COHERENT RANDOMNESS TESTS AND COMPUTING THE K-TRIVIAL SETS

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Abstract. We show that a Martin-Löf random set for which the effective version of the Lebesgue density theorem fails computes every K-trivial set. Combined with a recent result by Day and Miller, this gives a positive solution to the ML-covering problem (Question 4.6 in Randomness and computability: Open questions. Bull. Symbolic Logic, 12(3):390–410, 2006). On the other hand, we settle stronger variants of the covering problem in the negative. We show that any witness for the solution of the covering problem, namely an incomplete random set which computes all K-trivial sets, must be very close to being Turing complete. For example, such a random set must be LR-hard. Similarly, not every K-trivial set is computed by the two halves of a random set.

The work passes through a notion of randomness which characterises computing K-trivial sets by random sets. This gives a "smart" K-trivial set, all randoms from whom this set is computed have to compute all K-trivial sets.

1. Introduction

Turing reducibility captures the intuitive concept of relative information content. A set B of natural numbers is reducible to another set A, or is *computable from* A, if A has at least as much information as B does. Using this yardstick, a set is considered complicated if it is useful as an oracle, i.e. if it computes many sets.

Algorithmic randomness gives another measure for complexity of sets. A set is considered complicated if it is hard to detect patterns in its characteristic function:[1] if it passes all effective statistical tests. A major programme in the field of algorithmic randomness is to investigate the relationship between these two concepts of complexity. Can random sets be useful as oracles? On the one hand, random sets, lacking any patterns, should be hard to compute; but on the other, since they mostly contain "white noise", they should not be able to compute many sets.

The notion of compressibility is often a conduit between randomness and computability. The (plain) Kolmogorov complexity $C(\sigma)$ of a string σ is, roughly, the length of the shortest computer programme which outputs σ. We say that a string σ is incompressible if its complexity $C(\sigma)$ is close to the length of σ, which roughly says that the only way to describe σ is by writing it down in its entirety. At the other end of the spectrum, a string σ of length n is very compressible if it contains as much information as the string of n zeros; that is, if its complexity is the same as the complexity of its length, $C(\sigma) \sim C(n)$. Chaitin [7] showed that an infinite sequence X is computable if and only if every initial segment of X is very compressible. In this way, computability itself is characterised by compressibility.

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[1] Throughout, we identify a subset of $\mathbb{N}$ with its characteristic function, an element of Cantor space 2^ω.

The most useful notion of randomness, due to Martin-Löf, is not described precisely by plain complexity. Shifting to *prefix-free* Kolmogorov complexity K, which is defined by using self-delimiting machines, allows us to apply measure-theoretic tools to the study of complexity of finite strings. We then have analogous notions of compressibility: a string σ is K-incompressible if $K(\sigma) \sim |\sigma|$, and very K-compressible if $K(\sigma) \sim K(|\sigma|)$. Schnorr ([46], see [6]) showed that an infinite sequence X is ML-random if and only if each initial segment of X is K-incompressible. Solovay, on the other hand, showed that Chaitin's theorem does not hold for K: there are non-computable sequences X, all of whose initial segments are very K-compressible. He called these sequences *K-trivial*. These are the sequences that are as far from being random as possible.

The K-trivial sets turned out to be in central to the investigations into the interactions between computability and randomness. One example is related to an early result of Kučera's [29]: every ML-random set which is Δ^0_2-definable (equivalently, is computable from the halting problem $\varnothing'$) is Turing above a non-computable, computably enumerable set. This is surprising: in general, computably enumerable (c.e.) sets are very far from random and one would expect there is no interaction between these sets and random sets. Kučera's result, though, gives an example of precisely such an interaction. Hirschfeldt, Nies and Stephan [25] showed that the c.e. sets given by Kučera's argument, in the case that the random set is incomplete, must be K-trivial.

A hint that K-triviality is related to computable enumerability was first given by Chaitin [8], who showed that all K-trivial sets are Δ^0_2. Nies [39] extended this result significantly by showing that every K-trivial set is computable from a c.e. K-trivial set. Thus, K-trivial sets are inherently enumerable, and unlike random sets, cannot be constructed by a forcing argument. Nies's result was a corollary to a deep investigation of K-triviality (initiated in [15] and performed in [39]) which clarified the central role played by K-trivial sets. Nies showed that the K-trivial sets are computationally weak, and that they can be characterised by a variety of concepts, beyond compressibility. For example, the K-trivial sets are precisely those which are *low for ML-randomness*: the sets which cannot detect any patterns in (and thus derandomise) ML-random sets.

The result by Hirschfeldt, Nies and Stephan mentioned above did not pertain only to Kučera's construction: they showed that if Y is any ML-random set which is Turing incomplete (i.e. does not compute $\varnothing'$) then every c.e. set computable from Y is K-trivial. In light of this work, Stephan asked whether the converse holds:

> *Is every K-trivial set computed by a ML-random set that is Turing incomplete?*

The inherent enumerability of K-triviality implies that this is indeed a converse to the Hirschfeldt-Nies-Stephan result. Miller and Nies [36, 4.6] included this question, which became known as the *ML covering problem*, as one of the four major questions in their 2006 survey of open problems in algorithmic randomness. Combining the work in this paper with a recent result by Day and Miller [9] (obtained after the research described here was done) gives the affirmative solution to the problem (see [1]). However, we also show here that any ML-random set computing all K-trivial sets must be very close to being Turing complete.

Stephan also asked, as a second part of his question (see [36, 4.6]), whether every K-trivial set is computable from a *low* ML-random set. Further evidence for the plausibility of the existence of such a set was later given by Kučera and Slaman [30] who showed that there is a low PA-complete set which computes all K-trivial sets; on-going work by Kučera gave the impression that techniques of coding into PA-complete sets can often be strengthened to coding into ML-random sets. We answer this question in the negative. We also answer in the negative another strong variant of the ML-covering question asked by Stephan ([36, 4.7]): whether every K-trivial set is computable from both halves of a random set (that is, from both the bits in the even, and in the odd positions). A result which underlies this work is the construction of a "smart" K-trivial set: a K-trivial set A such that every ML-random set which computes A must compute all K-trivial sets.

The solution of the ML-covering problem passes through the analytic concept of density. Let λ denotes Lebesgue measure on $\mathbb{R}$. For measurable sets $P, A \subseteq \mathbb{R}$ with A non-null, $\lambda(P|A) = \lambda(P \cap A)/\lambda(A)$ is the conditional measure (probability) of P given A. Recall that the lower density of a measurable set $P \subseteq \mathbb{R}$ at a point $z \in \mathbb{R}$ is

$$\rho(P|z) = \liminf_{h\to 0} \{\lambda(P|I) : I \text{ is an open interval, } z \in I \;\&\; |I| < h\}.$$

Intuitively, $\rho(P|z)$ gauges the fraction of space filled by P around z if we "zoom in" arbitrarily close to z.

Lebesgue's density theorem [34, p. 407] says that for any measurable set P, for almost all $z \in P$ we have $\rho(P|z) = 1$. An effective version of this theorem is given by identifying a collection of effectively presented sets P and the collection of random point z for which $\rho(P|z) = 1$ for all sets P in the collection containing z as an element. Since the theorem is immediate for open sets, the simplest nontrivial effective version is obtained by choosing P to range over the collection of effectively closed subsets of $\mathbb{R}$. We call a real number $z \in \mathbb{R}$ a *density-one point* if for every effectively closed set P containing z we have $\rho(P|z) = 1$.

The randomness enhancement principle [42] states that beyond Martin-Löf, randomness strength corresponds inversely to proximity to $\varnothing'$. That is, among the ML-random sets, failing stronger randomness properties means being closer to being Turing complete. In light of Lebesgue's density theorem, being a density-one point (in conjunction with ML-randomness) is a randomness property which turns out to be strictly stronger than ML-randomness. An instance of the randomness enhancement principle is a by-product of the work presented here:

Theorem 1.1. *A Martin-Löf random set which is not a density-one point computes every K-trivial set.*

Day and Miller [9] constructed an incomplete ML-random real z which is not a density-one point. Theorem 1.1 says that z computes all K-trivial sets, thereby giving a positive answer to the ML-covering question. We remark that the notion of density plays an even greater role in the proof of the Day-Miller theorem. Franklin and Ng introduced a notion of randomness called *difference randomness* and showed that this notion is equivalent to being ML-random and incomplete. A result of Bienvenu, Hölzl, Miller and Nies [2] says that a ML-random real z is difference random if and only if $\rho(P|z) > 0$ for all effectively closed sets P containing z. Day and Miller's construction produced a ML-random set with the latter property, and they use the Franklin-Ng and the Bienvenu-Hölzl-Miller-Nies results to conclude that the set they produced is Turing incomplete.

Recall that an oracle $Y \in 2^\omega$ is *LR-hard* if every set which is ML-random relative to Y is 2-random, i.e., random relative to $\varnothing'$. Intuitively, such an oracle is "nearly" Turing complete. An instance of a formulation of this intuition is Cole and Simpson's result [49] that every LR-hard set is superhigh ($\varnothing'' \leqslant_{\mathrm{tt}} Y'$). Certainly, no LR-hard set is low ($Y' \equiv_{\mathrm{T}} \varnothing'$). A lower bound on the complexity of a solution to the covering problem is given by the following result, which is yet another instance of the randomness enhancement principle.

Theorem 1.2. *There is a K-trivial set A such that every ML-random set computing A is LR-hard.*

By the *halves* of a set X we mean the sets X_0 and X_1 where $X = X_0 \oplus X_1$; that is, the bits in the even and the bits in the odd positions.

Similar to Theorem 1.2, we show

Theorem 1.3. *There is a K-trivial set which is not computable from both halves of any random set.*

The K-trivial sets given by Theorems 1.3 and 1.2 can be taken to be the same, as the K-trivial sets are closed under join. The similarity between their properties, though, is really due to the fact that a single construction produces a set with both properties. Indeed, this is the "smart" K-trivial set mentioned above.

Theorem 1.4. *There is a K-trivial set A such that any ML-random set computing A also computes all K-trivial sets.*

Theorem 1.4 explains why the eventual solution to the covering problem was strong, in the sense that rather than showing that every K-trivial set is computed by an incomplete random set, it is shown that a single incomplete random set computes all K-trivial sets.

The proofs of all the theorems above are based on a randomness notion slightly stronger than Martin-Löf's.[2] We call this notion *Oberwolfach randomness* in appreciation of our two-week Research in Pairs stay at the Mathematisches Forschungsinstitut Oberwolfach in early 2012, where this research began.

Recall that a *Martin-Löf test* is a sequence $(\mathcal{U}_m)_{m\in\omega}$ of uniformly effectively open sets such that $\lambda(\mathcal{U}_m) \leqslant 2^{-m}$ for each $m \in \omega$. A set $Z \subseteq \omega$ *fails* the test if $Z \in \bigcap_m \mathcal{U}_m$, otherwise Z *passes* the test. Demuth [10] introduced the idea of increasing the power of a ML-test by allowing a computably bounded number of changes to the whole Σ^0_1 set $\mathcal{U}_m$ (see [33] for background). Oberwolfach tests use this idea, but in a very restricted fashion. The changes of components of the test have to be coherent across the levels of the test. Every two successive changes in $\mathcal{U}_{m+1}$ must be accompanied by a change in $\mathcal{U}_m$. In Section 2 we give other classes of tests which capture the same notion of randomness. We can show directly (Proposition 2.5) that Oberwolfach randomness implies difference randomness (which recall is the same as incomplete randomness); we will see that Day and Miller's construction separates between Oberwolfach randomness and difference randomness.

Facts about Oberwolfach randomness, which are of independent interest, are combined to give proofs of the theorems above.

The smart K-trivial set (Theorem 1.4) is given by the conjunction of the following two results:

- Theorem 4.9: there is a K-trivial set which is computable from no Oberwolfach random set.
- Theorem 4.4: every Martin-Löf random set which is not Oberwolfach random computes every K-trivial set.

Similarly, Theorem 1.2 follows from Theorem 4.9 and:

- Theorem 3.1: Every Martin-Löf random set which is not Oberwolfach random is LR-hard.

Theorem 1.3 follows from Theorem 4.9, using the fact (Proposition 4.6), deduced from a result in [17], that for any random set X, at least one of the halves X_0 or X_1 is Oberwolfach random.

[2] We note though, that after learning about Theorem 1.1, Miller proved it directly; see [2].

The proof of Theorem 1.1 follows from Theorem 4.4, together with:

- Corollary 5.10: every Oberwolfach random real is a density-one point.

The proof of Corollary 5.10 passes through other analytic concepts. In fact we give two proofs; both run along similar lines, but use different analytic concepts. The first proof uses martingale convergence. In Section 5 we show (Theorem 5.6) that if $Z \in 2^\omega$ is Oberwolfach random, then for every left-c.e. martingale M the sequence $\langle M(Z\upharpoonright_n)\rangle_{n<\omega}$ has a limit. We then show that this implies Corollary 5.10.

For the second proof we use differentiability of a class of effectively presented functions called *interval-c.e.* functions. A non-decreasing lower semicontinuous function f on $[0,1]$ with $f(0) = 0$ is called interval-c.e. if $f(y) - f(x)$ is a left-c.e. real, uniformly in rational numbers $x < y$. This class of functions was introduced in [20], where the authors show that the continuous interval-c.e. functions are precisely the variation functions of computable functions.

Demuth (again see [33] for background) started the program of analyzing how much randomness of a real z is needed to make effective functions of a certain type differentiable at z, when we know classically that they are differentiable at almost every real. In full generality, this program tries to identify the randomness strength needed to make effective versions of "almost-everywhere" theorems hold; the discussion of Lebesgue density above is an instance of this program, as are recent investigations into the effective content of ergodic theorems. A goal is to characterize known randomness notions by effective versions of classical theorems of analysis. Recent activity followed Demuth's original question and has focussed on differentiability theorems; see for example [5, 45].

We contribute to this program by showing (Theorem 6.8) that every interval-c.e. function is differentiable at every Oberwolfach random point. The proof of this theorem is an extension of the proof of Theorem 5.6, which was concerned with martingale convergence. This is not surprising because we show that in fact differentiability of interval-c.e. functions implies convergence of left-c.e. martingales as described above (Proposition 6.7(i)).

In particular, Theorem 6.8 implies Theorem 5.6. But because Theorem 6.8 is potentially stronger than Theorem 5.6, we can derive Corollary 5.10 from it more easily. If $\mathcal{P}$ is effectively closed then the function $f(x) = \lambda([0,x]\backslash\mathcal{P})$ is interval-c.e. We show in Proposition 6.7(ii) that for ML-random points z, differentiability of f at z implies that the density of $\mathcal{P}$ at z is 1.

We note that the combination of Theorem 3.1 and Corollary 5.10 gives a new proof of a result of Bienvenu et al. [3, Thm. 3.5], which states that a ML-random point which is not a density-one point must be LR-hard.

The ML covering problem is now solved, but the work described suggests possibly more fundamental questions. The characterisation of density-one points within the random reals is still open, as is their relationship to martingale convergence, differentiability, and LR-hardness. For example, we ask:

- Is there a density-one ML random point which is not Oberwolfach random?
- Is there an LR-hard Oberwolfach random set?

2. Oberwolfach randomness

Recall that a G_δ set is the intersection $\bigcap \mathcal{U}_n$ of a nested sequence $\langle \mathcal{U}_n \rangle$ of open sets; nested means that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$. The G_δ set is null if and only if $\lim \lambda(\mathcal{U}_n) = 0$. There are two ways to measure the complexity of such null sets.

- Via definability: an effectiveness condition is placed on the sequence $\langle \mathcal{U}_n \rangle$. In all cases we are concerned with, this results in the intersection being Π^0_2

(effectively G_δ). Most commonly the sequence $\langle \mathcal{U}_n \rangle$ is uniformly Σ^0_1 (effectively open), but it is possible to relax this condition; indeed sometimes the sets $\mathcal{U}_n$ may not be open.
- By calibrating the speed of convergence of $\lambda(\mathcal{U}_n)$ to zero. The most common way is to require that $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$.

We say that a sequence Z is *captured* by a test $\langle \mathcal{U}_n \rangle$ if $Z \in \bigcap \mathcal{U}_n$. Otherwise it *passes* the test. Because the test is nested, this passing condition is equivalent to Solovay's notion of escaping co-finitely many test components $\mathcal{U}_n$.

If $\mathfrak{C}$ is a countable collection of tests, then we say that a real is $\mathfrak{C}$-*random* if it passes every test in $\mathfrak{C}$. For example, a *difference test* (Franklin and Ng [19]) is a nested sequence of classes $\mathcal{U}_n = \mathcal{A}_m \cap \mathcal{B}$ (with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$), where the sequence $\langle \mathcal{A}_m \rangle$ is uniformly Σ^0_1 (effectively open) and the class $\mathcal{B}$ is Π^0_1 (effectively closed). A real is difference random if it passes every difference test. As mentioned in the introduction, Oberwolfach randomness implies difference randomness and is very close to, but distinct from, difference randomness.

The statistical tests that define Oberwolfach randomness can be presented in a variety of ways.

