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COHOMOGENEITY ONE SPECIAL LAGRANGIAN SUBMANIFOLDS IN THE COTANGENT BUNDLE OF THE SPHERE

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Abstract. We classify cohomogeneity one special Lagrangian submanifolds in the cotangent bundle of the sphere S^n invariant under $SO(p) \times SO(n+1-p)$ with respect to the Stenzel metric and a Ricci-flat cone Kähler metric. Moreover, we describe the asymptotic behavior and singularities of such special Lagrangian submanifolds.

1. Introduction. In their pioneering paper [5], Harvey and Lawson gave an important method to study homologically volume minimizing submanifolds by using calibrations. Let M be a Calabi-Yau manifold with a complex volume form Ω. Then naturally $\mathrm{Re}\,\Omega$ is a calibration on M, and a calibrated submanifold is called a *special Lagrangian* (SL) submanifold. Strominger, Yau and Zaslow [19] suggested that, from the view point of geometry, the mirror symmetry between Calabi-Yau 3-folds should be explained in terms of dual fibrations by special Lagrangian 3-tori, that is the so-called SYZ conjecture. According to the relationship with string theory, today many mathematicians pay attention to special Lagrangian submanifolds, especially their singularities. In his series of papers ([10], [11], [12], [13] and [15]), Joyce constructed many interesting examples of special Lagrangian submanifolds in $\boldsymbol{C}^n$, using various methods. Castro and Urbano [3] gave a construction of special Lagrangian immersions with cohomogeneity one or two in $\boldsymbol{C}^n$, from minimal Legendrian submanifolds in S^{2n-1}. Haskins [6] studied special Lagrangian cones in $\boldsymbol{C}^3$. There are fruitful results in $\boldsymbol{C}^n$, although $\boldsymbol{C}^n$ is flat and its holonomy group is trivial.

On the other hand, now we know interesting examples of non-flat Calabi-Yau manifolds. In 1993, Stenzel [18] constructed complete Ricci-flat Kähler metrics on the cotangent bundles of compact rank one symmetric spaces. Karigiannis and Min-Oo [17] proved that the conormal bundle of a submanifold M of the sphere S^n is an SL submanifold in the cotangent bundle T^*S^n, equipped with the Stenzel metric, if and only if M is austere. That is a natural generalization of the construction of special Lagrangian conormal bundles in $T^*\boldsymbol{R}^n \cong \boldsymbol{C}^n$ due to Harvey and Lawson. The Lie group $SO(n+1)$ acts on T^*S^n with cohomogeneity one preserving the Stenzel metric. Using this large symmetry, Anciaux [1] constructed SL submanifolds in T^*S^n invariant under $SO(n)$. Using the moment map technique, Ionel and Min-Oo [9] studied SL submanifolds in T^*S^3 invariant under the 2-torus T^2 or $SO(3)$. In the

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present paper, as a generalization of the above results, we study cohomogeneity one special Lagrangian submanifolds in T^*S^n invariant under $SO(p) \times SO(q)$ $(p+q=n+1)$. First we construct Lagrangian submanifolds by the moment map technique. Since these Lagrangian submanifolds are of cohomogeneity one, the condition to be special Lagrangian is reduced to certain ordinary differential equations (ODE). We analyze the solutions of the ODE, and investigate the asymptotic behavior and singularities of the corresponding SL submanifolds. In general, the condition to be special Lagrangian is given by a partial differential equation (PDE). Let G be a Lie group with Lie algebra $\mathfrak{g}$. Assume that G acts on a Calabi-Yau n-fold with a moment map μ preserving the Calabi-Yau structure. If the G-action on the level set $\mu^{-1}(c)$ for some element c of the center $Z(\mathfrak{g}^*)$ of $\mathfrak{g}^*$ has $(n-1)$-dimensional principal orbits, then the PDE to be special Lagrangian is reduced to a (first order) ODE defined on the orbit space $\mu^{-1}(c)/G$ of the G-action on $\mu^{-1}(c)$. On a Calabi-Yau manifold, the calibration is given as an S^1-family $\mathrm{Re}\,(e^{\sqrt{-1}\theta}\Omega)$ in general, here θ is called the phase. It is quite non-trivial to describe cohomogeneity one SL submanifolds of arbitrary phase θ in our cases, because we work in non-flat Calabi-Yau manifolds other than $\boldsymbol{C}^n$. We illustrate how our solution curves change when the phase varies (see Examples 5.3 and 5.5). When $SO(p) \times SO(q)$ is abelian (i.e., $p=q=2$ or $p=1, q=2$), then T^*S^n admits an S^1-family of special Lagrangian foliations, however there exist singular leaves (Remarks 3.7 and 3.9). We note that SL submanifolds with this kind of symmetry were also constructed by Kanemitsu [16] independently. Recently Haskins and Kapouleas [7] investigated SL cones in $\boldsymbol{C}^n$ invariant under $SO(p) \times SO(q)$.

This paper is organized as follows. In Section 2, we prepare some basics of special Lagrangian geometry, and explain the moment map technique to construct cohomogeneity one special Lagrangian submanifolds. Applying this, in Section 3, we construct special Lagrangian submanifolds in T^*S^n invariant under $SO(p) \times SO(q)$. To describe the asymptotic behavior of those SL submanifolds in T^*S^n, in Section 4, we give a (singular) Calabi-Yau metric on the complex cone Q_0^n as the limit of the Stenzel metric, and construct special Lagrangian submanifolds in Q_0^n. In Section 5, we observe the asymptotic behavior of the ends and singularities of SL submanifolds in T^*S^n.

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2. Preliminaries.

2.1. Calabi-Yau manifolds and special Lagrangian submanifolds.

We shall review some definitions and basic notions of Calabi-Yau manifolds and special Lagrangian submanifolds. See [14] for details.

There are several different definitions of Calabi-Yau manifolds. In this paper, we use the following definition.

DEFINITION 2.1. Let $n \geq 2$. An *almost Calabi-Yau n-fold* is a quadruple (M, J, ω, Ω) such that (M, J, ω) is a Kähler manifold of complex dimension n with a complex structure J and a Kähler form ω, and Ω is a nonvanishing holomorphic $(n, 0)$-form on M. In addition, if ω and Ω satisfy

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \bar{\Omega}\,, \tag{2.1}$$

then we call (M, J, ω, Ω) a *Calabi-Yau n-fold*.

If ω and Ω satisfy (2.1), then the Kähler metric g of (M, J, ω) is Ricci-flat. Its holonomy group $\mathrm{Hol}(g)$ is a subgroup of $SU(n)$, and this is another definition of a Calabi-Yau manifold.

A closed p-form φ on a Riemannian manifold (M, g) is called a *calibration* if $\varphi|_V \leq \mathrm{vol}_V$ for any oriented p-plane $V \subset T_x M$ for all $x \in M$. A p-dimensional submanifold N of M is said to be *calibrated* by a calibration φ if $\varphi|_{T_x N} = \mathrm{vol}_{T_x N}$ for all $x \in N$.

REMARK 2.2. The constant factor in (2.1) is chosen so that $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ is a calibration for any $\theta \in \boldsymbol{R}$.

DEFINITION 2.3. Let (M, J, ω, Ω) be a Calabi-Yau n-fold and L be a real n-dimensional submanifold of M. Then, for $\theta \in \boldsymbol{R}$, L is called a *special Lagrangian submanifold* of phase θ if it is calibrated by the calibration $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$.

We often abbreviate special Lagrangian by SL. Harvey and Lawson gave the following alternative characterization of SL submanifolds.

PROPOSITION 2.4 ([5]). *Let (M, J, ω, Ω) be a Calabi-Yau n-fold and L be a real n-dimensional submanifold of M. Then L is a special Lagrangian submanifold of phase θ if and only if $\omega|_L \equiv 0$ and $\mathrm{Im}(e^{\sqrt{-1}\theta}\Omega)|_L \equiv 0$.*

2.2. Stenzel metric on the cotangent bundle of the sphere. In [18], Stenzel constructed complete Ricci-flat Kähler metrics on the cotangent bundles of compact rank one symmetric spaces. For our use, here we shall recall the Stenzel metric on the cotangent bundle of the sphere. We describe the cotangent bundle of the n-sphere $S^n \cong SO(n+1)/SO(n)$ by

$$T^*S^n = \{(x, \xi) \in \boldsymbol{R}^{n+1} \times \boldsymbol{R}^{n+1};\ \|x\| = 1,\ \langle x, \xi\rangle = 0\}\,.$$

We identify the tangent bundle and the cotangent bundle of S^n by the Riemannian metric on S^n. Since any unit cotangent vector of S^n can be translated to another one, the Lie group $SO(n+1)$ acts on T^*S^n with cohomogeneity one by $g \cdot (x, \xi) = (gx, g\xi)$ for $g \in SO(n+1)$. Let Q^n be a complex quadric in $\boldsymbol{C}^{n+1}$ defined by

$$Q^n = \left\{ z = (z_1, \dots, z_{n+1}) \in \boldsymbol{C}^{n+1};\ \sum_{i=1}^{n+1} z_i^2 = 1 \right\}.$$

The Lie group $SO(n+1, \boldsymbol{C})$ acts on Q^n transitively, hence $Q^n \cong SO(n+1, \boldsymbol{C})/SO(n, \boldsymbol{C})$. According to Szöke [20], we can identify T^*S^n with Q^n by the diffeomorphism

$$\begin{array}{rcl} \Phi : T^*S^n & \longrightarrow & Q^n \\ \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid \\ (x, \xi) & \longmapsto & x\cosh(\|\xi\|) + \sqrt{-1}\dfrac{\xi}{\|\xi\|}\sinh(\|\xi\|)\,. \end{array}$$

The diffeomorphism Φ is equivariant under the action of $SO(n+1)$. Thus we frequently identify T^*S^n with Q^n. We give the complex structure on T^*S^n by pulling back the complex structure J of Q^n via the map Φ. With respect to this complex structure, Stenzel [18] constructed a complete Ricci-flat Kähler metric on Q^n, whose Kähler form is given by

$$\omega_{\mathrm{Stz}} = \sqrt{-1}\partial\bar{\partial}u(r^2) = \sqrt{-1}\sum_{i,j=1}^{n+1}\frac{\partial^2}{\partial z_i \partial \bar{z}_j}u(r^2)dz_i \wedge d\bar{z}_j\,,$$

where $r^2 = \|z\|^2 = \sum_{i=1}^{n+1} z_i\bar{z}_i$ and u is a smooth real function satisfying the differential equation

$$\frac{d}{dt}(U'(t))^n = cn(\sinh t)^{n-1} \qquad (c > 0)\,, \tag{2.2}$$

where $U(t) = u(\cosh t)$. In the case of $n = 2$, the Stenzel metric coincides with the hyperkähler metric on T^*S^2 discovered by Eguchi and Hanson [4]. The Calabi-Yau metric on $Q^3 \cong T^*S^3$ was first studied by Candelas and de la Ossa [2].

