar and vectorial cases. In particular, the modal decomposition of a source with a diagonal BCP matrix is derived. In Section 3, such results are applied to the particular case of a source for which closed-form results are achievable. Finally, Section 4 is devoted to the conclusions.

2. THEORETICAL ANALYSIS

A. Preliminaries and Notation

Let $\mathcal{H}$ represent a Hilbert space. Elements of $\mathcal{H}$ are denoted by Dirac's ket vectors $|\varphi\rangle$, $|\psi\rangle$,..., and the inner product between two vectors $|\varphi\rangle$, $|\psi\rangle$ is devoted by $\langle\varphi|\psi\rangle$. Let $\hat{J}$ be a linear, Hermitian, positive-semidefinite operator acting on $\mathcal{H}$.

Hermiticity means that

$$\langle\varphi|\hat{J}|\psi\rangle = \langle\psi|\hat{J}|\varphi\rangle^* \tag{1}$$

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for all $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$, the asterisk denoting complex conjugation, whereas nonnegativity demands that

$$\langle \varphi|\hat{J}|\varphi\rangle \geqslant 0 \tag{2}$$

for every $|\varphi\rangle \in \mathcal{H}$. Furthermore, if $\hat{J}$ satisfies the additional condition $\mathrm{Tr}\{\hat{J}^2\} < \infty$, $\mathrm{Tr}\{\cdot\}$ being the trace operator, then $\hat{J}$ is called a Hilbert–Schmidt operator.

As is well-known,[30,31] every Hilbert–Schmidt operator has the spectral representation or decomposition

$$\hat{J} = \sum_n \lambda_n |n\rangle\langle n|, \tag{3}$$

where the λ_n are nonnegative numbers and the kets $|n\rangle$ are orthonormal, i.e.,

$$\langle m|n\rangle = \delta_{m,n}, \tag{4}$$

$\delta_{m,n}$ being the Kronecker symbol. Equation (3) means that $\hat{J}$ is a convex combination of orthogonal one-dimensional projection operators in $\mathcal{H}$.

Using the orthonormality property (4), we can rewrite the above representation in the form

$$\hat{J}|n\rangle = \lambda_n|n\rangle, \qquad n = 0, 1, 2, \ldots, \tag{5}$$

so that the λ_n are the eigenvalues of $\hat{J}$, the corresponding eigenfunctions are (or can be chosen to be) orthonormal, and such eigenfunctions form a basis for the subspace of $\mathcal{H}$ over which $\hat{J}$ is a nonzero operator. In the orthogonal subspace of $\mathcal{H}$, the Hilbert–Schmidt operator acts as a null operator. Clearly, $\mathrm{Tr}\{\hat{J}^2\} = \Sigma_n {\lambda_n}^2$, and the Hilbert–Schmidt property gets transcribed into the statement $\Sigma_n {\lambda_n}^2 < \infty$.

The index n runs over a subset of the integer set or the natural numbers. However, in some particular applications, symmetry or other insights may make it natural and convenient to label this subset with two or more discrete variables. Further, a given physical situation may be such that the Hilbert–Schmidt operator of interest is conveniently described in a particular basis, which will be referred to as natural. In that case, the problem will be to determine the eigenvectors of $\hat{J}$ as a linear combination of these preferred, or natural, basis vectors. However, the natural basis may be labeled by one or more discrete variables, and in other cases it is labeled by one or more continuous variables. Indeed, in the vector case, one needs two continuous variables to label the natural basis.