(1) Oberwolfach tests are a "coherent" form of the balanced tests introduced in [17].
(2) Interval tests are uniformly Σ^0_1 classes indexed by rational intervals with certain measure and monotonicity conditions, and a left-c.e. real picking the Σ^0_1 classes that have to be avoided.
(3) Left-c.e. bounded tests are Π^0_2 (effectively G_δ) null classes of the form $\bigcap_n \mathcal{V}_n$, where the convergence to 0 of $\lambda(\mathcal{V}_n)$ is quantified by an additive cost function.

In this section we introduce the three test notions and show they all determine the same randomness class. Each of the three test concepts is intended for a different type of application. (1) is used to build the smart K-trivial set. (2) is mainly needed for the application to differentiability and density; in particular, for showing that effectively closed sets have density one at Oberwolfach random points. (3) is useful to show that any ML-random set that is not Oberwolfach random is close to Turing complete, and to show that every such random set computes all K-trivial sets. We now put some work into introducing these test notions and showing that they are equivalent. Their conceptual closeness to the intended applications will make that work pay off later on.

Remark 2.1. We will work in three computable (metric) measure spaces: Cantor space 2^ω, the unit interval $[0,1]$, and sometimes the real line $\mathbb{R}$. The equivalence of the first two is given by a the "near isomorphism" $\Theta\colon 2^\omega \twoheadrightarrow [0,1]$ given by $\Theta(Z) = \sum_{n<\omega} Z(n) 2^{-n-1}$. The map Θ is computable, continuous and closed, is measure-preserving, and injective when restricted to infinite, co-infinite sets (with image containing all irrational numbers in the unit interval). If $\Theta(Z) = z$ we say that Z is a *binary expansion* of z.

A randomness notion can be defined in any of these spaces, and will usually be invariant. For example, an ML-test is a sequence $\langle \mathcal{U}_n \rangle$ of uniformly effectively open sets with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$. This definition makes sense in both Cantor space, the unit interval and the real line, and so we get a notion of ML-randomness in each of these spaces. Because Θ is computable and measure-preserving, if $\langle \mathcal{U}_n \rangle$ is a ML-test in the unit interval, then $\langle \Theta^{-1} \mathcal{U}_n \rangle$ is an ML-test in Cantor space. In the other direction, let R be the set of sequences in Cantor space which are eventually constant. Then $\Theta \restriction_{2^\omega \setminus R}$ is an open map. If $\langle \mathcal{V}_n \rangle$ is a ML-test in Cantor space, then $\langle \Theta[\mathcal{V}_n \setminus R] \rangle$ is a ML-test in the unit interval. Since $\Theta[R]$ is the set of binary rational numbers, none of which are ML-random, altogether we see that for all $Z \in 2^\omega$, Z is ML-random if and only if $\Theta(Z)$ is ML-random.

We take the same approach when defining Oberwolfach randomness. The test notions we introduce below make sense in every computable probability space, and the argument above will show that for all $Z \in 2^\omega$, Z is Oberwolfach random if and only if $\Theta(Z)$ is Oberwolfach random.

2.1. **Oberwolfach tests.** We introduce tests which are a special case of weak limit tests (and in fact weak Demuth tests); see for instance [32]. In this context, we require that $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$, but the sequence $\langle \mathcal{U}_n \rangle$ need not be given effectively. Let $\langle \mathcal{W}_e \rangle_{e<\omega}$ be an effective list of all Σ^0_1 classes. We are interested in tests of the form $\left\langle \mathcal{W}_{f(n)} \right\rangle$, where $f \leqslant_{\mathrm{T}} \varnothing'$.

A computable approximation $\langle f_s \rangle$ for f gives an approximation for the test. We write:

- $\mathcal{U}_n\langle s \rangle = \mathcal{W}^{(\leqslant 2^{-n})}_{f_s(n)}$, where $\mathcal{W}^{(\leqslant \varepsilon)}_e$ is the result of enumerating $\mathcal{W}_e$ up to the point at which its measure reaches ε; and
- $\mathcal{U}_n[s] = \mathcal{U}_{n,s}\langle s \rangle = \mathcal{W}^{(\leqslant 2^{-n})}_{f_s(n),s}$, where $\mathcal{W}_{e,s}$ is the clopen set which is the result of enumerating $\mathcal{W}_e$ for s steps.

The set $\mathcal{U}_n\langle s \rangle$ is called a *version* of $\mathcal{U}_n$. We can require that at every stage s, $\mathcal{U}_{n+1}\langle s \rangle \subseteq \mathcal{U}_n\langle s \rangle$. We say that the version of $\mathcal{U}_n$ *changes* at a stage s if $f_s(n) \neq f_{s-1}(n)$. We write, though, $\mathcal{U}_n\langle s-1 \rangle \neq \mathcal{U}_n\langle s \rangle$ in this event, even if it is not technically true. That is, a version changes if its description (its index) changes, even if extensionally, the Σ^0_1 classes described are the same.

To be pedantic, the test $\langle \mathcal{U}_n \rangle$ does not contain all the information above; different choices of f and of the approximation $\langle f_s \rangle$ for f may yield the same test. Below, we always assume that a test comes with its approximation.

For background, we recall the following.

Definition 2.2. A test $\langle \mathcal{U}_n \rangle = \left\langle \mathcal{W}_{f(n)} \right\rangle$ is a *weak Demuth test* if the index function f is ω-c.a.: the number of stages s at which the version $\mathcal{U}_n\langle s \rangle$ of $\mathcal{U}_n$ changes is bounded by a computable function. If this computable bound is $O(2^n)$, then the test is called a *balanced test* [17].

In [17, Rmk. 18] it is shown that imposing the bound 2^n on the number of version changes of the n-th component results in the same notion of randomness, *balanced randomness*.

An Oberwolfach test is a balanced test for which the changes are coherent between the levels.

Definition 2.3. A weak Demuth test $\langle \mathcal{U}_n \rangle$ is an *Oberwolfach test* if for all n, for every interval I of stages, if $\mathcal{U}_n\langle s \rangle$ is constant on I, then there is at most one stage s in I at which $\mathcal{U}_{n+1}\langle s \rangle$ changes.

It is easily observed that every Oberwolfach test is a balanced test. Hence:

Proposition 2.4. *Every balanced random set is Oberwolfach random.*

The notions do not coincide: in [17] the authors construct a low ML-random set which is not balanced random. Such a set must be Oberwolfach random by Theorem 3.1 below.

Franklin and Ng [19] showed that difference randomness is also captured by the class of "version-disjoint" weak Demuth tests. In fact, these tests are naturally Oberwolfach tests. To wit, if Z is ML-random and not difference random, then it is Turing complete. So it computes Chaitin's complete random set Ω. Let Γ be a Turing functional such that $\Gamma(Z) = \Omega$. By a result of Levin [35], and

Miller and Yu [37] (also see [40, Prop. 5.1.14]), there is a constant c such that $2^{-m} \geqslant \lambda\{Z : \Omega\upharpoonright_{m+c} \prec \Gamma(Z)\}$ for each m. The version-disjoint weak Demuth test capturing Z defined by Franklin and Ng is defined by letting

$$\mathcal{U}_m\langle s\rangle = \{Z : \Omega_s\upharpoonright_{m+c} \prec \Gamma(Z)\}^{(\leqslant 2^{-n})}.$$

This test is in fact an Oberwolfach test, since two changes in $\Omega_t\upharpoonright_{n+1}$ necessitate a change in $\Omega_t\upharpoonright_n$. To sum up, difference randomness is captured by so-called "version-disjoint" Oberwolfach randomness. Hence:

Proposition 2.5. *Every Oberwolfach random set is difference random.*

The notions do not coincide. This follows from Day and Miller's construction [9] of a difference random real which is not a density-one point. Corollary 5.10 below shows this real is not Oberwolfach random.

Below, when considering an Oberwolfach test, we often assume without further mention that $\mathcal{U}_0\langle s\rangle$ never changes (we can simply start the test late enough). Also note that by delaying enumerations, we may assume that for all s, for all $n \geqslant s$, $\mathcal{U}_n[s] = \varnothing$.

2.2. **Interval tests.** The very general notion of a statistical test that we defined can be in fact further generalised, by replacing the natural numbers by indices coming from some partial ordering, and slightly less generally, from a filter in a separative partial ordering. To avoid excess abstraction, we consider a useful collection of such generalised tests. They will be useful for our intended application of Oberwolfach randomness in effective analysis in Section 6.

In this section, let $\mathcal{X}, \mathcal{Y} \in \{2^\omega, [0,1]\}$, considered as computable probability spaces. A *rational open ball* in 2^ω is a sub-basic clopen subset of the form $[\sigma]$ for some $\sigma \in 2^{<\omega}$, and in $[0,1]$ is an open interval with rational endpoints (including $[0,a)$ and $(b,1]$).

Definition 2.6. An *interval array* (in $\mathcal{Y}$, indexed by $\mathcal{X}$) is a effective map G from the collection of rational balls in $\mathcal{X}$ to the effectively open subsets of $\mathcal{Y}$ such that:

(a) For all I, $\lambda(G(I)) \leqslant \lambda(I)$; and
(b) If $I \subseteq J$ then $G(I) \subseteq G(J)$.

An *interval test* consists of an interval array G and a left-c.e. real $\alpha \in \mathcal{X}$. The set of reals in $\mathcal{Y}$ which are captured by the test (G, α) is

$$\bigcap_{\alpha \in I} G(I).$$

In the case $\mathcal{X} = 2^\omega$, an interval array G is an effective mapping $\sigma \mapsto \mathcal{G}_\sigma$ such that $\lambda(\mathcal{G}_\sigma) \leqslant 2^{-|\sigma|}$ and $\mathcal{G}_\tau \subseteq \mathcal{G}_\sigma$ if τ extends σ. The set of reals captured by an interval test (G, α) is $\bigcap_n \mathcal{G}_{\alpha\upharpoonright_n}$. It is not hard to see that in fact in this case, $\langle \mathcal{G}_{\alpha\upharpoonright_n}\rangle$ is an Oberwolfach test. Below we will see that all Oberwolfach tests are of this form.

Remark 2.7. It is sometimes convenient to extend an interval array to be defined on all open subsets of $\mathcal{X}$. If G is an interval array, then for open $\mathcal{U} \subseteq \mathcal{X}$ we let

$$G(\mathcal{U}) = \bigcup G(I)[\![I \text{ is a rational open ball contained in } \mathcal{U}]\!].$$

This function certainly extends G, and satisfies conditions (a) and (b) from Definition 2.6. The reason that (a) holds is that every open subset $\mathcal{U}$ of 2^ω equals the disjoint union of the maximal rational balls contained in $\mathcal{U}$; and that every open subset of $[0,1]$ is the disjoint union of the maximal open intervals contained in $\mathcal{U}$, while every open interval in $[0,1]$ is the increasing union of its rational subintervals. We note that if (G,α) is an interval test, then the reals captured by (G,α) are precisely the reals Z such that $Z \in G(\mathcal{U})$ for all open subsets $\mathcal{U}$ of $\mathcal{X}$ containing α.

Remark 2.8. In Theorem 6.8 below we make use of the fact that in (a) of Definition 2.6 for the case $\mathcal{X} = [0,1]$ we could also require the weaker condition that $\lambda(G(I)) \leqslant D\lambda(I)$ for some constant $D \in \mathbb{Q}^+$ while retaining the same randomness notion. For let I^* be a rational interval of length $< 1/D$ containing α. We only need to consider $G(J)$ for subintervals J of I^*. Let f be the increasing linear map sending I^* to $[0,1]$. We define a new interval array by $\widetilde{G}(I) = G(f^{-1}(I))$. Then the new array satisfies (a), and the test $(\widetilde{G}, f(\alpha))$ captures the same reals as (G,α).

Proposition 2.9. *Every real which is not Oberwolfach random is captured by an interval test indexed by* 2^ω.

The converse of Proposition 2.9 is proved in the next section.

Proof. Let $\mathcal{U}_n\langle s\rangle$ be an approximation for an Oberwolfach test. Let $\alpha_0 = 0^\omega$. Inductively define α_s by letting $\alpha_s(n) = 1$ for the least n such that $\mathcal{U}_n\langle s\rangle$ changes at stage s; $\alpha_s \upharpoonright_n = \alpha_{s-1} \upharpoonright_n$ and $\alpha_s(m) = 0$ for $m > n$. If no version changes at stage s then $\alpha_s = \alpha_{s-1}$. We then let $\alpha = \lim \alpha_s$ and define $\mathcal{G}_\sigma$ to be empty until we see a stage s at which σ is an initial segment of α_s; then we let $\mathcal{G}_\sigma = \mathcal{U}_{|\sigma|}\langle s\rangle$. Then $(\sigma \mapsto \mathcal{G}_\sigma, \alpha)$ is an interval test which captures the same reals captured by $\langle \mathcal{U}_n\rangle$. □

Proposition 2.10. *Every real which is captured by an interval test indexed in* 2^ω *is also captured by an interval test indexed in* $[0,1]$.

Proof. Let $(G = \langle \mathcal{G}_\sigma\rangle, \alpha)$ be an interval test indexed by 2^ω. We extend it to all open subsets of 2^ω, as in Remark 2.7. Let $\Theta\colon 2^\omega \to [0,1]$ be the canonical near-isomorphism (Remark 2.1). We push the array $\langle G(\mathcal{U})\rangle$ forward by Θ: we let $(\Theta_* G)(I) = G_{\Theta^{-1}I}$. Then $\Theta_* G$ is an interval array indexed in $[0,1]$, and $(\Theta_* G, \Theta(\alpha))$ captures every real captured by (G,α). □

2.3. **Cost functions.** The third test notion which captures Oberwolfach randomness uses the notion of an additive cost function. We review relevant material concerning cost functions.

As in [40, Section 5.3], a *cost function* is a computable function

$$\mathbf{c} : \omega \times \omega \to \{x \in \mathbb{Q} \colon\ x \geqslant 0\}.$$

We say $\mathbf{c}$ is *monotonic* if $\mathbf{c}(x+1,s) \leqslant \mathbf{c}(x,s) \leqslant \mathbf{c}(x,s+1)$ for each $x < s$. In this paper, all cost functions we encounter will be monotonic, and so we omit mentioning this adjective from now on.

When building a computable approximation of a Δ^0_2 set A, we view $\mathbf{c}(x,s)$ as the cost of changing $A(x)$ at stage s. We also write $\mathbf{c}_s(x)$ instead of $\mathbf{c}(x,s)$ to indicate it is the cost of a change at x at stage s. We can then express that the *total* cost of changes, taken over all x, is finite [40, Section 5.3]. We say that a computable approximation $\langle A_s\rangle_{s\in\omega}$ *obeys* a cost function $\mathbf{c}$ if

$$\infty > \sum_{x,s} \{\mathbf{c}_s(x) \colon\ x < s \wedge x \text{ is least such that } A_{s-1}(x) \neq A_s(x)\}.$$

We say that a Δ^0_2 set A obeys $\mathbf{c}$ if *some* computable approximation of A obeys $\mathbf{c}$.

We write $\mathbf{c}(x) = \sup_s \mathbf{c}(x,s)$ and call $\mathbf{c}(x)$ a *limit cost function*. We say that a cost function $\mathbf{c}$ fulfills the *limit condition* if $\lim_x \mathbf{c}(x) = 0$. The by-now classic cost-function construction states that every cost function with the limit condition is obeyed by some promptly simple c.e. set. The cost function construction originated in [31, 15] and was formulated in the present generality first in [40, Section 5.3]. Again, all cost functions considered in this paper satisfy the limit condition, and so we omit mentioning it below.

Let $g\colon \omega \to \omega$. A cost function $\mathbf{c}$ is called *g-benign* if $g(n)$ bounds the length of any finite sequence $x_0 < x_1 < \ldots < x_k$ such that $\mathbf{c}(x_i, x_{i+1}) \geqslant 2^{-n}$ for each $i < k$. A cost function is benign if it is g-benign for some computable function g.

The following was defined in [41].

Definition 2.11. We call a cost function $\mathbf{c}$ *additive* if $x < y < t$ implies $\mathbf{c}(x,t) = \mathbf{c}(x,y) + \mathbf{c}(y,t)$.

We see that if $\mathbf{c}$ is additive then $\mathbf{c}(x,t) = \sum_{a\in[x,t)} \mathbf{c}(a, a+1)$. Thus the additive cost functions are of the form $\mathbf{c}_\beta(x,s) = \beta_s - \beta_x$ for some left-c.e. real $\beta \in [0,\infty)$ (simply let $\beta_s = \mathbf{c}(0,s)$). We note that every additive cost function is $o(2^n)$-benign.

Obedience to cost functions characterises lowness classes by results in [39, 23, 11].

Theorem 2.12.