The Kähler form ω_{Stz} is exact and $\omega_{\mathrm{Stz}} = d\alpha_{\mathrm{Stz}}$ for the 1-form $\alpha_{\mathrm{Stz}} = -\mathrm{Im}\left(\bar{\partial}u(r^2)\right)$. We give the Liouville form α_0 on $\boldsymbol{C}^{n+1}$ by $\alpha_0(v) = \langle Jz, v\rangle$, where $\langle\ ,\ \rangle$ and J are the standard real inner product and complex structure on $\boldsymbol{C}^{n+1}$, respectively. Then one can show that $\alpha_{\mathrm{Stz}} = u'(r^2)\alpha_0$. Hence α_{Stz} has the expression

$$\alpha_{\mathrm{Stz}}(v) = u'(r^2)\alpha_0(v) = u'(r^2)\langle Jz, v\rangle \qquad (v \in T_zQ^n, z \in Q^n)\,.$$

From this, ω_{Stz} can be evaluated as

$$\begin{aligned} \omega_{\mathrm{Stz}}(v, w) &= d\alpha_{\mathrm{Stz}}(v, w) \\ &= v(\alpha_{\mathrm{Stz}}(w)) - w(\alpha_{\mathrm{Stz}}(v)) - \alpha_{\mathrm{Stz}}([v, w]) \\ &= 2u'(r^2)\langle Jv, w\rangle + 2u''(r^2)(\langle z, v\rangle\langle Jz, w\rangle - \langle z, w\rangle\langle Jz, v\rangle) \end{aligned} \tag{2.3}$$

for $v, w \in T_zQ^n$ and $z \in Q^n$.

The holomorphic $(n, 0)$-form Ω_{Stz} on Q^n is given by

$$\frac{1}{2}d(z_1^2 + z_2^2 + \cdots + z_{n+1}^2 - 1) \wedge \Omega_{\mathrm{Stz}} = \Omega_0\,,$$

where $\Omega_0 = dz_1 \wedge \cdots \wedge dz_{n+1}$ is the standard holomorphic $(n+1, 0)$-form on $\boldsymbol{C}^{n+1}$. We can express Ω_{Stz} as

$$\Omega_{\mathrm{Stz}}(v_1, \ldots, v_n) = \Omega_0(z, v_1, \ldots, v_n)$$

and also

$$\Omega_{\mathrm{Stz}}(v_1, \ldots, v_n) = \frac{1}{\|z\|^2}\Omega_0(\bar{z}, v_1, \ldots, v_n)\,,$$

where $v_1, \dots, v_n \in T_z Q^n$, $z \in Q^n$ and $z = z_1 \partial/\partial z_1 + \cdots + z_{n+1} \partial/\partial z_{n+1}$, $\bar{z} = \bar{z}_1 \partial/\partial z_1 + \cdots + \bar{z}_{n+1} \partial/\partial z_{n+1}$.

Clearly the action of $SO(n+1)$ on Q^n preserves J, ω_{Stz} and Ω_{Stz}. Moreover one can show that there exists a constant $\lambda \in \boldsymbol{R}$ such that

$$\frac{\omega_{\mathrm{Stz}}^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2} \right)^n \lambda^2 \Omega_{\mathrm{Stz}} \wedge \bar{\Omega}_{\mathrm{Stz}} .$$

Hence $(T^*S^n \cong Q^n, J, \omega_{\mathrm{Stz}}, \lambda\Omega_{\mathrm{Stz}})$ is a cohomogeneity one Calabi-Yau manifold with respect to the action of $SO(n+1)$.

2.3. Moment maps and Lagrangian submanifolds. Let (M, ω) be a symplectic manifold, and G be a Lie group acting on M. We denote the Lie algebra of G by $\mathfrak{g}$. Let X^* denote the fundamental vector field of $X \in \mathfrak{g}$ on M, i.e.,

$$X_x^* = \frac{d}{dt}\Big|_{t=0} \exp(tX)x \qquad (x \in M) .$$

Now we suppose that the action of G on M is Hamiltonian with the moment map $\mu : M \to \mathfrak{g}^*$. We define the center of $\mathfrak{g}^*$ to be $Z(\mathfrak{g}^*) = \{X \in \mathfrak{g}^*;\ \mathrm{Ad}^*(g)X = X \text{ for all } g \in G\}$. It is easy to see that the inverse image $\mu^{-1}(c)$ of $c \in \mathfrak{g}^*$ is G-invariant if and only if $c \in Z(\mathfrak{g}^*)$.

PROPOSITION 2.5. *Let L be a connected isotropic submanifold, i.e., $\omega|_L \equiv 0$, of M invariant under the action of G. Then $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

PROOF. For $X \in \mathfrak{g}$, we define a function μ_X on M by $\mu_X(x) = (\mu(x))(X)$. Then, from the definition of the moment map, μ_X is the Hamiltonian function of X^*. Since L is an isotropic submanifold of M, we have

$$\mathcal{L}_Y(\mu_X) = d\mu_X(Y) = \omega(X_x^*, Y) = 0$$

for all $X \in \mathfrak{g}$, $Y \in T_x L$ and $x \in L$. Since L is connected, this implies that μ_X is constant on L for all $X \in \mathfrak{g}$, hence $\mu : M \to \mathfrak{g}^*$ is also constant on L. Thus $L \subset \mu^{-1}(c)$ for some $c \in \mathfrak{g}^*$. Moreover, since L is G-invariant, we have $c \in Z(\mathfrak{g}^*)$. □

PROPOSITION 2.6. *Let L be a connected submanifold of M invariant under the action of G. Suppose that the action of G on L is cohomogeneity one* (*possibly transitive*). *Then L is an isotropic submanifold, i.e., $\omega|_L \equiv 0$, if and only if $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

PROOF. By Proposition 2.5, we know that $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$ if L is isotropic. So it suffices to prove the converse.

Suppose that $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$. This means that μ is constant on L, so μ_X is also constant on L for all $X \in \mathfrak{g}$. Therefore

$$\omega(X_x^*, Y) = \mathcal{L}_Y(\mu_X) = 0$$

for all $X \in \mathfrak{g}$, $Y \in T_x L$ and $x \in L$. Let $x \in L$ be a regular point of the action of G on L. It is known that the set of regular points is open dense in L. Since the action of G on L is cohomogeneity one, if we take $Y_1 \in T_x L$ which is transverse to the orbit of G at x, then

$T_xL = \mathrm{span}\{X_x^*, Y_1;\ X \in \mathfrak{g}\}$. Therefore $\omega|_{T_xL} \equiv 0$. Since ω vanishes on an open dense subset of L, it vanishes on L entirely. Thus L is isotropic. □

3. Construction of cohomogeneity one special Lagrangian submanifolds in T^*S^n. In this section we shall construct cohomogeneity one special Lagrangian submanifolds in T^*S^n with respect to the Stenzel metric, using the moment map technique. Since the zero-section S^n of T^*S^n is a Lagrangian submanifold, a hypersurface N in S^n is an $(n-1)$-dimensional isotropic submanifold in T^*S^n. In particular when N is homogeneous, evolving it to an n-dimensional submanifold in T^*S^n, we can construct a cohomogeneity one Lagrangian submanifold. For such a Lagrangian submanifold, the condition to be special Lagrangian can be described by an ordinary differential equation.

Let G be a compact Lie subgroup of $SO(n+1)$ and $\mathfrak{g}$ its Lie algebra. Then the action of G on Q^n is Hamiltonian, and its moment map $\mu : Q^n \to \mathfrak{g}^*$ is given by

$$(\mu(z))(X) = \mu_X(z) = \alpha_{\mathrm{Stz}}(X_z^*) = \alpha_{\mathrm{Stz}}(Xz) = u'(r^2)\langle Jz, Xz\rangle \quad (z \in Q^n, X \in \mathfrak{g})\,. \tag{3.1}$$

In this paper we shall study special Lagrangian submanifolds invariant under

$$G = \left(\begin{array}{c|c} SO(p) & O \\ \hline O & SO(q) \end{array}\right) \cong SO(p) \times SO(q) \quad (p+q = n+1,\ 1 \le p \le q \le n)\,.$$

In this case, the G-action on S^n is cohomogeneity one, and its principal orbits are diffeomorphic to $S^{p-1} \times S^{q-1}$. Let us take

$$X_{ij} = E_{ji} - E_{ij} \in \mathfrak{so}(n+1)\,,$$

where E_{ij} denotes the $(n+1) \times (n+1)$-matrix whose (i, j)-component is 1 and all others are 0. Then

$$\{X_{ij};\ 1 \le i < j \le p\} \cup \{X_{ij};\ p+1 \le i < j \le n+1\}$$

forms a basis of the Lie algebra $\mathfrak{g} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ of G. We denote by $\{\theta_{ij}\}$ the dual basis of $\{X_{ij}\}$. Then the moment map $\mu : Q^n \to \mathfrak{g}^*$ of the G-action on Q^n can be expressed as

$$\mu(z) = \sum_{i,j} \mu_{ij}(z)\theta_{ij}\,,$$

where μ_{ij} is defined by $\mu_{ij}(z) = \mu_{X_{ij}}(z) = (\mu(z))(X_{ij})$ and the summation is over $1 \le i < j \le p$ and $p+1 \le i < j \le n+1$. From (3.1) we have

$$\mu_{ij}(z) = u'(r^2)\langle Jz, X_{ij}z\rangle = 2u'(r^2)\mathrm{Im}(z_i\bar{z}_j)\,.$$

Thus, using the basis $\{\theta_{ij}\}$ of $\mathfrak{g}^*$, the moment map $\mu : Q^n \to \mathfrak{g}^*$ of the G-action on Q^n can be evaluated as

$$\mu(z) = 2u'(r^2)(\mathrm{Im}(z_i\bar{z}_j)_{1\le i<j\le p}\,,\ \mathrm{Im}(z_i\bar{z}_j)_{p+1\le i<j\le n+1})\,.$$

From Proposition 2.6, a Lagrangian submanifold of Q^n invariant under G should be contained in $\mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$. In the case of $p = 2$ or $q = 2$, since $SO(2)$ is abelian, $\mathfrak{g}^*$ has the non-trivial center. In the case of $p = 1$, the orbit space of the G-action on the zero-section S^n is different from the case of $p \geq 2$. Therefore we shall discuss the following five cases individually.

(1) $3 \leq p \leq q$, (2) $p = 1, q \geq 3$, (3) $p = 2, q \geq 3$, (4) $p = q = 2$, (5) $p = 1, q = 2$.

In the case of $p = 1$, we have SL submanifolds invariant under $SO(n)$, which were first studied by Anciaux [1]. Ionel and Min-Oo [9] investigated SL submanifolds in Q^3 invariant under $SO(2) \times SO(2)$ or $SO(3)$.

