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AUTHOR(S) Michael Martin Nieto

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Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

COHERENT STATES WITH CLASSICAL MOTION: FROM AN ANALYTIC METHOD COMPLEMENTARY TO GROUP THEORY

Michael Martin Nieto
Theoretical Division, Los Alamos National Laboratory, University of California
Los Alamos, New Mexico 87545, USA

ABSTRACT

From the motivation of Schrödinger, that of finding states which follow the motion which a classical particle would in a given potential, we discuss generalizations of the coherent states of the harmonic oscillator. We focus on a method which is the analytic complement to the group theory point of view. It uses a minimum uncertainty formalism as its basis. We discuss the properties and time evolution of these states, always keeping in mind the desire to find quantum states which follow the classical motion.

1. INTRODUCTION

In 1926 Schrödinger[1] discovered what have come to be known as the coherent states of the harmonic oscillator. His motivation was to find states which follow the motion that a classical particle would in an harmonic oscillator potential. He succeeded by finding a restricted class of Gaussian wave packets which can analytically be shown to have a shape independent of time, and whose centroid oscillates back and forth in the potential the same as a classical particle with an energy ($\langle H \rangle - E_0$) would, E_0 being the ground state energy.

Amusingly, at the end of his paper[1] Schrödinger commented that, "We can definitely foresee that, in a similar way, wave groups can be constructed which move round highly quantised Kepler ellipses and are the representation by wave mechanics of the hydrogen electron. But the technical difficulties in the calculation are greater than in the especially simple case which we have treated here." Schrödinger did not pursue the problem.

In the 1960's the coherent states came into wide usage through the new field of quantum optics, and many authors popularized their use.[2-13] However, despite the many advances, generalizations concentrated on systems whose eigenspectra are equally spaced. Thus, the idea of generalizing coherent states to more general types of potentials, as Schrödinger envisioned, did not reach full fruition.

This fundamental question of Schrödinger fascinated me and my coworkers (L. M. Simmons, Jr. and V. P. Gutschick). Is it possible to find quantum states which follow the motion that a classical particle would in any given potential?

Giving the intuitive answer now, I will show that it is, "Yes, up to a point, and depending..." Effectively, the more highly bound a particle state is and, given that, the greater number of eigenstates with which the "coherent state" has significant overlap and, finally, the closer to equally spaced (i.e., harmonic) these eigenstates are, then the better and longer the "coherent state" will follow the classical motion without dispersing.

The harmonic oscillator is, of course, that system which is the best of all possible worlds. It's coherent states never change their shape and follow the classical motion forever. When we began this problem we knew the harmonic oscillator is very special. However, its very special properties became clearer and clearer to us as we understood the problem better.

We wanted to find out how important it is to have systems with equally spaced eignevalues. We also wanted to know if coherent states could be found for systems which had non-equally spaced eigenvalues and/or had continuums and/or could not be solved analytically. We found a general method to handle such systems,[14-19] and at the end of the study[14-24] we came to the intuitive answer I have just given above.

What I intend to do here is first to review the harmonic oscillator and the three standard equivalent methods for defining its coherent states. In Sec. 3 I will show that systems with equally spaced eigenvalues which are not the harmonic oscillator do not in general have coherent states which follow the classical motion forever.[22-24] They almost do, but not exactly.

We then go to general potentials. It turns out that for non-harmonic systems, the generalizations of the three equivalent methods of defining harmonic oscillator coherent states lead to states which are no longer equivalent. In particular, although one definition of coherent states is easy to generalize from the group theory point of view, the method which we have most closely investigated (and which follows Schrödinger's philosophy) is best dealt with from an analytic point of view. Thus, what I will discuss can be viewed as the complementary viewpoint to group theory, just as Schrödinger's wave mechanics is the complementary viewpoint to Heisenberg's matrix mechanics. (I shall also mention further generalizations to multidimensional systems and time dependent potentials.)

To explicitly show how the formalism works, I will go through in detail its application to the $\cosh^{-2}$ potential. Finally, I will show how well these states do follow the classical motion by reviewing the results of a computer generated movie which displays the time evolution of these states for many potentials.