(1) *A set is K-trivial if and only if it obeys every additive cost function if and only if it obeys the cost function $\mathbf{c}_\alpha$ for some left-c.e. random real α.*
(2) *A set is strongly jump-traceable if and only if it obeys every benign cost function.*

2.4. **Left-c.e. bounded tests.** For Oberwolfach tests (as well as Martin-Löf, weak Demuth and limit tests) $\langle \mathcal{U}_n \rangle$ we require that $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$. As we mentioned above, more general notions of tests (such as weak 2-random tests) allow $\lambda(\mathcal{U}_n)$ to approach 0 more slowly. In this section we require that $\langle \mathcal{U}_n \rangle$ is a uniformly Σ^0_1 sequence (that is, $\mathcal{U}_n = \mathcal{W}_{f(n)}$ for a computable function f), but allow $\lambda(\mathcal{U}_n)$ to approach 0 more slowly than computable functions do. The speed at which $\lambda(\mathcal{U}_n)$ tends to 0 is calibrated by *cost functions* discussed in Subsection 2.3. Recall that we assume the limit condition $\mathbf{c}(n) \to 0$ for all cost functions $\mathbf{c}$.

Definition 2.13. Let $\mathbf{c}$ be a limit cost function. A (uniformly Σ^0_1) test $\langle \mathcal{V}_n \rangle$ is a *$\mathbf{c}$-test* if $\lambda(\mathcal{V}_n) \leqslant \mathbf{c}(n)$ for all n.

Definition 2.14. A *left-c.e. bounded test* is a $\mathbf{c}$-test for $\mathbf{c}$ the limit of an additive cost function.

Thus, a left-c.e. bounded test is a nested sequence $\langle \mathcal{V}_n \rangle$ of uniformly Σ^0_1 classes such that for some left-c.e. approximation $\langle \beta_s \rangle$ of a real β we have $\lambda(\mathcal{V}_n) \leqslant \beta - \beta_n$ for all n. By delaying enumeration into $\mathcal{V}_n$, we may assume that $\lambda(\mathcal{V}_n[s]) \leqslant \beta_s - \beta_n$ for all n and s. Below we will assume this throughout.

Proposition 2.15. *The following are equivalent for a real Z:*

(1) *Z is captured by some Oberwolfach test.*
(2) *Z is captured by some interval test.*
(3) *Z is captured by some left-c.e. bounded test.*

That is, Oberwolfach, interval and left-c.e. bounded randomness coincide.

For comparison, a set is weakly Demuth random if and only if it passes every $\mathbf{c}$-test when $\mathbf{c}$ is benign, and is balanced random if and only if it passes every $\mathbf{c}$-test when $\mathbf{c}$ is 2^n-benign. On the other hand, a set is ML-random if and only if it passes every $\mathbf{c}_\beta$-test for additive cost function c_β where the real β is computable.

Proof. (1)$\Longrightarrow$(2): This is Proposition 2.9.

(2)$\Longrightarrow$(3): Using Proposition 2.10, we may assume that Z is captured by an interval test (G, α) indexed by $[0,1]$. We may assume that G is defined on all open subsets of $[0,1]$ (Remark 2.7). We let I_n be the open interval $(\alpha_n, \alpha + 2^{-n})$, which is Σ^0_1, and let $\mathcal{V}_n = G(I_n)$. Then

$$\lambda(\mathcal{V}_n) \leqslant \lambda(I_n) = \alpha - (\alpha_n - 2^{-n}),$$

and we note that $\langle \alpha_s - 2^{-s} \rangle$ is also a left-c.e. approximation of α. So $\langle \mathcal{V}_n \rangle$ is a left-c.e. bounded test. If Z is captured by (G, α), then, as observed in Remark 2.7 above, $Z \in G(\mathcal{U})$ for all open $\mathcal{U}$ containing α, and so Z is captured by $\langle \mathcal{V}_n \rangle$.

(3)$\Longrightarrow$(1): Let $\langle \mathcal{V}_n \rangle$ be a left-c.e. bounded test, with $\lambda(\mathcal{V}_n) \leqslant \alpha - \alpha_n$ for some left-c.e. real α, which we may assume is irrational and lies in the open interval $(0,1)$. For all $n < \omega$ and $s \leqslant \omega$, we let $k_s(n)$ be the greatest integer k such that $\alpha_s \geqslant k/2^{-n}$; and we let $t_s(n)$ be the least stage $t \leqslant s$ such that $\alpha_t \geqslant k_s(n)/2^{-n}$. We let

$$\mathcal{U}_n\langle s \rangle = \mathcal{V}_{t_s(n)}^{(\leqslant 2^{-n})}.$$

It is easy to see that $\mathcal{U}_n$ is an Oberwolfach test, that $t_\omega(n)$ is a non-decreasing and unbounded sequence, and that $\mathcal{U}_n = \mathcal{V}_{t_\omega(n)}$. Because $\langle \mathcal{V}_n \rangle$ is nested, so is $\langle \mathcal{U}_n \rangle$, and $\bigcap \mathcal{U}_n = \bigcap \mathcal{V}_n$. $\square$

3. A Martin-Löf random set which is not Oberwolfach random is LR-hard

In this section we show that a Martin-Löf random set that is not Oberwolfach random is close to Turing complete. We provide two formal interpretations of the latter condition. The first is being LR-hard as discussed in the introduction. The second is tracing every partial computable function relative to $\varnothing'$, where the size of the n-th tracing set is bounded by $2^{K(n)}$. Note that usually trace bounds are computable. In our case, the bound is merely upper semicomputable. We also discuss existence of Turing incomplete sets that are close to Turing complete in the second sense. Interestingly, these cannot be obtained through pseudo-jump inversion in the sense of Jockusch and Shore.

3.1. LR-hardness.

Theorem 3.1. *Suppose that a Martin-Löf random set Y is not Oberwolfach random. Then Y is LR-hard.*

As mentioned in the introduction, this theorem improves a result of [3] where the hypothesis was that Y is not a density-one point. The proof relies on the technique used to prove this earlier result.

Proof. Dobrinen and Simpson [12] called a set X almost everywhere (a.e.) dominating if for almost every oracle B, every function $g \leqslant_{\mathrm{T}} B$ is dominated by some function $h \leqslant_{\mathrm{T}} Y$. Kjos-Hanssen, Miller and Solomon [26, 28] proved that X is LR-hard iff X is a.e. dominating.

Now suppose that Y is not LR-hard. As in [3], there is a positive measure class of oracles B such that some function $g \leqslant_{\mathrm{T}} B$ is not dominated by any function $h \leqslant_{\mathrm{T}} Y$. Pick a set B in this class such that B is ML-random relative to Y. By van Lambalgen's theorem, Y is ML-random relative to B.

Suppose also that there is a left-c.e. bounded test $\langle \mathcal{V}_x \rangle$ capturing Y, with $\lambda \mathcal{V}_x \leqslant \beta - \beta_x$ for some left-c.e. real β. We show that Y is not ML-random relative to B, which is a contradiction.

We define a function $f \leqslant_{\mathrm{T}} Y$. Let $f(0)$ be the least s such that $Y \in \mathcal{V}_0[s]$. If $f(n)$ has been defined, let $f(n+1)$ be the least $s > \max(f(n), n)$ such that $Y \in \mathcal{V}_{f(n)}[s]$.

Fix a function $g \leqslant_T B$ such that $\exists^\infty n\, g(n) > f(n+1)$.
Case 1: $g(n) > f(n+1)$ for almost all n. Let $h(r) = g^{(r)}(0)$, and let

$$\mathcal{S}_r = \mathcal{V}_{h(r)}[h(r+1)].$$

For almost every r, $h(r+1) = g(h(r)) > f(h(r)+1)$, so $Y \in \mathcal{V}_{f(h(r))}[h(r+1)]$ by definition. By construction, for every $r > 0$, $h(r) \leqslant f(h(r))$. So

$$\mathcal{S}_r \supseteq \mathcal{V}_{f(h(r))}[h(r+1)],$$

and $Y \in \mathcal{S}_r$ for almost every r. $\sum_r \lambda \mathcal{S}_r \leqslant \sum_r \beta_{h(r+1)} - \beta_{h(r)} = \beta - \beta_{h(0)}$, so $\langle \mathcal{S}_r \rangle$ is a Solovay test. Hence, Y is not Martin-Löf random relative to B.

Case 2: Otherwise. Then there are infinitely many n such that $g(n) \leqslant f(n+1)$ and $f(n+2) \leqslant g(n+1)$. Let instead

$$\mathcal{S}_n = \mathcal{V}_{g(n)}[g(n+1)].$$

For such n,

$$\mathcal{S}_n \supseteq \mathcal{V}_{f(n+1)}[f(n+2)],$$

so $Y \in \mathcal{S}_n$. Again, $\langle \mathcal{S}_n \rangle$ is a Solovay test, so Y is not Martin-Löf random relative to B. □

3.2. **JT-hardness for upper c.e. bounds.** Let $h\colon \omega \to \omega - \{0\}$. We say that an oracle Y is h-JT-hard if every function f that is partial computable in $\varnothing'$ has an Y-c.e. trace $\langle T_x \rangle$ that is bounded by h. That is, $|T_x| \leqslant h(x)$, and $f(x)\downarrow$ implies $f(x) \in T_x$.

In the following we show that if Y is a Martin-Löf random set that is not Oberwolfach random, then Y is h-JT hard for functions h such as $h(n) = 2^{K(n)}$. We use the following "measure-bounding" lemma, which reveals a salient property of Oberwolfach randomness. Although stated for left-c.e. bounded tests, it isolates the key difference between Oberwolfach tests and balanced tests: in the former, the opponent cannot let small components of the test "gang up" and amass much measure.

Lemma 3.2. *Let $\langle \mathcal{V}_n \rangle$ be a left-c.e. bounded test, where $\lambda(\mathcal{V}_n) \leqslant \alpha - \alpha_n$ for some left-c.e. real α. Suppose $\langle t_i \rangle$ is an increasing sequence in ω with $t_0 = 0$. Consider the sets $\mathcal{V}_{t_0}[t_1], \mathcal{V}_{t_1}[t_2], \dots$, and let $\mathcal{W}(k)$ be the $\mathbf{\Sigma^0_1}$ class consisting of reals Y which occur in at least k of these sets. Then $\lambda(\mathcal{W}(k)) \leqslant \alpha/k$.*

Proof. For all $i \in \omega$, let p_i be the characteristic function of $\mathcal{V}_{t_i}[t_{i+1}]$. Let $p = \sum_i p_i$. Then $Y \in \mathcal{W}(k) \iff p(Y) \geqslant k$. Also,

$$\begin{aligned} \int p \,\mathrm{d}\lambda &= \sum_i \int p_i \,\mathrm{d}\lambda \\ &= \sum_i \lambda \mathcal{V}_{t_i}[t_{i+1}] \\ &\leqslant \sum_i \alpha_{t_{i+1}} - \alpha_{t_i} \\ &= \alpha. \end{aligned}$$

Since p is non-negative, $\int p \,\mathrm{d}\lambda \geqslant k \cdot \lambda \mathcal{W}(k)$. The lemma follows. □

A function $h\colon \omega \to \omega$ is called *upper c.e.* if it has a computable approximation $h(x) = \lim_s h_s(x)$ with $h_s(x) \geqslant h_{s+1}(x)$ for each x, s.

Theorem 3.3. *Let $h\colon \omega \to \omega - \{0\}$ be an upper c.e. function such that $\sum_k 1/h(k) < \infty$. Suppose that a Martin-Löf random set Y is not Oberwolfach random. Then Y is h-JT-hard.*

Note that by the machine existence theorem (see e.g. [40, Theorem 2.2.17]), for an upper c.e. function h, the hypothesis $\sum_k 1/h(k) < \infty$ is equivalent to $2^{K(n)} = O(h(n))$. Note, though, that traceability is not invariant to multiplying bounds by constants.

The idea for the following proof originates in [17, Theorem 23].

Proof. Let f be any partial $\varnothing'$-computable function. We let $\langle f_s \rangle$ be a computable sequence of total functions converging to f in a Σ^0_2-fashion. That is, if $f(n)\downarrow$, then $\lim_s f_s(n) = f(n)$, while if $f(n)\uparrow$, then $\lim_s f_s(n)$ does not exist.

Fix a left-c.e. bounded test $\langle \mathcal{V}_n \rangle$ with associated left c.e. real α which Y fails. First we construct a sequence of c.e. operators $\langle T_k \rangle$. Then we will verify that $\langle T^Y_k \rangle$ eventually traces f and has size bounded by h.

In our strategy for constructing T_k, we keep an auxiliary value $t(k, s)$. We begin by defining $t(k, 0) = 0$. When not otherwise defined at the end of stage s, we define $t(k, s + 1) = t(k, s)$. In the following, s is always the current stage.

(1) While $f_s(k) = f_{t(k,s)}(k)$, enumerate $f_s(k)$ into T^Y_k for all $Y \in \mathcal{V}_{t(k,s)}[s]$.
(2) When $f_s(k) \neq f_{t(k,s)}(k)$, define $t(k, s + 1) = s$.
(3) Return to step 1.

Now, suppose $f(k)\downarrow$. Then the $f_s(k)$ $(s \in \omega)$ take on only finitely many values, and so the strategy for T_k reaches step 2 only finitely many times. Let $t(k) = \lim_s t(k, s)$. Once $f(k)$ has converged and Y has entered $\mathcal{V}_{t(k)}[s]$, $f(x)$ will be enumerated into T^Y_k. So $\langle T^Y_k \rangle_{k\in\omega}$ traces f.

Finally, we must show that $|T^Y_k| \leqslant h(k)$. We let $\mathcal{W}_k$ be the Σ^0_1 class consisting of all reals X with $\exists s\, |T^X_{k,s}| > h_s(k)$. Let $\langle t_i \rangle$ be the sequence of values $t(k, s)$ takes as s ranges over all stages of the construction. Now, note that while $t(k, s)$ is constant, $f_s(k)$ is constant. So to enter $\mathcal{W}_k$, X must enter at least $h(k)$ different $\mathcal{V}_{t_i}[t_{i+1}]$. By the measure-bounding Lemma 3.2, we thus know that $\lambda(\mathcal{W}_k) \leqslant \alpha/h(k)$. By assumption, $\sum_k \alpha/h(k) = \alpha \sum_k 1/h(k) < \infty$, so since Y is Martin-Löf random, Y must occur in only finitely many of the $\mathcal{W}_k$. So $|T^Y_k| \leqslant h(k)$ for all but finitely many k. □

3.3. **Discussion.** By random pseudo-jump inversion (Kučera, Nies, independently; see [47] and [40, Cor. 6.3.9], respectively), it is known that there is an incomplete, LR-hard ML-random set, which can in fact be chosen to be Δ^0_2. By Theorem 3.3 and the Day-Miller theorem [9] that some difference random real is not a density-one point, we now know that there is an incomplete, Δ^0_2 random set satisfying the highness condition of Theorem 3.3 – being $\alpha 2^{K(n)}$-JT-hard for all rational $\alpha > 0$. However, we do not know at present how to directly construct such a ML-random set. We can directly build such a Δ^0_2 set if we discard the requirement to be ML-random.

Proposition 3.4 (with Hirschfeldt)**.** *There is an incomplete Δ^0_2 set which is h-JT hard for some function $h = o(2^{K(n)})$.*

Proof. By the existence of compression functions in the sense of [44] and its extension to K in [40, 3.6.16], there is a low set A and a function $h \leqslant_{\mathrm{T}} A$ such that $h(n) = o(2^{K(n)})$ but $\lim h(n) = \infty$. For every order function g there is a non-computable c.e. set which is g-jump-traceable. In fact, the requirement that the bound function g be monotone is not necessary; all we need is that $\lim g(n) = \infty$. Relativizing this fact to A, and then to every $B \geqslant_{\mathrm{T}} A$, we get an A-computable pseudo-jump operator W such that $W(A \oplus X)$ is h-jump traceable in, and properly Turing above, $A \oplus X$ for each set X. Now relativize the usual Jockusch-Shore pseudo-jump inversion to A, and obtain $Y <_T \varnothing'$ such that $W(A \oplus Y) \equiv_T A' \equiv \varnothing'$. The set $Y \oplus A$ is as required. □

Note that the set Y constructed in this proof can be chosen to be ML-random in A, by using the relativisation to A of the ML-random pseudo-jump inversion theorem mentioned above. So if we start with a ML-random set A then $Y \oplus A$ is ML-random. However, we do not know whether there is a low ML-random computing a compression function for K. If we replace K by plain complexity C, then we see that in fact there is no low ML-random set A computing a compression function for C, because such a function would be of PA degree by [27, Theorem 4.1].

For a c.e. set Y, the highness condition actually implies Turing completeness. Because of the pseudo-jump inversion theorems, it is not common to see a highness property shared by some incomplete random sets but not by some incomplete c.e. sets.

Proposition 3.5 (F. Stephan)**.** *Suppose a c.e. set Y is $2^{K(n)}$-JT hard. Then $\varnothing' \leqslant_{\mathrm{T}} Y$.*

Proof. Assume for a contradiction that Y is $2^{K(n)}$-JT hard via a Y-c.e. trace $\langle T_x^Y \rangle$, but $\varnothing' \nleqslant_{\mathrm{T}} Y$. Then the size of the T_x^Y is unbounded in x. Define a Turing functional Γ^Z as follows: given n, search for p and stage s such that $|T_{p,s}^Z| > 2^{2n}$ and output p at that stage.

