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We explicitly describe cohomology of complete intersections in compact simplicial toric varieties.

In this paper we will study intersections of hypersurfaces in compact simplicial toric varieties $\mathbf{P}_\Sigma$. The main purpose is to relate naturally the Hodge structure of a complete intersection $X_{f_1}\cap\ldots\cap X_{f_s}$ in $\mathbf{P}_\Sigma$ to a graded ring. Originally this idea appears in [**Gr**], [**St**], [**Dol**], [**PS**]. The case of a hypersurface in a toric variety has been treated in [**BC**]. Also the Hodge structure of complete intersections in a projective space was described in [**Te**], [**Ko**], [**L**], [**Di**], [**Na**]. The common approach was to reduce studying of the Hodge structure on a complete intersection to studying of the Hodge structure on a hypersurface in a higher dimensional projective variety. This is the idea of a "Cayley trick". About a Cayley trick in the toric context see [**GKZ**], [**DK**], [**BB**]. A special case of a complete intersection (when it is empty) in a complete simplicial toric variety was elaborated in [**CCD**]. The basic references on toric varieties are [**F1**], [**O**], [**Da**], [**C**].

The paper is organized as follows:

Section 1 establishes notation and studies cohomology of subvarieties in a complete simplicial toric variety. In Section 2 we describe a Cayley trick for toric varieties. In Section 3 we prove the main result where we relate the Hodge components $H^{d-s-p,p}(X_{f_1}\cap\cdots\cap X_{f_s})$ in the middle cohomology group to homogeneous components of a graded ring. Section 4 treats a special case of complete intersections: a nondegenerate intersection.

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1. Quasi-smooth intersections.

We first fix some notation. Let M be a lattice of rank d, $N=\mathrm{Hom}(M,Z)$ the dual lattice; $M_{\mathbf{R}}$ (resp. $N_{\mathbf{R}}$) denotes the $\mathbf{R}$-scalar extension of M (resp. of N). Let Σ be a rational simplicial complete d-dimensional fan in $N_{\mathbf{R}}$ [**BC**], $\mathbf{P}_\Sigma$ a complete simplicial toric variety associated with this fan.

Such a toric variety can be described as a geometric quotient [**C**]. Let $S(\Sigma)=\mathbf{C}[x_1,\ldots,x_n]$ be the polynomial ring over $\mathbf{C}$ with variables $x_1,\ldots,x_n$

corresponding to the integral generators $e_1, \dots, e_n$ of the 1-dimensional cones of Σ. For $\sigma \in \Sigma$ let $\hat{x}_\sigma = \prod_{e_i \notin \sigma} x_i$, and let $B(\Sigma) = \langle \hat{x}_\sigma : \sigma \in \Sigma \rangle \subset S$ be the ideal generated by the $\hat{x}_\sigma$'s. This ideal gives the variety $Z(\Sigma) = \mathbf{V}(B(\Sigma)) \subset \mathbf{A}^n$. The toric variety $\mathbf{P} = \mathbf{P}_\Sigma$ will be a geometric quotient of $U(\Sigma) := \mathbf{A}^n \setminus Z(\Sigma)$ by the group $\mathbf{D} := \mathrm{Hom}_{\mathbf{Z}}(A_{d-1}(\mathbf{P}), \mathbf{C}^*)$, where $A_{d-1}(\mathbf{P})$ is the Chow group of Weil divisors modulo rational equivalence.

Each variable x_i in the coordinate ring $S(\Sigma)$ corresponds to a torus-invariant irreducible divisor D_i of $\mathbf{P}$. As in [C], we grade $S = S(\Sigma)$ by assigning to a monomial $\prod_{i=1}^n x_i^{a_i}$ its degree $[\sum_{i=1}^n a_i D_i] \in A_{d-1}(\mathbf{P})$. A polynomial f in the graded piece S_α corresponding to $\alpha \in A_{d-1}(\mathbf{P})$ is said to be $\mathbf{D}$-homogeneous of degree α.

Let $f_1, \dots, f_s$ be $\mathbf{D}$-homogeneous polynomials. They define a zero set $\mathbf{V}(f_1, \dots, f_s) \subset \mathbf{A}^n$, moreover $\mathbf{V}(f_1, \dots, f_s) \cap U(\Sigma)$ is stable under the action of $\mathbf{D}$ and hence descends to a closed subset $X \subset \mathbf{P}$, because $\mathbf{P}$ is a geometric quotient.