2. THE HARMONIC OSCILLATOR AND ITS COHERENT STATES

Let us begin by recalling the properties of a bound classical system of total energy E. The Hamiltonian equation is

$$E = \frac{1}{2m} p^2 + V(x) = \frac{1}{2} m\dot{x}^2 + V(x) \quad , \tag{2.1}$$

or

$$\dot{x} = (2/m)^{\frac{1}{2}}[E - V(x)]^{\frac{1}{2}} \quad . \tag{2.2}$$

For the simple harmonic oscillator with

$$V(x) = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2 x^2 \quad , \tag{2.3}$$

the solution for x(t) is

$$x(t) = (2E/m\omega^2)^{\frac{1}{2}} \sin \omega t \quad , \tag{2.4}$$

so that

$$p(t) = m\dot{x} = m(2E/m)^{\frac{1}{2}} \cos \omega t \quad . \tag{2.5}$$

For the quantum problem, with the Hamiltonian equation becoming the Schrödinger equation, $p \rightarrow (\hbar/i)d/dx$,

$$H\psi = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right]\psi = \hbar\omega\left[a^+ a^- + \frac{1}{2}\right]\psi = E\psi \quad , \tag{2.6}$$

the eigensolutions and eigenvalues are

$$\psi_n = \left(\frac{a_0}{\pi^{\frac{1}{2}} 2^n n!}\right)^{\frac{1}{2}} \exp(-\tfrac{1}{2}a_0^2 x^2) H_n(a_0 x) \quad , \tag{2.7}$$

$$a_0 = (m\omega/\hbar)^{\frac{1}{2}} = 1/(2^{\frac{1}{2}} x_0) \quad , \tag{2.8}$$

$$E_n = \hbar\omega(n + \tfrac{1}{2}) \quad . \tag{2.9}$$

Two of the most important properties of this system are that the energy levels are independent of n (equally spaced) and that the raising and lowering operators are independent of n,

$$a^{\pm}|n\rangle = (n \pm \tfrac{1}{2})^{\frac{1}{2}}|n \pm 1\rangle \quad , \tag{2.10}$$

$$x = [\hbar/(2m\omega)]^{\frac{1}{2}}(a^{-} + a^{+}) \quad , \tag{2.11}$$

$$p = [m\hbar\omega/2]^{\frac{1}{2}}(a^{-} - a^{+})/i \quad , \tag{2.12}$$

$$a^{\pm} = (2m\hbar\omega)^{-\frac{1}{2}}(m\omega x \mp ip) \quad , \tag{2.13}$$

$$[a^{-},a^{+}] = 1 \quad . \tag{2.14}$$

The implications of the above are many. But for us what is foremost is that any wave packet, no matter what its shape, will return to its original shape after one classical period of oscillation

$$\tau = 2\pi/\omega \quad . \tag{2.15}$$

This can be seen by decomposing any time-dependent state into eigenstates,

$$\Psi(x,t) = \sum_{n} a_n \psi_n(x) \exp[-i\omega t(n + \tfrac{1}{2})] \quad , \tag{2.16}$$

and observing that the equal spacing of the levels means that

$$\Psi^{*}(x,t_0)\Psi(x,t_0) = \Psi^{*}(x,t + 2\pi j/\omega + t_0)\Psi(x,t + 2\pi j/\omega + t_0) \quad , \tag{2.17}$$

where j is any integer.

This property will be useful in the next section, but for now we mention that for the harmonic oscillator the coherent states not only return to their original shape after one period of oscillation, they retain their original shape for all time and have a centroid which follows the classical motion.

How are these coherent states defined? In modern language, they are standardly defined in three equivalent ways. The first way is essentially what Schrödinger discovered.[1]

i. Minimum Uncertainty Coherent States (MUCS). From Eq. (2.6), the Hamiltonian of the harmonic oscillator is quadratic in p and x. From the commutation relation

$$[x,p] = i\hbar \quad , \tag{2.18}$$

there is an uncertainty relation

$$(\Delta x)^2(\Delta p)^2 \geq \tfrac{1}{4}\hbar^2 \quad . \tag{2.19}$$

Now, any commutation relation

$$[A,B] = iG \tag{2.20}$$

has an uncertainty relation

$$(\Delta A)^2(\Delta B)^2 \geq \tfrac{1}{4}\langle G\rangle^2 \tag{2.21}$$

whose equality can be satisfied by that three-parameter set of states which satisfies the eigenvalue equation[5,25,26]

$$\left(A + \frac{i\langle G\rangle}{2(\Delta B)^2}B\right)\psi = \left(\langle A\rangle + \frac{i\langle G\rangle}{2(\Delta B)^2}\langle B\rangle\right)\psi \quad . \tag{2.22}$$

Note that the four parameters $\langle A\rangle$, $\langle B\rangle$, $\langle B^2\rangle$, and $\langle G\rangle$ are not independent because they satisfy the equality in Eq. (2.21).