There is a prefix free machine that, if n enters $\varnothing'$ at stage s and $p = \Gamma_s^{Y_s}(n)\downarrow$, ensures $K(p) \leqslant n + O(1)$. We show $\varnothing' \leqslant_{\mathrm{T}} Y$. Given n, since Y is c.e., we can, using Y as an oracle, compute s such that $\Gamma_s^{Y_s}(n)\downarrow$ with Y stable on the use. Then $n \in \varnothing' \leftrightarrow n \in \varnothing'_s$. □

In particular, $\varnothing' \leqslant_{LR} Y$ does not in general imply that Y is $2^{K(n)}$-JT hard, so Theorems 3.1 and 3.3 are independent. By a result of [28], $\varnothing' \leqslant_{LR} Y$ implies that Y is h-JT hard for any *computable* function h with $\sum_n 1/h(n)$ finite (also see [40, Theorem 8.4.15]). A closer look at the proof reveals that the weaker hypothesis that h be computable from both Y and $\varnothing'$ is sufficient. For instance, if Y is Δ_2^0 we could let $h(n) = 2^{K^g(n)}$ where $g \leqslant_{\mathrm{T}} Y$ is a time bound.

4. Oberwolfach randomness and computing K-trivial sets

4.1. **Sets that are Martin-Löf, but not Oberwolfach random, compute all K-trivial sets.**

Our goal in this subsection is to show that every K-trivial is computed by every Martin-Löf random set Z that is not Oberwolfach random.

Proposition 4.1. *Every c.e. K-trivial set A obeys every additive cost function.*

In fact, every K-trivial set, whether c.e. or not, obeys every additive cost function [43]; but this fact relies on the golden run method in the form of the Main Lemma [40, 5.5.1]. For the c.e. case, a short direct proof that every K-trivial obeys the standard cost function $\mathbf{c}_{\mathcal{K}}$ was first given in [40, Thm. 5.3.27]. It can be easily adapted to the case of additive cost function (see [43]). To be self-contained, we give here a direct short proof.

Proof. Let $\mathbf{c}$ be an additive cost function; $\mathbf{c} = \mathbf{c}_\beta$ for some left-c.e. real β, and without loss of generality, $0 < \beta < 1$. Let $\langle \beta_s \rangle$ be a left-c.e. approximation for β so that $\beta_0 = 0$ and $\beta_s < \beta_{s+1}$ for all s.

Let $f(s) = -\log(\beta_s - \beta_{s-1})$. Because $\sum 2^{-f(s)} = \beta$, we have $K \leqslant^+ f$. Let A be K-trivial; so $K(A \upharpoonright_n) \leqslant^+ K(n)$, and together, $K(A \upharpoonright_n) \leqslant f(n) + b$ for some constant b. By speeding up, we can find an enumeration $\langle A_s \rangle$ of A so that for all $n \leqslant s$,

$$K_s(A_s \upharpoonright_n) \leqslant f(n) + b.$$

For each s, let x_s be the least x such that $A_s(x) \neq A_{s-1}(x)$ if such x exists, otherwise $x_s = s$. Let $S = \{s : x_s < s\}$. Note that if $x_s \geqslant s$ then $\mathbf{c}(x_s, s) = 0$. So we need to show that $\sum_{s \in S} (\beta_s - \beta_{x_s})$ is finite.

For $s \in S$, let $T_s = \{\sigma \prec A_s : x_s < |\sigma| \leqslant s\}$. Since $K(\sigma) \leqslant f(|\sigma|) + b$ for all $\sigma \in T_s$, we have

$$\sum_{\sigma \in T_s} 2^{-K(\sigma)} \geqslant \sum_{n = x_s + 1}^{s} 2^{-f(n)+b} = 2^{-b}(\beta_s - \beta_{x_s}).$$

The definition of x_s shows that the sets T_s are pairwise disjoint, and so

$$\sum_{s \in S} \sum_{\sigma \in T_s} 2^{-K(\sigma)} < \Omega. \qquad \square$$

As already mentioned in the introduction, Kučera [29] showed that every Δ^0_2 ML-random is Turing above an incomputable c.e. set. We use a key fact which comes from a result by Hirschfeldt-Miller (see [40, 5.3.15]) extending Kučera's argument. Again, for being self-contained, we give a proof using our notation.

Proposition 4.2. *Let $\mathbf{c}$ be a cost function, and let $\langle \mathcal{V}_n \rangle$ be a $\mathbf{c}$-test. If A is a Δ^0_2 set which obeys $\mathbf{c}$, then A is computable from every ML-random set in $\bigcap_n \mathcal{V}_n$.*

Proof. Define a functional Γ by letting $\Gamma^X(n) = A_s(n)$ for all $X \in \mathcal{V}_{n,s} - \mathcal{V}_{n,s-1}$. Here $\langle A_s \rangle$ is an enumeration of A which witnesses that A obeys $\mathbf{c}$, and $\langle \mathcal{V}_{n,s} \rangle$ is an enumeration of $\mathcal{V}_n$ so that for all s, $\lambda(\mathcal{V}_{n,s}) \leqslant \mathbf{c}_s(n)$.

Certainly, if Z is captured by $\langle \mathcal{V}_n \rangle$ then Γ^Z is total. We show that if Z is captured by $\langle \mathcal{V}_n \rangle$ but Z is ML-random, then $\Gamma^Z(n) = A(n)$ for all but finitely many n.

Let x_s be the least x such that $A_s(x) \neq A_{s-1}(x)$ if such x exists, otherwise $x_s = s$. So $\sum_s \mathbf{c}_s(x_s)$ is finite. Now consider the sequence $\langle \mathcal{V}_{x_s,s} \rangle$. For all s, we have $\lambda(\mathcal{V}_{x_s,s}) \leqslant \mathbf{c}_s(x_s)$, and so $\sum \lambda(\mathcal{V}_{x_s,s})$ is finite. That is, $\langle \mathcal{V}_{x_s,s} \rangle$ is a Solovay test.

Suppose that $\Gamma^Z(n)\downarrow \neq A(n)$. Let s be the stage at which $Z \in \mathcal{V}_{n,s} - \mathcal{V}_{n,s-1}$. So $A(n) \neq A_s(n) = \Gamma^Z(n)$; this means that there is some $t > s$ such that $x_t \leqslant n$. So $Z \in \mathcal{V}_{n,t} \subseteq \mathcal{V}_{x_t,t}$. This shows that if $\Gamma^Z(n) \neq A(n)$ for infinitely many n, then Z is captured by the Solovay test $\langle \mathcal{V}_{x_s,s} \rangle$. $\square$

Remark 4.3. The proof of Proposition 4.2 actually shows that Z computes a modulus for A, and so a c.e. set which computes A. This is not surprising, since every Δ^0_2 set which obeys a cost function is computable from a c.e. set which obeys the same cost function [40, 5.3.6].

Theorem 4.4. *Let A be a K-trivial set. Let Z be a Martin-Löf random set which is not Oberwolfach random. Then $A \leqslant_{\mathrm{T}} Z$.*

Proof. Every K-trivial set is computable from a c.e. K-trivial set [39], so we may assume that A is c.e. By Proposition 2.15, Z fails a left c.e. bounded test $\langle \mathcal{V}_n \rangle$ with associated additive cost function $\mathbf{c}$. By Proposition 4.1, A obeys $\mathbf{c}$. Hence $A \leqslant_{\mathrm{T}} Z$ by Proposition 4.2. $\square$

4.2. **There is a K-trivial set not computable from any Oberwolfach random set.** In this subsection we build a c.e. K-trivial set A which is computable from no Oberwolfach random set (Theorem 4.9). Intuitively, A is a relatively complicated K-trivial set in that the only ML-random sets able to compute it are close to Turing complete, or, equivalently, not very random. As mentioned in the introduction, in conjunction with the result of the previous subsection, we see that a Martin-Löf random set computes A if and only if it computes all the K-trivial sets.

Recall the notion of balanced randomness from Definition 2.2. As a warm-up to the proof, we first provide the simpler proof of a result which is not, in fact, implied by the main theorem of this subsection. This is because some K-trivial set does not obey every 2^n-benign cost function by Corollary 4.10 below.

Theorem 4.5. *For any 2^n-benign cost function $\mathbf{c}$ there is some c.e. set which obeys $\mathbf{c}$ and is computable from no balanced random set.*

Kučera and Nies have shown that for any benign cost function $\mathbf{c}$, there is some c.e. set which obeys $\mathbf{c}$ and is computable from no weak Demuth random set [32]. We emphasize the strong parallels with this theorem: balanced tests are a special case of weak Demuth tests, obtained by allowing at most 2^n version changes instead of allowing an arbitrary computable bound; similarly, 2^n-benign cost functions are a special cast of benign cost functions, obtained by allowing sequences of a certain type to have length at most 2^n instead of allowing an arbitrary computable bound.

Proof. We fix a Turing functional Υ that is universal in the sense that $\Upsilon(0^e1\hat{\ }X) = \Phi_e(X)$ for each X, e. We enumerate a c.e. set A. To show that A is not computable from any balanced random set, we show that no element of $\{X \in 2^\omega : \Upsilon(X) = A\}$ is balanced random. Then: if X computes A then there is some e such that $\Upsilon(0^e1\hat{\ }X) = A$. So $0^e1\hat{\ }X$ is not balanced random; it follows that X is not balanced random either because we allowed the $O(2^n)$ bound on changes in Definition 2.2.

Construction. We define an approximation $\langle \mathcal{G}_n\langle s\rangle\rangle$ for a test $\langle \mathcal{G}_n\rangle$. We also enumerate a Σ^0_1 class $\mathcal{E}$, which is a permanent error class. The key idea for giving a bound on the number of changes to each $\mathcal{G}_n\langle s\rangle$ and showing that A obeys $\mathbf{c}$ is by tying the cost of enumerating elements into A to the measure these enumerations add to $\mathcal{E}$.

We define diagonalisation witnesses $v_{n,s}$ targeted for A, which have the purpose of showing that lots of oracles compute the wrong set. To be precise, by $v_{n,s}$, A_s, $\mathcal{E}_s$ and $\mathcal{G}_n\langle s\rangle$ we mean the values of these objects at the *beginning* of stage s.

When we start a new version of $\mathcal{G}_n$ at a stage t, we choose $v_{n,t+1}$ to be large, and let

$$\mathcal{G}_n\langle t+1\rangle = \{X \in 2^\omega \backslash \mathcal{E}_{t+1} : \Upsilon(X) \succeq A_t \restriction_{v_{n,t+1}+1}\}.$$

Whenever we start a new version of $\mathcal{G}_n$, we also start a new version of $\mathcal{G}_m$ for $m \geqslant n$. So $v_{n,s} < v_{n+1,s}$ for all n and s. It follows that the sequence $\langle \mathcal{G}_n\langle s\rangle\rangle_{n<\omega}$ is nested.

We decide to start a new version of $\mathcal{G}_n$ at a stage s if $\lambda(\mathcal{G}_n[s]) > 2^{-n}$. If there is such n at the stage s, we choose the least such. Then, there are three cases.

1. $\lambda(\mathcal{E} \cap \mathcal{G}_n)[s] > 2^{-n-1}$;
2. $\lambda(\mathcal{E} \cap \mathcal{G}_n)[s] \leqslant 2^{-n-1}$ and $\mathbf{c}(v_n)[s] > 2^{-n-1}$; and
3. Cases (1) and (2) fail.

If either case (1) or case (2) holds, then we just start a new version of $\mathcal{G}_n$ (and $\mathcal{G}_m$ for all $m > n$). If case (3) holds, that is, if both $\lambda(\mathcal{E} \cap \mathcal{G}_n)[s] \leqslant 2^{-n-1}$ and $\mathbf{c}(v_n)[s] \leqslant 2^{-n-1}$, then we enumerate $v_{n,s}$ into A_{s+1}, $\mathcal{G}_n[s]$ into $\mathcal{E}_{s+1}$, and start new versions of $\mathcal{G}_m$ for $m \geqslant n$.

Verification. The main task is to obtain the $O(2^n)$ bound the number of times each version of $\mathcal{G}_n$ changes. First we note that $\mathcal{G}_0$ never changes. Now let $n > 0$, let s be a stage at which $\mathcal{G}_n$ changes but $\mathcal{G}_{n-1}$ does not, and let t be the stage at which the version $\mathcal{G}_n\langle s\rangle$ was defined.

One of the cases (1), (2) and (3) above hold at stage s.

1. In this case, since we know that $\mathcal{G}_n\langle s\rangle = \mathcal{G}_n\langle t+1\rangle$, and by definition $\mathcal{G}_n\langle t+1\rangle$ is disjoint from $\mathcal{E}_{t+1}$, we can conclude that $\lambda(\mathcal{E}_s - \mathcal{E}_{t+1}) > 2^{-n-1}$. This shows that case (1) can hold at at most 2^{n+1} many stages s.

(2) In this case, since $v_{n,s} = v_{n,t+1}$ is chosen to be large at stage t, the benignity bound on c shows that this case can hold at at most 2^{n+1} many stages.
(3) Finally, in this case, the failure of (1), the fact that $\lambda(\mathcal{G}_n)[s] \geqslant 2^{-n}$, and the action taken in this case, together show that $\lambda(\mathcal{E}_{s+1} - \mathcal{E}_s) \geqslant 2^{-n-1}$. So this case too can happen at most 2^{n+1} many times.

Altogether, we see that the number of stages s at which $\mathcal{G}_n$ changes but $\mathcal{G}_{n-1}$ does not is at most $6 \cdot 2^n$. By induction, we see that the total number versions of $\mathcal{G}_n$ is bounded by

$$1 + 6 \cdot 2 + 6 \cdot 4 + \cdots 6 \cdot 2^n \leqslant 12 \cdot 2^n.$$

Hence the approximation $\langle \mathcal{G}_n\langle s\rangle\rangle$ stabilizes at some value $\mathcal{G}_n$, and the sequence $\langle \mathcal{G}_n\rangle$ is nested. Certainly, for all n, $\lambda(\mathcal{G}_n) \leqslant 2^{-n}$.

Claim 4.5.1. For all $X \in \mathcal{E}$, $\Upsilon(X) \neq A$.

Proof. Suppose that $\mathcal{G}_n[s]$ is enumerated into $\mathcal{E}$ at stage s. This enumeration is accompanied by the enumeration of $v_{n,s}$ into A_{s+1}, while for every $X \in \mathcal{G}_n[s]$ we have $\Upsilon(X, v_{n,s}) = A_t(v_{n,s}) = 0$ (for the stage t at which this version was defined). □

Claim 4.5.2. For all X, if $\Upsilon(X) = A$ then $X \in \bigcap \mathcal{G}_n$.

Proof. Let $n < \omega$; let t be the stage at which the final version of $\mathcal{G}_n$ was defined; let $v_n = v_{n,t+1}$ be the final value of $\langle v_{n,s}\rangle$. Let $\alpha = A_{t+1}\restriction_{v_n+1}$; so

$$\mathcal{G}_n = \{X \in 2^\omega \backslash \mathcal{E}_{t+1} \,:\, \Upsilon(X) \geqslant \alpha\}.$$

The fact that none of the versions $\mathcal{G}_m$ for $m \leqslant n$ change after stage t, and that the sequence $\langle v_{m,t}\rangle_{m<\omega}$ is strictly increasing, shows that $\alpha = A\restriction_{v_n+1}$. By Claim 4.5.1, if $\Upsilon(X) = A$ then $X \notin \mathcal{E}_{t+1}$. Hence such X must be an element of $\mathcal{G}_n$. □

Our final task is to show that A obeys c; of course, the enumeration of A witnessing this will be the enumeration given by the construction. Suppose that $A_{s+1} \neq A_s$; then at stage s, some (unique) $v_{n,s}$ is enumerated into A_{s+1}. We know that in this case, $\mathbf{c}(v_n)[s] \leqslant 2^{-n-1}$ and $\lambda(\mathcal{E}_{t+1} - \mathcal{E}_t) \geqslant 2^{-n-1}$. This shows that the total cost

$$\sum \mathbf{c}_s(x) \; [\![x \text{ is enumerated into } A \text{ at stage } s]\!]$$

is bounded by

$$\sum_s \lambda(\mathcal{E}_{s+1} - \mathcal{E}_s) = \lambda(\mathcal{E}) \leqslant 1.$$

Hence the total cost is bounded as required. □

Note a feature of this construction: unlike most cost-function constructions, we cannot bound, for each n, the contribution of the n^{th} actor to the total cost $\mathbf{c}(\langle A_s\rangle)$. The only possible calculation is global. We will see that, in some sense, this is even more so when Oberwolfach randomness is concerned: the relationship between the total cost and the measure of $\mathcal{E}$ is tighter.

Recall that by the two "halves" of a set $Y \subseteq \omega$ we mean the bits in the even, and the bits in the odd positions. Miller and Nies [36, Question 4.7] asked whether every K-trivial is computable from both halves of some ML-random set. We answer this in the negative. We need one fact that is implicit in [17].