Definition 1.1. We say that X is a *quasi-smooth intersection* if $\mathbf{V}(f_1, \dots, f_s) \cap U(\Sigma)$ is either empty or a smooth subvariety of codimension s in $U(\Sigma)$.

Remark 1.2. This notion generalizes a nonsingular complete intersection in a projective space. Notice that since the $(n-d)$-dimensional group $\mathbf{D}$ has only zero dimensional stabilizers [**BC**], X is of pure dimension $d-s$ or empty.

We can now relate this notion to a V-submanifold (see Definition 3.2 in [**BC**]).

Proposition 1.3. *If $X \subset \mathbf{P}$ is a closed subset of codimension s defined by $\mathbf{D}$-homogeneous polynomials $f_1, \dots, f_s$, then X is a quasi-smooth intersection if and only if X is a V-submanifold of $\mathbf{P}$.*

The proof of this is very similar to the proof of the Proposition 3.5 in [**BC**].

The next result is a Lefschetz-type theorem.

Proposition 1.4. *Let $X \subset \mathbf{P}$ be a closed subset, defined by $\mathbf{D}$-homogeneous polynomials $f_1, \dots, f_s$, in a complete simplicial toric variety $\mathbf{P}$. If $f_1, \dots, f_s \in B(\Sigma)$, then the natural map $i^* : H^i(\mathbf{P}) \to H^i(X)$ is an isomorphism for $i < d-s$ and an injection for $i = d-s$. In particular, this is valid if X is an intersection of ample hypersurfaces.*

Proof. We can present $X = X_{f_1} \cap \dots \cap X_{f_s}$, where $X_{f_i} \subset \mathbf{P}$ is a hypersurface defined by f_i. As it was shown in the proof of the Proposition 10.8 [**BC**], if $f \in B(\Sigma)$ then $\mathbf{P} \setminus X_f = (\mathbf{A}^n \setminus \mathbf{V}(f))/\mathbf{D}(\Sigma)$ is affine, hence $H^i(\mathbf{P} \setminus X_f) = 0$ for $i > d$. We will prove by induction on s that $H^i(\mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_s})) = 0$ for $i > d+s-1$. Consider the Mayer-Vietoris sequence

$$\cdots \to H^i(U \cap V) \to H^{i+1}(U \cup V) \to H^{i+1}(U) \oplus H^{i+1}(V) \to H^{i+1}(U \cap V) \to \cdots$$

with $U = \mathbf{P}\backslash(X_{f_1}\cap\ldots\cap X_{f_{s-1}})$, $V = \mathbf{P}\backslash X_{f_s}$. Notice that $U\cup V = \mathbf{P}\backslash(X_{f_1}\cap \ldots\cap X_{f_s})$ and $U\cap V = \cup_{i=1}^{s-1}\mathbf{P}\setminus(X_{f_i}\cup X_{f_s}) = \mathbf{P}\setminus(X_{f_1\cdot f_s}\cap\ldots\cap X_{f_{s-1}\cdot f_s})$. So, using the induction and the above sequence, we obtain that $H^i(\mathbf{P}\backslash X) = 0$ for $i > d+s-1$. As a consequence of this, X is nonempty unless $s > d$ because the dimension $h^{2d}(\mathbf{P}) = 1$. Since $\mathbf{P}\setminus X$ is a V-manifold, Poincaré duality implies that $H^i_c(\mathbf{P}\setminus X) = 0$ for $i \le d-s$. Now the desired result follows from the long exact sequence of the cohomology with compact supports (X and $\mathbf{P}$ are compact):

$$\cdots \to H^i_c(\mathbf{P}\setminus X) \to H^i_c(\mathbf{P}) \to H^i_c(X) \to H^{i+1}_c(\mathbf{P}\setminus X) \to H^{i+1}_c(\mathbf{P}) \to \cdots .$$

If X is an intersection of ample hypersurfaces defined by $f_1,\ldots,f_s$, then Lemma 9.15 [**BC**] gives us that $f_1,\ldots,f_s$ belong to $B(\Sigma)$. □

Corollary 1.5. *A quasi-smooth intersection $X = X_{f_1}\cap\ldots\cap X_{f_s}$, defined by $f_1,\ldots,f_s \in B(\Sigma)$, has pure dimension $d-s$.*

Since the dimension of $H^0(X,\mathbf{C})$ is the number of connected components of X, we obtain another important result.

