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Cohomology and the Resolution of the Nilpotent Variety

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1. Let G be a split reductive linear algebraic group over a field k of characteristic zero. Consider the variety N of the nilpotent elements of the Lie algebra $\mathfrak{g}$ of G. It is a normal variety cf. [14] Theorem 16. It is isomorphic to the variety of the unipotent elements of G, cf. [17]. The theorem of Brieskorn-Steinberg-Tits states that the rational singularities are dense in the singular locus of N, see [1] and [18] (3.10). Here we shall prove that N has only rational singularities, cf. [12] p. 50, i.e. we prove

Theorem A. *There exists a proper birational morphism $\tau: Y \to N$ such that Y is smooth over k, that $\tau_*(\mathcal{O}_Y) = \mathcal{O}_N$ and $R^p\tau_*(\mathcal{O}_Y) = 0$ for $p \geqq 1$.*

This theorem admits a generalization which will be stated and proved in Section 5. In the complex analytic situation the same assertions follow by Theorem 5 of [3] exp. II. For rational singularities in that case see [2]. In [11] we investigated the local structure of N for the classical groups.

2. By [17] (2.2) we may assume that G is semi-simple and simply connected. Let G be split with respect to a maximal torus T and a Borel group B.

If E is a finite dimensional vector space over k and $\varrho: B \to Gl(E)$ is a morphism of algebraic groups then $E = (E, \varrho)$ is called a B-module. The contracted product $G \times {}^B E$ is defined as the quotient of $G \times E$ under the right action of B given by $(g, e)b = (gb, \varrho(b^{-1})e)$ for all $g \in G$, $e \in E$, $b \in B$. The morphism $\psi: G \times {}^B E \to G/B$ given by $\psi(g, e)B = gB$, is a vector bundle over G/B. The sheaf of sections of ψ is a locally free $\mathcal{O}_{G/B}$-module. It is denoted by $\mathcal{L}(E)$. See [7] p. 55, 56.

Let $\mathfrak{u}$ be the Lie algebra of the unipotent part U of B. As $\mathfrak{u}$ is a B-module we can define $Y = G \times {}^B \mathfrak{u}$. The adjoint action $\mathrm{Ad}: G \times \mathfrak{g} \to \mathfrak{g}$ induces a surjective morphism $\tau: Y \to N$. Since $\psi: Y \to G/B$ is a vector bundle, Y is an irreducible smooth variety. The morphism τ is easily identified with the G-equivariant proper morphism τ considered in [17] (the proof of 2.1), which is birational, cf. [17] (2.4). Since N is normal, it follows that $\mathcal{O}_N = \tau_*(\mathcal{O}_Y)$. By [10] (1.4.11) it suffices now to prove that $H^p(Y, \mathcal{O}_Y) = 0$ for all $p \geqq 1$. This is a special case of the corollary in Section 5.

Remark. The action of B on $\mathfrak{u}$ is not completely reducible, so we cannot apply the theorem in [13].

3. Let $X(T)$ be the character group of T. Let R be the root system of G with respect to T and let W be the Weyl group. Conform [5] and [6] a root α is called positive if its eigenspace $\mathfrak{g}_\alpha$ is not contained in $\mathfrak{u}$. The set of positive roots is denoted by R_+. For each root α we have an associated co-root α^* in the dual $\mathbb{Z}$-module $X(T)^*$. A weight $\chi \in X(T)$ is called regular (resp. dominant) if we have $\langle \alpha^*, \chi \rangle \neq 0$ (resp. $\langle \alpha^*, \chi \rangle \geqq 0$) for all $\alpha \in R_+$. The set of dominant weights is denoted by $X(T)_+$. If V is a subset of R we write $|V| = \sum_{\alpha \in V} \alpha$. The number of elements of V is denoted by $\#(V)$. The weight ϱ is defined by $2\varrho = |R_+|$. It is well known that $\varrho + X(T)_+$ is the set of regular dominant weights. The length of $w \in W$ is denoted by $n(w)$, cf. [6].

If E is a B-module we write $H^p(E) = H^p(G/B, \mathscr{L}(E))$. As $\mathscr{L}$ is an exact functor, the functors H^p form an exact delta-functor from the category of B-modules to the category of G-modules. For $\chi \in X(T)$ let $E(\chi)$ be the one-dimensional B-module corresponding to the induced morphism $B \to T \to Gl(1)$. We shall use the following version of Bott's theorem, cf. [6].

If $H^p(E(\chi)) \neq 0$, then $\chi + \varrho$ is regular and $p = n(w)$, where w is the unique element of W such that $w(\chi + \varrho)$ is dominant (and regular).

4. Definition. If $\mu \in X(T)$, let $p(\mu)$ be the maximal value of $n(w) - \#(V)$, where $w \in W$ and V is a subset of R_+ such that $w(\mu + \varrho - |V|)$ is dominant and regular.