Proposition 4.6. *Suppose X is ML-random. Let X_0, X_1 be the two halves of X. Then at least one of X_0, X_1 is balanced random.*

Proof. Greenberg and Nies [23] defined a set Z to be *ω-c.a.-tracing* if each function $f \leqslant_{\mathrm{wtt}} \varnothing'$ has a Z-c.e. trace $(T^Z_x)_{x\in\omega}$ such that $|T^Z_x| \leqslant 2^x$ for each x. By [17, Theorem 23], if X_0 is not ω-c.a. tracing then X_0 is balanced random. Otherwise, by [17, Prop. 32], any set ML-random in X_0 is weakly Demuth random; thus by van Lambalgen's Theorem, X_1 is weakly Demuth random, which implies that it is balanced random. □

The following is now immediate from Theorem 4.5. Note that it implies Theorem 1.3 in the introduction.

Corollary 4.7. *There is a c.e., K-trivial set which is not computable from both halves of any random set. In fact, for every 2^n-benign cost function* $\mathbf{c}$*, there is a c.e. set of this kind that obeys* $\mathbf{c}$*.*

We proceed to the main result of this subsection, the construction of a smart K-trivial set. To justify the use of a universal Turing functional Υ, we prove an analogue of the fact used above that balanced randomness is preserved under adding a finite string at the beginning of a bit sequence.

Lemma 4.8. *Let Z be an Oberwolfach random set. Then $\rho\hat{}Z$ is Oberwolfach random for each string ρ.*

Proof. Recall that for a Σ^0_1 class $\mathcal{W}$ and a string ρ we let

$$\mathcal{W}|[\rho] = \{X \in 2^\omega \,:\, \rho\hat{}X \in \mathcal{W}\}\,.$$

Suppose that $\rho\hat{}Z$ is not Oberwolfach random. Let $(\sigma \mapsto \mathcal{G}_\sigma, \alpha)$ be an interval test capturing $\rho\hat{}Z$ (see the proof of Proposition 2.9). Let $\tau = \alpha\upharpoonright_{|\rho|}$. For all σ, let $\mathcal{V}_\sigma = \mathcal{G}_{\tau\hat{}\sigma}|[\rho]$. Let β be such that $\alpha = \tau\hat{}\beta$. Then $(\sigma \mapsto \mathcal{V}_\sigma, \beta)$ is an interval test which captures Z. □

Theorem 4.9. *There is a K-trivial set A such that no set $Y \geqslant_{\mathrm{T}} A$ is Oberwolfach random.*

We note that this yields another, albeit circuitous, proof of Proposition 2.5 that every Oberwolfach random set is difference random. A ML-random which is not difference random is complete, and so computes all K-trivial sets. Hence it is not Oberwolfach random.

Proof. We actually prove an exact analogue of Theorem 4.5: for any additive cost function $\mathbf{c}$ there is some c.e. set obeying $\mathbf{c}$ which is computed by no Oberwolfach random set. Of course now the point is that the K-trivial sets are characterised as those which obey all additive cost functions, and in fact some additive cost function such as $\mathbf{c}_\Omega$ characterises K-triviality on its own. So fix an additive cost function $\mathbf{c}$.

As in the proof of Theorem 4.5, we enumerate a c.e. set A, and make sure that A obeys $\mathbf{c}$. Again we fix a universal Turing functional Υ, and show that $\{X \in 2^\omega \,:\, \Upsilon(X) = A\}$ is covered by an Oberwolfach test $\langle \mathcal{G}_n \rangle$ which we approximate during the construction. Now Lemma 4.8 ensures that A is computable from no Oberwolfach-random set.

The components $\langle \mathcal{G}_n\langle s\rangle\rangle$ of our approximation will be identical to those of the proof of Theorem 4.5; we again enumerate the error class $\mathcal{E}$, define markers $v_{n,s}$, and when redefining $\mathcal{G}_n$ at a stage t, we choose $v_{n,t+1}$ to be large and let

$$\mathcal{G}_n\langle t+1\rangle = \{X \in 2^\omega \backslash \mathcal{E}_{t+1} \,:\, \Upsilon(X) \succeq A_t\upharpoonright_{v_{n,t+1}+1}\}.$$

Certainly, to make $\langle \mathcal{G}_n\langle s\rangle\rangle$ an approximation showing that $\langle \mathcal{G}_n \rangle$ is an Oberwolfach test, whenever we start a new version of $\mathcal{G}_n$, we also start a new version of $\mathcal{G}_{n+1}$; so again $v_{n,s} < v_{n+1,s}$ for all n and s. The only difference is the timing of the changes and the choice when to enumerate numbers into A, which has to be slightly more delicate.

In general, we need to start a new version of $\mathcal{G}_n$ at a stage s if:

(1) either $\lambda(\mathcal{G}_n[s]) \geqslant 2^{-n}$; or

(2) $\lambda(\mathcal{E} \cap \mathcal{G}_n) + \mathbf{c}(v_n) \geqslant 2^{-n}$ $[s]$.

And the aim is to restart with no enumeration in case (2), and enumerate $v_{n,s}$ into A (and $\mathcal{G}_n[s]$ into $\mathcal{E}$) if (1) (but not (2)) holds.

We note the similarity with the proof of Theorem 4.5. In the previous setting, the only prompt to changing $\mathcal{G}_n\langle s\rangle$ was if the measure bound $\lambda(\mathcal{G}_n) \leqslant 2^{-n}$ was exceeded. In the current construction we also need to pre-empt situations which entail multiple changes of $\mathcal{G}_n$ while $\langle \mathcal{G}_{n-1}\rangle$ is stable. For example, in the previous construction we could see about 2^{n+1} many stages during which $\mathcal{G}_{n-1}$ is stable, but at which $\mathcal{G}_n$ changes are accompanied by high $\mathbf{c}(v_n)$ costs and so by no enumerations into $\mathcal{E}$. However, to ensure the overall coherence of moves of various levels, we need to consider the following scenario. Suppose that at stage s, $\mathcal{G}_n$ requests an "aggressive" change which results in enumerations. The added mass into $\mathcal{E}$ will now trigger an earlier $\mathcal{G}_p$, say $\mathcal{G}_{n-1}$, to want to change as well. In the previous construction, $\mathcal{G}_{n-1}$ could wait until the next stage to act. But our analysis below will show that in fact the change in $\mathcal{G}_n$ can be the second change during the time $\mathcal{G}_{n-1}$ is fixed, which is not allowed: $\mathcal{G}_{n-1}$ has to change immediately.

So here is the construction. At stage s, first see if there is some $n < s$ with $\lambda(\mathcal{G}_n[s]) \geqslant 2^{-n}$ but $\lambda(\mathcal{E} \cap \mathcal{G}_n) + \mathbf{c}(v_n) < 2^{-n}$ $[s]$. If there is such an n, we pick the least such, enumerate $v_{n,s}$ into A and $\mathcal{G}_n[s]$ into $\mathcal{E}$. If there is no such n, then $A_{s+1} = A_s$ and $\mathcal{E}_{s+1} = \mathcal{E}_s$.

Then, we look to see if there is some p such that $\lambda(\mathcal{E}_{s+1} \cap \mathcal{G}_p[s]) + \mathbf{c}(v_p)[s] \geqslant 2^{-p}$. If so, we choose the least such p and restart $\mathcal{G}_k$ for all $k \geqslant \min\{n, p\}$. If neither n nor p are found, then no test component is restarted.

To verify the construction, this time, we first show that the total cost $\mathbf{c}(\langle A_s\rangle)$ is finite. Suppose that $v_{n,s}$ is enumerated into A at stage s. Again, the finite bound on the total cost is obtained once we show:

Claim 4.9.1.
$$\mathbf{c}(v_n)[s] \leqslant \lambda(\mathcal{E}_{s+1} - \mathcal{E}_s).$$

Proof. At stage s we see that $\lambda(\mathcal{G}_n[s]) \geqslant 2^{-n}$ but $\lambda(\mathcal{E}_s \cap \mathcal{G}_n[s]) + \mathbf{c}(v_n)[s] < 2^{-n}$. This means that
$$\lambda\left(\mathcal{G}_n[s] - \mathcal{E}_s\right) \geqslant 2^{-n} - \lambda\left(\mathcal{G}_n[s] \cap \mathcal{E}_s\right) > \mathbf{c}(v_n)[s].$$
But at stage s we enumerate $\mathcal{G}_n[s]$ into $\mathcal{E}$, so
$$\mathcal{G}_n[s] - \mathcal{E}_s \subseteq \mathcal{E}_{s+1} - \mathcal{E}_s,$$
proving the claim. □

Now we turn to the main task of showing that $\langle \mathcal{G}_n\langle s\rangle\rangle$ converges to an Oberwolfach test. To show that there are only finitely many versions of $\mathcal{G}_0$ one can argue directly, as we shall soon do for $\mathcal{G}_n$, but a quick way is to start our sequence with $\mathcal{G}_{-1}$ instead of $\mathcal{G}_0$, and to scale $\mathbf{c}$ so that $\mathbf{c}_s(x) < 1$ for all s and x. Then, we easily see that $\mathcal{G}_{-1}$ never changes.

For the rest of the argument we make a simple observation:

Claim 4.9.2. Let $n \geqslant -1$, and let $u < w$ be successive stages at which $\mathcal{G}_{n+1}$ is restarted. Suppose that $\mathcal{G}_n$ is not restarted at stage w. Then
$$\mathbf{c}(v_{n+1})[w] + \lambda\left(\mathcal{G}_n[w] \cap (\mathcal{E}_{w+1} - \mathcal{E}_{u+1})\right) \geqslant 2^{-n-1}.$$

Proof. Let $q = \mathbf{c}(v_{n+1})[w]$. There are two cases. In both cases we use the facts that $\mathcal{G}_{n+1}[w] \subseteq \mathcal{G}_n[w]$, and that $\mathcal{G}_{n+1}[w] \cap \mathcal{E}_{u+1} = \varnothing$.

If $v_{n+1,w}$ is not enumerated into A at stage w, then we know that

$$q + \lambda(\mathcal{E}_{w+1} \cap \mathcal{G}_{n+1}[w]) \geqslant 2^{-n-1},$$

and the result follows.

Otherwise, at stage w we enumerate all of $\mathcal{G}_{n+1}[w]$ into $\mathcal{E}_{w+1} \cap \mathcal{G}_n[w]$, and we know that $\lambda(\mathcal{G}_{n+1}[w]) \geqslant 2^{-n-1}$; then in fact we get

$$\lambda(\mathcal{G}_n[w] \cap (\mathcal{E}_{w+1} - \mathcal{E}_{u+1})) \geqslant 2^{-n-1},$$

without q's aid. □

Fix $n \geqslant -1$, and let $r < s < t$ be successive stages at which a new version of $\mathcal{G}_{n+1}$ is defined; suppose, for contradiction, that $\mathcal{G}_n$ is restarted at neither stage s nor stage t.

For brevity, let $x = v_{n,r+1} = v_{n,s} = v_{n,t}$; and let $y = v_{n+1,r+1} = v_{n+1,s}$ and $z = v_{n+1,s+1} = v_{n+1,t}$. So $x < y < s < z < t$.

Now we apply Claim 4.9.2 twice, at stages s and t. We obtain:

$$\mathbf{c}_t(z) + \lambda(\mathcal{G}_n[t] \cap (\mathcal{E}_{t+1} - \mathcal{E}_{s+1})) \geqslant 2^{-n-1},$$

and because $\mathcal{G}_n[s] \subseteq \mathcal{G}_n[t]$ we obtain

$$\mathbf{c}_s(y) + \lambda\left(\mathcal{G}_n[t] \cap (\mathcal{E}_{s+1} - \mathcal{E}_{r+1}\right) \geqslant 2^{-n-1}.$$

Additivity and monotony of $\mathbf{c}$ imply that

$$\mathbf{c}(v_n)[t] = \mathbf{c}_t(x) \geqslant \mathbf{c}_t(y) \geqslant \mathbf{c}_s(y) + \mathbf{c}_t(z),$$

and so putting the two inequalities together we obtain

$$\mathbf{c}(v_n)[t] + \lambda\left(\mathcal{G}_n[t] \cap \mathcal{E}_{t+1}\right) \geqslant 2^{-n}.$$

Thus, n would play the role of p in the second step of the construction at stage t, and so we would be instructed to pick a new version for $\mathcal{G}_n$ at stage t, contrary to assumption.

The rest of the proof follows the proof of Theorem 4.5 verbatim; we see that every X for which $\Upsilon(X) = A$ is in $\bigcap_n \mathcal{G}_n$ and so is not Oberwolfach random. □

K-triviality is characterized by obeying the standard cost function $\mathbf{c}_{\mathcal{K}}$ [39]. This cost function is $o(2^n)$ benign. We obtain a corollary to Theorem 4.9 specifying a different but related sense in which the K-trivial set A constructed there is smart.

Corollary 4.10. *There is a c.e., K-trivial set which does not obey some $o(2^n)$-benign cost function.*

Proof. Each computable approximation $\langle Y_s \rangle$ of a Δ^0_2 ML-random set Y yields a cost function $\mathbf{c}_Y$ such that any set A obeying it is Turing below Y [23] (or see [40, 5.3.13]). By [17, Theorem 11] there is a low ML-random set Y with a computable approximation $\langle Y_s \rangle$ such that $Y_s \upharpoonright_n$ changes only $o(2^n)$ many times. Then by its definition, $\mathbf{c}_Y$ is $o(2^n)$ benign. Thus, using the fact that each ML-random non-Oberwolfach random set is high, the smart K-trivial constructed in Theorem 4.9 does not obey $\mathbf{c}_Y$. □

Turetsky [51] has built a c.e., K-trivial set A that is complex in the sense that it is not $o(\log n)$ jump traceable. We do not know at present whether the smart K-trivial built in Theorem 4.9 must have this property.

4.3. **Diamond classes and ML-reducibility.** Recall that for any class $\mathcal{C} \subseteq 2^\omega$, we let $\mathcal{C}^\diamond$ denote the c.e. sets Turing below every Martin-Löf random member of $\mathcal{C}$. Usually $\mathcal{C}$ is arithmetical. By the foregoing results, together with [9], the c.e. K-trivial sets form a diamond class:

Corollary 4.11. *Let $\mathcal{C}$ be a nonempty class of difference random, non-Oberwolfach random sets. Then $\mathcal{C}^\diamond$ coincides with the c.e. K-trivial sets.*

For instance, we can let Y be a difference random set that is not a density-one point [9] and hence not Oberwolfach random; then $\{Y\}^\diamond$ equals the class of c.e. K-trivial sets. Thus, the whole class of K-trivial sets is encoded in a single random set, which can in fact be chosen to be Δ^0_2.

Proof. $\mathcal{C}^\diamond$ is contained in the K-trivial sets by the aforementioned result of [25]. On the other hand, each K-trivial set is in $\mathcal{C}^\diamond$ by Theorem 4.4. □

We let JTH denote the class of sets that are h-JT-hard for some (computable) order function h. Every LR-hard set is in JTH via an $o(2^n)$ order function by [48] (or see [40, 8.4.15]).

The class $\mathit{JTH}^\diamond$ was discussed in [40, 8.5.12]. In particular, by [23] there is a single benign cost function $\mathbf{c}$ such that any set obeying $\mathbf{c}$ is in $\mathit{JTH}^\diamond$. As a consequence, $\mathit{JTH}^\diamond$ strictly contains the class of c.e. strongly jump traceable sets. As a consequence of the previous two theorems, we separate $\mathit{JTH}^\diamond$ from the c.e. K-trivials.

Corollary 4.12. *$\mathit{JTH}^\diamond$ is a proper subclass of the c.e. K-trivial sets.*

Proof. By [18] there is a c.e. jump traceable set W that is not jump traceable at order n^2. Then, by pseudo-jump inversion for random sets [40, Thm. 6.3.9] there is a ML-random Δ^0_2 set Z which is JT-hard, but not n^2-JT hard. Then Z is Oberwolfach random by Theorem 3.3. Thus the smart K-trivial constructed in Theorem 4.9 is not Turing below Z. □

The investigations on diamond classes such as in [22], together with the results in this section, suggest a new reducibility coarser than $\leqslant_{\mathrm{T}}$ among the K-trivials.

Definition 4.13. For K-trivial sets A and B, we write $B \leqslant_{\mathrm{ML}} A$ if $A \leqslant_{\mathrm{T}} Y$ implies $B \leqslant_{\mathrm{T}} Y$ for any ML-random set Y.

This reducibility gauges complexity via the paradigm of [22] that being low means easy to compute, in the sense that many oracles compute the set. Clearly, $\leqslant_{\mathrm{T}}$ implies $\leqslant_{\mathrm{ML}}$, and the ML-degrees form an upper semilattice where the least upper bound of K-trivial sets C and D is given by the K-trivial set $C \oplus D$. The set A constructed in Theorem 4.9 is smart in that it satisfies $B \leqslant_{\mathrm{ML}} A$ for every K-trivial set B.

Consider now the ML-degrees of K-trivial sets. Each diamond class induces an ideal of this degree structure (an initial segment closed under join). Within the ML-degrees of c.e. sets, any principal ideal $\{B\colon\ B \leqslant_{\mathrm{ML}} A\}$ is the diamond class of the Σ^0_3 class

$$\mathcal{C}_A = \{Y\colon\ A \leqslant_{\mathrm{T}} Y\}.$$

Thus, $B \leqslant_{\mathrm{ML}} A$ if and only if B lies in every (Σ^0_3) diamond class that contains A.