3.1. The case of $3 \leq p \leq q$**.** We give a parametrization of the orbit space of the action of $G = SO(p) \times SO(q)$ on $T^*S^n = \{(x, \xi) \in \boldsymbol{R}^{n+1} \times \boldsymbol{R}^{n+1};\ \|x\| = 1, \langle x, \xi \rangle = 0\}$. First, $x \in S^n$ can be moved to

$$x = (\overset{(1)}{\cos t}, 0, \dots, 0, \overset{(p+1)}{\sin t}, 0, \dots, 0) \qquad (t \in \boldsymbol{R})$$

by the action of G. Furthermore $\xi \in T^*_x S^n$ can be moved to

$$\xi = (\overset{(1)}{-\xi_1 \sin t}, \overset{(2)}{\xi_2}, 0, \dots, 0, \overset{(p+1)}{\xi_1 \cos t}, \overset{(p+2)}{\xi_3}, 0, \dots, 0) \qquad (\xi_1, \xi_2, \xi_3 \in \boldsymbol{R})$$

by the action of the isotropy subgroup $G_x = \{g \in G;\ g \cdot x = x\}$ at x. Therefore we define a subset Σ of T^*S^n by

$$\Sigma = \left\{ (x, \xi)\ ;\ \begin{array}{l} x = (\cos t, 0, \dots, 0, \sin t, 0, \dots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, 0, \dots, 0, \xi_1 \cos t, \xi_3, 0, \dots, 0) \end{array} \right\}.$$

Then every G-orbit in T^*S^n meets Σ, i.e., $G \cdot \Sigma = T^*S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \{0\}$. We determine the subset $\mu^{-1}(0) \cap \Phi(\Sigma)$ of Q^n. Now $z = \Phi(x, \xi) \in \Phi(\Sigma)$ can be expressed as

$$\begin{aligned} z = \Bigg(&\cos t \cosh \rho - \sqrt{-1} \frac{\xi_1 \sin t}{\rho} \sinh \rho, \sqrt{-1} \frac{\xi_2}{\rho} \sinh \rho, 0, \dots, 0, \\ &\sin t \cosh \rho + \sqrt{-1} \frac{\xi_1 \cos t}{\rho} \sinh \rho, \sqrt{-1} \frac{\xi_3}{\rho} \sinh \rho, 0, \dots, 0 \Bigg), \end{aligned}$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Then $\mu(z) = 0$ if and only if

$$\begin{aligned} 0 = \mathrm{Im}(z_1 \bar{z}_2) &= -\frac{\xi_2}{\rho} \cos t \sinh \rho \cosh \rho, \\ 0 = \mathrm{Im}(z_{p+1} \bar{z}_{p+2}) &= -\frac{\xi_3}{\rho} \sin t \sinh \rho \cosh \rho. \end{aligned}$$

So we have $\xi_2 = \xi_3 = 0$, hence

$$z = \big(\cos(t + \sqrt{-1}\xi_1), 0, \dots, 0, \sin(t + \sqrt{-1}\xi_1), 0, \dots, 0 \big).$$

Consequently, we obtain

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \big\{ (\cos \tau, 0, \dots, 0, \sin \tau, 0, \dots, 0)\ ;\ \tau = t + \sqrt{-1}\xi_1\ (t, \xi_1 \in \boldsymbol{R}) \big\}.$$

Since $\mu^{-1}(0)$ is G-invariant, we have

$$\mu^{-1}(0) = G \cdot (\mu^{-1}(0) \cap \Phi(\Sigma)).$$

Thus the orbit space $\mu^{-1}(0)/G$ of the G-action on $\mu^{-1}(0)$ is parametrized by t and ξ_1.

REMARK 3.1. The (t, ξ_1)-plane can be regarded as the covering space of the orbit space $\mu^{-1}(0)/G$. In fact, we can take $t \in [0, \pi/2]$, and $\mu^{-1}(0)/G \cong \boldsymbol{C}/(\boldsymbol{Z} \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$, where the action of $\boldsymbol{Z}$ on $\boldsymbol{C}$ is the parallel translation of period 2π and the actions of $\boldsymbol{Z}_2$ are reflections across the points $(t, \xi_1) = (0, 0)$ and $(\pi/2, 0)$, respectively. Principal orbits of the G-action on $\mu^{-1}(0)$ are diffeomorphic to $S^{p-1} \times S^{q-1}$. There are two singular orbits S^{p-1} and S^{q-1} at $(t, \xi_1) = (0, 0)$ and $(\pi/2, 0)$, respectively. This implies that the orbit space $\mu^{-1}(0)/G$ is an orbifold with two singular points. (See Figures in Example 5.3.)

THEOREM 3.2. *Let τ be a regular curve in the complex plane $\boldsymbol{C}$. We define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = (\cos\tau(s), 0, \dots, 0, \sin\tau(s), 0, \dots, 0)\,.$$

Then the G-orbit $L = G \cdot \sigma$ through σ is a Lagrangian submanifold (*possibly with singularities*) *in Q^n. Moreover*, *the smooth part of L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c \in \boldsymbol{R}$ such that τ satisfies*

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\int_0^{\tau(s)}(\cos w)^{p-1}(\sin w)^{q-1}dw\right) = c\,. \tag{3.2}$$

PROOF. Since $L = G\cdot\sigma$ is a cohomogeneity one (possibly homogeneous) submanifold of dimension n contained in $\mu^{-1}(0)$, it follows from Proposition 2.6 that L is a Lagrangian submanifold in Q^n. We look for σ such that L is a special Lagrangian submanifold in Q^n. We take a basis of the tangent space $T_{\sigma(s)}L$ of L at $\sigma(s)$ as follows:

$$\begin{aligned}
X^*_{1,2} &= X_{1,2}\sigma(s) = (0, \overset{(2)}{\cos}\tau(s), 0, \dots, 0)\,,\\
&\ \ \vdots\\
X^*_{1,p} &= X_{1,p}\sigma(s) = (0, \dots, 0, \overset{(p)}{\cos}\tau(s), 0, \dots, 0)\,,\\
X^*_{p+1,p+2} &= X_{p+1,p+2}\sigma(s) = (0, \dots, 0, \overset{(p+2)}{\sin}\tau(s), 0, \dots, 0)\,,\\
&\ \ \vdots\\
X^*_{p+1,n+1} &= X_{p+1,n+1}\sigma(s) = (0, \dots, 0, \overset{(n+1)}{\sin}\tau(s))\,,\\
\sigma'(s) &= (\overset{(1)}{-\tau'(s)\sin\tau(s)}, 0, \dots, 0\,, \overset{(p+1)}{\tau'(s)\cos\tau(s)}, 0, \dots, 0)\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\Omega_{\mathrm{Stz}}(X^*_{1,2}, \dots, X^*_{1,p}, \sigma'(s), X^*_{p+1,p+2}, \dots, X^*_{p+1,n+1})\\
&\quad = (dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1})(\sigma(s), X^*_{1,2}, \dots, X^*_{1,p}, \sigma'(s), X^*_{p+1,p+2}, \dots, X^*_{p+1,n+1})
\end{aligned}$$

$$
= \begin{vmatrix}
\cos\tau(s) & 0 & \cdots & 0 & -\tau'(s)\sin\tau(s) & 0 & \cdots & 0 \\
0 & \cos\tau(s) & & \vdots & 0 & \vdots & & \vdots \\
\vdots & 0 & \ddots & 0 & \vdots & \vdots & & \vdots \\
0 & \vdots & & \cos\tau(s) & 0 & \vdots & & \vdots \\
\sin\tau(s) & \vdots & & 0 & \tau'(s)\cos\tau(s) & 0 & & \vdots \\
0 & \vdots & & \vdots & 0 & \sin\tau(s) & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & 0 & \ddots & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \sin\tau(s)
\end{vmatrix}
$$

$$= \tau'(s)(\cos\tau(s))^{p-1}(\sin\tau(s))^{q-1} .$$

Thus L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\operatorname{Im}\bigl(e^{\sqrt{-1}\theta}\tau'(s)(\cos\tau(s))^{p-1}(\sin\tau(s))^{q-1}\bigr) = 0 . \tag{3.3}$$

This condition is equivalent to (3.2) for some $c \in \boldsymbol{R}$. □

For a curve τ in the complex plane $\boldsymbol{C}$, L coincides with the image of the map

$$
\begin{array}{ccl}
\Psi : I \times S^{p-1} \times S^{q-1} & \longrightarrow & Q^n \\
\Psi & & \Psi \\
(s, x, y) & \longmapsto & (\cos\tau(s)x_1, \dots, \cos\tau(s)x_p, \sin\tau(s)y_1, \dots, \sin\tau(s)y_q) .
\end{array}
$$

Here I is an open interval in $\boldsymbol{R}$. When τ passes through $m\pi/2$ ($m \in \boldsymbol{Z}$), the map Ψ degenerates at that point. If τ does not pass through $m\pi/2$ ($m \in \boldsymbol{Z}$), then L is diffeomorphic to $I \times S^{p-1} \times S^{q-1}$ and immersed in Q^n by the map Ψ. (See Figures in Example 5.3.)

3.2. The case of $p = 1, q \geq 3$. The orbit space of the action of

$$G = \left(\begin{array}{c|c} 1 & O \\ \hline O & SO(n) \end{array}\right) \cong SO(n)$$

on T^*S^n is parametrized as

$$\Sigma = \left\{ (x, \xi) \ ; \begin{array}{l} x = (\cos t, \sin t, 0, \dots, 0) \\ \xi = (-\xi_1 \sin t, \xi_1 \cos t, \xi_2, 0, \dots, 0) \end{array} \right\} .$$

Then every G-orbit in T^*S^n meets Σ, i.e., $G \cdot \Sigma = T^*S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \{0\}$. For $z = \Phi(x, \xi) \in \Phi(\Sigma)$, the equality $\mu(z) = 0$ is satisfied if and only if $\xi_2 = 0$. So we have

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \left\{ (\cos\tau, \sin\tau, 0, \dots, 0) \ ; \ \tau = t + \sqrt{-1}\xi_1 \ (t, \xi_1 \in \boldsymbol{R}) \right\} ,$$

and

$$\mu^{-1}(0) = G \cdot (\mu^{-1}(0) \cap \Phi(\Sigma)) .$$

Thus the orbit space $\mu^{-1}(0)/G$ of the G-action on $\mu^{-1}(0)$ is parametrized by t and ξ_1.

REMARK 3.3. In this case, we can take $t \in [0, \pi]$, and $\mu^{-1}(0)/G \cong \boldsymbol{C}/(\boldsymbol{Z} \times \boldsymbol{Z}_2)$. Principal orbits of the G-action on $\mu^{-1}(0)$ are diffeomorphic to S^{n-1}. There are two singular orbits at $(t, \xi_1) = (0, 0)$ and $(\pi, 0)$, that is, fixed orbits at the north pole and the south pole of the zero-section S^n. Thus the orbit space $\mu^{-1}(0)/G$ is an orbifold with two singular points. (See Figures in Example 5.5.)

THEOREM 3.4. *Let τ be a regular curve in the complex plane $\boldsymbol{C}$. We define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by*

$$\sigma(s) = (\cos\tau(s), \sin\tau(s), 0, \dots, 0)\,.$$

Then the G-orbit $L = G \cdot \sigma$ through σ is a Lagrangian submanifold (possibly with singularities) in Q^n. Moreover, the smooth part of L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c \in \boldsymbol{R}$ such that τ satisfies

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\int_0^{\tau(s)}(\sin w)^{n-1}dw\right) = c\,. \tag{3.4}$$

PROOF. Similarly to Theorem 3.2, we can see that L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta}\tau'(s)(\sin\tau(s))^{n-1}\big) = 0\,. \tag{3.5}$$

This condition is equivalent to (3.4) for some $c \in \boldsymbol{R}$. □

For a curve τ in the complex plane $\boldsymbol{C}$, L coincides with the image of the map

$$\begin{array}{ccc} \Psi : I \times S^{n-1} & \longrightarrow & Q^n \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (s, y) & \longmapsto & (\cos\tau(s), \sin\tau(s)y_1, \dots, \sin\tau(s)y_n)\,. \end{array}$$

When τ passes through $m\pi$ ($m \in \boldsymbol{Z}$), the map Ψ degenerates at that point. If τ does not pass through $m\pi$ ($m \in \boldsymbol{Z}$), then L is diffeomorphic to $I \times S^{n-1}$ immersed in Q^n by Ψ. (See Figures in Example 5.5.)