Applying this to x and p yields

$$\psi_{CS}(x) = [2\pi(\Delta x)^2]^{-\frac{1}{4}} \exp\left\{-\left[\frac{x - \langle x\rangle}{2(\Delta x)^2}\right]^2 + \frac{i}{\hbar}\langle p\rangle x\right\} \quad . \tag{2.23}$$

Now demanding that the ground-state ($n = 0$) wave function be a special case of (2.23) for $\langle x\rangle = \langle p\rangle = 0$ gives the additional restriction that

$$(\Delta x/\Delta p)^2 = 1/(m\omega)^2 \quad , \tag{2.24}$$

or

$$(\Delta x)^2 = (2\alpha_0)^{-1} = x_0^2 \quad , \tag{2.25}$$

yielding a two-parameter set of states. This last restriction is physically necessary because it corresponds to a classical particle at rest. With this restriction the coherent-state wave packets will follow the motion of a classical particle and retain their shape. If a different value of $(\Delta x/\Delta p)$ were to be chosen, the packet would not keep its shape.

It will turn out that the generalization of this method is the one we will concentrate on when we generalize from our classical motion point of view.

ii. Annihilation-Operator Coherent States (AOCS). These states are defined as the eigenvalues of the destruction operator with complex eigenvalue α:

$$a^- |\alpha\rangle = \alpha|\alpha\rangle \quad . \tag{2.26}$$

They can be written in terms of the number states as

$$|\alpha\rangle = \exp(-\tfrac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{\frac{1}{2}}} |n\rangle \quad , \tag{2.27}$$

and further expressed as the set of Gaussians,

$$|\alpha\rangle = [2\pi(\Delta x)^2]^{-\frac{1}{4}} \exp\left[-\frac{x^2}{4(\Delta x)^2} + \frac{x\alpha}{\Delta x} - \frac{1}{2}(\alpha^2 + |\alpha|^2)\right] \quad . \tag{2.28}$$

With the physical restrictions above, α can be shown to be

$$\alpha = \frac{\langle x\rangle}{2\Delta x} + \frac{i}{\hbar}\langle p\rangle\Delta x = \frac{1}{2}\left[\frac{\langle x\rangle}{\Delta x} + i\frac{\langle p\rangle}{\Delta p}\right] , \tag{2.29}$$

so that

$$|\alpha\rangle = \exp[-i\langle p\rangle\langle x\rangle/(2\hbar)]|\psi_{CS}\rangle \tag{2.30}$$

Thus the AOCS are the MUCS up to an irrelevant phase factor.

iii. Displacement Operator Coherent States (DOCS). These states are defined as those states which are created by the unitary displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a^-) \tag{2.31}$$

acting on the ground state. With the aid of the Baker-Campbell-Hausdorff identity, one can show that

$$D(\alpha)|0\rangle = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\alpha a)|0\rangle$$

$$= \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^+)|0\rangle$$

$$= \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n(a^+)^n}{n!}|0\rangle$$

$$= |\alpha\rangle \quad . \qquad (2\ 32)$$

Thus, the DOCS are equivalent to the AOCS and MUCS.

This method is the one which is most easily generalized from the group theory point of view, since it involves the exponential of the anti-Hermitian operator $(\alpha a^+ - \alpha^* a)$. Indeed, Perelomov[11] has championed and greatly explored this point of view.

It is from these MUCS and DOCS starting points that the complementarity of the analytic vs. group theory points of view is seen. As always, an advantage of the group theory point of view is that the symmetry is a useful tool to simplify the concepts. An advantage of the analytic point of view will be that one can construct states and numerically see how they evolve in time.

3. QUANTUM AND CLASSICAL HARMONIC POTENTIALS

As mentioned in Sec. 2, two of the most striking properties of the harmonic oscillator potential are that (i) in the classical system, the classical period of oscillation $\tau = 2\pi/\omega$ is independent of energy and that (ii) in the quantum system, the eigenvalues are equally spaced by $\hbar\omega$. Any potential which satisfies property (i) I call a "classical harmonic potential," and any potential which satisfies property (ii) I call a "quantum harmonic potential."[22-24]

As is discussed in the standard work of Landau and Lifshitz,[27] there are many potentials which are classical harmonic potentials; in fact, an uncountable number. In Refs. 22-24 it was also shown that the set of quantum harmonic potentials is not the same as the set of classical harmonic potentials, even though they are the same in WKB approximation.[22]

What was found is that there are three distinct classes: a) potentials which are classical harmonic but not quantum harmonic, b) those which are quantum harmonic but not classical harmonic, and c) those which are both. Further, each class contains an uncountable number. I refer people to Refs. 22-24 but the

The demonstration that quantum harmonic and classical harmonic potentials are not equivalent classes was a proof by example. What was used was an idea of Abraham and Moses[28] (AM) based on the Gel'fand-Levitan formalism.[29] One can take an exactly solvable potential with a known spectra and generate an exactly solvable potential with the same spectra, but with one or, indeed, any number of eigenstates removed.