Technical questions on $\leqslant_{\mathrm{ML}}$ abound. For instance, is $\leqslant_{\mathrm{ML}}$ arithmetical? Is the ordering of ML-degrees linear? Within the c.e. ML-degrees, one can equivalently ask: are there incomparable diamond classes? To show non-linearity, one would need to build K-trivial sets A_0, A_1 and ML-random sets Y_0, Y_1 such that $A_i \leqslant_{\mathrm{T}} Y_i$ yet $A_i \not\leqslant_{\mathrm{T}} Y_{1-i}$ $(i = 0, 1)$.

5. Density, martingale convergence, and Oberwolfach randomness

Recall that in the introduction, we discussed the concept of density: for measurable $S \subseteq \mathbb{R}$ and $z \in \mathbb{R}$, we define the *lower density* of S at z to be

$$\rho(S|z) = \liminf_{h \to 0} \{\lambda(S|I) : I \text{ is an open interval, } z \in I \ \& \ |I| < h\},$$

where $\lambda(S|U) = \lambda(S \cap U)/\lambda(U)$ is the conditional measure of S given U. For brevity, we sometimes use the notation

$$\liminf_{I \to z} \lambda(S|I)$$

to denote the limit above. The *upper density* of S at z is defined similarly, but using the limit superior instead of the inferior: it is $\limsup_{I \to z} \lambda(S|I)$. If the upper and lower densities of S at z are equal, then their common value is known as the *density* of S at z.

When working in Cantor space, it is more natural to work with *dyadic* density, which is defined analogously. For a set $S \subseteq 2^\omega$ and $Z \in 2^\omega$, the *lower dyadic density* of S at Z is

$$\rho_2(S|Z) = \liminf_{n \to \infty} \lambda(S|[Z \upharpoonright_n]),$$

The *upper dyadic density* of S at Z is $\limsup_n \lambda(S|[Z \upharpoonright_n])$. If the lower and upper dyadic densities of S at Z are equal, then their common value is the *dyadic density* of S at Z.

Remark 5.1. Even though it is natural to use dyadic density in Cantor space and full density in the real line, we nonetheless can use both notions in either space, by using the "near-isomorphism" Θ between the two described in Remark 2.1. We can then extend the notion from the unit interval to all of $\mathbb{R}$ by using rational shifts. If $z \in \mathbb{R}$ is not a dyadic rational, then the dyadic density of a set $S \subseteq \mathbb{R}$ at z is the limit, as $|I| \to 0$, of $\lambda(S|I)$, where I is a dyadic open interval (an interval of the form $(k2^{-n}, (k+1)2^{-n})$ for $k \in \mathbb{Z}$ and $n < \omega$) which contains z. Since we mostly consider random points, we are not concerned about rational numbers. Thus, for any irrational number z and any measurable set S, $\rho(S|z) \leqslant \rho_2(S|z)$.

Recall that the Lebesgue density theorem [34, p. 407] says that for any measurable set $S \subseteq [0,1]$, for almost all points $z \in S$, the density of S at z is 1. As mentioned in the introduction, an expanding project of algorithmic randomness is to understand the effective content of "almost everywhere" theorems of analysis by associating with each theorem the class of random sets which makes every effective instance of this theorem work. Usually, different choices for the effective version of the theorem would yield different classes of random sets.

To state an effective version of Lebesgue's density theorem, we need to choose a class $\mathfrak{C}$ of effectively presented subsets of $[0,1]$ and ask, for which random points z, is the density of S at z equal 1, for all $S \in \mathfrak{C}$ containing z? Choosing $\mathfrak{C}$ to be the class of effectively open sets will yield trivial answers, and so we concentrate on the class of effectively closed sets.

As mentioned, a closely related result of Bienvenu, Hölzl, Miller and Nies characterizes nonzero density of ML-random sets.

Theorem 5.2 ([2], Remark 3.4)**.** *The following are equivalent for a Martin-Löf random set $Z \in 2^\omega$:*

1. *Z is difference random;*
2. *$\rho_2(P|Z) > 0$ for every effectively closed subset P of 2^ω which contains Z;*
3. *$\rho(P|z) > 0$ for every effectively closed subset P of $[0,1]$ which contains the real z with binary expansion given by Z.*

The main question – for which random points z, is $\rho(P|z) = 1$ for all effectively closed sets P containing z? – remains open. As we mentioned in the introduction, following Bienvenu et al., we say that a real $z \in [0, 1]$ is a *density-one point* if $\rho(P|z) = 1$ for every effectively closed set containing z. So the question is: which notion of randomness (if any) is equivalent to being a ML-random, density-one point? A possible answer would be given by a test notion analogous to but stronger than the one in Theorem 5.2(1).

As mentioned in the introduction, Day and Miller [9] showed that difference randomness does not suffice. They construct a Martin-Löf random set Z such that $\rho_2(P|Z) > 0$ for every effectively closed set containing Z, but such that $\rho_2(P|Z) < 1$ for some effectively closed set containing Z. The first part guarantees that Z is difference random (Theorem 5.2).

Remark 5.3. Bienvenu et al. noted that there are density-one points which are not random. For example, every 1-generic point is a density-one point, since it lies in the interior of any effectively closed set containing it. No 1-generic real is Martin-Löf random, or even Schnorr random. This is why in this investigation, we focus on classifying the density-one points within the ML-random points.

We show that every Oberwolfach random set is a density-one point. The argument naturally filters through a possibly stronger property concerning martingale convergence. In the next section we give another proof, using differentiability. We do not know, though, whether there is a density-one random point which is not Oberwolfach random.

5.1. **Martingale convergence.** In the theory of algorithmic randomness, by a martingale one means a function $M\colon 2^{<\omega} \to [0, \infty)$ with the usual averaging condition $M(\sigma 0) + M(\sigma 1) = 2M(\sigma)$ for each string σ. For background on martingales in this sense see for instance [14, Section 5.3]. An important fact frequently used is *Kolmogorov's inequality*: if $M(\langle\rangle) < b$, then

$$\lambda\{Z \in 2^\omega \colon \exists n\, [M(Z\restriction_n) \geqslant b]\} \leqslant M(\langle\rangle)/b.$$

See for instance [40, 7.1.9] or [14, Section 5.3].

A martingale M is called *left-c.e.* if $M(\sigma)$ is a left-c.e. real uniformly in σ; *right-c.e.* martingales are defined analogously.

For a set $Z \in 2^\omega$, we say that a martingale M *converges on* Z if the sequence $\langle M(Z\restriction_n)\rangle_{n<\omega}$ has a (finite) limit. The savings trick (see for example [14, 5.3.1]) shows that a set Z is computably random if and only if every computable martingale converges on Z. The analogous fact fails for left-c.e. martingales and ML-randomness. If every left-c.e. martingale converges on Z then certainly no left-c.e. martingale can succeed on Z, and so Z is ML-random. However, the converse may fail.

To see this, we note that dyadic density translates to martingales. For any measurable set $\mathcal{A} \subseteq 2^\omega$, the function $M_{\mathcal{A}}(\sigma) = \lambda(\mathcal{A}|\sigma)$ is a martingale. By definition, the dyadic density of $\mathcal{A}$ at Z exists if and only if $M_{\mathcal{A}}$ converges on Z. If $\mathcal{P} \subseteq 2^\omega$ is effectively closed, then $M_{\mathcal{P}}$ is a right-c.e. martingale bounded by 1; $\sigma \mapsto 1 - M(\sigma)$ is a left-c.e. martingale.

For a random set Z and an effectively closed set $\mathcal{P}$ containing Z, the existence of density is equivalent to having lower density 1; the same holds for dyadic density.

Proposition 5.4. *Let $\mathcal{P} \subseteq 2^\omega$ be an effectively closed set and let $Z \in \mathcal{P}$ be Martin-Löf random. Then the upper density of $\mathcal{P}$ at Z is 1.*

Proof. Fix a Π^0_1-class $\mathcal{P} \subseteq 2^\omega$. Let $Y \in \mathcal{P}$ such that the upper density of $\mathcal{P}$ at Y is less than some rational $q < 1$. We define a Martin-Löf test $\langle \mathcal{U}_n \rangle$ which captures Y.

The components $\mathcal{U}_n$ are defined by induction on n. We let $\mathcal{U}_0 = [Y \upharpoonright_k]$, where k is sufficiently large so that $\lambda(\mathcal{P}|Y \upharpoonright_m) < q$ for all $m \geqslant k$. Given $\mathcal{U}_n$, let U_n be a c.e. antichain of strings generating $\mathcal{U}_n$; note that all of these strings will extend $Y \upharpoonright_k$. We let $\mathcal{U}_{n+1}$ be the union of the sets $\mathcal{P}_t \cap [\sigma]$ where $\sigma \in U_n$ and t is the least stage such that $\lambda(\mathcal{P}_t|[\sigma]) < q$. Note that each such set is clopen, so $\mathcal{U}_n$ is indeed (effectively) open. By induction, we see that $Y \in \mathcal{U}_{n+1}$. Also, we note that by definition, for each $\sigma \in U_n$, $\lambda(\mathcal{U}_{n+1}|[\sigma]) < q$ (it is of course possible that $\mathcal{U}_{n+1} \cap [\sigma] = \varnothing$) and so $\lambda(\mathcal{U}_{n+1}) \leqslant q\lambda(\mathcal{U}_n)$. So by induction, $\lambda(\mathcal{U}_n) \leqslant q^n$. Replacing $\mathcal{U}_n$ by $\mathcal{U}_{rn}$ for an appropriate $r \in \omega$ yields a ML-test as required. □

The translation of dyadic density to martingales yields:

Corollary 5.5. *Let $Z \in 2^\omega$. If every left-c.e. martingale M converges on Z, then for every effectively closed set $\mathcal{P} \subseteq 2^\omega$ containing Z we have $\rho_2(\mathcal{P}|Z) = 1$.*

Thus, the construction of Day and Miller [9] shows that there is some difference random set Z and a left-c.e. martingale M which does not converge on Z. Doob's martingale convergence theorem [13] states that every martingale M converges on almost every $Z \in 2^\omega$. Hence *some* notion of randomness ensures the convergence of left-c.e. martingales. We show that Oberwolfach randomness suffices.

Theorem 5.6. *If Z is Oberwolfach random, then every left-c.e. martingale converges on Z.*

Corollary 5.7. *Let $z \in [0,1]$ be Oberwolfach random. Then for any effectively closed set $\mathcal{P}$ containing z, the dyadic density of $\mathcal{P}$ at z is 1.*

For the proof of Theorem 5.6 we observe the oscillations in the value of the martingale along an element of Cantor space. We need a fact that follows from the upcrossing inequality for martingales (see, for instance, [16, pg. 235]); the proof of this fact is short, so we give it for completeness.

Let M be a martingale. Let $a < b$ be real numbers, and let $n < \omega$. Let $\mathcal{O}_n = \mathcal{O}_n(M,a,b)$ be the set of all sequences $X \in 2^\omega$ for which there is a sequence $m_1 < k_1 < m_2 < k_2 < \cdots < m_n < k_n$ such that for $i = 1,\dots,n$ we have $M(X \upharpoonright_{m_i}) < a$ and $M(X \upharpoonright_{k_i}) > b$.

Lemma 5.8. $\lambda(\mathcal{O}_n(M,a,b)) \leqslant (a/b)^n$.

Proof. We consider "smallest" oscillations. We define antichains of strings U_n and V_n by induction, with V_n refining U_n and U_{n+1} refining V_n. We start with $V_{-1} = \langle\rangle$. Given V_n, we let U_{n+1} be the collection of minimal strings τ extending some string in V_n such that $M(\tau) < a$. Given U_n, we let V_n be the collection of minimal strings τ extending some string in U_n such that $M(\tau) > b$. Let $\mathcal{U}_n$ be the open set generated by U_n and $\mathcal{V}_n$ be the open set generated by V_n. So $2^\omega = \mathcal{V}_{-1} \supseteq \mathcal{U}_0 \supseteq \mathcal{V}_0 \supseteq \mathcal{U}_1 \supseteq \mathcal{V}_1 \supseteq \dots$. Kolmogorov's inequality tells us that for every $\sigma \in U_n$, $\lambda(\mathcal{V}_n|\sigma) \leqslant a/b$. Hence by induction we see that $\lambda(\mathcal{V}_n) \leqslant (a/b)^n$. But we also see that $\mathcal{V}_n = \mathcal{O}_n$: certainly $\mathcal{V}_n \subseteq \mathcal{O}_n$; for the other inclusion, take $X \in \mathcal{O}_n$ and by induction let m_1 be the least m such that $M(X \upharpoonright_m) < a$; k_1 be the least $k > m_1$ such that $M(X \upharpoonright_k) > b$; m_2 be the least $m > k_1$ such that $M(X \upharpoonright_m) < a$; and so on. By induction we see that $X \upharpoonright_{m_i} \in U_i$ and $X \upharpoonright_{k_i} \in V_i$. □

Let M be a left-c.e. martingale. The set $\mathcal{O}_n = \mathcal{O}_n(M,a,b)$ is open, but may not be effectively open. The point, of course, is that we can discover that $M(X \upharpoonright_k) > b$ at some stage, but if we see at some stage that $M(X \upharpoonright_m) < a$, there is no guarantee that the value of $M(X \upharpoonright_m)$ will not increase beyond a at some later stage. However, when an observed oscillation "goes bad", i.e. ceases to be a true oscillation, there is a necessary corresponding increase to $M(\langle\rangle)$. We use an interval test to let components of the test guess approximate values for $M(\langle\rangle)$, and so limit the amount of badness.

Proof of Theorem 5.6. Let M be a left-c.e. martingale. After applying a rational scaling factor to M, we may assume that $M(\langle\rangle) \in (0,1)$. Suppose that $\langle M(Z\upharpoonright_n)\rangle_{n<\omega}$ does not converge; find rational numbers $a < b$ such that $\liminf_n M(Z\upharpoonright_n) < a < b < \limsup_n M(Z\upharpoonright_n)$. We will define an interval test that captures Z.

Let $\langle M_t\rangle$ be an increasing, uniformly computable sequence of (rational-valued) martingales which converges (pointwise) to M. We define an interval test using the left-c.e. real $M(\langle\rangle)$. We later calculate a constant $C > 0$. For a rational open interval $I \subseteq [0,1]$ we let $n_I = 1 + \lfloor C \cdot (-\log_2 |I|)\rfloor$ and define

$$G(I) = \bigcup \mathcal{O}_{n_I}(M_t, a, b)\ [\![M_t(\langle\rangle) \in I]\!].$$

That is, we enumerate into $G(I)$ all the sets X on which we see an (a,b)-oscillation of length n_I in $\langle M_t(X\upharpoonright_n)\rangle_{n<\omega}$, while $M_t(\langle\rangle) \in I$. For all n, for all sufficiently large t, we have $Z \in \mathcal{O}_n(M_t, a, b)$. If $M(\langle\rangle) \in I$ then for almost all t, $M_t(\langle\rangle) \in I$ and so $Z \in G(I)$. Thus, the interval test $(G, M(\langle\rangle))$ captures Z. It remains to show that G is an interval array, as in Definition 2.6.

Suppose that $I \subseteq I'$ are rational open intervals. Then $|I| \leqslant |I'|$ implies that $n_I \geqslant n_{I'}$. Hence, for all t, $\mathcal{O}_{n_I}(M_t, a, b) \subseteq \mathcal{O}_{n_{I'}}(M_t, a, b)$. If $M_t(\langle\rangle) \in I$ then $M_t(\langle\rangle) \in I'$, and we conclude that $G(I) \subseteq G(I')$.

It remains to bound the measure of $G(I)$; by Remark 2.8 it suffices to show that $\lambda(G(I))$ is bounded by a constant multiple of $|I|$. Fix a rational open interval I, and let $T = T(I)$ be the set of stages t at which $M_t(\langle\rangle) \in I$; so T is an interval (or ray) of stages. We bound the measure of $G(I)$ by considering two parts: oscillations which "go bad", and oscillations which remain more or less good.

Let $c = (a+b)/2$ (any number in the interval (a,b) would do, but it cannot depend on I). Let $t^* = \sup T$ (if $M(\langle\rangle) \in I$ then $t^* = \omega$, and below we let $M_\omega = M$). We let $G(I)^{\text{bad}} = G(I)\backslash\mathcal{O}_{n_I}(M_{t^*}, c, b)$, that is, all sequences X which appear to be oscillating n_I times between a and b at some stage $t \in T$, but by stage t^* we see that they no longer oscillate even between c and b. We let $G(I)^{\text{good}} = G(I) \cap \mathcal{O}_{n_I}(M_{t^*}, c, b)$.

First, we observe that as $G(I)^{\text{good}} \subseteq \mathcal{O}_{n_I}(M_{t^*,c,b})$, Lemma 5.8 shows that $\lambda(G(I)^{\text{good}}) \leqslant (c/b)^{n_I}$.

