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# Coherent States of Fermi Operators and the Path Integral*)

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Coherent states of Fermi operators are explicitly constructed with the aid of Grassmann numbers, and their properties are discussed in detail. A rigorous form of the path integral for Fermi fields is formulated by means of the coherent states. A simple example is also discussed.

## § 1. Introduction

It is well known that the classical picture of quantum-mechanical amplitudes of Bose oscillators is given by a set of complex eigenvalues of the annihilation operators. The eigenstate corresponding to it is called the coherent state. In this connection, if one wishes to get a classical analog of Fermi operators, it seems reasonable to consider their coherent states. In this case, however, even if a coherent state exists, corresponding eigenvalues are no longer made complex numbers because of the anticommutation property of Fermi operators. Thus to be consistent the eigenvalues must be anticommutable with each other. Such peculiar anticommuting $c$-numbers are known as the Grassmann numbers or the Grassmann variables.[2] Schwinger[3] is perhaps the first who introduced the Grassmann variable in field theory. He used it as a source of fermion field in his theory of Green's function. Some systematic and elegant studies on applications of the Grassmann variables have been made by Berezin.[4),5),6]

The purpose of the present paper is to construct coherent states of fermions***) with the aid of the Grassmann numbers and to apply them for a description of fermion systems.

Now suppose that $\xi_1$, $\xi_2$, $\cdots$, $\xi_n$ are $n$ independent Grassmann numbers. They are defined by

$$\{\xi_i, \xi_j\} = 0 \quad (i, j = 1, 2, \cdots, n) \qquad (1\cdot 1)$$

and

*) Lectures[1] based on a preliminary version of this work were given by one of the authors (Y. O.) at National Laboratory of High Energy Physics (KEK), Tsukuba, on 7th and 8th of July, 1977.

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***) After completion of this work the authors knew that almost the same idea had already been proposed by Martin.[7] Though the formalism is not complete, his paper may be the first in which coherent states of fermions are explicitly discussed. Montonen[8] also introduced the coherent states which correspond to our $|(\xi)_n\rangle$ and $\langle(\xi)_n{}^*|$.

$$\xi_1\xi_2\cdots\xi_n \neq 0\,. \tag{1·2}$$

On the basis of these relations we shall in § 2 study some general properties of the Grassmann variables, which will play fundamental roles in our later discussion. With these preparations we shall in § 3 explicitly construct coherent states of Fermi operators. It is shown that they satisfy an orthogonality relation which is written in terms of $\delta$-functions defined on the Grassmann variables. Several types of the completeness relation are also given by means of the coherent states. In § 4, as representatives of the ordinary state vector of fermions we shall introduce some kinds of *wavefunctions* whose arguments are Grassmann variables. All formalisms for any fermion system are shown to be rewritten in terms of these wavefunctions. Especially a rigorous expression of the path integral for time development operator will be given. We shall also write down the trace of $e^{-\beta H}$ in a form of the path integral of the Grassmann variables, in which the antiperiodic boundary condition usually imposed on the initial and final variables is shown to be an automatic consequence of our theory.

## § 2. Integrals, δ-functions and transformations of variables

By analogy with the usual theory we define a $\delta$-function $\delta(\xi, \xi')$ for Grassmann variables $\xi$ and $\xi'$ by the relation

$$\int f(\xi)\,\delta(\xi, \xi')\,d\xi = f(\xi')\,, \tag{2·1}$$

with $f(\xi)$ being an arbitrary function of $\xi$. As usual the $\delta(\xi, \xi')$ is assumed to be a function of $\xi-\xi'$. The integral is also defined by (2·1). Since no infinitesimal quantity is definable for Grassmann variables, the symbol $\int\cdots d\xi$ is to be understood as a kind of mapping, which is assumed to satisfy the linearity

$$\int\{\alpha_1 f_1(\xi) + \alpha_2 f_2(\xi)\}\,d\xi = \alpha_1\int f_1(\xi)\,d\xi + \alpha_2\int f_2(\xi)\,d\xi\,, \tag{2·2}$$

where $\alpha_1$ and $\alpha_2$ are, in general, functions of the Grassmann numbers other than $\xi$. Though the notation $d\xi$ can have a significance as a mapping only when combined with the notation $\int$, we shall, for simplicity, sometimes drop out the latter notation in discussing the properties of the mapping and deal with $d\xi$ as if it were a kind of number.

By virtue of $\xi^2=0$ we may write the general form of $f(\xi)$ as $\alpha_1+\alpha_2\xi$. So Eq. (2·1) is equivalent to the set of equations

$$\int\delta(\xi, \xi')\,d\xi = 1\,, \quad \int\xi\,\delta(\xi, \xi')\,d\xi = \xi'\,. \tag{2·3}$$

Now writing the $\delta$-function as $\beta_1+\beta_2(\xi-\xi')$, we can easily solve Eqs. (2·3) and obtain $\delta(\xi, \xi') = (\xi-\xi')/\alpha$, $\int\xi d\xi = \alpha$ and $\int d\xi = 0$. Here $\alpha$ is a *constant* which has its own inverse and is commutable with $\xi$ and $\xi'$. Although we can develop a consistent theory with any given $\alpha$, we shall for later convenience put $\alpha=i$ and then we have

$$\delta(\xi,\xi')=\frac{1}{i}(\xi-\xi'), \tag{2·4}$$

$$\int\xi d\xi=i \quad \text{and} \quad \int d\xi=0\,. \tag{2·5}^{*)}$$

We note that, when use is made of these relations and $\exp\{-(\xi-\xi')\xi''\}=1-(\xi-\xi')\xi''$, the $\delta$-function is expressed in such a form of *Fourier transformation*, that is,

$$\delta(\xi,\xi')=\int e^{-(\xi-\xi')\xi''}d\xi''\,. \tag{2·6}$$

It is not a difficult task to generalize the above discussion to the case of multi-variable. Applying $\xi'$ to both the sides of Eqs. (2·5) from left or right, we get $\xi' d\xi=-d\xi\cdot\xi'$. Thus we may write consistently

$$\{\xi_i, d\xi_j\}=0\,, \tag{2·7}^{**)}$$

and so from (2·4) we have

$$\int\!\!\int\xi_1\xi_2\cdots\xi_n d\xi_n d\xi_{n-1}\cdots d\xi_1=(i)^n$$

and (2·8)

$$\int\!\!\int\xi_{j_1}\xi_{j_2}\cdots\xi_{j_r}d\xi_n d\xi_{n-1}\cdots d\xi_1=0 \quad \text{for } 0\leq r<n\,,$$

which also imply

$$\{d\xi_i, d\xi_j\}=0\,. \quad (i\neq j) \tag{2·9}$$

The notation $\int\!\int$ stands for the multiple integral.