Lemma. *Let $\mu \in X(T)$.*

(a) *$p(\mu)$ is the maximal value of $\#(V \cap -wR_+) - \#(V \cap wR_+)$, where $w \in W$ and V is a subset of R_+ such that $w(\mu) - |V|$ is dominant.*

(b) *We have $0 \leqq p(\mu) \leqq \#(R_+)$.*

(c) *If $\mu = 0$ then $p(\mu) = 0$.*

Proof. Consider subsets P of R satisfying $P \cap -P = \emptyset$ and $P \cup -P = R$. The relations $V = R_+ \cap P$, $P = V \cup -(R_+ \setminus V)$ define a one-to-one correspondence between these subsets of R and the subsets V of R_+. The natural action of W on the collection of the subsets P induces an action of W on the power set of R_+, which is given by

$$w * V = R_+ \cap (wV \cup -w(R_+ \setminus V)).$$

If V corresponds to P then $\varrho - |V| = -\frac{1}{2}|P|$. This implies

$$w(\varrho - |V|) = \varrho - |w * V|.$$

It follows that $w(\mu + \varrho - |V|)$ is dominant and regular if and only if $w(\mu) - |w * V|$ is dominant. As $n(w) = \#(R_+ \cap -wR_+)$ we have

$$n(w) - \#(V) = \#((w * V) \cap -wR_+) - \#((w * V) \cap wR_+).$$

Now (a) follows immediately. (b) is a consequence of (a). If V is a non-empty subset of R_+ then $-|V|$ is not dominant. So (c) follows from (a). Compare [4] and [15] (2.13).

Remark. It seems that $\mu \in \varrho + X(T)_+$ implies $p(\mu) = 0$. If R is a root system of type A_l and α is the highest root, then we have $\alpha \in X(T)_+$ and $p(\alpha) \geqq l - 4$.

5. The notations are as before. G is semi-simple and simply connected. Let $\mathfrak{p}$ be a linear subspace of $\mathfrak{g}$ which is B-invariant and contains $\mathfrak{u}$. Consider $Z = G \times^B \mathfrak{p}$ and the canonical morphism $\psi: Z \to G/B$.

Theorem B. *Let* $\mu \in X(T)$ *and* $p > p(\mu)$. *Then* $H^p(Z, \psi^* \mathscr{L}(E(\mu))) = 0$.

As $\mathcal{O}_Z = \psi^*(\mathcal{O}_{G/B}) = \psi^* \mathscr{L}(E(0))$ we obtain by 4. Lemma (c) immediately the following

Corollary. $H^p(Z, \mathcal{O}_Z) = 0$ *for all* $p \geqq 1$.

Proof of Theorem B. If E is a B-module, let $S(E^*) = \sum_q S_q(E^*)$ be the graded symmetrical algebra on the dual E^* of E. It may be considered as the ring of polynomial functions on E. The summands $S_q(E^*)$ are B-modules. Using [9] (9.4) one verifies that $\psi_*(\mathcal{O}_Z) = S((\mathscr{L}(\mathfrak{p}))^*) = \sum_q \mathscr{L}(S_q(\mathfrak{p}^*))$ and hence

$$\psi_* \psi^* \mathscr{L}(E(\mu)) = \psi_*(\mathcal{O}_Z) \otimes_{G/B} \mathscr{L}(E(\mu)) = \sum_q \mathscr{L}(S_q(\mathfrak{p}^*) \otimes_k E(\mu)).$$

Now it follows from [10] (1.3.3) and [8] Chapter II (3.10) that

$$H^p(Z, \psi^* \mathscr{L}(E(\mu))) = \sum_q H^p(S_q(\mathfrak{p}^*) \otimes E(\mu)).$$

Consider the $S(\mathfrak{g}^*)$-module $M = S(\mathfrak{g}^*) \otimes_k E(\mu)$. Let J be the kernel of the canonical surjection $\mathfrak{g}^* \to \mathfrak{p}^*$. Let $\boldsymbol{x}$ be a basis of J. As $\boldsymbol{x}$ is an M-regular sequence we have the long exact sequence

$$\ldots K_2 \to K_1 \to K_0 \to S(\mathfrak{p}^*) \otimes E(\mu) \to 0$$

where $K_. = K(\boldsymbol{x}; M)$ is the exterior complex, cf. [16] IV $A2$. Instrinsically the complex $K_.$ may be defined by

$$K_n = M \otimes_k \Lambda^n J, \qquad d_n: K_{n+1} \to K_n,$$

$$d_n(m \otimes (a_0 \wedge \ldots \wedge a_n)) = \sum_{i=0}^n (-1)^i a_i m \otimes (a_0 \wedge \ldots \hat{a}_i \ldots \wedge a_n).$$

It is canonically graded by $K_. = \sum_q K_.^q$ where

$$K_n^q = S_{q-n}(\mathfrak{g}^*) \otimes E(\mu) \otimes \Lambda^n J.$$

Thus we have the long exact sequences of B-modules

$$\ldots \to K_2^q \to K_1^q \to K_0^q \to S_q(\mathfrak{p}^*) \otimes E(\mu) \to 0.$$

As $H^\cdot$ is an exact delta-functor it suffices now to prove that $H^{p+n}(K_n^q) = 0$ for $q, n \geqq 0$.