Next, we consider $G(I)^{\text{bad}}$. Here we note that for all $X \in G(I)^{\text{bad}}$ there is some k and some $t < t^*$ in T such that $M_t(X\upharpoonright_k) < a$ but $M_{t^*}(X\upharpoonright_k) \geqslant c$. Let $t_* = \min T$ and let K be the set of minimal strings σ such that $M_{t_*}(\sigma) < a$ and $M_{t^*}(\sigma) \geqslant c$. Thus, $G(I)^{\text{bad}}$ is contained in the open set $\mathcal{K}$ generated by K. Since K is an antichain, Kolmogorov's inequality applied to the martingale $M_{t^*} - M_{t_*}$ shows that

$$\lambda(\mathcal{K}) \leqslant \frac{M_{t^*}(\langle\rangle) - M_{t_*}(\langle\rangle)}{c-a} \leqslant \frac{|I|}{c-a}.$$

Thus, overall,

$$\lambda(G(I)) \leqslant \left(\frac{c}{b}\right)^{n_I} + \frac{|I|}{c-a}.$$

As mentioned above, by Remark 2.8 is suffices to choose C so that $(c/b)^{n_I}$ is bounded by a constant multiple of $|I|$. Since $n_I \geqslant -C\log_2(|I|)$, we have

$$\left(\frac{c}{b}\right)^{n_I} \leqslant (c/b)^{-C\log_2(|I|)} = |I|^{-C\log_2(c/b)}.$$

So we choose $C = -1/\log_2(c/b)$ and are done. $\square$

5.2. **Martingale convergence and full density.** We use the following lemma to lift Corollary 5.7 to full (non-dyadic) density.

Proposition 5.9. *Let $\mathcal{C} \subseteq \mathbb{R}$ be effectively closed, and let $z \in \mathbb{R}$ be irrational. Then $\mathcal{C}$ has density 1 at $z \Leftrightarrow \mathcal{C}$ has dyadic density 1 at z and $\mathcal{C} + 1/3$ has dyadic density 1 at $z + 1/3$.*

Proof. $\Rightarrow$: Full density is translation-invariant. So, by hypothesis that $\mathcal{C}$ has Lebesgue density 1 at z, $\mathcal{C} + 1/3$ has density 1 at $z + 1/3$. Hence $\mathcal{C}$ has dyadic density 1 at z and $\mathcal{C} + 1/3$ has dyadic density 1 at $z + 1/3$.
$\Leftarrow$: We rely on a geometric fact already used in [38]. For $m \in \omega$ let D_m be the collection of open intervals of the form

$$(k2^{-m}, (k+1)2^{-m})$$

where $k \in \mathbb{Z}$. Let D'_m be the set of intervals of the form $I - 1/3$ where $I \in \mathsf{D}_m$.

Fact 5.9.1. Let $m \geqslant 1$. If $I \in \mathsf{D}_m$ and $J \in \mathsf{D}'_m$, then the distance between an endpoint of I and an endpoint of J is at least $1/(3 \cdot 2^m)$.

To see this, assume that $|k2^{-m} - (p2^{-m} - 1/3)| < 1/(3 \cdot 2^m)$ for $k, p \in \mathbb{Z}$. This yields $|3k - 3p + 2^m| < 1$, and hence $3|2^m$ (or $2^m = 0$), a contradiction.

For every $m \geqslant 1$, let I_m be the unique interval in D_m which contains z, and let I'_m be the unique interval in D_m which contains $z + 1/3$. Given $\varepsilon > 0$, by hypothesis we may choose $m^* \in \omega$ so that for each $n \geqslant m^*$ we have

$$2^n \lambda(I_n \cap \mathcal{C}) \geqslant 1 - \varepsilon \text{ and } 2^n \lambda(I'_n \cap (\mathcal{C} + 1/3)) \geqslant 1 - \varepsilon.$$

Fix $n \geqslant m^*$. Let $J_n = I'_n - 1/3$, so $I_n \in \mathsf{D}_n$ and $J_n \in \mathsf{D}'_n$. Then $z \in I_n \cap J_n$ and $\lambda((I_n \cup J_n) \cap \mathcal{C}) \geqslant (1-\varepsilon)|I_n \cup J_n|$ (to see this, note that for interval L, $\lambda(L \cap \mathcal{C})/\lambda L$ is the slope of the function $x \to \lambda([v, x] \cap \mathcal{C})$ at L, where $v < \min L$; the slope at the interval $I \cup J$ is at least the minimum of the slopes at I and at J).

Suppose that K is an open interval containing z such that $|K| < 2^{-m^*-2}$. Find $n \geqslant m^*$ such that $2^{-n-3} < |K| \leqslant 2^{-n-2}$. Fact 5.9.1 shows that the distance of z to either endpoint of $I_n \cup J_n$ is greater than $2^{-n}/3$, and so $K \subseteq I_n \cup J_n$. However, $|K| > 2^{-n}/8 > |I_n \cup J_n|/16$, and so

$$\frac{\lambda(K \cap \mathcal{C})}{|K|} \geqslant \frac{\lambda((I_n \cup J_n) \cap \mathcal{C})}{|K|} - \frac{|I_n \cup J_n| - |K|}{|K|} \geqslant$$
$$\frac{(1-\varepsilon)|I_n \cup J_n|}{|K|} - \frac{|I_n \cup J_n|}{|K|} + 1 \geqslant 1 - 16\varepsilon. \quad \square$$

Corollary 5.10. *Every Oberwolfach random real is a density-one point.*

Proof. In view of Proposition 5.9, it suffices to show that if $z \in \mathbb{R}$ is Oberwolfach random, then so is $z + 1/3$. If $\langle \mathcal{U}_n \rangle$ is a left-c.e. bounded test which captures $z + 1/3$, then

$$\langle \mathcal{U}_n - 1/3 \rangle = \langle \{x - 1/3 : x \in \mathcal{U}_n\} \rangle$$

is a left-c.e. bounded test which captures z. $\square$

6. Oberwolfach randomness and differentiability

A classic result of Lebesgue states that every non-decreasing function $f\colon [0,1] \to \mathbb{R}$ is differentiable almost everywhere. As mentioned above, almost-everywhere theorems invite effectivization. In this case, for a given class $\mathfrak{F}$ of effective non-decreasing functions, we ask how random must a real z be so that every function in $\mathfrak{F}$ is differentiable at z. For example, Brattka, Miller and Nies [5] studied the case that $\mathfrak{F}$ is the class of non-decreasing computable functions (which, for continuous non-decreasing functions, coincides with the class of functions f mapping a rational number q to a computable real, uniformly in q). They showed that the randomness notion corresponding to this effective version of the differentiability theorem is computable randomness, a notion properly implied by Martin-Löf randomness.

Here we consider the larger class $\mathfrak{F}$ of interval-c.e. functions. We observe that the corresponding randomness notion – the collection of reals z at which every interval-c.e. function is differentiable – implies ML-randomness; indeed, it implies that every left-c.e. martingale converges on the binary expansion of z. However, a short and direct argument, avoiding Proposition 5.9, shows that each such point is a density-one point. We then show that Oberwolfach randomness implies this randomness property. This gives us a modified proof of Corollary 5.10.

6.1. **Interval-c.e. functions.** Recall that a function $f\colon [0,1] \to \mathbb{R}$ is *lower semicontinuous* if for every $q \in \mathbb{R}$, the inverse image $f^{-1}(q,\infty)$ is open. Upper semicontinuity is defined analogously, with $f^{-1}(-\infty,q)$ being open instead.

The effective version of lower semicontinuity is lower semicomputability. A function $f\colon [0,1] \to \mathbb{R}$ is *lower semicomputable* if for every rational number q, $f^{-1}(q,\infty)$ is effectively open, uniformly in q. A function f is lower semi-computable if and only if it has an approximation from below; an increasing computable sequence $\langle f_s \rangle_{s<\omega}$ of rational-valued step functions – linear combinations of characteristic functions of rational intervals – such that $f = \lim_s f_s$ pointwise. The notion of upper semicomputable functions is defined analogously; a function f is lower semicomputable if and only if $-f$ is upper semicomputable.

Lower semi-continuous functions to the extended real line can be used, for example, to characterise Martin-Löf randomness, via so called "integral tests" [52, 21]: $z \in [0,1]$ is ML-random if and only if for every integrable lower semicomputable function $f\colon [0,1] \to [0,\infty]$ we have $f(z) < \infty$. When f is universal among the integral tests, $f(z)$ can be thought of as the "randomness deficiency" of z, analogous to the index of the least component of a universal ML-test omitting z.

Identifying the variations of computable functions, Freer, Kjos-Hanssen, Nies and Stephan [20] studied a class of monotone, continuous, lower semicomutable functions which they called *interval-c.e.*

Let $g\colon [0,1] \to \mathbb{R}$. For $0 \leqslant x < y \leqslant 1$ define the *variation* of g in $[x,y]$ by

$$V(g,[x,y]) = \sup\left\{ \sum_{i=1}^{n-1} \left| g(t_{i+1}) - g(t_i) \right| : x \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant y \right\}.$$

The function g is of bounded variation if $V(g,[0,1])$ is finite. If g is a continuous function of bounded variation then the function $f(x) = V(g,[0,x])$ is also continuous. If g is computable then the function $f(x) = V(g,[0,x])$ is lower semicomputable (but may fail to be computable). A further property of this "variation function" comes from the observation that $V(g,[x,y]) + V(g,[y,z]) = V(g,[x,z])$ for $x < y < z$ (see [4, Prop. 5.2.2]).

Definition 6.1. A non-decreasing, lower semicontinuous function $f\colon [0,1] \to \mathbb{R}$ is *interval-c.e.* if $f(0) = 0$, and $f(y) - f(x)$ is a left-c.e. real, uniformly in rationals $x < y$.

Thus, the variation function of each computable function of bounded variation is interval-c.e. Freer et al. [20], together with Rute, showed that conversely, every continuous interval-c.e. function is the variation of a computable function.

Note that if the assumption of lower semicontinuity is dropped from Definition 6.1 then we obtain an uncountable class of functions; if an interval-c.e. function f is discontinuous at an irrational point a, then changing the value of $f(a)$ to any number between $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(a)$ results in a function in that class.

We mention that nonetheless, the differentiability results in this section all hold for this wider class of functions.

On the other hand, once we require lower semicontinuity, we see that every interval-c.e. function is lower semicomputable: $f(x) > q$ if and only if there is some rational number $r < x$ such that $f(r) - f(0) > q$, and this is an effectively open condition.

A simple example of a continuous interval-c.e. function is the function $f(x) = \lambda(\mathcal{U} \cap [0, x))$, where $\mathcal{U} \subseteq [0, 1]$ is effectively open: $f(y) - f(x) = \lambda(\mathcal{U} \cap (x, y))$ is the measure of a uniformly given, effectively open set, and so is uniformly a left c.e. real.

6.1.1. *Measures and martingales.* In general, the non-decreasing and lower semicontinuous functions $f\colon [0, 1] \to \mathbb{R}$ with $f(0) = 0$ correspond to Borel measures μ on $[0, 1)$ by letting $f_\mu(x) = \mu([0, x))$; in the other direction, given f, we let μ_f be the measure generated by letting $\mu_f([x, y)) = f(y) - f(x)$. The measure μ_f is atomless if and only if f is continuous.

Recall that a martingale M is atomless if the measure μ_M on 2^ω generated by letting $\mu_M([\sigma]) = 2^{-|\sigma|}M(\sigma)$ has no atoms; that is, if for all $X \in 2^\omega$, $M(X \upharpoonright_n) = o(2^n)$.

There is a correspondence between continuous interval-c.e. functions f and atomless left-c.e. martingales M. Given a continuous interval-c.e. function f we define the martingale M_f by letting $M_f(\sigma) = \mu_f(\Theta[\sigma]) = f(0.\sigma 1) - f(0.\sigma 0)$ (see Remark 2.1 for the definition of the near-isomorphism Θ). In the other direction, given M, by pushing forward by Θ, we identify μ_M and $\Theta^*\mu_M$ and so view μ_M as a measure on $[0, 1]$. The corresponding function, denoted by f_M, turns out to be interval-c.e.: $\mu_M([x, y])$ is the supremum of $\mu_M(A)$, where A is a finite union of dyadic intervals contained in $[x, y]$.

6.2. **Characterizing ML-randomness via the existence of the upper derivative.** Using a result of [20] and a combinatorial lemma in [5], we will characterize Martin-Löf randomness of a real z by the condition that all interval-c.e. functions have a finite upper derivative at z.

We introduce some convenient notation. Let $f\colon [0, 1] \to \mathbb{R}$. For reals $x < y$ we let

$$\Delta_f(x, y) = f(y) - f(x)$$

and let

$$S_f(x, y) = \frac{f(y) - f(x)}{y - x}$$

denote the slope of f between x and y. Sometimes we write $\Delta_f(I)$ and $S_f(I)$ to denote $\Delta_f(x, y)$ and $S_f(x, y)$ where $I = (x, y)$. Below, it is sometimes important that we work with *open* intervals.

In this section and below, we make use of the notion of (p, q)*-intervals*. For rational numbers $p > 0$ and q, we call an image of a basic dyadic interval under the map $y \mapsto py + q$ a (p, q)-interval. That is, the (p, q)-intervals in $\mathbb{R}$ are the intervals whose closure is of the form $[p \cdot m2^{-n} + q, p \cdot (m + 1)2^{-n} + q]$ for some $n \geqslant 0$ and $m \in \mathbb{Z}$. As is shown in [5], (p, q)-intervals allow us to reduce analytic questions on the real line to arguments in the relatively simple setting of Cantor space. For a set L of rational numbers, an L-interval is a (p, q)-interval for some $p, q \in L$.

The main combinatorial lemma concerning (p, q)-intervals is Lemma 4.2 from [5]:

Lemma 6.2. *For any real $\alpha > 1$ there is a finite set L of rationals as follows: for every interval $A \subset \mathbb{R}$ there are L-intervals B and C such that*

- *$A \subset B$ and $|B|/|A| < \alpha$;*
- *$C \subset A$ and $|A|/|C| < \alpha$.*

Let $f\colon [0,1] \to \mathbb{R}$ and $z \in [0,1]$. In the notation introduced above,

$$\overline{D}f(z) = \limsup_{h\to 0} S_f(z, z+h)$$

and

$$\underline{D}f(z) = \liminf_{h\to 0} S_f(z, z+h),$$

where of course if $h < 0$ then $(z, z+h) = (z+h, z)$.

Since z may be noncomputable, we need to approximate these quantities by looking at rational intervals close to z. Even this may be too complicated, for example when we want to capture non-differentiability by martingales. We prefer to work with (p,q) intervals. Slightly modifying notation from [5], for shorthand we let

$$D^{(p,q)}f\,(z) = \limsup_{|A|\to 0} S_f(A)\ [\![A \text{ is a } (p,q)\text{-interval and } z \in A]\!],$$

and

$$D_{(p,q)}f\,(z) = \liminf_{|A|\to 0} S_f(A)\ [\![A \text{ is a } (p,q)\text{-interval and } z \in A]\!].$$

The following lemma is related to Lemma 4.3 in [5].

Lemma 6.3. *Let $f\colon [0,1] \to \mathbb{R}$ be nondecreasing. Let $z \in [0,1]$, and suppose that $\overline{D}f(z) = \infty$. Then there are rationals $p > 0$ and q such that $D^{(p,q)}f(z) = \infty$.*

Proof. Find a finite set L given by Lemma 6.2 for $\alpha = 2$. Let $h > 0$, and let A be either $[x, x+h]$ or $[x-h, x]$. Find an L-interval B with $A \subset B$ and $|B| < 2|A|$. Then $S_f(B) \geqslant S_f(A)/2$. Since L is finite, the pigeonhole principle, applied as $h \to 0$, gives a single pair (p,q) from L which witnesses that $D^{(p,q)}f(z) = \infty$. □

Theorem 6.4. *Let $z \in [0,1]$. Then z is ML-random $\Leftrightarrow$*
$\overline{D}f(z) < \infty$ for each interval-c.e. function f.

Proof. ($\Leftarrow$) Let $\mathbb{U}$ be a universal prefix-free machine (see for instance [40, Chapter 2]). Freer et al. [20, Prop 2.6] show that the function $f_{\mathbb{U}}(x) = \lambda[\{\sigma\colon \mathbb{U}(\sigma) < x\}]^{\prec}$ is interval-c.e., and that $\overline{D}f_{\mathbb{U}}(z) < \infty$ implies that z is ML-random.

($\Rightarrow$) Suppose that f is interval c.e. and $\overline{D}f(z) = \infty$. Applying Lemma 6.3, let $p > 0$ and q be rationals such that $D^{(p,q)}f(z) = \infty$. If z is on the boundary of some (p,q)-interval, then z is rational and so not Martin-Löf random. So we may assume that z is in the interior of any (p,q)-interval which contains it.