Now let us consider a general transformation of the set of the Grassmann variables $\{\xi_1, \xi_2, \cdots, \xi_n\}$ into a new set $\{\xi_1', \xi_2', \cdots, \xi_n'\}$. We may write it down in the form

$$\xi_i'=\alpha_i+\sum_j\alpha_i{}^j\xi_j+\sum_{j_1<j_2}\alpha_i{}^{j_1j_2}\xi_{j_1}\xi_{j_2} + \sum_{j_1<j_2<j_3}\alpha_i{}^{j_1j_2j_3}\xi_{j_1}\xi_{j_2}\xi_{j_3}+\cdots+\alpha_i{}^{12\cdots n}\xi_1\xi_2\cdots\xi_n\,. \tag{2·10}$$

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*) Berezin[4] defined the integral with $\alpha=1$.

**) This can be shown only for $i\neq j$. But for the sake of simplicity we have assumed here the anticommutability of $\xi_i$ and $d\xi_i$, which gives a definition of integral of the form $\int d\xi\cdot\xi$.

The coefficients $\alpha_i^{\cdots}$ are, in general, functions of Grassmann numbers independent of $\xi_i (i=1, 2, \cdots, n)$,*) and in order that the new variables also obey the Grassmann algebra $\{\xi_i', \xi_j'\}=0$, they are assumed to satisfy

$$[\alpha_i^{\text{odd}}, \alpha_j^{\text{odd}}] = [\alpha_i^{\text{odd}}, \alpha_j^{\text{even}}] = [\alpha_i^{\text{odd}}, \xi_j] = 0$$

and

$$\{\alpha_i^{\text{even}}, \alpha_j^{\text{even}}\} = \{\alpha_i^{\text{even}}, \xi_j\} = 0\,, \tag{2·11}$$

where $\alpha_i^{\text{even}} (\alpha_i^{\text{odd}})$ stands for $\alpha_i^{\cdots}$ with an even (odd) number of superscripts. Then from (2·10) and (2·11) we obtain

$$(\xi_1' - \alpha_1)(\xi_2' - \alpha_2) \cdots (\xi_n' - \alpha_n) = \det(\alpha_i^j) \cdot \xi_1 \xi_2 \cdots \xi_n\,. \tag{2·12}$$

This implies that, if $\det(\alpha_i^j)^{-1}$ exists, non-vanishingness of the product $\xi_1\xi_2\cdots\xi_n$ is equivalent to that of $(\xi_1'-\alpha_1)(\xi_2'-\alpha_2)\cdots(\xi_n'-\alpha_n)$, and hence of $\xi_1'\xi_2'\cdots\xi_n'$, since $\alpha_i$ is independent of $\xi_j$'s. In other words, if $\xi_1', \xi_2', \cdots, \xi_n'$ are independent of each other, so too are $\xi_1, \xi_2, \cdots, \xi_n$ under the condition that $\det(\alpha_i^j)^{-1}$ exists, and vice versa. Thus under this condition, which will be assumed in the following, $\xi_i$ $(i=1, 2, \cdots, n)$ are inversely expressed as functions of $\xi_j'$'s in a unique way, and each of the sets $\{\xi_1, \xi_2, \cdots, \xi_n\}$ and $\{\xi_1', \xi_2', \cdots, \xi_n'\}$ satisfies the relations (1·1) and (1·2). Consequently, all formulas are required to take the same forms for both sets of the Grassmann variables connected by transformations thus defined. In this respect we note that the following relation must hold true to keep the integral formulas (2·8) invariant:

$$d\xi_n' d\xi_{n-1}' \cdots d\xi_1' = \frac{1}{J} d\xi_n d\xi_{n-1} \cdots d\xi_1 \tag{2·13}$$

with the Jacobian

$$J \equiv \det\left(\frac{\partial \xi_i'}{\partial \xi_j}\right). \tag{2·14}$$ **)

By definition of $\alpha_i^{\cdots}$ it is evident that all elements of the determinant $J$ are commutable with $\alpha_j^{\cdots}$ and $\xi_j$ $(j=1, 2, \cdots, n)$. The existence of $J^{-1}$ is assured by the existence of $\det(\alpha_i^j)^{-1}$ (see Lemma 1 in the Appendix). Unlike the case of $c$-number variables the Jacobian now appears as a denominator on the right-hand side of (2·13).***) The proof of (2·13) is somewhat lengthy, so we shall give it in the Appendix.

*) We consider here that $\xi_1, \xi_2, \cdots, \xi_n$ are chosen from a set of more than $n$ independent Grassmann variables. If necessary we shall in the following regard the number of elements of this set sufficiently large.

**) For the definition of derivatives see Ref. 4). In the present paper we shall use, for definiteness, only left derivatives.

***) The simplest case, where $\xi_i' = \alpha_i + \sum_j \alpha_i^j \xi_j$, was examined by Berezin.[4]

## § 3. An extension of the Hilbert space and coherent states of Fermi operators

Let $a_i (i=1, 2, \cdots, n)$ be Fermi operators satisfying the ordinary anticommutation relations. The vacuum state is defined by

$$a_i|0\rangle=\langle 0|a_i^\dagger=0\,. \tag{3·1}$$

The Hilbert space $\mathfrak{H}$ is spanned by linear combinations of those vectors with $c$-number coefficients which are obtained by acting the $a_i^\dagger$'s on $|0\rangle$. We will extend $\mathfrak{H}$ by introducing functions of Grassmann numbers as the coefficients of state vectors. The extended space thus defined will be denoted as $\mathfrak{H}_G$. In the same way the dual space of $\mathfrak{H}$ can also be extended to $\mathfrak{H}_G^*$ which consists of vectors $\langle 0|a_i\cdots a_j$ with coefficients of functions of Grassmann numbers. For definiteness we assume that any Grassmann number is anticommutable with fermion operators and is commutable with the vacuum states $|0\rangle$ and $\langle 0|$;

$$\{\xi, a_i\} = \{\xi, a_i^\dagger\} =0 \tag{3·2}$$

and

$$\xi|0\rangle=|0\rangle\xi\,, \quad \xi\langle 0|=\langle 0|\xi\,. \tag{3·3}$$

By the use of these relations we can make a product of any element of $\mathfrak{H}_G^*$ and that of $\mathfrak{H}_G$. For instance the product of $\langle 0|a_i$ and $\xi a_j^\dagger|0\rangle$ is given by $\langle 0|a_i\xi a_j^\dagger|0\rangle =-\delta_{ij}\xi$, for $\xi a_i^\dagger|0\rangle$ is equal to $-a_j^\dagger|0\rangle\xi$ by virtue of (3·2) and (3·3). Furthermore, remembering that the right-hand sides of Eqs. (2·5) are $c$-numbers, we obtain from (3·2) and (3·3)

$$\{d\xi, a_i\} = \{d\xi, a_i^\dagger\} =0 \tag{3·4}$$

and

$$d\xi|0\rangle=|0\rangle d\xi\,, \quad d\xi\langle 0|=\langle 0|d\xi\,. \tag{3·5}$$