In particular, when applied to the harmonic oscillator, one can show that[22,28] if the harmonic oscillator ground state is removed, one has the exactly solvable AM[28] potential v (in dimensionless units)

$$z = a_0 x \quad , \quad a_0 = (m\omega/\hbar)^{\frac{1}{2}} \quad , \tag{3.1}$$

$$v = V/\hbar\omega \quad , \quad \varepsilon_n = E_n/\hbar\omega \quad , \tag{3.2}$$

$$v = v_0 + v_1 \quad , \tag{3.3}$$

$$v_0 = \tfrac{1}{2}z^2 \quad , \quad v_1 = \tfrac{1}{2}\phi(\phi - z) \quad , \tag{3.4}$$

$$\phi(z) = \frac{e^{-z^2}}{z^{\frac{1}{2}}\mathrm{erfc}(z)} \quad , \tag{3.5}$$

$$\mathrm{erfc}(z) = \frac{2}{\pi^{\frac{1}{2}}} \int_z^\infty e^{-t^2} dt \quad . \tag{3.6}$$

The solutions are

$$\varepsilon_n = n + \tfrac{1}{2} \quad , \quad n = 1,2,\ldots \tag{3.7}$$

$$\chi_n(z) \quad \psi_n(z) = \left(\frac{2}{n}\right)^{\frac{1}{2}} \phi(z)\psi_{n-1}(z) \quad , \tag{3.8}$$

where the ψ_n are the harmonic oscillator wave functions of Eq. (2.7). The shape of the potential is shown in Fig. 1.

The AM potential is thus a quantum harmonic potential. However, by numerically calculating the classical period $\tau(\varepsilon)$ as a function of dimensionless energy ε, one finds that the dimensionless quantity

$$P(\varepsilon) \quad \tau(\varepsilon)/\tau \quad \tau(\varepsilon)\omega/2\pi \tag{3.9}$$

varies with energy. This is shown in Fig. 2, which has two scales on it.

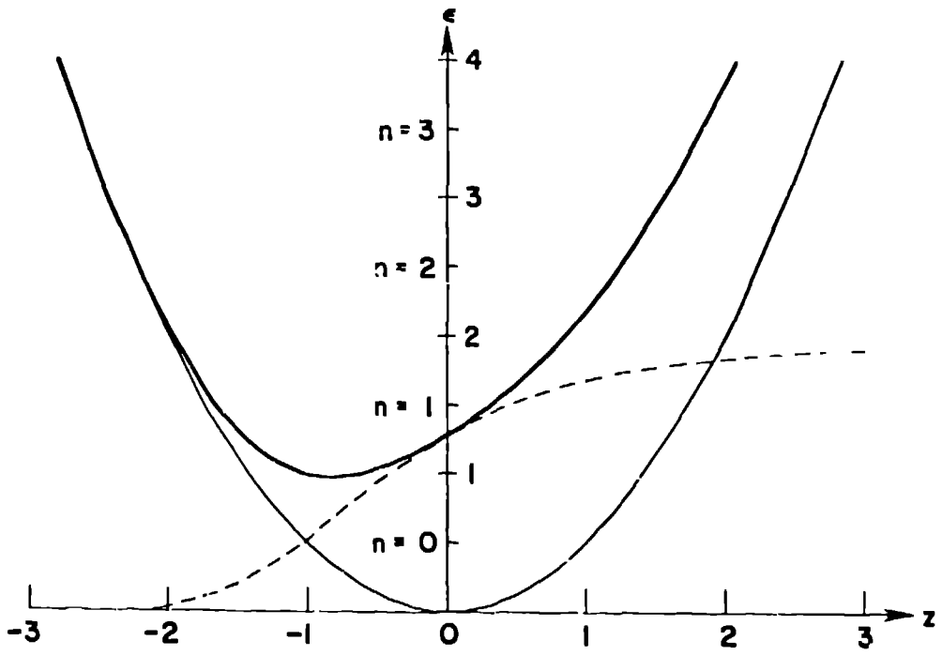

Figure 1.