3.3. The case of $p = 2, q \geq 3$. The orbit space of the action of

$$G = \left(\begin{array}{c|c} SO(2) & O \\ \hline O & SO(n-1) \end{array}\right) \cong SO(2) \times SO(n-1)$$

on T^*S^n is parametrized as

$$\Sigma = \left\{ (x, \xi)\,;\ \begin{array}{l} x = (\cos t, 0, \sin t, 0, \dots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, \xi_3, 0, \dots, 0) \end{array} \right\}.$$

Then every G-orbit in T^*S^n meets Σ, i.e., $G \cdot \Sigma = T^*S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \boldsymbol{R}\theta_{12}$. For $c_1 \in \boldsymbol{R}$, we determine the subset $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$ of Q^n. For $z = \Phi(x, \xi) \in \Phi(\Sigma)$, the equality $\mu(z) = c_1\theta_{12}$ is satisfied if and only if

$$c_1 = 2u'(r^2)\mathrm{Im}(z_1\bar{z}_2) = -2u'(\cosh(2\rho))\frac{\xi_2}{\rho}\cos t \sinh\rho\cosh\rho\,,$$

$$0 = \mathrm{Im}(z_3\bar{z}_4) = -\frac{\xi_3}{\rho}\sin t \sinh\rho\cosh\rho\,,$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}$. Thus $\xi_3 = 0$ and we obtain

$$\Phi^{-1}(\mu^{-1}(c_1\theta_{12}))\cap\Sigma = \left\{ (x,\xi)\ ;\ \begin{aligned} x &= (\cos t, 0, \sin t, 0, \dots, 0)\\ \xi &= (-\xi_1\sin t, \xi_2, \xi_1\cos t, 0, \dots, 0)\\ c_1 &= -u'(\cosh(2\rho))(\xi_2/\rho)\cos t\sinh(2\rho)\end{aligned}\right\}.$$

Since $\mu^{-1}(c_1\theta_{12})$ is G-invariant, we have

$$\mu^{-1}(c_1\theta_{12}) = G\cdot(\mu^{-1}(c_1\theta_{12})\cap\Phi(\Sigma))\,.$$

THEOREM 3.5. *Let σ be a regular curve in $\mu^{-1}(c_1\theta_{12})\cap\Phi(\Sigma)$. We express σ as*

$$\sigma(s) = (z_1(s), z_2(s), z_3(s), 0, \dots, 0)\,.$$

Then the G-orbit $L = G\cdot\sigma$ through σ is a Lagrangian submanifold (possibly with singularities) in Q^n. Moreover, the smooth part of L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c_2\in\boldsymbol{R}$ such that σ satisfies

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta}z_3(s)^{n-1}\big) = c_2\,. \tag{3.6}$$

PROOF. Since $L = G\cdot\sigma$ is a cohomogeneity one submanifold contained in $\mu^{-1}(c_1\theta_{12})$, it follows from Proposition 2.6 that L is a Lagrangian submanifold in Q^n. We look for σ such that L is a special Lagrangian submanifold in Q^n. We take a basis of the tangent space $T_{\sigma(s)}L$ of L at $\sigma(s)$ as follows:

$$\begin{aligned} X_{12}^* &= X_{12}\sigma(s) = (-z_2(s), z_1(s), 0, \dots, 0)\,,\\ X_{34}^* &= X_{34}\sigma(s) = (0,0,0,z_3(s),0,\dots,0)\,,\\ &\ \,\vdots\\ X_{3,n+1}^* &= X_{3,n+1}\sigma(s) = (0,\dots,0,z_3(s))\,,\\ \sigma'(s) &= (z_1'(s), z_2'(s), z_3'(s), 0, \dots, 0)\,.\end{aligned}$$

Since z is in Q^n, we note

$$\begin{aligned} &z_1(s)^2+z_2(s)^2+z_3(s)^2 = 1\,,\\ &z_1(s)z_1'(s)+z_2(s)z_2'(s)+z_3(s)z_3'(s) = 0\,.\end{aligned}$$

Using these equalities, we have

$$\begin{aligned} &\Omega_{\mathrm{Stz}}(X_{12}^*, \sigma'(s), X_{34}^*, \dots, X_{3,n+1}^*)\\ &\quad = (dz_1\wedge dz_2\wedge\cdots\wedge dz_{n+1})(\sigma(s), X_{12}^*, \sigma'(s), X_{34}^*, \dots, X_{3,n+1}^*)\end{aligned}$$

$$
= \begin{vmatrix}
z_1(s) & -z_2(s) & z_1'(s) & 0 & \cdots & 0 \\
z_2(s) & z_1(s) & z_2'(s) & 0 & \cdots & 0 \\
z_3(s) & 0 & z_3'(s) & 0 & \cdots & 0 \\
0 & \vdots & 0 & z_3(s) & & \vdots \\
\vdots & \vdots & \vdots & & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 & z_3(s)
\end{vmatrix}
$$

$$= z_3(s)^{n-2} z_3'(s) \,.$$

Therefore L is a special Lagrangian submanifold of phase θ in Q^n if and only if σ satisfies

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta} z_3(s)^{n-2} z_3'(s)\big) = 0 \,.$$

This condition is equivalent to (3.6) for some $c_2 \in \boldsymbol{R}$. □

For a curve σ in $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$, L coincides with the image of the map

$$
\begin{array}{rcl}
\Psi : I \times S^1 \times S^{n-2} & \longrightarrow & Q^n \\
\text{⋓} & & \text{⋓} \\
(s, x, y) & \longmapsto & (z_1(s)x_1 - z_2(s)x_2, z_1(s)x_2 + z_2(s)x_1, z_3(s)y_1, \dots, z_3(s)y_{n-1}) \,.
\end{array}
$$

When σ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, \dots, 0)$ or $z = (0, 0, \pm 1, 0, \dots, 0)$, the map Ψ degenerates at that point. If σ does not pass through the points of singular orbits, L is diffeomorphic to $I \times S^1 \times S^{n-2}$ and immersed in Q^n by Ψ.

3.4. The case of $p = q = 2$. The orbit space of the action of

$$G = \left(\begin{array}{c|c} SO(2) & O \\ \hline O & SO(2) \end{array}\right) \cong SO(2) \times SO(2)$$

on T^*S^3 is parametrized as

$$\Sigma = \left\{ (x, \xi) \;;\; \begin{array}{l} x = (\cos t, 0, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, \xi_3) \end{array} \right\}.$$

Then every G-orbit in T^*S^3 meets Σ, i.e., $G \cdot \Sigma = T^*S^3$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \boldsymbol{R}\theta_{12} + \boldsymbol{R}\theta_{34} = \mathfrak{g}^*$. For $c_1, c_2 \in \boldsymbol{R}$, we can verify

$$\Phi^{-1}(\mu^{-1}(c_1\theta_{12} + c_2\theta_{34})) \cap \Sigma = \left\{ (x, \xi) \;;\; \begin{array}{l} x = (\cos t, 0, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, \xi_1 \cos t, \xi_3) \\ c_1 = -u'(\cosh(2\rho))(\xi_2/\rho)\cos t \sinh(2\rho) \\ c_2 = -u'(\cosh(2\rho))(\xi_3/\rho)\sin t \sinh(2\rho) \end{array} \right\},$$

and

$$\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) = G \cdot (\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)) \,.$$

THEOREM 3.6. *Let σ be a regular curve in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$. We express σ as*

$$\sigma(s) = (z_1(s), z_2(s), z_3(s), z_4(s)) \,.$$

Then the G-orbit $L = G \cdot \sigma$ through σ is a Lagrangian submanifold (possibly with singularities) in Q^3. Moreover, the smooth part of L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c_3 \in \boldsymbol{R}$ such that σ satisfies

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta}(z_1(s)^2 + z_2(s)^2)\big) = c_3\,. \tag{3.7}$$

PROOF. The proof is similar to the previous theorem. We take a basis of the tangent space $T_{\sigma(s)}L$ of L at $\sigma(s)$ as follows:

$$\begin{aligned} X_{12}^* &= X_{12}\sigma(s) = (-z_2(s), z_1(s), 0, 0)\,,\\ X_{34}^* &= X_{34}\sigma(s) = (0, 0, -z_4(s), z_3(s))\,,\\ \sigma'(s) &= (z_1'(s), z_2'(s), z_3'(s), z_4'(s))\,. \end{aligned}$$

Then we have

$$\begin{aligned} \Omega_{\mathrm{Stz}}(X_{12}^*, X_{34}^*, \sigma'(s)) &= (dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4)(\sigma(s), X_{12}^*, X_{34}^*, \sigma'(s))\\ &= \begin{vmatrix} z_1(s) & -z_2(s) & 0 & z_1'(s)\\ z_2(s) & z_1(s) & 0 & z_2'(s)\\ z_3(s) & 0 & -z_4(s) & z_3'(s)\\ z_4(s) & 0 & z_3(s) & z_4'(s) \end{vmatrix}\\ &= z_1(s)z_1'(s) + z_2(s)z_2'(s)\,. \end{aligned}$$

Therefore L is a special Lagrangian submanifold of phase θ in Q^3 if and only if

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta}(z_1(s)z_1'(s) + z_2(s)z_2'(s))\big) = 0\,.$$

This condition is equivalent to (3.7) for some $c_3 \in \boldsymbol{R}$. □

REMARK 3.7. Since $G = SO(2) \times SO(2)$ is abelian and $Z(\mathfrak{g}^*) = \mathfrak{g}^*$, arbitrary $z \in Q^3$ lies in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34})$ for some $c_1, c_2 \in \boldsymbol{R}$. Furthermore, $z \in Q^3$ satisfies (3.7) for some $c_3 \in \boldsymbol{R}$. This yields that, for a fixed θ, the family of special Lagrangian submanifolds, which is constructed in Theorem 3.6, gives a foliation of $T^*S^3 \cong Q^3$ with singular leaves.

For a curve σ in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$, L coincides with the image of the map

$$\begin{aligned} \Psi : I \times S^1 \times S^1 &\longrightarrow Q^3\\ \Psi \qquad\qquad & \qquad \Psi\\ (s, x, y) &\longmapsto (z_1(s)x_1 - z_2(s)x_2, z_1(s)x_2 + z_2(s)x_1\,,\\ &\qquad\ z_3(s)y_1 - z_4(s)y_2, z_3(s)y_2 + z_4(s)y_1)\,. \end{aligned}$$

When σ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, 0)$ or $(0, 0, \pm\cosh(\xi_3), \sqrt{-1}\sinh(\xi_3))$, the map Ψ degenerates at that point. If σ does not pass through the points of singular orbits, then L is diffeomorphic to $I \times S^1 \times S^1$ and immersed in Q^3 by Ψ.

3.5. The case of $p = 1, q = 2$. The orbit space of the action of

$$G = \left(\begin{array}{c|c} 1 & O\\ \hline O & SO(2) \end{array}\right) \cong SO(2)$$

on T^*S^2 is parametrized as

$$\Sigma = \left\{ (x, \xi) \ ; \ \begin{array}{l} x = (\cos t, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_1 \cos t, \xi_2) \end{array} \right\} .$$

Then every G-orbit in T^*S^2 meets Σ, i.e., $G \cdot \Sigma = T^*S^2$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \boldsymbol{R}\theta_{23} = \mathfrak{g}^*$. For $c_1 \in \boldsymbol{R}$, we can verify

$$\Phi^{-1}(\mu^{-1}(c_1\theta_{23})) \cap \Sigma = \left\{ (x, \xi) \ ; \ \begin{array}{l} x = (\cos t, \sin t, 0) \\ \xi = (-\xi_1 \sin t, \xi_1 \cos t, \xi_2) \\ c_1 = -u'(\cosh(2\rho))(\xi_2/\rho) \sin t \sinh(2\rho) \end{array} \right\} ,$$

and

$$\mu^{-1}(c_1\theta_{23}) = G \cdot (\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)) .$$

THEOREM 3.8. *Let σ be a regular curve in $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$. We express σ as*

$$\sigma(s) = (z_1(s), z_2(s), z_3(s)) .$$

Then the G-orbit $L = G \cdot \sigma$ through σ is a Lagrangian submanifold (possibly with singularities) in Q^2. Moreover, the smooth part of L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c_2 \in \boldsymbol{R}$ such that σ satisfies

$$\mathrm{Im}\big(e^{\sqrt{-1}\theta} z_1(s)\big) = c_2 . \tag{3.8}$$

PROOF. The proof is similar to the previous theorems. □

REMARK 3.9. Since $G = SO(2)$ is abelian and $Z(\mathfrak{g}^*) = \mathfrak{g}^*$, for a fixed θ, the family of special Lagrangian submanifolds, which is constructed in Theorem 3.8, gives a foliation of $T^*S^2 \cong Q^2$ with singular leaves.