With these preparations we define the two vectors $|(\xi)_n\rangle$ and $\langle(\xi)_n|$ by

$$|(\xi)_n\rangle\equiv\exp(-\sum_{j=1}^{n}\xi_j a_j^\dagger)|0\rangle \tag{3·6}$$

and

$$\langle(\xi)_n|\equiv\langle 0|\delta(\xi_1, a_1)\,\delta(\xi_2, a_2)\cdots\delta(\xi_n, a_n) \tag{3·7}$$

with

$$\delta(\xi_j, a_j)\equiv\frac{1}{i}(\xi_j-a_j). \tag{3·8}$$

Taking account of the relations

$$e^{\xi a_j^\dagger}a_i\,e^{-\xi a_j^\dagger}=a_i+\delta_{ij}\xi \tag{3·9}$$

and

$$\delta(\xi, a_j)\, a_j = \delta(\xi, a_j)\, \xi\,, \tag{3·10}$$

one easily finds that the states $|(\xi)_n\rangle$ and $\langle(\xi)_n|$ are coherent states of the Fermi operators $a_j$, though the eigenvalues are now the Grassmann numbers:

$$a_j|(\xi)_n\rangle = \xi_j|(\xi)_n\rangle, \tag{3·11}$$

$$\langle(\xi)_n|a_j = \langle(\xi)_n|\xi_j\,. \tag{3·12}$$

It is noted that the coherent states have the *orthogonality* relation

$$\langle(\xi)_n|(\xi')_n\rangle = \delta(\xi_1, \xi_1')\,\delta(\xi_2, \xi_2')\cdots\delta(\xi_n, \xi_n'), \tag{3·13}$$

which can be shown by using Eqs. (3·8) and (3·9).

We next show the completeness of these states:

$$\int\!\!\int |(\xi)_n\rangle\langle(\xi)_n|\,(d\xi)_n = 1\,, \tag{3·14}$$

where

$$(d\xi)_n \equiv d\xi_n d\xi_{n-1}\cdots d\xi_1\,. \tag{3·15}$$

*Proof*:

Defining the state $|j\rangle$ by

$$|j\rangle \equiv a_j^\dagger a_{j-1}^\dagger \cdots a_1^\dagger|0\rangle, \tag{3·16}$$

we have

$$\int e^{-\xi_j a_j^\dagger}|j-1\rangle\langle j-1|\delta(\xi_j, a_j)\, d\xi_j = |j\rangle\langle j| + |j-1\rangle\langle j-1|. \tag{3·17}$$

This is because the integrand is explicitly calculated as

$$\begin{aligned} e^{-\xi_j a_j^\dagger}|j-1\rangle\langle j-1|\delta(\xi_j, a_j) &= (1-\xi_j a_j^\dagger)\,|j-1\rangle\langle j-1|\,(-i)\,(\xi_j - a_j) \\ &= (-i)\,(|j-1\rangle\langle j-1|\xi_j + a_j^\dagger|j-1\rangle\langle j-1|a_j\xi_j - |j-1\rangle\langle j-1|a_j) \end{aligned}$$

and the integration is performed by the use of Eqs. (2·5). By mathematical induction Eq. (3·17) can be generalized to

$$\int\!\!\int |(\xi)_n\rangle\langle(\xi)_n|\,(d\xi)_n = \sum_{p\,;\,\text{all possible basis vectors}} |p\rangle\langle p|\,,$$

which completes the proof of the completeness. q.e.d.

We note here that the following relations are derived for any Grassmann variable $\xi$ owing to (3·2) and (3·3):

$$\xi|(\xi)_n\rangle = |(\xi)_n\rangle\xi \tag{3·18}$$

and

$$\xi\langle(\xi)_n| = (-1)^n\langle(\xi)_n|\xi\,. \tag{3·19}$$

The similar relations evidently hold true for $d\xi$.

We next obtain coherent states of $a_j^\dagger$. To this end, following Berezin[4] we shall make use of the adjoint-operation denoted by $*$, which is a generalization of the ordinary adjoint $a_j \overset{*}{\leftrightarrow} a_j^\dagger$ and $|\rangle \overset{*}{\leftrightarrow} \langle|$ in the Fock space. It is rather trivial and we have only to keep the following rules:

$$\begin{aligned}
&(\xi)^* = \xi^*\,, \quad (\xi_i\xi_j)^* = \xi_j^*\xi_i^*\,,\\
&(\alpha_1\xi_i + \alpha_2\xi_j)^* = \xi_i^*\alpha_1^* + \xi_j^*\alpha_2^*\,,\\
&(d\xi_i d\xi_j)^* = d\xi_j^* d\xi_i^*\,, \quad (\xi_i a_j)^* = a_j^\dagger\xi_i^*\,.
\end{aligned} \tag{3·20}$$ *)

Especially we have

$$(\delta(\xi, \xi'))^* = \delta(\xi'^*, \xi^*)$$

and

$$\left(\int \xi d\xi\right)^* = \int d\xi^*\xi^* = -\int \xi^* d\xi^*\,. \tag{3·21}$$