B. Scalar Fields

In the case of quasi-monochromatic partially coherent light sources described by the mutual intensity $J(\mathbf{r}_1, \mathbf{r}_2)$,[1,32] the natural basis consists of the position eigenvectors $|x, y\rangle = |\mathbf{r}\rangle$, which satisfy the following relationships:

$$\hat{x}|\mathbf{r}\rangle = x|\mathbf{r}\rangle, \qquad \hat{y}|\mathbf{r}\rangle = y|\mathbf{r}\rangle, \tag{6}$$

$$\langle \mathbf{r}_1|\mathbf{r}_2\rangle = \delta^{(2)}(\mathbf{r}_1 - \mathbf{r}_2), \tag{7}$$

$$\int \mathrm{d}^2 r|\mathbf{r}\rangle\langle\mathbf{r}| = 1. \tag{8}$$

In Eqs. (6)–(8), $\mathbf{r}_1$ and $\mathbf{r}_2$ denote two typical points across the transverse section of the beam, while a reference frame (x, y, z), with the z axis coincident with the mean propagation direction of the beam, has been introduced. Furthermore, $\delta^{(2)}$ is the two-dimensional Dirac function.

Given a mutual intensity function $J(\mathbf{r}_1, \mathbf{r}_2)$, we represent it as a Hermitian nonnegative operator $\hat{J}$ in the Hilbert space $\mathcal{H} = L_2(\mathcal{R}^2)$, which consists of complex-valued functions that are square integrable over the two-dimensional plane $\mathcal{R}^2$. The matrix elements of $\mathcal{J}$ are defined as

$$J(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathbf{r}_1|\hat{J}|\mathbf{r}_2\rangle. \tag{9}$$

Using the completeness property of the position eigenvectors given in Eq. (8), we can immediately invert Eq. (9), i.e.,

$$\hat{J} = \iint \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 J(\mathbf{r}_1, \mathbf{r}_2)|\mathbf{r}_1\rangle\langle\mathbf{r}_2|. \tag{10}$$

Furthermore, it follows that the Hilbert–Schmidt property reads as

$$\mathrm{Tr}\{\hat{J}^2\} = \iint \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 |J(\mathbf{r}_1, \mathbf{r}_2)|^2 < \infty. \tag{11}$$

Let us assume that our mutual intensity function possesses such a property. Then the spectral representation theorem applies. On projecting Eq. (3) to the position representation, on using the completeness relation (8), and on denoting $\langle \mathbf{r}|n\rangle$, which is a complex-valued scalar function of $\mathbf{r}$, by $\Phi_n(\mathbf{r})$, we eventually find that

$$J(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \lambda_n \Phi_n(\mathbf{r}_1)\Phi_n^*(\mathbf{r}_2),$$

$$\lambda_n > 0, \quad (12)$$

$$\int \mathrm{d}^2 r \Phi_m^*(\mathbf{r})\Phi_n(\mathbf{r}) = \delta_{m,n}. \tag{13}$$

Similarly, Eq. (5) reads, in the position representation, as follows:

$$\int \mathrm{d}^2 r_2 J(\mathbf{r}_1, \mathbf{r}_2)\Phi_n(\mathbf{r}_2) = \lambda_n \Phi_n(\mathbf{r}_1); \tag{14}$$

i.e., it becomes an integral equation.

Equations (12)–(14) indeed coincide with the familiar equations of the coherent-mode representation for partially coherent scalar beams.[1,33]

C. Vectorial Case

In the case of vector beams, we need a *binary* variable, say α, to label the states of polarization. Although $|u\rangle$ and $|v\rangle$ may represent two generic orthogonal polarization states, for simplicity we will consider them as linear polarization states along x and y, respectively. An arbitrary (fully polarized) state can be written as

$$c_u|u\rangle + c_v|v\rangle = \sum_\alpha c_\alpha|\alpha\rangle, \tag{15}$$

where α runs over its binary values u and v. We wish to allow the possibility that the state of polarization may change with position, so that c_u and c_v are (independent) functions of position across the transverse plane. It follows that the Hilbert space of relevance to the present problem is the tensor product $\mathcal{C}^2 \otimes L_2(\mathcal{R}^2)$,[34] with $\mathcal{C}^2$ representing the two-dimensional complex vector space corresponding to states of polarization and $L_2(\mathcal{R}^2)$ again corresponding to the space of complex-valued functions that are square integrable over the plane $\mathcal{R}^2$, as in the scalar case. The natural basis is $\{|\alpha;\, \mathbf{r}\rangle = |\alpha\rangle \otimes |\mathbf{r}\rangle\}$, where α runs over $(u,\, v)$ and $\mathbf{r}$ runs over the transverse plane. The orthonormality and completeness relations become