The harmonic oscillator potential $v_0(z)$ is a light curve, the contribution $v_1(z)$ is a light dashed curve, and the complete AM potential $v(z)$ is a heavy curve. The number eigenstates are also indicated.

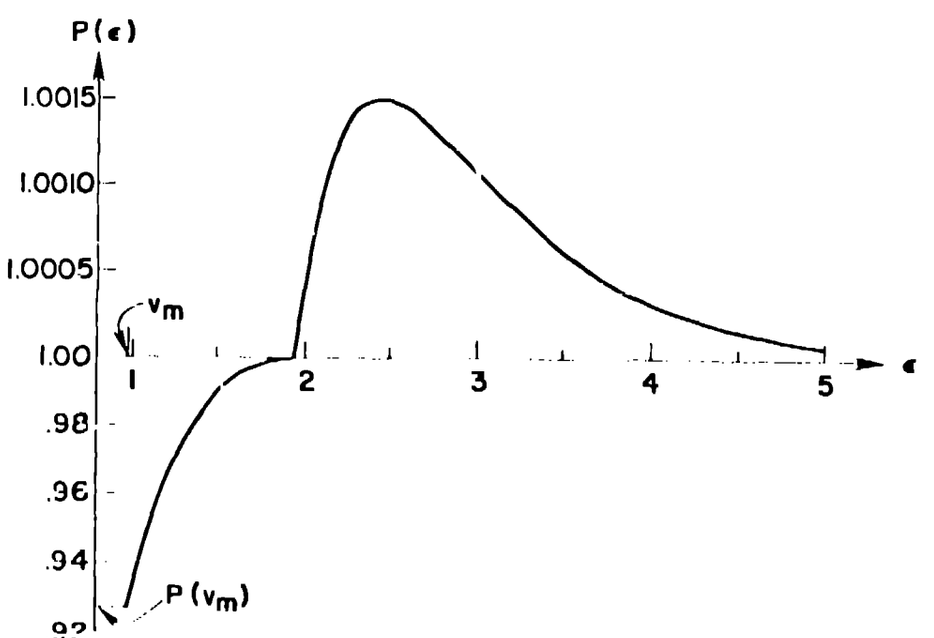

Figure 2.

$P(\varepsilon)$ plotted as a function of ε. There are two scales for $P(\varepsilon)$. The scale for $P(\varepsilon) > 1$ is greatly expanded.

Although the variation of the classical period with energy is very small, it does exist, and the AM potential is not a classical harmonic potential.

The above means a "coherent state" will not follow the classical motion for all time. As we mentioned at Eq. (2.17), any quantum system with equally spaced eigenvalues will have any wave packet return to its original shape after every time $\tau = 2\pi/\omega$. But for the AM potential the classical period of oscillation is not independent of energy. Therefore, the quantum and classical systems will diverge; slowly, but they will diverge.

Thus, we find that a potential with a little variation from the harmonic oscillator, i.e., one which is quantum harmonic but not classical harmonic, will no longer yield perfect classical motion in the quantum system.

4. ANALYTIC COHERENT STATES FOR GENERAL POTENTIALS

We have just seen that the more "harmonic oscillator-like" a potential is, the more nearly there is a quantum state that can follow the classical motion. The thing that produces the special features of the harmonic oscillator is that its Hamiltonian is quadratic in the variables x and p, which vary sinusoidally with time. We call these the "natural classical variables" for the harmonic oscillator. What are the "natural classical variables" for other potentials?

My first physical ansatz is that the "natural classical variables" X_c and P_c for any bound system are the variables which change the total energy equation into a quadratic equation in these two variables and which vary sinusoidally with time. This makes the equation "harmonic oscillator-like" in these variables. It changes the x-p phase space plot into an ellipse in an X_c-P_c plot. Specifically, there exist variables

$$X_c = A(E)\sin[\omega_c(E)t] \quad , \tag{4.1}$$

$$P_c = m\dot{X}_c = mX_c'(x)\dot{x} \tag{4.2}$$

$$= mA(E)\omega_c(E)\cos[\omega_c(E)t] \quad . \tag{4.3}$$

Because

$$\dot{x}^2 = 2(E - V)/m \quad , \tag{4.4}$$

$X_c(x)$ is the solution of the equation

$$X_c' = \frac{d}{dx} X_c(x) = \omega_c \left[\frac{m(A^2 - X_c^2)}{2E - V(x)} \right]^{1/2} \quad . \tag{4.5}$$