With these rules we can see that all fundamental relations that hold for $(a_i, \xi)$ also hold for $(a_i^\dagger, \xi^*)$. Thus defining $|(\xi)_n{}^*\rangle$ and $\langle(\xi)_n{}^*|$ by

$$|(\xi)_n{}^*\rangle \equiv \delta(a_n^\dagger, \xi_n^*)\,\delta(a_{n-1}^\dagger, \xi_{n-1}^*)\cdots\delta(a_1^\dagger, \xi_1^*)\,|0\rangle \tag{3·22}$$

and

$$\langle(\xi)_n{}^*| \equiv \langle 0|\exp\left(-\sum_{j=1}^n a_j\xi_j^*\right), \tag{3·23}$$

which are the adjoints of $\langle(\xi)_n|$ and $|(\xi)_n\rangle$ respectively, we find they are coherent states of $a_j^\dagger$, that is,

$$a_j^\dagger|(\xi)_n{}^*\rangle = \xi_j^*|(\xi)_n{}^*\rangle \tag{3·24}$$

and

$$\langle(\xi)_n{}^*|a_j^\dagger = \langle(\xi)_n{}^*|\xi_j^*\,. \tag{3·25}$$

In the same way, taking the adjoints of Eqs. (3·13) and (3·14), we obtain the following orthogonality and completeness relations:

$$\langle(\xi)_n{}^*|(\xi')_n{}^*\rangle = \delta(\xi_n^*, \xi_n'^*)\,\delta(\xi_{n-1}^*, \xi_{n-1}'^*)\cdots\delta(\xi_1^*, \xi_1'^*) \tag{3·26}$$

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*) The adjoint given here may be regarded as a very formal operation introduced merely for convenience, because unlike the ordinary hermitian conjugation we cannot actually discriminate whether or not two Grassmann numbers arbitrarily given are related by the adjoint. Without use of it we can build the same theory in a consistent way.

and

$$\iint (d\xi)_n^* |(\xi)_n^*\rangle\langle(\xi)_n^*| = 1 \tag{3·27}$$

with

$$(d\xi)_n^* \equiv d\xi_1^* d\xi_2^* \cdots d\xi_n^* . \tag{3·28}$$

Moreover the products of (3·14) and (3·27) give us other forms of the completeness:

$$\iint \exp(-\sum_j \xi_j \xi_j^*) |(\xi)_n^*\rangle\langle(\xi)_n| (d\xi)_n^* (d\xi)_n = 1 \tag{3·29}$$

and

$$\iint \exp(-\sum_j \xi_j^* \xi_j) |(\xi)_n\rangle\langle(\xi)_n^*| (d\xi)_n (d\xi)_n^* = 1 . \tag{3·30}$$ [8]

Here we have used the relations

$$\langle(\xi)_n^*|(\xi')_n\rangle = \exp(\sum_j \xi_j^* \xi_j'),$$

$$\langle(\xi')_n|(\xi)_n^*\rangle = \exp(\sum_j \xi_j' \xi_j^*), \tag{3·31}$$

and the anticommutability of $\xi'$ and $\xi^*$.

Before closing this section we remark that there holds the following relation:

$$\delta_{ll'} = \iint \exp(\sum_j \xi_j^* \xi_j) \langle(\xi)_n^*|l\rangle\langle l'|(\xi)_n\rangle (d\xi)_n^* (d\xi)_n \tag{3·32a}$$

$$= \iint (d\xi)_n \langle(-\xi)_n|l\rangle\langle l'|(\xi)_n\rangle , \tag{3·32b}$$

where $\{|l\rangle; l=1, 2, \cdots\}$ is a complete set of basis vectors of the Hilbert space $\mathfrak{H}$, with the orthonormality

$$\langle l|l'\rangle = \delta_{ll'} . \tag{3·33}$$

*Proof*:

Let us define a complete and orthonormal set of state vectors by

$$|N, r\rangle \equiv a_{j_N}^\dagger a_{j_{N-1}}^\dagger \cdots a_{j_1}^\dagger |0\rangle$$

with

$$j_1 < j_2 < \cdots < j_N \quad \text{and} \quad N \leq n ,$$

and

$$\langle N, r|N', r'\rangle = \delta_{NN'} \delta_{rr'} .$$

Here $r(=1, 2, \cdots, {}_N C_r)$ stands for a choice of the ${}_N C_r$ combinations of $N$ fermion operators out of the $n$ fermion operators $a_1, a_2, \cdots, a_n$. Without loss of generality we can use $|N, r\rangle$ in place of $|l\rangle$. Owing to Eq. (3·11) and the relation $\langle 0|(\xi)_n\rangle =1$, we have

$$\langle N, r|(\xi)_n\rangle = \xi_{j_1}\xi_{j_2}\cdots\xi_{j_N},$$

which leads us to

$$\langle(\xi)_n{}^*|N', r'\rangle\langle N, r|(\xi)_n\rangle = (-1)^{NN'}\langle N, r|(\xi)_n\rangle\langle(\xi)_n{}^*|N', r'\rangle, \quad (3\cdot 34)$$

since $\xi_j$ is commutable with $\langle(\xi)_n{}^*|$ by virtue of the adjoint relation of Eq. (3·18). Thus taking account of Eq. (3·34) as well as the relation

$$(d\xi)_n{}^*(d\xi)_n = (-1)^n(d\xi)_n(d\xi)_n{}^*,$$

we are led to

$$\begin{aligned}&\iint \exp(-\sum_j \xi_j{}^*\xi_j)\langle(\xi)_n{}^*|N', r'\rangle\langle N, r|(\xi)_n\rangle(d\xi)_n{}^*(d\xi)_n\\ &= (-1)^{n+NN'}\iint \exp(-\sum_j \xi_j{}^*\xi_j)\langle N, r|(\xi)_n\rangle\langle(\xi)_n{}^*|N', r'\rangle(d\xi)_n(d\xi)_n{}^*\\ &= (-1)^{n+N}\delta_{NN'}\delta_{rr'}. \qquad (3\cdot 35)\end{aligned}$$

Here we have used the completeness relation (3·30). Remembering that the inner product is invariant under the unitary transformation $a_i \to -a_i$ and $a_i{}^\dagger \to -a_i{}^\dagger$, we get

$$\langle N, r|(\xi)_n\rangle = (-1)^N\langle N, r|(-\xi)_n\rangle.$$

By this equation the left-hand side of Eq. (3·35) becomes as follows:

$$\begin{aligned}&\text{l.h.s. of } (3\cdot 35)\\ &= (-1)^N\iint \exp(-\sum_j \xi_j{}^*\xi_j)\langle(\xi)_n{}^*|N', r'\rangle\langle N, r|(-\xi)_n\rangle(d\xi)_n{}^*(d\xi)_n. \qquad (3\cdot 36)\end{aligned}$$

Then with the change of variables $\xi_i \to -\xi_i$ we get from (3·35) and (3·36)

$$\iint \exp(\sum_j \xi_j{}^*\xi_j)\langle(\xi)_n{}^*|N', r'\rangle\langle N, r|(\xi)_n\rangle(d\xi)_n{}^*(d\xi)_n = \delta_{NN'}\delta_{rr'}.$$

Since the state vectors $|N, r\rangle$ span the complete set of the Hilbert space $\mathfrak{H}$, this accomplishes the proof of (3·32a).