$$\langle \alpha;\, \mathbf{r}_1 | \beta;\, \mathbf{r}_2\rangle = \delta_{\alpha,\beta}\delta^{(2)}(\mathbf{r}_1 - \mathbf{r}_2), \tag{16}$$

$$\sum_{\alpha} \int \mathrm{d}^2 r |\alpha;\, \mathbf{r}\rangle\langle\alpha;\, \mathbf{r}| = 1, \tag{17}$$

respectively. Furthermore, in Eq. (17), the symbol 1 stands for the identity operator on the Hilbert space $\mathcal{C}^2 \otimes L_2(\mathcal{R}^2)$.

In place of the mutual intensity of the scalar case, we now have the BCP matrix,[14,26] and thus the BCP operator, denoted again by $\hat{J}$. Then we have the following relationship between the BCP matrix and the BCP operator:

$$J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2) = \langle \alpha;\, \mathbf{r}_1 | \hat{J} | \beta;\, \mathbf{r}_2\rangle, \tag{18}$$

together with the inverse relationship

$$\hat{J} = \sum_{\alpha,\beta} \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2) |\alpha;\, \mathbf{r}_1\rangle\langle\beta;\, \mathbf{r}_2|. \tag{19}$$

By allowing α, β to run over their binary values $(u,\, v)$ independently, we can write $J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2)$ symbolically as a 2×2 matrix:

$$\mathbf{J}(\mathbf{r}_1,\, \mathbf{r}_2) = \begin{bmatrix} J_{uu}(\mathbf{r}_1,\, \mathbf{r}_2) & J_{uv}(\mathbf{r}_1,\, \mathbf{r}_2) \\ J_{vu}(\mathbf{r}_1,\, \mathbf{r}_2) & J_{vv}(\mathbf{r}_1,\, \mathbf{r}_2) \end{bmatrix}. \tag{20}$$

In particular, the Hilbert–Schmidt property reads now as

$$\mathrm{Tr}\{\hat{J}^2\} = \sum_{\alpha,\beta} \int\int \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 |J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2)|^2 < \infty, \tag{21}$$

while the nonnegativity condition, which for the $\hat{J}$ operator turns out to be

$$\langle \varphi | \hat{J} | \varphi\rangle \geqslant 0 \qquad \forall | \varphi\rangle \in \mathcal{C}^2 \otimes L_2(\mathcal{R}^2), \tag{22}$$

leads, once projected to the $|\alpha;\mathbf{r}\rangle$ basis, to

$$\sum_{\alpha,\beta} \int\int \mathrm{d}^2 r_1 \mathrm{d}^2 r_2 J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2) \varphi_\alpha^*(\mathbf{r}_1)\varphi_\beta(\mathbf{r}_2) \geqslant 0 \tag{23}$$

for any pair of functions $\varphi_u(\mathbf{r})$, $\varphi_v(\mathbf{r})$. We assume henceforth that our BCP matrix (and hence the BCP operator) possesses such a property, so that the spectral-representation theorem, given in Eqs. (3)–(5), applies.