Equations (4.2) and (4.3) imply that the classical equations of motion are

$$\dot{X}_c = P_c/m \quad , \tag{4.6}$$

$$\ddot{X}_c = \frac{\dot{P}_c}{m} = -\omega_c^2(E)X_c \quad . \tag{4.7}$$

Thus, Eq. (2.1) is replaced by a form which is similar to the harmonic-oscillator equation for a given energy. That is, this transformation is equivalent to rewriting (2.1) as

$$\tfrac{1}{2}m\omega_c^2(E)A^2(E) = \frac{1}{2m}P_c^2 + \tfrac{1}{2}m\omega_c^2X_c^2 \quad . \tag{4.8}$$

I now define the analogous "natural quantum operators," $p = (\hbar/i)d/dx$,

$$X = X_c(x) \quad , \tag{4.9}$$

$$P = \tfrac{1}{2}(X_c'p + pX_c') \quad . \tag{4.10}$$

Now obtain those states which minimize the uncertainty relation associated with the commutator

$$[X,P] = iG \quad . \tag{4.11}$$

My second physical ansatz is that the coherent states are a two-parameter subset of the above minimum uncertainty states defined by restricting $\Delta X/\Delta P$ so that the ground state of the potential is a member of the set. (A priori it was not clear at first that this would be true.) Finally, observe that just as x and p can be written as sums and differences of raising and lowering operators for the harmonic oscillator, so too X and P can be written as the Hermitian sums and differences of the in-general n-dependent raising and lowering operators of the system $A_n^{\pm}$,

$$X = K_1(n)\{[A_n^- + (A_n')^\dagger] + [A_n^+ + (A_n^-)^\dagger]\} \quad , \tag{4.12}$$

$$P = \frac{1}{i}K_2(n)\{[A_n^- + (A_n')^\dagger] - [A_n' + (A_n^-)^\dagger]\} \quad , \tag{4.13}$$

where $K_1(n)$ and $K_2(n)$ are n-dependent c-numbers.

What we have is a chain in which you can go in either direction. If you can solve the classical problem, you will obtain the natural classical variables. From these you obtain the natural quantum operators, and find the coherent states as those which minimize the associated uncertainty relation subject to the ground state being a member of the set. Finally, these natural quantum operators can be written as the Hermitian sums and differences of the quantum raising and lowering operators.

Similarly, you can go in the reverse direction if you can solve the quantum problem. As a matter of fact, when we were discovering this method it was this reverse procedure which led us to the solution. Now it appears to us that for most examples it is simpler to proceed starting from the classical problem.

In our series[14-19] we also discussed analytic generalizations of annihilation-operator and displacement-operator coherent states. In general these are not the same as the generalized minimum-uncertainty coherent states. For the annihilation-operator coherent states, one uses the fact that just as a^- can be written in terms of x and p for the harmonic oscillator, so too A_n^- can be written in terms of X and P. This operator is used. Further, in general these states are equivalent to states from a displacement-operator which is not an exponential. One can understand this by realizing that for very confining systems, like a square well, you cannot displace the ground state unitarily to the side. It has nowhere to go.

I refer the reader to our series[14-19] for more details. I will now concentrate on the minimum-uncertainty coherent states, which we found to be the most physical and enlightening to study.

5. AN EXAMPLE

To show how the procedure works, I will use what I find a very illuminating example, the $\cosh^{-2}$ potential. Normalizing so that the potential's minimum is at the origin, the potential is

$$V(x) = U_0 \tanh^2 ax$$

$$= U_0 s(s+1)\tanh^2 z \ , \tag{5.1}$$

$$z = ax/2m \ . \tag{5.2}$$

This potential has a finite number of bound states, goes to zero at the origin, and goes to U_0 as $x \to \infty$.