Next we shall prove (3·32b). To do this we shall make use of the relation

$$\iint \exp(\sum_j \xi_j{}^*\xi_j)\langle(\xi)_n{}^*|(d\xi)_n{}^* = (-1)^n\langle(-\xi)|,$$

that can easily be derived from the definition of $\langle(\xi)_n{}^*|$ and Eq. (2·6) with replacement of $\xi'$ by $-a$. With the help of this relation we can explicitly perform the $(\xi)_n{}^*$-integration in (3·32a) and then arrive at (3·32b). q.e.d.

From (3·32a) and (3·32b) we obtain the following trace formula for an operator $F$, which is defined on the Hilbert space $\mathfrak{H}$, by taking the product of (3·32a) or (3·32b) with $\langle l|F|l'\rangle$ (being a $c$-number) and summing up it over $l$ and $l'$:

$$\mathrm{Tr}\, F=\iint \exp(\sum_j \xi_j{}^*\xi_j)\langle(\xi)_n{}^*|F|(\xi)_n\rangle(d\xi)_n{}^*(d\xi)_n \qquad (3\cdot 37\mathrm{a})^{8)}$$

$$=\iint(d\xi)_n\langle(-\xi)_n|F|(\xi)_n\rangle\,. \qquad (3\cdot 37\mathrm{b})$$

The minus sign appearing in the bra vector is considered to be characteristic of fermions, since in the case of bosons the trace is expressed as $\int dq\langle q|F|q\rangle$. As will be seen in the next section the formula (3·37b) is powerful to represent the statistical partition function $\mathrm{Tr}e^{-\beta H}$ in a form of path integral.

## § 4. Grassmann representatives of state vectors and the path integral

For any state vector in $\mathfrak{H}$ we can define the Grassmann representatives $\phi_A((\xi)_n)$ etc. by

$$\phi_A((\xi)_n)\equiv\langle(\xi)_n|A\rangle,\quad \phi_A((\xi)_n{}^*)\equiv\langle(\xi)_n{}^*|A\rangle, \qquad (4\cdot 1)$$

and their adjoints

$$\tilde{\phi}_A((\xi)_n{}^*)\equiv\langle A|(\xi)_n{}^*\rangle,\quad \tilde{\phi}_A((\xi)_n)\equiv\langle A|(\xi)_n\rangle. \qquad (4\cdot 2)$$

These are analogous to wavefunctions in the ordinary theory. There is a one to one correspondence between $\phi_A((\xi)_n)$ and $|A\rangle$ in virtue of the completeness condition. Namely, when $\phi_A((\xi)_n)$ is given, we can uniquely define $|A\rangle$ corresponding to it. It is evident that the same is true for other representatives. By definition the vacuum state $|0\rangle$ corresponds to the following representatives:

$$\phi_0((\xi)_n)=(-i)^n\xi_1\xi_2\cdots\xi_n\,,\qquad \phi_0((\xi)_n{}^*)=1 \qquad (4\cdot 3)$$

and

$$\tilde{\phi}_0((\xi)_n{}^*)=i^n\xi_n{}^*\cdots\xi_2{}^*\xi_1{}^*\,,\qquad \tilde{\phi}_0((\xi)_n)=1\,. \qquad (4\cdot 4)$$

The inner product of two state vectors $|A\rangle$ and $|B\rangle$ in $\mathfrak{H}$ is expressed in terms of the Grassmann representatives by sandwiching the completeness condition (3·30) between $\langle A|$ and $|B\rangle$:

$$\langle A|B\rangle=\iint\exp(-\sum_j\xi_j{}^*\xi_j)\tilde{\phi}_A((\xi)_n)\phi_B((\xi)_n{}^*)(d\xi)_n(d\xi)_n{}^*\,. \qquad (4\cdot 5)$$

Of course there are many other ways of expressing the inner products in terms of other pairs of the Grassmann representatives. For instance, if use is made of (3·32b), one finds

$$\langle A|B\rangle=\iint(d\xi)_n\phi_B((-\xi)_n)\tilde{\phi}_A((\xi)_n). \qquad (4\cdot6)$$

Now let $F$ be an operator, that is the product of the normal-ordered operators $F_k(a^\dagger, a)$ $(k=1, 2, \cdots, N)$, namely,

$$F=F_N(a^\dagger, a)\,F_{N-1}(a^\dagger, a)\cdots F_1(a^\dagger, a), \qquad (4\cdot7)$$

where

$$F_k(a^\dagger, a)=:F_k(a^\dagger, a):. \qquad (4\cdot8)$$

We insert the completeness condition (3·30) between $F_k$ and $F_{k-1}$ and rewrite $\langle(\xi_k)_n{}^*|F_k(a^\dagger, a)|(\xi_{k-1})_n\rangle$ by using the relation

$$\langle(\xi_k)_n{}^*|F_k(a^\dagger, a)|(\xi_{k-1})_n\rangle=\exp(\sum_j\xi^*_{k,j}\xi_{k-1,j})\,F_k(\xi_k{}^*, \xi_{k-1}), \qquad (4\cdot9)$$

which is obtained from Eqs. (3·11), (3·25), (3·18) and the first equation of (3·31). Then we can express the operator $F$ in the form

$$F=\iint|(\xi_N)_n\rangle\exp\{-\sum_{k=0}^{N}\sum_{j=1}^{n}\xi^*_{k,j}(\xi_{k,j}-\xi_{k-1,j})\}F_N(\xi_N{}^*, \xi_{N-1})$$
$$\times F_{N-1}(\xi^*_{N-1}, \xi_{N-2})\cdots F_1(\xi_1{}^*, \xi_0)\langle(\xi_0)_n{}^*|\prod_{k=0}^{N}(d\xi_k)_n(d\xi_k)_n{}^* \qquad (4\cdot10)$$

with $\xi_{-1,j}=0$. It is straightforward to get the path integral for the time development operator $U(t_F, t_I)$ with the aid of this relation. For this purpose, let us make the Hamiltonian normal-ordered,

$$H(a^\dagger, a, t)=:H(a^\dagger, a, t):. \qquad (4\cdot11)$$

The operator $U(t_F, t_I)$ is regarded as the limit of the product of infinitesimal time development operators:

$$U(t_F, t_I)=\lim_{N\to\infty}\{1-i\Delta t\cdot H(a^\dagger, a, t_{N-1})\}\{1-i\Delta t\cdot H(a^\dagger, a, t_{N-2})\}$$
$$\times\cdots\{1-i\Delta t\cdot H(a^\dagger, a, t_0)\}, \qquad (4\cdot12)$$

where the time interval $\Delta t$ is given by $(t_F-t_I)/N=t_k-t_{k-1}$ with $t_F=t_N$ and $t_I=t_0$ $(k=1, 2, \cdots, N)$, so we can use $\{1-i\Delta t\cdot H(a^\dagger, a, t_{k-1})\}$ in place of $F_k(a^\dagger, a)$ of Eq. (4·7). Thus from (4·10) we obtain

$$\langle A|U(t_F, t_I)|B\rangle$$
$$=\lim_{N\to\infty}\iint\tilde{\phi}_A((\xi_N)_n)\exp[\sum_{k=0}^{N}i\Delta t\cdot\mathcal{L}_k]\phi_B((\xi_0)_n{}^*)\prod_{k=0}^{N}(d\xi_k)_n(d\xi_k)_n{}^* \qquad (4\cdot13)$$

with

$$\mathcal{L}_k=i\sum_{j=1}^{n}\xi^*_{k,j}(\xi_{k,j}-\xi_{k-1,j})/\Delta t-H(\xi_k{}^*, \xi_{k-1}, t_{k-1}) \qquad (4\cdot14)$$

and

$$H(\xi_0^*, \xi_{-1}, t_{-1}) = 0 .$$

Here we have equated $\{1 - i\Delta t \cdot H(\xi_k^*, \xi_{k-1}, t_{k-1})\}$ to $\exp[-i\Delta t \cdot H(\xi_k^*, \xi_{k-1}, t_{k-1})]$ for infinitesimal $\Delta t$. Equation (4·13) is a correct form of the path integral, in which the initial and final states are given by the Grassmann representatives.

As a simple example let us consider a free Hamiltonian described in terms of the fermion operator $a$ and the anti-fermion operator $b$ together with their conjugates $a^\dagger$ and $b^\dagger$. In order to derive the Green's functions, we add source terms of Grassmann numbers to the Hamiltonian. We write such a Hamiltonian as

$$H(t) = \frac{m}{2}\{[\psi^\dagger, \sigma_3 \psi] + 2\} + \psi^\dagger \eta(t) + \eta^*(t)\psi \tag{4·15}$$

with

$$\psi = \begin{pmatrix} a \\ b^\dagger \end{pmatrix}, \quad \eta(t) = \begin{pmatrix} \eta^{(1)}(t) \\ \eta^{(2)}(t) \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4·16}$$

Since the Hamiltonian (4·15) is rewritten in the normal-ordered form

$$\begin{aligned} &H(a^\dagger, b^\dagger, a, b, \eta^*(t), \eta(t)) \\ &= m(a^\dagger a + b^\dagger b) + a^\dagger \eta^{(1)}(t) + b\eta^{(2)}(t) + \eta^{(1)*}(t) a + \eta^{(2)*}(t) b^\dagger , \end{aligned} \tag{4·17}$$

we obtain the following equation:

$$\begin{aligned} Z &= \langle 0 | U(t_F, t_I) | 0 \rangle \\ &= \lim_{N\to\infty} \int\int \exp[i \sum_{k=0}^{N} \Delta t \cdot \mathcal{L}_k] \prod_{k=0}^{N} (d\xi_k^{(2)} d\xi_k^{(1)} d\xi_k^{(1)*} d\xi_k^{(2)*}) \end{aligned} \tag{4·18}$$

with

$$\mathcal{L}_k = i \sum_{j=1,2} \xi_k^{(j)*} (\xi_k^{(j)} - \xi_{k-1}^{(j)}) / \Delta t - H(\xi_k^{(1)*}, \xi_k^{(2)*}, \xi_{k-1}^{(1)}, \xi_{k-1}^{(2)}, \eta_k^*, \eta_{k-1}), \tag{4·19}$$

where the Grassmann numbers $\xi^{(1)}$ and $\xi^{(2)}$ correspond to the operators $a$ and $b$ respectively. In deriving (4·18) we have used the relation $\phi_0(\xi^{(1)*}, \xi^{(2)*}) = \tilde{\phi}_0(\xi^{(1)}, \xi^{(2)}) = 1$ (cf., the second equations in (4·3) and (4·4)). It may be more convenient to express the path integral (4·18) in terms of the Grassmann numbers corresponding to the operators $\psi$ and $\psi^\dagger$. Then writing them as

$$\begin{pmatrix} \chi_k^{(1)} \\ \chi_k^{(2)} \end{pmatrix} = \begin{pmatrix} \xi_k^{(1)} \\ \xi_k^{(2)*} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \chi_k^{(1)*} \\ \chi_k^{(2)*} \end{pmatrix} = \begin{pmatrix} \xi_k^{(1)*} \\ \xi_k^{(2)} \end{pmatrix}, \tag{4·20}$$

we find that Eq. (4·18) turns out to be

$$Z = \lim_{N\to\infty} \int\int \exp[i \sum_{k=0}^{N} \Delta t \cdot \mathcal{L}_k] \prod_{k=0}^{N} (d\chi_k^{(1)} d\bar{\chi}_k^{(1)} d\chi_k^{(2)} d\bar{\chi}_k^{(2)}) \tag{4·21}$$

with

$$\Delta t\cdot\mathcal{L}_k=i\Big[\bar{\chi}_k\chi_k-\frac{1}{2}\bar{\chi}_k\{(1+\sigma_3)(1-i\Delta tm)\chi_{k-1}+(1-\sigma_3)(1-i\Delta tm)\chi_{k+1}\}$$
$$+\frac{i\Delta t}{2}\bar{\chi}_k\{(1+\sigma_3)\eta_{k-1}+(1-\sigma_3)\eta_{k+1}\}$$
$$+\frac{i\Delta t}{2}\bar{\eta}_k\{(1+\sigma_3)\chi_{k-1}+(1-\sigma_3)\chi_{k+1}\}\Big],\qquad(4\cdot22)$$

$\bar{\chi}_k=\chi_k{}^*\sigma_3$ and $\chi_{N+1}=\eta_{N+1}=0$.

It is interesting to note that the Lagrangian given here is different from the one usually taken. Evidently it is essentially due to the fact that the fermion annihilation and the anti-fermion creation are both described by the single operator $\psi$. Indeed, if the operator $\psi$ would annihilate the vacuum state, we were led to the usual form of Lagrangian.