Let us denote the function $\langle \alpha;\, \mathbf{r}|n\rangle$ by $\Phi_{n;\alpha}(\mathbf{r})$. When $\alpha = u$, this represents a fully coherent beam, fully polarized along the *x* direction. For $\alpha = v$ the polarization is in the *y* direction. On projecting Eq. (3) to the $|\alpha;\mathbf{r}\rangle$ basis and on using the completeness property (17), one obtains

$$J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2) = \sum_n \Lambda_n \Phi_{n;\alpha}(\mathbf{r}_1)\Phi_{n;\beta}^*(\mathbf{r}_2), \tag{24}$$

$$\sum_{\alpha} \int \mathrm{d}^2 r \Phi_{n;\alpha}(\mathbf{r})\Phi_{n;\beta}^*(\mathbf{r}) = \delta_{m,n}, \tag{25}$$

where Λ_n denotes the *n*th eigenvalue in the vector case.

The coherent modes now have the vector form

$$\mathbf{\Phi}_n(\mathbf{r}) = \begin{pmatrix} \Phi_{n;u}(\mathbf{r}) \\ \Phi_{n;v}(\mathbf{r}) \end{pmatrix}, \tag{26}$$

and $\mathbf{J}$ may be written in the alternative form

$$\mathbf{J}(\mathbf{r}_1,\, \mathbf{r}_2) = \sum_n \Lambda_n \Phi_n(\mathbf{r}_1)\Phi_n^\dagger(\mathbf{r}_2), \tag{27}$$

the dagger denoting the Hermitian conjugate of the vector (26). Furthermore, on projecting Eq. (5) to the $|\alpha;\mathbf{r}\rangle$ basis, we obtain

$$\sum_{\beta} \int \mathrm{d}^2 r_2 J_{\alpha\beta}(\mathbf{r}_1,\, \mathbf{r}_2)\Phi_{n;\beta}(\mathbf{r}_2) = \Lambda_n \Phi_{n;\alpha}(\mathbf{r}_1) \qquad (\alpha,\, \beta = u,\, v), \tag{28}$$

which represents a *pair* of coupled integral equations. This result generalizes the corresponding integral equation (14) (a homogeneous Fredholm integral) obtained for the scalar case. When polarization of light is taken into account, it will be necessary to solve this pair of equations.

For a given BCP matrix, solving the set of integral equations (28) may be nontrivial. However, in some favorable cases it may happen that a coherent-mode decomposition is known for J_{uu}, J_{vv}, and J_{uv} individually and that the integral equations (28) couple only a few of these coherent modes. This happens, for instance, when the BCP matrix has, or can be reduced to, a diagonal form, as we shall see in Subsection 2.D.

D. Diagonal Case

Let us suppose that the BCP matrix $\mathbf{J}_0$ has a diagonal form, i.e.,

$$\mathbf{J}_0(\mathbf{r}_1,\, \mathbf{r}_2) = \begin{bmatrix} J_{uu}(\mathbf{r}_1,\, \mathbf{r}_2) & 0 \\ 0 & J_{vv}(\mathbf{r}_1,\, \mathbf{r}_2) \end{bmatrix}. \tag{29}$$

As we shall see in a moment, for such a BCP matrix the modal decomposition can be easily achieved as follows. First, let the modal decompositions of the two diagonal terms of the BCP matrix (29) be

$$J_{uu}(\mathbf{r}_1,\, \mathbf{r}_2) = \sum_n \lambda_n \phi_n(\mathbf{r}_1)\phi_n^*(\mathbf{r}_2), \tag{30}$$

$$J_{vv}(\mathbf{r}_1,\, \mathbf{r}_2) = \sum_n \mu_n \psi_n(\mathbf{r}_1)\psi_n^*(\mathbf{r}_2), \tag{31}$$

respectively. In Eqs. (30) and (31), λ_n and μ_n are the eigenvalues associated with the two scalar mutual intensities J_{uu} and J_{vv}, respectively, while $\phi_n(\mathbf{r})$ and $\psi_n(\mathbf{r})$ are the corresponding eigenfunctions. Accordingly, the BCP matrix (29) can be given the form in Eq. (27) simply on letting, for instance,