The solutions for our natural classical variables X_c and P_c are

$$X_c = \sinh z = \left(\frac{E}{U_o - E}\right)^{\frac{1}{2}} \sin \omega_c t \quad , \tag{5.3}$$

$$P_c = p \cosh z = (2ma^2E)^{\frac{1}{2}} \cos \omega_c t \quad , \tag{5.4}$$

$$\omega_c(E) = \left[\frac{2a^2(U_o - E)}{m}\right]^{\frac{1}{2}} \quad . \tag{5.5}$$

One can verify that these variables obey classical equations of motion analogous to those for x and p in the harmonic oscillator,

$$\dot{X}_c = P_c/m \quad , \tag{5.6}$$

$$\dot{P}_c = -m\omega_c^2 X_c \quad . \tag{5.7}$$

Now we do our coherent-states procedure. The natural quantum operators are

$$X \quad X_c \quad \sinh z \quad , \tag{5.8}$$

$$p \quad \frac{\hbar}{2i} a^{-1}\left[\frac{d}{dz} \cosh z + \cosh z \frac{d}{dz}\right] \quad , \tag{5.9}$$

so that we are looking for states which satisfy the equality in the uncertainty relation

$$(\Delta X)^2(\Delta P)^2 = (\hbar^2/4)a^{-4}\cosh^2 z^2 \quad . \tag{5.10}$$

These normalized states are

$$\psi \quad \left[\frac{a\,\Gamma(B + \frac{1}{2} + iu)\,\Gamma(B + \frac{1}{2} - iu)}{\pi^{\frac{1}{2}}\,\Gamma(B)\,\Gamma(B + \frac{1}{2})}\right]^{\frac{1}{2}} (\cosh z)^{-B}$$

$$\cdot \exp[iu \sin^{-1}(\tanh z)] \quad , \tag{5.11}$$

$$B \quad \frac{1}{2} \cosh^2 z \quad \frac{1}{2} \quad , \tag{5.12}$$

$$(\Delta \sinh z)^2 + \frac{1}{2}$$

$$C = u + iv = B\langle \sinh z\rangle + \langle \cosh z \frac{d}{dz} \rangle \quad . \tag{5.13}$$

Finally, if

$$B = s \tag{5.14}$$

this set of states includes the ground state. Therefore, imposing (5.14) gives us our coherent states.

Now the quantum problem has exact eigenstates and eigenvalues

$$\psi_n \quad \left[\frac{a(s-n)\Gamma(2s-n+1)}{\Gamma(n+1)}\right]^{\frac{1}{2}} P_s^{n-s}(\tanh z) \quad , \tag{5.15}$$

$$E_n = \mathcal{E}_0(2ns - n^2 + s) \quad , \tag{5.16}$$

where the P_n^{n-s} are associated Legendre functions. Because of that one can figure out what the raising and lowering operators are for these eigenstates. They are

$$A_n^{\dagger} \quad (s-n)\sinh z - \cosh z \frac{d}{dz} \quad . \tag{5.17}$$

This verifies that indeed our natural quantum operators X and P can be expressed as the Hermitian sums and differences of the quantum raising and lowering operators, just as in Eqs. (4.12) and (4.13).

6. DISCUSSION

Given our minimum uncertainty coherent states (MUCS), just how well do they follow the classical motion? At the beginning I gave the intuitive answer, and I now repeat it. Effectively, "The more highly bound a particle state is and, given that, the greater number of eigenstates with which a coherent state has a significant overlap and, finally, the closer to equally spaced (i.e., harmonic) these eigenstates are, then the better and longer a coherent state will follow the classical motion without dispersing."

In the 5th article of our series,[18] we showed in detail photographs displaying the time evolution of MUCS in differing potentials and situations. I refer people to that for the detailed discussion which shows how we came to the above

intuitive answer. This article[18] discusses work which was the basis for a computer generated film[30] which graphically displays the time evolution of our states (this film to be shown at the present seminar).

In this printed version I wish to include one display of the time evolution of an MUCS, shown in Fig. 3. It is for the $\cosh^{-2}$ potential of the last section. s = 399.5 (400 bound states), the coherent state starts in the middle and has an <H> that is 1/10 the way up to the continuum at U_0.

Note that each time the wave packet hits a wall, part of it begins to disperse until, at the end, after 8-5/8ths classical oscillations, there is significant spread in the packet from the main hump and the packet's <x> has deviated from the classical position.

From this and other runs like this[18,30] we came to the intuitive description of the time evolution I have given.

However, we also considered the time evolution of other types of coherent states.[14-18,31] Our conclusion there was that our MUCS always do as well as or better than other states investigated. Specifically, the less confining a potential is, the more the various methods numerically yield about the same coherent states and time evolution. However, for highly confined systems (like the infinite square-well), our MUCS method still works well whereas some others can run into trouble.[18]

One can, by a slight generalization, apply this method to multidimensional systems[17] and systems where the commutation relations involved are complicated.[16] One can also apply this method to systems which cannot be solved analytically, either by analytic approximations or by numerically solving the differential equations involved. (In fact, for the Morse potential[16] our work involved analytic approximation techniques.) Finally, the method has also been applied to time dependent potentials.[32] Thus, I feel that this method is a general way of finding classical motion states in arbitrary potentials.