The same situation also occurs in the path integral for the relativistic Dirac fields. In this case, however, it is shown that in addition to the above feature the Lagrangian becomes inevitably non-local because of the non-locality existing in relativistic creation and annihilation operators in the configuration space.*)

Finally we try to get the path integral representation for the statistical partition function $\mathrm{Tr}\, e^{-\beta H}$. Again we assume the Hamiltonian to be of the normal order. Then the following formula is derived:

$$\mathrm{Tr}\, e^{-\beta H}=\lim_{N\to\infty}\int\!\!\int\exp[-\sum_{k=1}^{N}\{\sum_{j=1}^{n}\xi_{k,j}^*(\xi_{k+1,j}-\xi_{k,j})+\Delta\beta H(\xi_k{}^*,\xi_k)\}]|_{\xi_{N+1}=-\xi_1}$$
$$\times\prod_{k=1}^{N}(d\xi_k)_n{}^*(d\xi_k)_n\qquad(4\cdot23)$$

with

$$\Delta\beta=\beta/N.$$

*Proof*:

Let us substitute each of the factors $F_k(a^\dagger,a)$ of (4·7) by the operator $(1-\Delta\beta H(a^\dagger,a))$. Then from Eqs. (4·9) and (3·37b) we obtain

$$\mathrm{Tr}\, e^{-\beta H}=\lim_{N\to\infty}\int\!\!\int(d\zeta)_n\langle(-\zeta)_n|(\xi_N)_n\rangle K(\xi^*,\xi)\exp(-\sum_{j=1}^{n}\xi_{0,j}^*\xi_{0,j})$$
$$\times\langle(\xi_0)_n{}^*|(\zeta)_n\rangle\prod_{k=0}^{N}(d\xi_k)(d\xi_k)_n{}^*\qquad(4\cdot24)$$

with

$$K(\xi^*,\xi)=\exp[-\sum_{k=1}^{N}\{\sum_{j=1}^{n}\xi_{k,j}^*(\xi_{k,j}-\xi_{k-1,j})+\Delta\beta H(\xi_k{}^*,\xi_{k-1})\}].$$

*) Details of this problem will be discussed elsewhere together with other related problems.

Since $\langle(\xi_0)_n^*|(\zeta)_n\rangle$ is commutable with any Grassmann number, we can shift it to the leftmost in the integrand of (4·24). Furthermore, owing to Eqs. (2·7) and (3·13) the term $(d\zeta)_n\langle(-\zeta)_n|(\xi_N)_n\rangle$ in (4·24) is rewritten as $\langle(\zeta)_n|(-\xi_N)_n\rangle(d\zeta)_n$. With these arrangements of factors in the integrand, we can perform the $(\zeta)_n$-integration as follows:

$$\int\!\!\int\langle(\xi_0)_n^*|(\zeta)_n\rangle\langle(\zeta)_n|(-\xi_N)_n\rangle(d\zeta)_n=\langle(\xi_0)_n^*|(-\xi_N)_n\rangle$$

$$=\exp(-\sum_{j=1}^{n}\xi_{0,j}^*\xi_{N,j}).$$

Inserting this into (4·24) and taking account of the relation

$$\prod_{k=0}^{N}(d\xi_k)_n(d\xi_k)_n^*=(-1)^n(d\xi_0)_n^*(d\xi_N)_n\prod_{k=1}^{N}(d\xi_k)_n^*(d\xi_{k-1})_n,$$

we are led to

$$\mathrm{Tr}\,e^{-\beta H}=(-1)^n\lim_{N\to\infty}\int\!\!\int K(\xi^*,\xi)\exp\{\sum_{j=1}^{n}(\xi_{N,j}+\xi_{0,j})\xi_{0,j}^*\}$$
$$\times(d\xi_0)_n^*(d\xi_N)_n\prod_{k=1}^{N}(d\xi_k)_n^*(d\xi_{k-1})_n.$$

Since no $\xi_{0,j}^*$ is contained in $K(\xi^*,\xi)$, the $(\xi_0)_n^*$- and $(\xi_N)_n$-integrations in the above are easily performed with the aid of Eqs. (2·6) and (2·1). Consequently we obtain

$$\mathrm{Tr}\,e^{-\beta H}=\lim_{N\to\infty}\int\!\!\int K(\xi^*,\xi)|_{\xi_N=-\xi_0}\prod_{k=1}^{N}(d\xi_k)_n^*(d\xi_{k-1})_n,$$

which becomes Eq. (4·23) by the change of variables $\xi_{k,j}\to\xi_{k+1,j}$ $(k=0,1,\cdots,N)$. q.e.d.

It is to be noted that the indices of $\xi$ and the order of $(d\xi_k)_n$ and $(d\xi_k)_n^*$ in (4·23) are not the same as those of (4·13) and (4·14). The condition $\xi_{N+1}=-\xi_1$ in the above is the antiperiodic boundary condition which was pointed out by Berezin[5] and has been checked so far only in some simple cases,[9] whereas according to our formalism it is an automatic consequence which generally holds true for the trace formula in any fermion system.*)

## Appendix

——*Proof of (2·13)*——

In order to show the validity of Eq. (2·13) we begin with proving the following:

*) Very recently the authors have been informed by Soper[10] that he has also derived the antiperiodic boundary condition in his formalism of the fermion path integral.

*Lemma 1. If* $det(\alpha_i^j)^{-1}$ *exists, then there also exists the inverse of the Jacobian.*

*Proof*:

Let $\|\alpha\|$ and $\|J\|$ be the matrices whose $i$-$j$ elements are $\alpha_i^j$ and $\partial\xi_i'/\partial\xi_j$ respectively. Writing the transformation (2·10) as

$$\xi_i'=\alpha_i+\sum_{j=1}^{n}\alpha_i^j\xi_j+R_i(\xi) \tag{A·1}$$

with

$$R_i(\xi)\equiv\sum_{j_1<j_2}\alpha_i^{j_1j_2}\xi_{j_1}\xi_{j_2}+\cdots+\alpha_i^{12\cdots n}\xi_1\xi_2\cdots\xi_n\,,$$

we get

$$\|J\|=\|\alpha\|\,(1+\|\alpha\|^{-1}\|\partial R_i(\xi)/\partial\xi_j\|)\,.$$

By definition, $\partial R_i(\xi)/\partial\xi_j$ is either vanishing or a polynomial in $\xi$ whose lowest power is at least of first order. Hence we have

$$\|J\|^{-1}=\sum_{p=0}^{n}\{-\|\alpha\|^{-1}\|\partial R_i(\xi)/\partial\xi_j\|\}^p\|\alpha\|^{-1}\,,$$

which completes the proof. q.e.d.