$$\Lambda_{2n} = \lambda_n, \qquad \mathbf{\Phi}_{2n}(\mathbf{r}) = \begin{pmatrix} \phi_n(\mathbf{r}) \\ 0 \end{pmatrix},$$
$$\Lambda_{2n+1} = \mu_n, \qquad \mathbf{\Phi}_{2n+1}(\mathbf{r}) = \begin{pmatrix} 0 \\ \psi_n(\mathbf{r}) \end{pmatrix}, \qquad n = 0, 1, 2, \ldots, \tag{32}$$

which represents the modal decomposition of a diagonal BCP matrix. In such a way, the modes turn out to be alternately polarized along x and y. It should be noted that the diagonal case is of particular importance, since a wide class of BCP matrices can be reduced to a diagonal form simply by means of a suitable rotation of the reference frame.[26,27]

In Section 3, we present an example of modal decomposition of a partially polarized, partially coherent source where the problem can be solved in closed-form terms.

3. EXAMPLE OF COHERENT-MODE DECOMPOSITION

A. Preliminaries

For simplicity, in the following we shall limit ourselves to a two-dimensional problem.

Let us consider the following BCP matrix at the plane $z = 0$ of the (x, z) reference frame:

$$\mathbf{J}_0(x_1, x_2) = I_0 \exp\left[-\frac{\beta}{2}(x_1{}^2 + x_2{}^2)\right] \times \begin{bmatrix} \exp[-\gamma(x_1 - x_2)^2] & \exp[-\gamma(x_1 + x_2)^2] \\ \exp[-\gamma(x_1 + x_2)^2] & \exp[-\gamma(x_1 - x_2)^2] \end{bmatrix}, \tag{33}$$

where I_0, β, and γ are positive parameters. $\mathbf{J}_0$ is a bona fide BCP matrix, since it satisfies both the Hermiticity (the matrix is real and symmetric) and nonnegativeness conditions (see Appendix A). On the other hand, the correlation functions between the various components of the field take values having modulus ranging from 0 to 1.

The partially polarized, partially coherent source characterized by the BCP (33) is indistinguishable, as far as measurements with no anisotropic elements are concerned, from an ordinary, scalar Gaussian Schell-model (GSM) source.[1] In fact, the mutual intensity of the equivalent scalar partially coherent source[35] turns out to be

$$J_{\mathrm{eq}}(x_1, x_2) = 2I_0 \exp\left[-\frac{\beta}{2}(x_1{}^2 + x_2{}^2)\right] \times \exp[-\gamma(x_1 - x_2)^2], \tag{34}$$

where both the intensity and the degree of coherence are Gaussian.

Each of the elements of the matrix (33) has the structure of a mutual intensity function. The two diagonal terms simply correspond to the mutual intensity of a scalar GSM source. The antidiagonal ones, on the other hand, present a Gaussian intensity profile, just as for the diagonal elements, but their degree of coherence (which takes into account the cross correlation between E_x and E_y) has the form

$$g(x_1, x_2) = \exp[-\gamma(x_1 + x_2)^2], \tag{35}$$

i.e., depends only on the sum of the two coordinates x_1 and x_2. From Eq. (35), it is easily seen that the maximum value (unitary) of g is achieved for $x_1 = -x_2$, i.e., when the two considered points are symmetric with respect to the origin. Sources of this kind were extensively studied in the scalar case[29] and were termed sources endowed with specular mutual intensity (SMI). In the present case, Eq. (35) leads to the fact that, when $x_1 = -x_2$, a perfect correlation between the components E_x and E_y is achieved.