For a curve σ in $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$, L coincides with the image of the map

$$\begin{array}{ccc} \Psi : I \times S^1 & \longrightarrow &\, Q^2 \\ \Psi & & \Psi \\ (s, y) & \longmapsto & (z_1(s), z_2(s)y_1 - z_3(s)y_2, z_2(s)y_2 + z_3(s)y_1) . \end{array}$$

When σ passes through $z = (\pm 1, 0, 0)$, the map Ψ degenerates at that point. If σ does not pass through $z = (\pm 1, 0, 0)$, then L is diffeomorphic to $I \times S^1$ and immersed in Q^2 by Ψ.

3.6. Conormal bundle special Lagrangian submanifolds. Harvey and Lawson [5] introduced the notion of austere submanifolds in order to construct special Lagrangian submanifolds in $T^*\boldsymbol{R}^n \cong \boldsymbol{C}^n$ as the conormal bundles of submanifolds in $\boldsymbol{R}^n$. A submanifold M of a Riemannian manifold $\tilde{M}$ is said to be *austere* if the set of eigenvalues of the shape operator of M is invariant under the multiplication by -1 concerning the multiplicities. Clearly an austere submanifold is minimal. As a generalization of Harvey and Lawson's construction, Karigiannis and Min-Oo proved the following theorem.

THEOREM 3.10 ([17]). *The conormal bundle N^*M of a submanifold $M \subset S^n$ is special Lagrangian in T^*S^n equipped with the Stenzel metric if and only if M is austere in S^n.*

In [8], we determined all austere orbits of the isotropy representations of irreducible symmetric spaces of compact type. All austere orbits of the action of $SO(p) \times SO(q)$ $(p + q = n + 1)$ on S^n are the following (in this situation, however, the symmetric space is not irreducible).

(1) When $p = q$, a minimal principal orbit of the action of $SO(p) \times SO(p)$ on S^n, that is called a minimal Clifford hypersurface $S^{p-1}(1/\sqrt{2}) \times S^{p-1}(1/\sqrt{2}) \subset S^n(1)$.

(2) When $p = 1$, a minimal principal orbit of the action of $SO(1) \times SO(n)$ on S^n, that is a totally geodesic hypersphere $S^{n-1}(1) \subset S^n(1)$.

(3) Singular orbits of the action of $SO(p) \times SO(q)$ on S^n, that are totally geodesic spheres $S^{p-1}(1) \subset S^n(1)$ and $S^{q-1}(1) \subset S^n(1)$.

From Theorem 3.10, the conormal bundles of the above austere orbits are special Lagrangian in T^*S^n. In fact, we can describe these SL submanifolds by the construction we gave in this section.

(1) Let $\tau(s) = \pi/4 + \sqrt{-1}s$ and define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\overset{(1)}{\cos\tau(s)}, 0, \dots, 0, \overset{(p+1)}{\sin\tau(s)}, 0, \dots, 0)\,.$$

Then the orbit $L = G \cdot \sigma$ of the action of $G = SO(p) \times SO(p)$ through σ is the conormal bundle of a minimal Clifford hypersurface in $Q^n \cong T^*S^n$. In fact, τ satisfies (3.3) for $\theta = \pi/2$, hence L is an SL submanifold of phase $\pi/2$ in Q^n.

(2) Let $\tau(s) = \pi/2 + \sqrt{-1}s$ and define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\cos\tau(s), \sin\tau(s), 0, \dots, 0)\,.$$

Then the orbit $L = G \cdot \sigma$ of the action of $SO(1) \times SO(n)$ through σ is the conormal bundle of a totally geodesic hypersphere in $Q^n \cong T^*S^n$. In fact, τ satisfies (3.5) for $\theta = \pi/2$, hence L is an SL submanifold of phase $\pi/2$ in Q^n.

(3) Let $\tau(s) = 0 + \sqrt{-1}s$ and define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\overset{(1)}{\cos\tau(s)}, 0, \dots, 0, \overset{(p+1)}{\sin\tau(s)}, 0, \dots, 0)\,.$$

Then the orbit $L = G \cdot \sigma$ of the action of $SO(p) \times SO(q)$ through σ is the conormal bundle of a totally geodesic sphere S^{p-1} in $Q^n \cong T^*S^n$. In fact, when q is even (resp. odd), L is an SL submanifold of phase 0 (resp. $\pi/2$) in Q^n.

4. Ricci-flat Kähler metric and special Lagrangian submanifolds in the complex cone. We define the complex cone Q_0^n in $\boldsymbol{C}^{n+1}$ by

$$Q_0^n = \left\{ z = (z_1, \dots, z_{n+1}) \in \boldsymbol{C}^{n+1};\ \sum_{i=1}^{n+1} z_i^2 = 0 \right\}\,.$$

Q_0^n has a (unique) singularity at the origin of $\boldsymbol{C}^{n+1}$. As $r = \|z\|$ tends to ∞, Q^n is asymptotic to Q_0^n in $\boldsymbol{C}^{n+1}$. In this section, we give a (singular) Ricci-flat Kähler metric on Q_0^n as the limit of the Stenzel metric on Q^n.

The holomorphic $(n,0)$-form Ω_{cone} on Q_0^n is given by

$$\frac{1}{2}d(z_1^2+\cdots+z_{n+1}^2)\wedge\Omega_{\mathrm{cone}}=\Omega_0\,.$$

We can express Ω_{cone} as

$$\Omega_{\mathrm{cone}}(v_1,\dots,v_n)=\frac{1}{\|z\|^2}(dz_1\wedge\cdots\wedge dz_{n+1})(\bar{z},v_1,\dots,v_n)$$

for $v_1,\dots,v_n\in T_zQ^n$ and $z\in Q^n$.

As $t\to\infty$, the differential equation (2.2) is asymptotic to

$$\frac{d}{dt}(F'(t))^n=\left(\frac{1}{2}\right)^{n-1}nce^{t(n-1)}\qquad(c>0)\,.$$

Then

$$F(t)=\left(\frac{1}{2}\right)^{(n-1)/n}\left(\frac{n}{n-1}\right)^{(n+1)/n}c^{1/n}e^{(n-1)t/n}$$

is a solution of the above differential equation. Since $\cosh t\to(1/2)e^t$ as $t\to\infty$, we define a function f as $F(t)=f((1/2)e^t)$. Then we have

$$f(t)=\left(\frac{n}{n-1}\right)^{(n+1)/n}c^{1/n}t^{(n-1)/n}\,.$$

PROPOSITION 4.1. *Let $f_{\mathrm{cone}}(t)=c\,t^{(n-1)/n}$ $(c>0)$ and define a Kähler form ω_{cone} on Q_0^n by*

$$\omega_{\mathrm{cone}}=\sqrt{-1}\partial\bar{\partial}f_{\mathrm{cone}}(r^2)=\sqrt{-1}\sum_{i,j=1}^{n+1}\frac{\partial^2}{\partial z_i\partial\bar{z}_j}f_{\mathrm{cone}}(r^2)dz_i\wedge d\bar{z}_j\,.$$

Then ω_{cone} gives a Ricci-flat Kähler metric on Q_0^n.

REMARK 4.2. When $n=3$, the above Kähler metric coincides with the the Ricci-flat metric on Q_0^3 due to Candelas and de la Ossa [2].

PROOF OF PROPOSITION 4.1. Henceforth we write f for f_{cone}. In a way similar to (2.3), we can evaluate

$$\omega_{\mathrm{cone}}(v,w)=2f'(r^2)\langle Jv,w\rangle+2f''(r^2)(\langle v,z\rangle\langle Jz,w\rangle-\langle w,z\rangle\langle Jz,v\rangle)$$

for $v,w\in T_zQ_0^n$, $z\in Q_0^n$. From this, it follows

$$\omega_{\mathrm{cone}}(v,\bar{w})=2\sqrt{-1}\big(f'(r^2)(v,w)+2f''(r^2)(v,z)(z,w)\big)\,,$$

where $(\ ,\ )$ is the standard Hermitian inner product on $\boldsymbol{C}^{n+1}$.

Now we show that there exists a constant $\lambda\in\boldsymbol{R}$ such that

$$\frac{\omega_{\mathrm{cone}}^n}{n!}=(-1)^{n(n-1)/2}\left(\frac{\sqrt{-1}}{2}\right)^n\lambda\,\Omega_{\mathrm{cone}}\wedge\bar{\Omega}_{\mathrm{cone}}\,.\tag{4.1}$$

Let $v_1, \dots, v_n$ be a basis of $T_z Q_0^n$ which satisfies $(v_i, v_j) = \delta_{ij}$, and $\theta_1, \dots, \theta_n$ be its dual basis. Using this basis, we can express ω_{cone} as

$$\omega_{\mathrm{cone}} = \sum_{i,j=1}^{n} \omega_{ij} \theta_i \wedge \bar{\theta}_j \,,$$

where

$$\omega_{ij} = \omega_{\mathrm{cone}}(v_i, \bar{v}_j) = 2\sqrt{-1}\big(f'(r^2)\delta_{ij} + 2f''(r^2)(v_i, z)(z, v_j)\big) \,.$$

Then the left-hand side of (4.1) is

$$\frac{\omega_{\mathrm{cone}}^n}{n!} = (-1)^{n(n-1)/2} \det(\omega_{ij}) \theta_1 \wedge \cdots \wedge \theta_n \wedge \bar{\theta}_1 \wedge \cdots \wedge \bar{\theta}_n \,.$$

Here we can compute

$$\begin{aligned} \det(\omega_{ij}) &= (2\sqrt{-1})^n (f'(r^2))^n \left(1 + 2\frac{f''(r^2)}{f'(r^2)}(|(v_1, z)|^2 + \cdots + |(v_n, z)|^2 t)\right) \\ &= (2\sqrt{-1})^n \left(\frac{c(n-1)}{n}\right)^n \left(\frac{n-2}{n}\right) \frac{1}{r^2} \,. \end{aligned} \tag{4.2}$$

On the other hand, Ω_{cone} can be computed as follows:

$$\begin{aligned} \Omega_{\mathrm{cone}}(v_1, \dots, v_n) &= \frac{1}{\|z\|^2}(dz_1 \wedge \cdots \wedge dz_{n+1})(\bar{z}, v_1, \dots, v_n) \\ &= \frac{1}{\|z\|} \det\left(\frac{\bar{z}}{\|z\|}, v_1, \dots, v_n\right) . \end{aligned}$$

Here $\bar{z}$ is orthogonal to $v_1, \dots, v_n$ with respect to the Hermitian inner product, since $z \in Q_0^n$. Therefore $\bar{z}/\|z\|, v_1, \dots, v_n$ forms a unitary basis of $\boldsymbol{C}^{n+1}$. Hence we have

$$\Omega_{\mathrm{cone}} \wedge \bar{\Omega}_{\mathrm{cone}}(v_1, \dots, v_n, \bar{v}_1, \dots, \bar{v}_n) = \frac{1}{\|z\|^2} = \frac{1}{r^2} \,. \tag{4.3}$$

From (4.2) and (4.3), consequently we obtain

$$\frac{\omega_{\mathrm{cone}}^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{\sqrt{-1}}{2}\right)^n \left(\frac{4c(n-1)}{n}\right)^n \left(\frac{n-2}{n}\right) \Omega_{\mathrm{cone}} \wedge \bar{\Omega}_{\mathrm{cone}} \,.$$

Thus $(Q_0^n, J, \omega_{\mathrm{cone}}, \Omega_{\mathrm{cone}})$ is Calabi-Yau, hence it is Ricci-flat. $\square$

We shall construct cohomogeneity one special Lagrangian submanifolds in Q_0^n in a way similar to the previous section, using the moment map technique.