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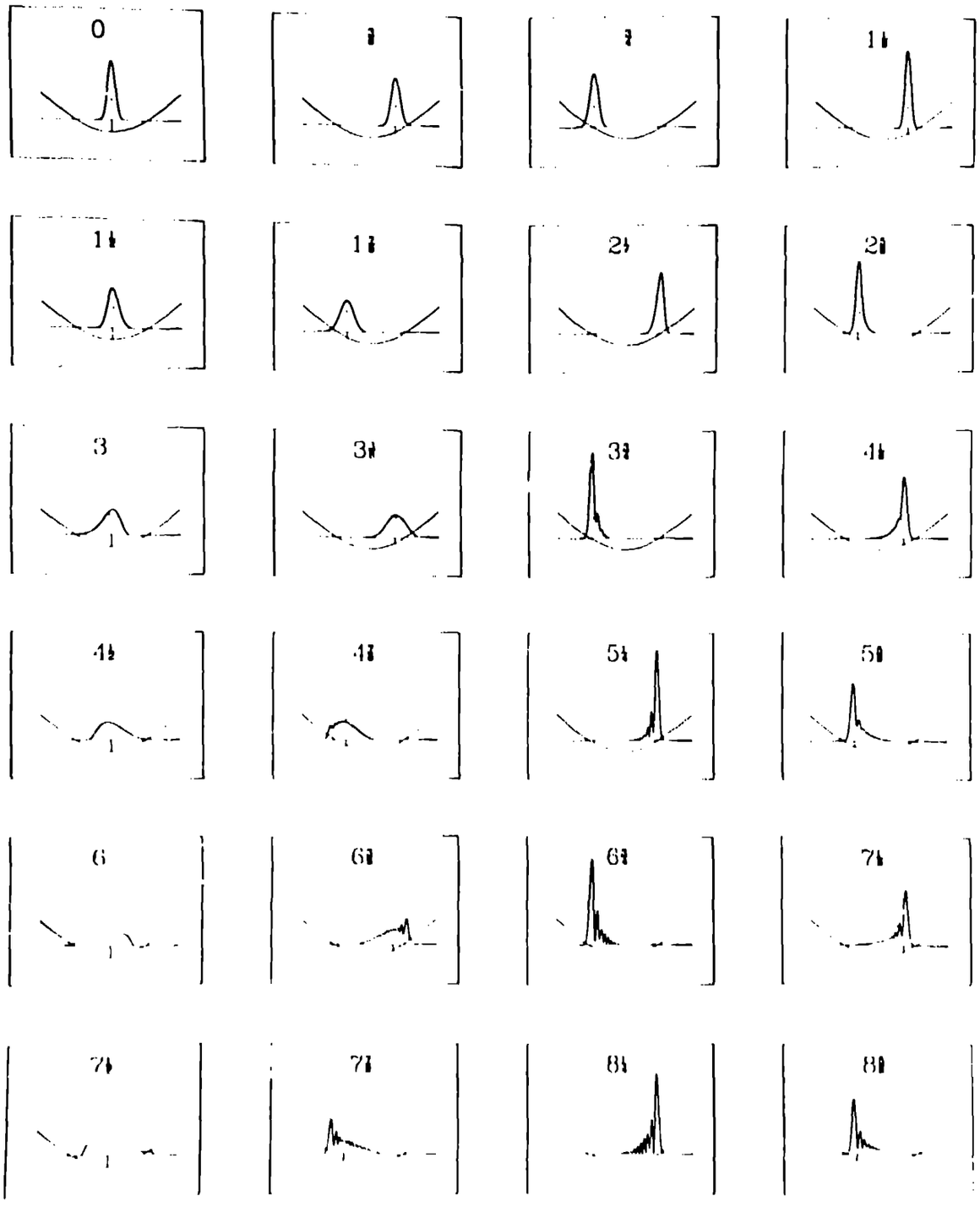

Figure 3.

Time evolution of a coherent state wave packet in the $\cosh^{-2}$ potential with $s = 399.5$ (400 bound state). The state has an "H" that is 1/10 the way up to the continuum at U_0. Shown is the potential, the wave packet at a height corresponding to "H", a vertical bar that shows the classical position, and a dot that represents "x". The numbers are the number of classical oscillations for each frame.

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