As is well-known, the polarization features of the source are determined by the *local* BCP matrix, i.e.,[26]

$$\mathbf{J}_0(x, x) = I_0 \exp(-\beta x^2)\begin{bmatrix} 1 & \exp(-4\gamma x^2) \\ \exp(-4\gamma x^2) & 1 \end{bmatrix}. \tag{36}$$

For instance, let us assume that $|u\rangle$ and $|v\rangle$ correspond to x and y linearly polarized states, respectively, so that the Stokes parameters at each point and the local degree of polarization P (Refs. 1 and 36) are easily calculated from Eq. (36) as

$$s_0 = J_{xx} + J_{yy} = 2I_0 \exp(-\beta x^2), \tag{37}$$

$$s_1 = J_{xx} - J_{yy} = 0, \tag{38}$$

$$s_2 = 2\,\mathrm{Re}(J_{xy}) = 2I_0 \exp[-(\beta + 4\gamma)x^2], \tag{39}$$

$$s_3 = 2\,\mathrm{Im}(J_{xy}) = 0, \tag{40}$$

$$P = \left[\frac{(J_{xx} - J_{yy})^2 + 4|J_{xy}|^2}{(J_{xx} + J_{yy})^2}\right]^{1/2} = \exp(-4\gamma x^2). \tag{41}$$

The source turns out to be partially polarized with a nonuniform degree of polarization P across the transverse section with a Gaussian profile. In particular, the degree of polarization P presents a Gaussian profile with the maximum value at the center, and the width of such a profile is proportional to $\sqrt{1/\gamma}$. On the other hand, from the previous equations it can be seen that if we decompose the wave into an unpolarized and a polarized portion that are mutually independent,[32] then its totally polarized component is linear with azimuth 45°.

An important property of the BCP matrix (33) is that it can be diagonalized simply by using a new reference frame, say (ξ, η), for representing the electric field, which is rotated by $\pi/4$ with respect to the (x, y) fraone. In fact, since in our case $J_{xx} = J_{yy}$ and $J_{xy} = J_{yx}$, the BCP matrix in the (ξ, η) reference frame turns out to be[26]

$$\mathbf{J}_0(x_1, x_2) = \begin{bmatrix} J_{\xi\xi}(x_1, x_2) & 0 \\ 0 & J_{\eta\eta}(x_1, x_2) \end{bmatrix}, \tag{42}$$

where

$$J_{\xi\xi}(x_1, x_2) = I_0 \exp\left[-\frac{\beta}{2}(x_1{}^2 + x_2{}^2)\right] \times \{\exp[-\gamma(x_1 - x_2)^2] + \exp[-\gamma(x_1 + x_2)^2]\},$$

$$J_{\eta\eta}(x_1, x_2) = I_0 \exp\left[-\frac{\beta}{2}(x_1{}^2 + x_2{}^2)\right] \times \{\exp[-\gamma(x_1 - x_2)^2] - \exp[-\gamma(x_1 + x_2)^2]\}. \tag{43}$$

As a consequence, the partially polarized source can be thought of as arising from the superposition of two independent sources, linearly polarized along the ξ and η axes, respectively, whose mutual intensities are given by Eqs. (43). Accordingly, results given in Subsection 2D pertinent to diagonal BCP matrices can now be applied to our source. This will be done in the next section.

B. Coherent-Mode Decomposition for Sources with Beam-Coherence-Polarization Matrix of the Form of Eq. (33)

First, let us introduce the functions $J_{\pm}(x_1, x_2)$, defined as

$$J_{\pm}(x_1, x_2) = I_0 \exp\left[-\beta \frac{{x_1}^2 + {x_2}^2}{2} - \gamma(x_1 \pm x_2)^2\right]. \tag{44}$$