As $S_{q-n}(\mathfrak{g}^*)$ is a G-module we have the following cartesian square.

$$\begin{array}{ccc} G \times^B S_{q-n}(\mathfrak{g}^*) & \longrightarrow & S_{q-n}(\mathfrak{g}^*) \\ \downarrow & & \downarrow \\ G/B & \longrightarrow & G/G = pt. \end{array}$$

Considering $S_{q-n}(\mathfrak{g}^*)$ as a locally free sheaf on G/G we have therefore $\mathscr{L}(S_{q-n}(\mathfrak{g}^*)) = f^*(S_{q-n}(\mathfrak{g}^*))$ where $f: G/B \to G/G$ is the canonical morphism. By [10] (0_{III} 12.2.3) this implies

$$H^{p+n}(K_n^q) = S_{q-n}(\mathfrak{g}^*) \otimes H^{p+n}(E(\mu) \otimes \Lambda^n J).$$

So it suffices to prove that $H^{p+n}(E(\mu)\otimes\Lambda^n J)=0$ for $n\geqq 0$.

The B-module $E(\mu)\otimes\Lambda^n J$ has a filtration of B-modules F_i, $0\leqq i\leqq r$, such that $F_0=0$ and $F_i/F_{i-1}\cong E(\chi_i)$ for some enumeration (χ_i) of the weights of $F_r=E(\mu)\otimes\Lambda^n J$. By convention the weights of $\mathfrak{u}$ are the negative roots. Now dual modules have opposite weights and $\mathfrak{p}$ contains $\mathfrak{u}$, so all positive roots are weights of $\mathfrak{p}^*$. Therefore the non-zero weights of J are negative roots with multiplicity one. Thus for every i there is a subset V_i of R_+ such that $\chi_i=\mu-|V_i|$ and $\#(V_i)\leqq n$. If $w\in W$ is such that $w(\chi_i+\varrho)$ is dominant and regular, then we have

$$n(w)\leqq p(\mu)+\#(V_i)<p+n\,.$$

By Bott's theorem as quoted in Section 3 this implies $H^{p+n}(E(\chi_i))=0$ for all i. It follows that $H^{p+n}(E(\mu)\otimes\Lambda^n J)=0$.

References

1. Brieskorn, E.: Singular elements of semi-simple algebraic groups. Actes du Congrès intern. des mathématiciens 1970, t. II, 279—284
2. Burns, D.: On rational singularities in dimensions > 2. Math. Ann. **211**, 237—244 (1974)
3. Séminaire H. Cartan, vol. 3, 1956—1957, New York: Benjamin 1967
4. Cartier, P.: Remarks on "Lie algebra cohomology and the generalized Borel-Weil theorem" by B. Konstant. Ann. Math. **74**, 388—390 (1961)
5. Chevalley, C.: Classification des groupes de Lie algébriques. Notes polycopiées, Inst. H. Poincaré, Paris 1956—1958
6. Demazure, M.: Une démonstration algébrique d'un théorème de Bott. Inventiones Math. **5**, 349—356 (1968)
7. Demazure, M.: Désingularisation des variétés de Schubert généralisées. Ann. Sc. Ec. Norm. Sup. 4ᵉ série, t. **7**, 53—88 (1974)
8. Godement, R.: Théorie des faisceaux. Paris: Hermann 1958
9. Grothendieck, A., Dieudonné, J. A.: Eléments de géométrie algébrique I. Grundlehren, 166. Berlin, Heidelberg, New York: Springer 1971
10. Grothendieck, A., Dieudonné, J. A.: Eléments de géométrie algébrique III. Publ. Math. de l'I.H.E.S. **11** (Paris 1961)
11. Hesselink, W. H.: Singularities in the nilpotent scheme of a classical group, to appear in Trans. A.M.S.
12. Kempf, G. et al.: Toroidal Embeddings I. Lecture Notes in Math. 339. Berlin, Heidelberg, New York: Springer 1973
13. Kempf, G. R.: Images of homogeneous vector bundles and varieties of complexes. Bull. A.M.S. **81**, 900—901 (1975)
14. Kostant, B.: Lie group representations on polynomial rings. Amer. J. Math. **85**, 327—404 (1963)
15. Macdonald, I. G.: The Poincaré series of a Coxeter group. Math. Ann. **199**, 161—174 (1972)
16. Serre, J.-P.: Algébre locale—multiplicités. Lecture Notes in Math. 11. Berlin, Heidelberg, New York: Springer 1965
17. Springer, T. A.: The unipotent variety of a semisimple algebraic group. Proc. Bombay Coll. on Alg. Geometry 1968
18. Steinberg, R.: Conjugacy classes in algebraic groups. Lecture Notes in Math. 366. Berlin, Heidelberg, New York: Springer 1974

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