We write the transformation $\{\xi_1,\xi_2,\cdots,\xi_n\}\rightarrow\{\xi_1',\xi_2',\cdots,\xi_n'\}$ simply as $\{\xi'\}=f\cdot\{\xi\}$ or $\xi_i'=f_i\cdot\{\xi\}$, where the mapping $f$ is completely given corresponding to a given set of transformation coefficients $\alpha_i^{\cdots}$. The composition of two successive transformations, say, $\{\xi\}\xrightarrow{f}\{\xi'\}\xrightarrow{g}\{\xi''\}$ is denoted by the product $gf$; $\{\xi''\}=gf\cdot\{\xi\}\equiv g\cdot\{\xi'\}=g\cdot\{f\cdot\{\xi\}\}$.

Consider the two transformations $\{\xi\}\rightarrow\{\xi'\}$ and $\{\xi'\}\rightarrow\{\xi''\}$. Owing to the assumption (2·11) we can easily show

$$\frac{\partial\xi_i''}{\partial\xi_j}=\sum_{k=1}^{n}\frac{\partial\xi_i''}{\partial\xi_k'}\frac{\partial\xi_k'}{\partial\xi_j}\,,$$

from which we obtain:

*Lemma 2. If each of transformations f and g satisfies Eq.* (2·13), *so does the product gf as well.*

It is noted that general transformation (A·2) is expressed by the product of $(n+1)$ transformations, such as $f^{(n+1)}f^{(n)}f^{(n-1)}\cdots f^{(1)}$. Here the $f$'s are defined by

$$f_i^{(k)}\cdot\{\xi\}\equiv\xi_i+\delta_{ki}\tilde{R}_i(\xi)\,,\quad(k=1,2,\cdots,n)$$

$$f_i^{(n+1)}\cdot\{\xi\}\equiv\alpha_i+\sum_{j=1}^{n}\alpha_i^j\xi_j\,, \tag{A·2}$$

where $\tilde{R}_i(\xi)$ are given by

$$\tilde{R}_1(\xi)=S_1(\xi)\quad\text{and}\quad\tilde{R}_i(\xi)=S_i([f^{(i)}f^{(i-1)}\cdots f^{(1)}]^{-1}\{\xi\})\quad(n\geq i\geq 2)$$

with

$$S_i(\xi)=\sum_{j=1}^{n}\beta_i^jR_j(\xi)\quad\text{and}\quad\sum_{k=1}^{n}\alpha_k^j\beta_i^k=\delta_{ij}\,. \tag{A·3}$$

Thus, according to Lemma 2 it is sufficient for us to show the validity of (2·13) for every transformation given in (A·2). To this end we will show that the integral formulas (2·8) with replacement of $\{\xi\}$ by $\{\xi'\}=f\cdot\{\xi\}$ are derived from (2·8) only when use is made of (2·13).

As for the transformation $f^{(n+1)}$ it is rather trivial, since in this case $J=\det(\alpha_i^j)$, $\xi_1'\xi_2'\cdots\xi_n'=Q^{(n-1)}(\xi)+\det(\alpha_i^j)\xi_1\xi_2\cdots\xi_n$, and $\xi'_{j_1}\xi'_{j_2}\cdots\xi'_{j_r}=Q^{(r)}(\xi)\ (r<n)$, where $Q^{(s)}(\xi)$ stands for a polynomial in $\xi$ whose highest power is equal to or less than $s$. So we shall examine the transformation $f^{(k)}$ $(k=1, 2, \cdots, n)$ in the following.

We write it as $\xi_i'=f_i^{(k)}\cdot\{\xi\}$, corresponding to which the Jacobian is denoted by $J^{(k)}$. Then from (A·2) we get $J^{(k)}=1+\partial\tilde{R}_k(\xi)/\partial\xi_k$. The $\tilde{R}_k(\xi)$ is the sum of two parts; the one contains $\xi_k$ and the other does not. We factorize $\xi_k$ to the left in the former and write $\tilde{R}_k(\xi)$ as

$$\tilde{R}_k(\xi)=\xi_k A^{(k)}(\xi)+B^{(k)}(\xi). \tag{A·4}$$

Notice that no $\xi_k$ exists in $A^{(k)}(\xi)$ and $B^{(k)}(\xi)$. Hence the Jacobian takes the form

$$J^{(k)}=1+A^{(k)}(\xi). \tag{A·5}$$

On other hand, since $\xi_i'=\xi_i$ for $i\neq k$ and $\xi_k'=\xi_k(1+A^{(k)})+B^{(k)}$, we have

$$\begin{aligned}\xi_1'\xi_2'\cdots\xi_n'=&\{1+A^{(k)}(\xi)\}\xi_1\xi_2\cdots\xi_n\\&+B^{(k)}(\xi)\xi_1\xi_2\cdots\xi_{k-1}\xi_{k+1}\cdots\xi_n.\end{aligned} \tag{A·6}$$

No $\xi_k$ is contained in the second term. Thus we find that Eq. (2·13) is a sufficient condition*) for the first formula of (2·8) to be invariant under the transformation $f^{(k)}$.

We next examine the second formula of (2·8). There are two cases: (i) $\xi'_{j_1}\xi'_{j_2}\cdots\xi'_{j_r}$ $(r<n)$ containing $\xi_k'$ in itself and (ii) otherwise. In case (i) we put $\xi'_{j_1}=\xi_k'$ for simplicity. Then we obtain

$$\xi'_{j_1}\xi'_{j_2}\cdots\xi'_{j_r}=\begin{cases}\{1+A^{(k)}(\xi)\}\xi_k\xi_{j_2}\xi_{j_3}\cdots\xi_{j_r}+B^{(k)}(\xi)\xi_{j_2}\xi_{j_3}\cdots\xi_{j_r} & \text{for case (i)},\\ \xi_{j_1}\xi_{j_2}\cdots\xi_{j_r}\quad(\xi_{j_i}\neq\xi_k;\ i=1,2,\cdots,r<n) & \text{for case (ii)}.\end{cases}$$

Consequently, if taking account of (A·5), we easily find that Eq. (2·13) is a necessary and sufficient condition for the second formula of (2·8) to remain invariant under any transformation $f^{(k)}$ given in (A·2).

Thus we have proved Eq. (2·13).

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*) This is not a necessary condition since $A^{(k)}(\xi)\xi_1\xi_2\cdots\xi_n=0$.

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