Let $T^\circ S^n$ denote the subset of $T^* S^n$ excluding the zero-section. Then we can identify $T^\circ S^n$ and $Q_0^n \backslash \{0\}$ by the diffeomorphism (see [21])

$$\begin{array}{rcl} \Pi : T^\circ S^n & \longrightarrow & Q_0^n \setminus \{0\} \\ \Psi & & \Psi \\ (x, \xi) & \longmapsto & \|\xi\| x + \sqrt{-1}\xi \,. \end{array}$$

The diffeomorphism Π is equivariant under the action of $SO(n+1)$.

Here we consider a Lie subgroup

$$G = \left(\begin{array}{c|c} SO(p) & O \\ \hline O & SO(q) \end{array} \right) \cong SO(p) \times SO(q) \quad (p + q = n + 1,\ 1 \le p \le q \le n)$$

of $SO(n+1)$. The action of G on Q_0^n is Hamiltonian, and its moment map $\mu : Q_0^n \to \mathfrak{g}^*$ can be expressed as

$$\mu(z) = 2f'(r^2)(\mathrm{Im}(z_i \bar{z}_j)_{1 \le i < j \le p},\ \mathrm{Im}(z_i \bar{z}_j)_{p+1 \le i < j \le n+1})$$

using the basis $\{\theta_{ij}\}$ of $\mathfrak{g}^*$.

From Proposition 2.6, an SL submanifold of Q_0^n invariant under G should be contained in the inverse image $\mu^{-1}(c)$ of some $c \in Z(\mathfrak{g}^*)$. Although we should discuss each type of the center $Z(\mathfrak{g}^*)$ individually, here we shall work on the generic case, $3 \le p \le q$. For other cases, we can study in a way similar to the previous section.

4.1. The case of $3 \le p \le q$**.** The orbit space of the action of $G = SO(p) \times SO(q)$ on $T^\circ S^n$ is parametrized as

$$\Sigma = \left\{ (x, \xi)\ ;\ \begin{array}{l} x = (\cos t, 0, \dots, 0, \sin t, 0, \dots, 0) \\ \xi = (-\xi_1 \sin t, \xi_2, 0, \dots, 0, \xi_1 \cos t, \xi_3, 0, \dots, 0) \\ (\xi_1, \xi_2, \xi_3) \ne (0, 0, 0) \end{array} \right\}.$$

Then every G-orbit in $T^\circ S^n$ meets Σ, i.e., $G \cdot \Sigma = T^\circ S^n$.

In this case, the center of $\mathfrak{g}^*$ is $Z(\mathfrak{g}^*) = \{0\}$. We determine the subset $\mu^{-1}(0) \cap \Pi(\Sigma)$ of Q_0^n. Now $z \in \Pi(x, \xi) \in \Pi(\Sigma)$ can be expressed as

$$z = (\rho \cos t - \sqrt{-1}\xi_1 \sin t, \sqrt{-1}\xi_2, 0, \dots, 0, \rho \sin t + \sqrt{-1}\xi_1 \cos t, \sqrt{-1}\xi_3, 0, \dots, 0),$$

where $\rho = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. Then $\mu(z) = 0$ if and only if

$$0 = \mathrm{Im}(z_1 \bar{z}_2) = -\xi_2 \rho \cos t,$$
$$0 = \mathrm{Im}(z_{p+1} \bar{z}_{p+2}) = -\xi_3 \rho \sin t.$$

Thus $\xi_2 = \xi_3 = 0$ and we obtain

$$\begin{aligned} &\mu^{-1}(0) \cap \Pi(\Sigma) \\ &\quad = \left\{ \left(|\xi_1| \cos t - \sqrt{-1}\xi_1 \sin t, 0, \dots, 0, |\xi_1| \sin t + \sqrt{-1}\xi_1 \cos t, 0, \dots, 0 \right)\ ;\ \xi_1 \ne 0 \right\}. \end{aligned}$$

Since $\mu^{-1}(0)$ is G-invariant, we have

$$\mu^{-1}(0) = G \cdot (\mu^{-1}(0) \cap \Pi(\Sigma)).$$

Thus the orbit space $\mu^{-1}(0)/G$ of the G-action on $\mu^{-1}(0)$ is parametrized by t and ξ_1.

PROPOSITION 4.3. *Let σ be a curve in $\mu^{-1}(0) \cap \Pi(\Sigma)$. We express σ as*

$$\sigma(s) = (z_1(s), 0, \dots, 0, z_{p+1}(s), 0, \dots, 0).$$

Then the G-orbit $L = G \cdot \sigma$ through σ is a Lagrangian submanifold in Q_0^n. Moreover, L is a special Lagrangian submanifold of phase θ if and only if there exists a constant $c \in \boldsymbol{R}$ such *that σ satisfies*

$$\operatorname{Im}\bigl(e^{\sqrt{-1}\theta}(-1)^{q/2}z_1(s)^{n-1}\bigr) = c\,. \tag{4.4}$$

PROOF. Since $L = G\cdot\sigma$ is a cohomogeneity one submanifold of dimension n contained in $\mu^{-1}(0)$, it follows from Proposition 2.6 that L is a Lagrangian submanifold in Q_0^n. We look for σ such that L is an SL submanifold in Q_0^n. Since $\sigma(s) \in Q_0^n$, we note

$$\begin{aligned} &z_1^2(s) + z_{p+1}^2(s) = 0\,,\\ &z_1(s)z_1'(s) + z_{p+1}(s)z_{p+1}'(s) = 0\,. \end{aligned}$$

We take a basis of the tangent space $T_{\sigma(s)}L$ of L at $\sigma(s)$ as follows:

$$\begin{aligned} X_{1,2}^* &= X_{1,2}\sigma(s) = (0, \overset{(2)}{z_1(s)}, 0, \dots, 0)\,,\\ &\vdots\\ X_{1,p}^* &= X_{1,p}\sigma(s) = (0, \dots, 0, \overset{(p)}{z_1(s)}, 0, \dots, 0)\,,\\ X_{p+1,p+2}^* &= X_{p+1,p+2}\sigma(s) = (0, \dots, 0, \overset{(p+2)}{z_{p+1}(s)}, 0, \dots, 0)\,,\\ &\vdots\\ X_{p+1,n+1}^* &= X_{p+1,n+1}\sigma(s) = (0, \dots, 0, \overset{(n+1)}{z_{p+1}(s)})\,,\\ \sigma'(s) &= (\overset{(1)}{z_1'(s)}, 0, \dots, 0\,, \overset{(p+1)}{z_{p+1}'(s)}, 0, \dots, 0)\,. \end{aligned}$$

Then we have

$$\begin{aligned} &\Omega_{\text{cone}}(X_{1,2}^*, \dots, X_{1,p}^*, \sigma'(s), X_{p+1,p+2}^*, \dots, X_{p+1,n+1}^*)\\ &\quad = \frac{1}{\|z\|^2}(dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+1})(\overline{\sigma(s)}, X_{1,2}^*, \dots, X_{1,p}^*, \sigma'(s), X_{p+1,p+2}^*, \dots, X_{p+1,n+1}^*)\\ &\quad = \frac{1}{\|z\|^2}\begin{vmatrix} \bar z_1(s) & 0 & \cdots & 0 & z_1'(s) & 0 & \cdots & 0\\ 0 & z_1(s) & & \vdots & 0 & \vdots & & \vdots\\ \vdots & 0 & \ddots & 0 & \vdots & \vdots & & \vdots\\ 0 & \vdots & & z_1(s) & 0 & \vdots & & \vdots\\ \bar z_{p+1}(s) & \vdots & & 0 & z_{p+1}'(s) & 0 & & \vdots\\ 0 & \vdots & & \vdots & 0 & z_{p+1}(s) & & \vdots\\ \vdots & \vdots & & \vdots & \vdots & 0 & \ddots & 0\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & z_{p+1}(s) \end{vmatrix}\\ &\quad = (-1)^{q/2}z_1(s)^{n-2}z_1'(s)\,. \end{aligned}$$

Thus L is a special Lagrangian submanifold of phase θ if and only of σ satisfies

$$\operatorname{Im}\bigl(e^{\sqrt{-1}\theta}(-1)^{q/2}z_1(s)^{n-2}z_1'(s)\bigr) = 0\,.$$

This condition is equivalent to (4.4) for some $c \in \boldsymbol{R}$. □

We express $z_1 = |\xi_1|\cos t - \sqrt{-1}\xi_1 \sin t$. When $\xi_1 > 0$, the condition (4.4) becomes

$$\operatorname{Im}\bigl((-1)^{q/2}e^{\sqrt{-1}(\theta-(n-1)t)}\bigr) = c \tag{4.5}$$

for some $c \in \boldsymbol{R}$. In particular, when $c = 0$ we have

$$\theta - (n-1)t = \begin{cases} 0 & (\operatorname{mod} \pi) \qquad (q : \text{even})\,, \\ \pi/2 & (\operatorname{mod} \pi) \qquad (q : \text{odd})\,. \end{cases}$$

When $c \neq 0$, solution curves of (4.5) are asymptotic to the lines

$$\left\{\tau = t + \sqrt{-1}\xi_1\,;\ t = \frac{\theta - k\pi}{n-1},\ \xi_1 \in \boldsymbol{R}\right\} \quad (k \in \boldsymbol{Z}) \qquad (q : \text{even})\,,$$
$$\left\{\tau = t + \sqrt{-1}\xi_1\,;\ t = \frac{2\theta - (2k+1)\pi}{2(n-1)},\ \xi_1 \in \boldsymbol{R}\right\} \quad (k \in \boldsymbol{Z}) \qquad (q : \text{odd})\,.$$

Therefore, when $c = 0$, the cones over the orbits of the action of $SO(p) \times SO(q)$ through

$$\frac{1}{\sqrt{2}}\bigl(e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \dots, 0\,, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \dots, 0\bigr) \qquad (q : \text{even})\,,$$
$$\frac{1}{\sqrt{2}}\bigl(e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \dots, 0\,, \sqrt{-1}e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \dots, 0\bigr) \qquad (q : \text{odd})$$

are special Lagrangian cones of phase θ in Q_0^n. When $c \neq 0$, SL submanifolds are diffeomorphic to $\boldsymbol{R} \times S^{p-1} \times S^{q-1}$, and their ends are asymptotic to the above SL cones.

5. Asymptotic behavior of cohomogeneity one special Lagrangian submanifolds in T^*S^n. Cohomogeneity one special Lagrangian submanifolds in Q^n which we constructed in Section 3 are diffeomorphic to $\boldsymbol{R} \times S^{p-1} \times S^{q-1}$ generically. In this section, we shall study the asymptotic behavior of their ends and the singular sets.

5.1. The case of $3 \leq p \leq q$. We shall analyze solution curves of the differential equation (3.3), that is equivalent to (3.2). In the phase space $\boldsymbol{C}$, the orbit space of the G-action on $\mu^{-1}(0)$ can be reduced to

$$\left\{\tau = t + \sqrt{-1}\xi_1\,;\ 0 \leq t \leq \frac{\pi}{2},\ \xi_1 \in \boldsymbol{R}\right\}.$$

In this area, (3.3) has singularities at 0 and $\pi/2$. When $\theta = 0$, the real segment $[0, \pi/2]$ is a trivial solution, and its corresponding SL submanifold is the zero-section S^n of T^*S^n.