Note that the mutual intensities given in Eqs. (43) can be written in terms of J_+ and J_- as follows:

$$J_{\xi\xi}(x_1, x_2) = J_-(x_1, x_2) + J_+(x_1, x_2),$$

$$J_{\eta\eta}(x_1, x_2) = J_-(x_1, x_2) - J_+(x_1, x_2). \tag{45}$$

It is evident that J_- corresponds to the mutual intensity of a scalar GSM source, so that its coherent-mode decomposition reads[37,38]

$$J_-(x_1, x_2) = \left(\frac{c}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{\lambda_0 q^n}{2^n n!} H_n(x_1\sqrt{c}) H_n(x_2\sqrt{c}) \times \exp\left(-c\frac{{x_1}^2 + {x_2}^2}{2}\right), \tag{46}$$

where H_n is the nth-order Hermite polynomial[39] and

$$c = 2(\beta^2 + 2\beta\gamma)^{1/2},$$

$$q = \gamma/(\beta + \gamma + c),$$

$$\lambda_0 = I_0\left(\frac{\pi}{(\beta + \gamma + c)}\right)^{1/2}. \tag{47}$$

On the other hand, we can obtain an analogous expansion for J_+ starting from Eq. (46), replacing x_2 by $-x_2$, so that

$$J_+(x_1, x_2) = J_-(x_1, -x_2) = \left(\frac{c}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{\lambda_0(-q)^n}{2^n n!} H_n(x_1\sqrt{c}) \times H_n(x_2\sqrt{c}) \exp\left(-c\frac{{x_1}^2 + {x_2}^2}{2}\right), \tag{48}$$

where use has been made of the following property of the Hermite polynomials[39]:

$$H_n(-s) = (-1)^n H_n(s). \tag{49}$$

On substituting from Eqs. (46) and (48) into Eq. (45), we finally obtain

$$J_{\xi\xi}(x_1, x_2) = 2\lambda_0\left(\frac{c}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{q^{2n}}{2^{2n} 2n!} \times H_{2n}(x_1\sqrt{c}) H_{2n}(x_2\sqrt{c}) \times \exp\left(-c\frac{{x_1}^2 + {x_2}^2}{2}\right),$$

$$J_{\eta\eta}(x_1, x_2) = 2\lambda_0\left(\frac{c}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{2^{2n+1}(2n+1)!} \times H_{2n+1}(x_1\sqrt{c}) H_{2n+1}(x_2\sqrt{c}) \times \exp\left(-c\frac{{x_1}^2 + {x_2}^2}{2}\right), \tag{50}$$

which correspond to the expansions in Eqs. (30) and (31). Accordingly, the modal decomposition (32) holds, on letting

$$A_n = 2\lambda_0 q^n,$$

$$\phi_n(x) = \left(\frac{c}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^{2n} 2n!}} H_{2n}(x\sqrt{c}) \exp\left(-\frac{cx^2}{2}\right),$$

$$\psi_n(x) = \left(\frac{c}{\pi}\right)^{1/4} \frac{1}{[2^{2n+1}(2n+1)!]^2} H_{2n+1}(x\sqrt{c}) \times \exp\left(-\frac{cx^2}{2}\right), \tag{51}$$

which constitutes the coherent-mode decomposition of the partially polarized SMI source. We recall that if the modes of the source are linearly polarized Hermite Gaussian ones, the knowledge of the modal expansion allows us to obtain in an easy way all the beam caraczteristics upon propagation through a typical first-order $ABCD$ paraxial optical system.[40] In particular, since Hermite Gaussian beams are shape invariant upon propagation, the same holds for the beams radiated by sources having a BCP matrix of the form (33). Moreover, in this case the state of polarization, which is described across the source by the Stokes parameters (37)–(40) and by the local transverse degree of polarization, remain unchanged under propagation through first-order optical systems and, in particular, under free propagation. This suggests a method to measure, in a simple way, the coherence features across the source: Simply measure the distribution of polarization [see Eq. (41)] in the transverse section at any plane z = constant.

4. CONCLUSIONS

In this paper, a general investigation about the coherent-mode decomposition of partially polarized, partially coherent sources has been presented. In doing so, we have applied a formalism based on Hilbert operators to the BCP matrix, which has recently been proposed as a tool for characterizing partially polarized, partially coherent sources. In particular, we have shown that, under very general hypotheses, any partially polarized, partially coherent source can be represented through a superposition of coherent modes with orthogonal polarizations, which have to be determined by solving a system of two coupled integral equations. Such a system decouples in the case of diagonal BCP matrices, so that the problem reduces to two scalar modal decomposition problems.