Define the Σ^0_1 set $\mathcal{U}_n\langle s\rangle$ as follows: enumerate into $\mathcal{U}_n$ all *open* (p,q)-intervals A such that $S_f(A) > 2^n$ (by our earlier argument, open intervals will suffice). This is indeed a Σ^0_1 set since the real $S_f(A)$ is left-c.e. uniformly in A. Let U_n be the set of maximal (with respect to inclusion) (p,q)-intervals enumerated into $\mathcal{U}_n$. Importantly, these are disjoint. Let μ_f be the measure defined on intervals by $\mu_f([x,y]) = f(y) - f(x)$. Then we have

$$\lambda(\mathcal{U}_n) = \sum_{A\in U_n} \lambda(A) = \sum_{A\in U_n} \mu_f(A)/S_f(A) \leqslant 2^{-n} \sum_{A\in U_n} \mu_f(A) \leqslant f(1)\cdot 2^{-n},$$

with the last inequality coming from the fact that the elements of U_n are pairwise disjoint. Since by construction each $\mathcal{U}_n$ contains z, we conclude that z is not Martin-Löf random. □

The direction ($\Leftarrow$) of Theorem 6.4 has an alternative proof based on the following two facts of interest on their own. We obtain connections between differentiability and martingale convergence.

Proposition 6.5. *Let M be an atomless left-c.e. martingale, and let f_M be the corresponding interval-c.e. function (from Subsection 6.1.1). Let $X \in 2^\omega$ and let x be the real with binary expansion X.*

1. *If M succeeds on X then $\overline{D}f_M(x) = \infty$.*
2. *If f_M is differentiable at x then M converges on X.*

Proof. Let $\tau \prec X$ and let $I = \Theta[[\tau]] = [a, b]$; so $a \leqslant x \leqslant b$. We have $M(\tau) = S_f(I)$, and

$$\min\{S_f(a, x), S_f(x, b)\} \leqslant S_f(I) \leqslant \max\{S_f(a, x), S_f(x, b)\}$$

(see [5, Fact 2.4]). □

A universal left-c.e. martingale introduced by Stephan is atomless. For a string τ, let E_τ be the martingale which starts with 1, doubles its capital along τ, and then rests. So $E_\tau(\sigma) = 0$ if τ and σ are incomparable; for $n \leqslant |\tau|$, $E_\tau(\tau\restriction_n) = 2^n$; and $E_\tau(\sigma) = 2^{|\tau|}$ for σ extending τ. Stephan [50] showed that the martingale $M = \sum_\tau 2^{-K(\tau)}E_\tau$ is universal (also see [40, Thm. 7.2.8]). In [24] it is shown that Stephan's martingale is atomless.

Now ($\Leftarrow$) of Theorem 6.4 can be proved as follows. Suppose that $z \in [0, 1]$ is not ML-random, and let Z be a binary expansion of z. Then Z is not ML-random, and so Stephan's martingale succeeds on M. By Proposition 6.5, $\overline{D}f_M(z) = \infty$. Indeed, since f_M is continuous, we obtain the following strengthening of Theorem 6.4:

Corollary 6.6. *The following are equivalent for $z \in [0, 1]$:*

1. *z is ML-random.*
2. *$\overline{D}f(z) < \infty$ for every interval-c.e. function f.*
3. *$\overline{D}f(z) < \infty$ for every continuous interval-c.e. function f.*

6.3. **Being a point of differentiability of every interval-c.e. function.** In the introduction we discussed the program of determining the randomness strength needed to make effective versions of "almost-everywhere" theorems hold. Recall that a main result in [5, Section 4] states that a real z is computably random iff (a) every nondecreasing computable function g satisfies $\overline{D}g(z) < \infty$ iff (b) every nondecreasing computable function f is differentiable at z. In our setting the effectiveness condition is being interval c.e. The analog of (a) is equivalent to ML-randomness by Theorem 6.4. In contrast, the analog of (b) is stronger than ML-randomness by the following.

Proposition 6.7. *Suppose that every continuous interval-c.e. function is differentiable at z. Then the following hold.*
(i) Every left-c.e. martingale converges on the binary expansion $Z = \Theta^{-1}(z)$, and so z is ML-random.
(ii) z is a density-one point.

Proof. (i) First, we note that if f is differentiable at z then $\overline{D}f(z) < \infty$, and so we conclude that z is ML-random (Theorem 6.4). Suppose that some left-c.e. martingale N does not converge on Z. Since $\{N(Z\restriction_n) : n < \omega\}$ is bounded, we can produce a bounded martingale M which does not converge on Z (when $N(\sigma)$ exceeds a bound on $N(Z\restriction_n)$, stop betting). The martingale M is certainly atomless. Proposition 6.5 shows that the continuous interval-c.e. function f_M is not differentiable at z.

(ii) We could combine (i) with the argument in the proof of Corollary 5.10. For a direct proof, let $\mathcal{P}$ be an effectively closed set containing z. Let $g(x) = \lambda([0, x)]\backslash\mathcal{P})$. Then the function g is interval-c.e. and continuous (see Subsection 6.1.1). By hypothesis, $g'(z)$ exists. The differentiability of g at z implies that $g'(z)$ is in fact the limit, as $|I| \to 0$, of $S_g(I)$ for *open* intervals I containing z (see the proof of

Proposition 6.5). Hence, $g'(z)$ is the density of $[0,1]\backslash\mathcal{P}$ at z. Thus, we conclude that the upper and lower density of $\mathcal{P}$ at z are the same. Since $z \in \mathcal{P}$ and z is ML-random, Proposition 5.4 says that the upper density of $\mathcal{P}$ at z is 1. Hence the lower density of $\mathcal{P}$ at z is 1. □

We turn to the main task of this subsection.

Theorem 6.8. *Let z be an Oberwolfach random real. Then every interval-c.e. function is differentiable at z.*

The proof of Theorem 6.8 is a more complex variant of the proof of Theorem 5.6. For an interval-c.e. function f and an Oberwolfach random real z, we need to show that the upper and lower derivatives of f at z are finite and equal. Finiteness follows from Theorem 6.4. If $\underline{D}f(z) < \overline{D}f(z)$ then we want to capture z by an interval test with associated left-c.e. real $f(1)$; this will be done by enumerating intervals on which we observe long oscillations of the slope S_f.

We note two new problems.

1. We have no access to z directly, and so cannot measure the slopes $S_f(z, z+h)$ which presumably oscillate beyond two rationals values. We may assume though that f is continuous at z, and so we can find oscillations in $S_f(I)$ for rational intervals I containing z. Even this, though, is insufficient for finding the bound on the measure of $G(I)^{\mathrm{bad}}$. In the analogous calculation in the proof of Theorem 5.6, we made use of the antichain K of minimal strings on which M rises from a to c; in the current proof this will be a set of intervals on which the slope grows from a to c. But we may have two overlapping intervals of this kind, where the union is not so. As in the proof of Theorem 6.4, we want to mimic the structure of Cantor space. This is again done with the aid of (p,q)-intervals.

2. The previous argument, in particular bounding the measure of $G(I)^{\mathrm{bad}}$, relied on the existence of a nice effective approximation for the martingale M, namely the increasing sequence of rational-valued martingales M_t. There may be no full analogue of this approximation for the function f. Even though $S_f(x,y)$ is left-c.e., uniformly in rationals x and y, the stage t approximations to these values need not be coherent with each other. There may be no computable sequence of functions f_t increasing to f such that f_t is defined on all rational numbers and $\Delta_{f_t}(x,y)$ is non-decreasing for all rationals $x < y$. We restrict ourselves to partially defined approximating functions.

Let $Q_0 \subset Q_1 \subset Q_2 \subset \dots$ be an increasing computable sequence of finite sets whose union is $\mathbb{Q} \cap [0,1]$; we assume that $0, 1 \in Q_0$. For rationals $x < y$ let $\langle \alpha_t^{x,y} \rangle_{t<\omega}$ be a computable increasing sequence of rational numbers whose limit is $\Delta_f(x,y)$. For $t < \omega$ and $x \in Q_t$, we let

$$f_t(x) = \max \sum_{i=1}^{m} \alpha_t^{x_{i-1},x_i} \; [\![0 = x_0 < x_1 < \dots < x_m = x \text{ are in } Q_t]\!].$$

We extend the slope notation to $S_{f_t}(I)$, where I is an interval with endpoints in Q_t.

- For any rational x, $\langle f_t(x) \rangle_{x \in Q_t}$ is a non-decreasing sequence of rationals which converges to $f(x)$.
- For any interval I with endpoints in Q_t, $S_{f_t}(I) \leqslant S_{f_{t+1}}(I)$.

To tackle issue (1) above, we invoke Lemma 4.3 of [5].

Lemma 6.9. *Suppose $f\colon [0,1] \to \mathbb{R}$ is nondecreasing and continuous at $z \in [0,1]$, but that $\underline{D}f(z) < \overline{D}f(z)$. Then there are pairs of rationals (p,q) and (r,s) such that $D_{(r,s)}f(z) < D^{(p,q)}f(z)$.*

We need to define oscillations in the context of the functions f_t and f. Let g be a partial function from $[0,1]$ to $\mathbb{R}$ (we will use $g = f_t$ for various stages t). Fixing parameters, pairs (p,q) and (r,s), rationals $a < b$, and $n < \omega$, we let $\mathcal{O}_n = \mathcal{O}_n(g,a,b) = \mathcal{O}_n(g,a,b;r,s,p,q)$ be the set of real numbers $z \in (0,1)$ for which there is a sequence of open intervals $I_1 \supset J_1 \supset I_2 \supset J_2 \supset \cdots \supset I_n \supset J_n$ such that $z \in J_k$ and for all $k = 1,\dots,n$,

- I_k is an (r,s)-interval with endpoints in $\operatorname{dom} g$ and $S_g(I_k) < a$; and
- J_k is a (p,q)-interval with endpoints in $\operatorname{dom} g$ and $S_g(J_k) > b$.

We note that $\mathcal{O}_n$ is open; it is the union of intervals which appear as J_k in such an oscillating sequence.

We again have to bound the measure of all truly oscillating sequences. This is a calculation which is classical (it contains no consideration of effectiveness). The calculation is very similar to that proving Lemma 5.8.

Lemma 6.10. $\lambda(\mathcal{O}_n(g,a,b)) \leqslant (a/b)^n$.

Proof. Shortest strings correspond to maximal intervals. Here U_n is a set of pairwise disjoint (r,s)-intervals and V_n is a set of pairwise disjoint (p,q)-intervals. All intervals mentioned have endpoints in $\operatorname{dom} g$. V_n refines U_n (in the sense that every interval in V_n is contained in an interval in U_n) and U_{n+1} refines V_n. We let $V_{-1} = (0,1)$. Given V_n, we let U_{n+1} be the collection of maximal (r,s)-intervals I contained in some interval in V_n such that $S_g(I) < a$. Given U_n, we let V_n be the collection of maximal (p,q)-intervals J contained in some interval in V_n such that $S_g(J) > b$. By the same process of "maximisation", we observe that $\mathcal{O}_n$ is generated by V_n, and by induction show that $\lambda(\mathcal{O}_n) \leqslant (a/b)^n$. Instead of using Kolmogorov's inequality, we note that if $S_g(I) < a$ and V is a collection of pairwise-disjoint intervals $J \subseteq I$ with $S_g(J) > b$ then $\lambda(\bigcup V) \leqslant |I| \cdot (a/b)$. To see why, observe that for any finite $V' \subseteq V$ we have $\Delta_g(I) \geqslant \sum_{J \in V'} \Delta_g(J)$. $\square$

Proof of Theorem 6.8. Let $f\colon [0,1] \to \mathbb{R}$ be interval-c.e. Suppose that f is not differentiable at z. We show that z is not Oberwolfach random. If $\overline{D}f(z) = \infty$ then z is not ML-random (Theorem 6.4), and so certainly not Oberwolfach random. So we may assume that the upper and lower derivatives $\overline{D}f(z)$ and $\underline{D}f(z)$ are both finite and that $\underline{D}f(z) < \overline{D}f(z)$. Since f is non-decreasing and $\overline{D}f(z) < \infty$, we conclude that f is continuous at z (in fact, in [24] it is shown that every point of discontinuity of an interval-c.e. function is computable). By Lemma 6.9, there are pairs (r,s) and (p,q), and rationals $a < b$, such that

$$D_{(r,s)}f(z) < a < b < D^{(p,q)}f(z).$$

We note that if I is any open interval and $x \in I$ is irrational, then $z \in J$ for some open (p,q)-subinterval J of I (and similarly for (r,s)). Since we may assume that z is irrational, we see that for every n, $z \in \mathcal{O}_n(f,a,b)$ (from now we fix the parameters r,s,p,q and do not mention them again). We fix an approximation $\langle f_t \rangle$ to f as described in (2) above. We note that for all n, for almost all t, $z \in \mathcal{O}_n(f_t,a,b)$.

After applying a rational scaling factor to f, we may assume that $f(1) < 1$. We define an interval test associated with the left-c.e. real $f(1)$. Again let $C = -1/\log_2(c/b)$. For an interval $I \subseteq [0,1]$, we let $n_I = 1 + \lfloor C \cdot (-\log_2 |I|) \rfloor$ and define

$$G(I) = \bigcup \mathcal{O}_{n_I}(f_t,a,b)\ [\![f_t(1) \in I]\!].$$

If $f(1) \in G(I)$ then $z \in G(I)$ so $(G, f(1))$ captures z. It remains to show that G is an interval array.

The argument follows that of the proof of Theorem 5.6. The proof that G respects inclusion is verbatim. To bound the measure of $G(I)$, we again separate to good and bad parts; we again let $T = T(I)$ be the interval (or ray) of stages t

such that $f_t(1) \in I$, $c = (a+b)/2$, $G(I)^{\text{bad}} = G(I) \setminus \mathcal{O}_{n_I}(f_{\sup T}, c, b)$ and $G(I)^{\text{good}} = G(I) \cap \mathcal{O}_{n_I}(f_{\sup T}, c, b)$. We use Lemma 6.10 to see that $\lambda(G(I)^{\text{good}}) \leqslant (c/b)^{n_I}$ which is bounded by $|I|$ by the choice of C and n_I.

Let $t^* = \sup T$ and $t_* = \min T$. We note that $G(I)^{\text{bad}}$ is contained in the union of (r,s)-intervals J such that for some $t \in [t_*, t^*)$, J's endpoints lie in Q_t and $S_{f_t}(J) < a$ but $S_{f_{t^*}}(J) \geqslant c$. We let K be the set of maximal such intervals; the intervals in K are pairwise disjoint, and $G(I)^{\text{bad}} \subseteq \mathcal{K}$ where $\mathcal{K}$ is the open set generated by K. We again want to show that $\lambda(\mathcal{K}) \leqslant (f_{t^*}(1) - f_{t_*}(1))/(c-a)$, but the fact that intervals in K may not have endpoints in Q_{t_*} makes our life a bit harder.

Claim 6.10.1. Let $s < \omega$, and let L be a finite set of pairwise disjoint intervals with endpoints in Q_s. For all $t \geqslant s$,

$$f_t(1) - f_s(1) \geqslant \sum_{J \in L} |J| \cdot (S_{f_t}(J) - S_{f_s}(J)).$$

Proof. Enumerate L as $\{J_1, J_2, \ldots, J_m\}$ with $J_i = (x_i, y_i)$ and $y_i \leqslant x_{i+1}$. For $i = 1, \ldots, m$ we have $f_t(y_i) - f_s(y_i) \geqslant (f_t(x_i) - f_s(x_i)) + |J_i| \cdot (S_{f_t}(J_i) - S_{f_s}(J_i))$, and we have $f_t(x_{i+1}) - f_s(x_{i+1}) \geqslant f_t(y_i) - f_s(y_i)$. By induction, we see that

$$f_t(1) - f_s(1) \geqslant f_t(y_m) - f_s(y_m) \geqslant \sum_{i=1}^{m} |J_i| \cdot (S_{f_t}(J_i) - S_{f_s}(J_i))$$

as required. □

Let $K' \subseteq K$ be finite. For each $J \in K'$, let $t(J)$ be the least stage $t \in [t_*, t^*)$ such that the endpoints of J are in Q_t. For $t \in [t_*, t^*)$, let K_t be the set of intervals $J \in K'$ with $t(J) \leqslant t$. Claim 6.10.1 tells us that for each t,

$$f_{t+1}(1) - f_t(1) \geqslant \sum_{J \in K_t} |J| \cdot (S_{f_{t+1}}(J) - S_{f_t}(J)).$$

Summing for $t \in [t_*, t^*)$, we get

$$f_{t^*}(1) - f_{t_*}(1) \geqslant \sum_{J \in K'} |J| \cdot (S_{f_{t^*}}(J) - S_{f_{t(J)}}(J)) \geqslant (c-a) \sum_{J \in K'} |J|.$$

Taking larger and larger $K' \subseteq K$ (if K is infinite) shows that

$$f_{t^*}(1) - f_{t_*}(1) \geqslant \lambda(\mathcal{K}) \cdot (c-a)$$

as required.

This concludes the proof. We remark again on the difference between this last calculation and the corresponding one in the proof of Theorem 5.6. If we knew that all the intervals in K had endpoints in Q_{t_*}, then we could take the function $f_{t^*} \upharpoonright_{Q_{t_*}} - f_{t_*}$ and build a martingale from the slopes of this function on (r,s)-intervals; then, the inequality would follow from Kolmogorov's inequality. This is precisely where the absence of a "nice" approximation $\langle f_t \rangle$ (as in problem (2) above) makes us work harder.

On the other hand, for Claim 6.10.1, we could have restricted ourselves to (r,s)-intervals and used this approach. We preferred to give a direct proof which does not pass through martingales. □

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