As ξ_1 tends to ∞, $\cos\tau$ and $\sin\tau$ are asymptotic to

$$\cos\tau \to \frac{1}{2}e^{-\sqrt{-1}\tau}\,, \quad \sin\tau \to \frac{\sqrt{-1}}{2}e^{-\sqrt{-1}\tau}\,.$$

Then (3.3) is asymptotic to

$$\operatorname{Im}\left(\sqrt{-1}^{q-1}\tau'(s)e^{\sqrt{-1}(\theta-(n-1)\tau(s))}\right)=0\,.$$

This condition becomes

$$\operatorname{Im}\left(\sqrt{-1}\tau'(s)e^{\sqrt{-1}(\theta-(n-1)\tau(s))}\right)=0 \qquad (q:\text{even})\,,$$
$$\operatorname{Im}\left(\tau'(s)e^{\sqrt{-1}(\theta-(n-1)\tau(s))}\right)=0 \qquad (q:\text{odd})\,,$$

and it is equivalent to the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}(\theta-(n-1)\tau)}\right)=c \qquad (q:\text{even})\,, \tag{5.1}$$
$$\operatorname{Re}\left(e^{\sqrt{-1}(\theta-(n-1)\tau)}\right)=c \qquad (q:\text{odd})$$

for some $c \in \boldsymbol{R}$. In particular, when $c=0$ we have

$$\theta-(n-1)t=0 \pmod{\pi} \qquad (q:\text{even})\,,$$
$$\theta-(n-1)t=\frac{\pi}{2} \pmod{\pi} \qquad (q:\text{odd})\,.$$

When $c \neq 0$, solution curves of (5.1) are asymptotic to these lines. Therefore, as $\xi_1 \to \infty$, solution curves of (3.3) are asymptotic to the lines

$$\left\{\tau=t+\sqrt{-1}\xi_1\,;\ t=\frac{\theta-k\pi}{n-1},\ \xi_1\in\boldsymbol{R}\right\} \quad (k\in\boldsymbol{Z}) \qquad (q:\text{even})\,,$$
$$\left\{\tau=t+\sqrt{-1}\xi_1\,;\ t=\frac{2\theta-(2k+1)\pi}{2(n-1)},\ \xi_1\in\boldsymbol{R}\right\} \quad (k\in\boldsymbol{Z}) \qquad (q:\text{odd})\,.$$

A special Lagrangian submanifold L in Q^n is given as the orbit through a curve

$$\sigma(s)=(\overset{(1)}{\cos}\tau(s),0,\dots,0,\overset{(p+1)}{\sin}\tau(s),0,\dots,0)$$

in $\mu^{-1}(0)\cap\Phi(\Sigma)$ by the action of $SO(p)\times SO(q)$. The unit vector is

$$\frac{\sigma}{\|\sigma\|}\to\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}(k\pi-\theta)/(n-1)},0,\dots,0,\sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)/(n-1)},0,\dots,0\right) \qquad (q:\text{even})\,,$$
$$\frac{\sigma}{\|\sigma\|}\to\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))},0,\dots,0,\sqrt{-1}e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))},0,\dots,0\right)$$
$$(q:\text{odd})$$

as $\xi_1 \to \infty$.

As τ approaches 0, $\cos\tau$ and $\sin\tau$ are asymptotic to

$$\cos\tau \to 1\,, \quad \sin\tau\to\tau\,.$$

Then (3.3) is asymptotic to

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)\tau(s)^{q-1}\right)=0\,,$$

and it is equivalent to the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau^{q}\right)=c$$

for some $c \in \boldsymbol{R}$. In particular, when $c = 0$ solutions of the above equation are the half-lines

$$\left\{\tau = t + \sqrt{-1}\xi_1 \ ; \ \arg(\tau) = \frac{k\pi - \theta}{q}\right\} \qquad (k = 0, 1, 2, \dots, 2q - 1)\,.$$

Therefore the solution of (3.3) branches to $2q$ curves at 0, and those curves are asymptotic to the above half-lines around 0. The orbit of the action of $SO(p) \times SO(q)$ through $z = (1, 0, \dots, 0)$ is a singular orbit, which is diffeomorphic to S^{p-1}.

As $\tau \to \pi/2$, $\cos\tau$ and $\sin\tau$ are asymptotic to

$$\cos\tau \to \frac{\pi}{2} - \tau\,, \quad \sin\tau \to 1\,.$$

Then (3.3) is asymptotic to

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)\left(\frac{\pi}{2} - \tau(s)\right)^{p-1}\right) = 0\,,$$

and it is equivalent to the equation

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\left(\tau - \frac{\pi}{2}\right)^{p}\right) = c$$

for some $c \in \boldsymbol{R}$. In particular, when $c = 0$, solutions of the above equation are the half-lines

$$\left\{\tau = t + \sqrt{-1}\xi_1 \ ; \ \arg\left(\tau - \frac{\pi}{2}\right) = \frac{k\pi - \theta}{p}\right\} \qquad (k = 0, 1, 2, \dots, 2p - 1)\,.$$

Therefore the solution of (3.3) branches to $2p$ curves at $\pi/2$, and those curves are asymptotic to the above half-lines around $\pi/2$. The orbit of the action of $SO(p) \times SO(q)$ through

$$z = (0, \dots, 0, \overset{(p+1)}{1}, 0, \dots, 0)$$

is a singular orbit, which is diffeomorphic to S^{q-1}.

Consequently we obtain the following observations.

THEOREM 5.1. *In the case of* $3 \leq p \leq q$, *cohomogeneity one special Lagrangian submanifolds* L *invariant under* $SO(p) \times SO(q)$ *are diffeomorphic to* $I \times S^{p-1} \times S^{q-1}$ *and embedded in* $T^*S^n \cong Q^n$ *generically.*

(1) *Two ends of* L *in* Q^n *are asymptotic to special Lagrangian cones in* Q_0^n *which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \dots, 0\,, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \dots, 0\big) \quad (k \in \boldsymbol{Z}) \qquad (q : \text{even})\,,$$

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \dots, 0\,, \sqrt{-1}e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \dots, 0\big) \quad (k \in \boldsymbol{Z}) \qquad (q : \text{odd})$$

by the action of $SO(p) \times SO(q)$.

(2) *When the curve* τ *passes through* 0, *the map* $\Psi : I \times S^{p-1} \times S^{q-1} \to Q^n$ *degenerates, and* q *special Lagrangian submanifolds of* Q^n *meet at the singular set* S^{p-1}.

(3) *When τ passes through $\pi/2$, the map $\Psi : I \times S^{p-1} \times S^{q-1} \to Q^n$ degenerates, and p special Lagrangian submanifolds of Q^n meet at the singular set S^{q-1}.*

Furthermore we observe the following.

REMARK 5.2. A smooth solution of (3.2) approaches a singular one as $c \to 0$. This shows that a smooth SL submanifold is deformed to a singular one. In other words, a branched SL submanifold can be deformed to a smooth one.

EXAMPLE 5.3. In the case of $n = 6$, $p = 3$, $q = 4$, the equation (3.2) is

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\left(\frac{1}{5}\cos^5\tau - \frac{1}{3}\cos^3\tau\right)\right) = c\,.$$

Figures 1, 2 and 3 show solution curves of this equation when $\theta = 0, \pi/4$ and $\pi/2$. Each solution curve corresponds to a special Lagrangian submanifold in Q^n.

In each of these figures, there exist two branched points at 0 and $\pi/2$, which shows that four smooth special Lagrangian submanifolds meet at a singular set S^2 and three meet at S^3. In Figure 1, a real segment $[0, \pi/2]$ corresponds to the zero-section S^6 of T^*S^6, which is the only compact SL submanifold in this case. When θ varies, the branches around each branched point 0 and $\pi/2$ rotate clockwise.

5.2. The case of $p = 1, q \geq 3$. In the phase space $\boldsymbol{C}$, the orbit space of the G-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1\ ;\ 0 \leq t \leq \pi,\ \xi_1 \in \boldsymbol{R}\}\,.$$

In this area, (3.5) has singularities at 0 and π. When $\theta = 0$, the real segment $[0, \pi]$ is a trivial solution, and its corresponding special Lagrangian submanifold is the zero-section S^n of T^*S^n.

In a way similar to the previous case, we see that solution curves of (3.5) are asymptotic to the lines

$$\left\{\tau = t + \sqrt{-1}\xi_1\ ;\ t = \frac{\theta - k\pi}{n-1},\ \xi_1 \in \boldsymbol{R}\right\} \quad (k \in \boldsymbol{Z}) \qquad (n : \text{even})\,,$$
$$\left\{\tau = t + \sqrt{-1}\xi_1\ ;\ t = \frac{2\theta - (2k+1)\pi}{2(n-1)},\ \xi_1 \in \boldsymbol{R}\right\} \quad (k \in \boldsymbol{Z}) \qquad (n : \text{odd})$$

as $\xi_1 \to \infty$.

The solution of (3.5) branches to $2n$ curves at 0 and π, and these curves are asymptotic to the half-lines

$$\left\{\tau = t + \sqrt{-1}\xi_1\ ;\ \arg(\tau) = \frac{k\pi - \theta}{n}\right\}\,,$$
$$\left\{\tau = t + \sqrt{-1}\xi_1\ ;\ \arg(\tau - \pi) = \frac{k\pi - \theta}{n}\right\} \qquad (k = 0, 1, 2, \ldots, 2n-1)$$

around 0 and π, respectively. The orbits of the action of $SO(n)$ through $z = (\pm 1, 0, \ldots, 0)$ are singular orbits, that is, fixed orbits.

Therefore we obtain the following observations.

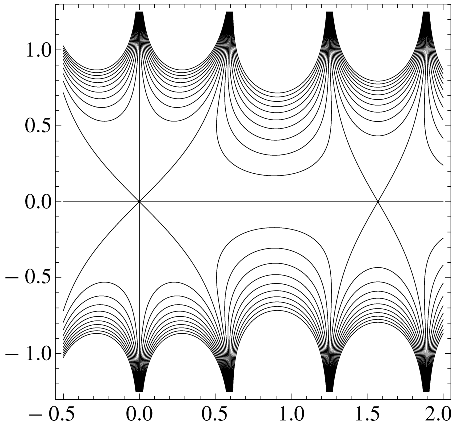

FIGURE 1. $\theta = 0$.

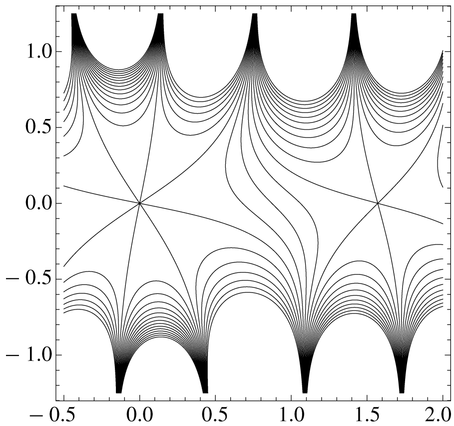

FIGURE 2. $\theta = \pi/4$.

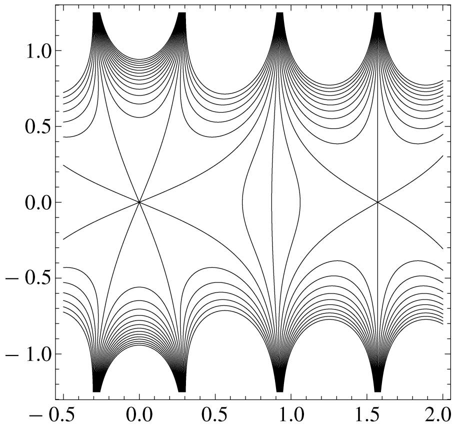

FIGURE 3. $\theta = \pi/2$.