As a particular case, sources characterized by BCP matrices having GSM diagonal elements and SMI antidiagonal elements have been considered. The beams radiated from these sources are, in general, partially and/or nonuniformly polarized in their transverse section. It turns out that such sources can be expressed as the superposition of fully coherent and linearly polarized Hermite Gaussian modes. As a consequence, the beam keeps the same transverse intensity shape as well as the same local state and degree of polarization under paraxial propagation. Furthermore, it has been shown that the degree of polarization P presents, at any transverse plane, a Gaussian distribution, whose width is related to the coherence features of the source. This fact suggests a method to obtain information about the coherence properties of the source by measuring the sole local degree of polarization at different transverse planes upon free propagation.

APPENDIX A: PROOF OF NONNEGATIVITY

Let us start from the nonnegativity condition (23), i.e.,

$$\begin{aligned}\langle \psi|\hat{J}|\psi\rangle = \iint & \mathrm{d}x_1\mathrm{d}x_2\{[g_1^*(x_1)g_1(x_2) \\ & + g_2^*(x_1)g_2(x_2)]\exp[-\alpha(x_1 - x_2)^2] \\ & + [g_1^*(x_1)g_2(x_2) + g_2^*(x_1)g_1(x_2)] \\ & \times \exp[-\alpha(x_1 + x_2)^2]\}, \qquad (A1)\end{aligned}$$

where we set $g_j(x) = \psi_j(x)\exp(-\beta x^2)$ $(j = 1, 2)$.

If we take into account that

$$\begin{aligned}\exp[-\alpha(x_1 \pm x_2)^2] = & \sqrt{\frac{\pi}{\alpha}}\int \mathrm{d}u \exp\left(-\frac{\pi^2 u^2}{\alpha}\right) \\ & \times \exp[i2\pi(x_1 \pm x_2)u], \qquad (A2)\end{aligned}$$

Eq. (A1) becomes

$$\begin{aligned}\langle \psi|\hat{J}|\psi\rangle = & \sqrt{\frac{\pi}{\alpha}}\int \mathrm{d}u \exp\left(-\frac{\pi^2 u^2}{\alpha}\right) \\ & \times [\tilde{g}_1^*(u)\tilde{g}_1(u) + \tilde{g}_2^*(u)\tilde{g}_2(u) \\ & + \tilde{g}_1^*(u)\tilde{g}_2(-u) + \tilde{g}_2^*(u)\tilde{g}_1(-u)], \qquad (A3)\end{aligned}$$

where $\tilde{g}_j(u)$ $(j = 1, 2)$ denotes the Fourier transform of the function $g_j(x)$ $(j = 1, 2)$. Let us introduce the even and odd parts of the $\tilde{g}_j$ functions, namely,

$$\tilde{g}_j(u) = \tilde{E}_j(u) + \tilde{O}_j(u), \qquad j = 1, 2, \qquad (A4)$$

where, of course, $\tilde{E}_j(-u) = \tilde{E}_j(u)$ and $\tilde{O}_j(-u) = -\tilde{O}_j(u)$. On substituting from Eq. (A4) into Eq. (A3), we eventually obtain, after lengthy but straightforward algebra,

$$\begin{aligned}\langle \psi|\hat{J}|\psi\rangle = & \sqrt{\frac{\pi}{\alpha}}\int \mathrm{d}u \exp\left(-\frac{\pi^2 u^2}{\alpha}\right)[|\tilde{E}_1(u) + \tilde{E}_2(u)|^2 \\ & + |\tilde{D}_1(u) - \tilde{D}_2(u)|^2], \qquad (A5)\end{aligned}$$

which turns out to be positive for any choice of $f_j(x)$.

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