THEOREM 5.4. *In the case of $p = 1, q \geq 3$, cohomogeneity one special Lagrangian submanifolds L invariant under $SO(n)$ are diffeomorphic to $I \times S^{n-1}$ and embedded in $T^*S^n \cong Q^n$ generically.*

(1) *Two ends of L in Q^n are asymptotic to special Lagrangian cones in Q_0^n which are the cones over the orbit through*

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \dots, 0\big) \quad (k \in \mathbf{Z}) \qquad (n : \text{even}),$$

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, \sqrt{-1}e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \dots, 0\big) \quad (k \in \mathbf{Z})$$

$$(n : \text{odd})$$

by the action of $SO(n)$.

(2) *When the curve* τ *passes through* 0 *or* π, *the map* $\Psi : I \times S^{n-1} \to Q^n$ *degenerates, and* n *special Lagrangian submanifolds of* Q^n *meet at the singular point* $z = (\pm 1, 0, \dots, 0)$.

EXAMPLE 5.5. In the case of $n = 4$, $p = 1$, $q = 4$, the equation (3.4) is

$$\mathrm{Im}\left(e^{\sqrt{-1}\theta}\left(\frac{1}{3}\cos^3\tau + \cos\tau\right)\right) = c\,.$$

Figures 4, 5 and 6 show solution curves of this equation when $\theta = 0$, $\pi/4$ and $\pi/2$.

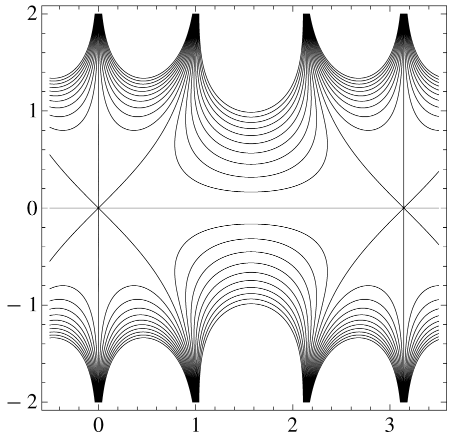

FIGURE 4. $\theta = 0$.

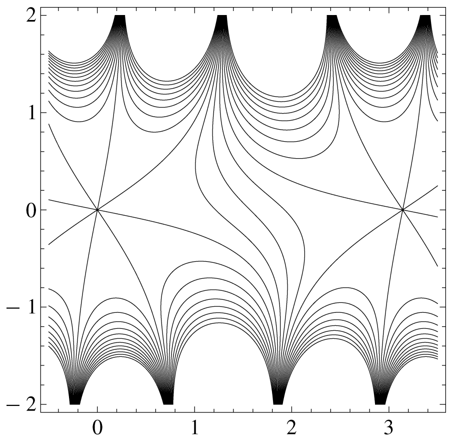

FIGURE 5. $\theta = \pi/4$.

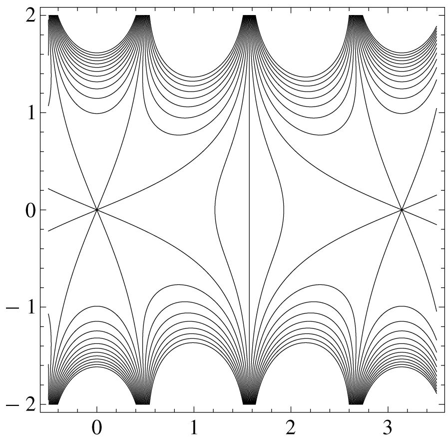

FIGURE 6. $\theta = \pi/2$.

5.3. The case of $p = 2, q \geq 3$. We express $z \in \Phi(\Sigma)$ as

$$z = (z_1, z_2, z_3, 0, \dots, 0)\,,$$

where

$$z_1 = \cos t \cosh \rho - \sqrt{-1}\frac{\xi_1}{\rho} \sin t \sinh \rho\,,$$

$$z_2 = \sqrt{-1}\frac{\xi_2}{\rho} \sinh \rho\,,$$

$$z_3 = \sin t \cosh \rho + \sqrt{-1}\frac{\xi_1}{\rho} \cos t \sinh \rho\,.$$

Then the condition for that $z \in \mu^{-1}(c_1\theta_{12})$ is

$$c_1 = -u'(\cosh(2\rho))\frac{\xi_2}{\rho} \cos t \sinh(2\rho)\,.$$

This equation approaches the condition for that $z \in \mu^{-1}(0)$ as $\rho \to \infty$. Thus $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$ is asymptotic to $\mu^{-1}(0) \cap \Phi(\Sigma)$ as $\rho \to \infty$. Therefore, we shall describe the asymptotic behavior of SL submanifolds in the case of $c_1 = 0$.

When $c_1 = 0$, the orbit space $\mu^{-1}(0)/G$ of the G-action on $\mu^{-1}(0)$ is parametrized as

$$\mu^{-1}(0) \cap \Phi(\Sigma) = \{(\cos\tau, 0, \sin\tau, 0, \dots, 0)\ ;\ \tau = t + \sqrt{-1}\xi_1\ (t, \xi_1 \in \boldsymbol{R})\}\,.$$

Let τ be a regular curve in the complex plane $\boldsymbol{C}$. We define a curve σ in $\mu^{-1}(0) \cap \Phi(\Sigma)$ by

$$\sigma(s) = (\cos\tau(s), 0, \sin\tau(s), 0, \dots, 0)\,.$$

Then the G-orbit $L = G \cdot \sigma$ through σ is a Lagrangian submanifold in Q^n. For a curve τ, L coincides with the image of the map

$$\begin{array}{cccc} \Psi_0 : & I \times S^1 \times S^{n-2} & \longrightarrow & Q^n \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (s, x, y) & \longmapsto & (\cos\tau(s)x_1, \cos\tau(s)x_2, \sin\tau(s)y_1, \dots, \sin\tau(s)y_{n-1})\,. \end{array}$$

When τ passes through $m\pi/2$ ($m \in \boldsymbol{Z}$), the map Ψ_0 degenerates at that point. If τ does not pass through $m\pi/2$ ($m \in \boldsymbol{Z}$), then L is diffeomorphic to $I \times S^1 \times S^{n-2}$ and immersed in Q^n by the map Ψ_0. Moreover, L is a special Lagrangian submanifold of phase θ if and only if τ satisfies

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\tau'(s)\cos\tau(s)(\sin\tau(s))^{n-2}\right) = 0\,. \tag{5.2}$$

This condition is equivalent to the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}(\sin\tau)^{n-1}\right) = c_2$$

for some $c_2 \in \boldsymbol{R}$. In the phase space $\boldsymbol{C}$, the orbit space of the G-action on $\mu^{-1}(0)$ can be reduced to

$$\{\tau = t + \sqrt{-1}\xi_1\ ;\ 0 \leq t \leq \frac{\pi}{2},\ \xi_1 \in \boldsymbol{R}\}\,.$$

In this area, (5.2) has singularities at 0 and $\pi/2$. When $\theta = 0$, the real segment $[0, \pi/2]$ is a trivial solution, and its corresponding SL submanifold is the zero-section S^n of T^*S^n.

Then, similarly to the previous cases, we obtain the following observations.

THEOREM 5.6. *In the case of $p = 2, q \geq 3$, cohomogeneity one special Lagrangian submanifolds L invariant under $SO(2) \times SO(n-2)$ are diffeomorphic to $I \times S^1 \times S^{n-2}$ and embedded in $T^*S^n \cong Q^n$ generically.*

(1) *Two ends of L in Q^n are asymptotic to special Lagrangian cones in Q_0^n which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)/(n-1)}, 0, \dots, 0\big) \quad (k \in \mathbf{Z}) \qquad (n : \text{odd}),$$

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \sqrt{-1}e^{\sqrt{-1}((2k+1)\pi-2\theta)/(2(n-1))}, 0, \dots, 0\big) \quad (k \in \mathbf{Z}) \qquad (n : \text{even})$$

by the action of $SO(2) \times SO(n-1)$.

(2) *When the curve σ in $\mu^{-1}(c_1\theta_{12}) \cap \Phi(\Sigma)$ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, \dots, 0)$, the map $\Psi : I \times S^1 \times S^{n-2} \to Q^n$ degenerates at that point. Especially when σ passes through $z = (\pm 1, 0, \dots, 0)$, then $(n-1)$ special Lagrangian submanifolds of Q^n meet at the singular set S^1.*

(3) *When σ passes through $z = (0, 0, \pm 1, 0, \dots, 0)$, the map $\Psi : I \times S^1 \times S^{n-2} \to Q^n$ degenerates, and 2 special Lagrangian submanifolds of Q^n meet at the singular set S^{n-2}.*

5.4. The case of $p = q = 2$. Similarly to the previous case, we can see that $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$ is asymptotic to $\mu^{-1}(0) \cap \Phi(\Sigma)$ as $\rho \to \infty$ for any $c_1, c_2 \in \boldsymbol{R}$. Then we obtain the following observations.

THEOREM 5.7. *In the case of $p = q = 2$, cohomogeneity one special Lagrangian submanifolds L invariant under $SO(2) \times SO(2)$ are diffeomorphic to $I \times S^1 \times S^1$ and embedded in $T^*S^3 \cong Q^3$ generically.*

(1) *Two ends of L in Q^3 are asymptotic to special Lagrangian cones in Q_0^3 which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\big(e^{\sqrt{-1}(k\pi-\theta)/2}, 0, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)/2}, 0\big) \quad (k \in \mathbf{Z})$$

by the action of $SO(2) \times SO(2)$.

(2) *When the curve σ in $\mu^{-1}(c_1\theta_{12} + c_2\theta_{34}) \cap \Phi(\Sigma)$ passes through $z = (\pm\cosh(\xi_2), \sqrt{-1}\sinh(\xi_2), 0, 0)$ or $(0, 0, \pm\cosh(\xi_3), \sqrt{-1}\sinh(\xi_3))$, the map $\Psi : I \times S^1 \times S^1 \to Q^3$ degenerates at that point. Especially when σ passes through $z = (\pm 1, 0, 0, 0)$ or $(0, 0, \pm 1, 0)$, then 2 special Lagrangian submanifolds of Q^3 meet at the singular set S^1.*

5.5. The case of $p = 1, q = 2$.

THEOREM 5.8. *In the case of $p = 1, q = 2$, cohomogeneity one special Lagrangian submanifolds L invariant under $SO(2)$ are diffeomorphic to $I \times S^1$ and embedded in $T^*S^2 \cong Q^2$ generically.*

(1) *Two ends of L in Q^2 are asymptotic to special Lagrangian cones in Q_0^2 which are the cones over the orbits through*

$$\frac{1}{\sqrt{2}}\left(e^{\sqrt{-1}(k\pi-\theta)}, \sqrt{-1}e^{\sqrt{-1}(k\pi-\theta)}, 0\right) \quad (k \in \mathbf{Z})$$

by the action of $SO(2)$.

(2) *When the curve σ in $\mu^{-1}(c_1\theta_{23}) \cap \Phi(\Sigma)$ passes through $z = (\pm 1, 0, 0)$, the map $\Psi : I \times S^1 \to Q^2$ degenerates, and 2 special Lagrangian submanifolds of Q^2 meet at the singular point $z = (\pm 1, 0, 0)$.*

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