Corollary 1.6. *An intersection $X_{f_1}\cap\ldots\cap X_{f_s}$, defined by $f_1,\ldots,f_s \in B(\Sigma)$, in a complete simplicial toric variety $\mathbf{P}_\Sigma$ is connected provided $s < \dim\mathbf{P}_\Sigma$.*

Remark 1.7. If the polynomials $f_1,\ldots,f_s$ have ample degrees, then this corollary follows from a more general statement in [**FL1**] (see also [**FL2**] and [**FH**] for connectedness theorems).

2. "Cayley trick".

We will explore a Cayley trick to reduce studying of the cohomology of quasi-smooth intersections to results already known for hypersurfaces.

Let $L_1,\ldots,L_s$ be line bundles on a complete d-dimensional toric variety $\mathbf{P} = \mathbf{P}_\Sigma$, and let $\pi : \mathbf{P}(E) \to \mathbf{P}$ be the projective space bundle associated to the vector bundle $E = L_1\oplus\cdots\oplus L_s$. Then the $\mathbb{P}^{s-1}$-bundle $\mathbf{P}(E)$ is a toric variety. The fan corresponding to it can be described as follows [**O**, p. 58]. Suppose that support functions $h_1,\ldots,h_s$ give rise to the isomorphism classes of line bundles $[L_1],\ldots,[L_s] \in \mathrm{Pic}(\mathbf{P})$, respectively. Introduce a $\mathbf{Z}$-module N' with a $\mathbf{Z}$-basis $\{n_2,\ldots,n_s\}$ and let $\tilde{N} := N\oplus N'$ and $n_1 := -n_2-\cdots-n_s$. Denote by $\tilde\sigma$ the image of each $\sigma\in\Sigma$ under the $\mathbf{R}$-linear map $N_\mathbf{R} \to \tilde{N}_\mathbf{R}$ which sends $y \in N_\mathbf{R}$ to $y - \sum_{j=1}^s h_j(y)n_j$. On the other hand, let σ'_i be the cone in $N'_\mathbf{R}$ generated by $n_1,\ldots,n_i,n_{i+1},\ldots,n_s$ and let Σ' be the fan in $N'_\mathbf{R}$ consisting of the faces of $\sigma'_1,\ldots,\sigma'_s$. Then $\mathbf{P}(E)$ corresponds to the fan $\tilde\Sigma := \{\tilde\sigma+\sigma' : \sigma\in\Sigma, \sigma'\in\Sigma'\}$. From this description it is easy to see that if Σ is a complete simplicial fan then $\mathbf{P}(L_1\oplus\cdots\oplus L_s)$ is a complete simplicial toric variety. We see that the integral generators of the 1-dimensional cones in $\tilde{\Sigma}$ are given by

$$\begin{aligned}\tilde{e}_i &= e_i - \sum_{1\le j\le s} h_j(e_i)n_j, \quad i = 1, \dots n,\\ \tilde{n}_1 &= -n_2 - \cdots - n_s,\\ \tilde{n}_j &= n_j, \quad j = 2, \dots, s,\end{aligned}$$

where $e_1, \dots, e_n$ are the integral generators of the 1-dimensional cones in Σ.

The homogeneous coordinate ring of $\mathbf{P}(E)$ is the polynomial ring

$$R = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_s],$$

where x_i corresponds to $\tilde{e}_i$ and y_j corresponds to $\tilde{n}_j$. This ring has a grading by the Chow group $A_{d+s-2}(\mathbf{P}(E))$. Since $\mathbf{P}$ is a normal variety, there is an embedding of the Picard group $\mathrm{Pic}(\mathbf{P}) \hookrightarrow A_{d-1}(\mathbf{P})$. We want to show that if some polynomials $f_j \in S(\Sigma) = \mathbf{C}[x_1, \dots, x_n]$ have the property $\deg(f_j) = [L_j] \in \mathrm{Pic}(\mathbf{P})$, then the polynomials $y_j f_j$ all have the same degree in R. This will allow us to consider a hypersurface defined by the homogeneous polynomial $F = \sum_{j=1}^{s} y_j f_j$.

Lemma 2.1. *Let $f_1, \dots, f_s \in S(\Sigma)$ be $\mathbf{D}$-homogeneous polynomials, such that* $\deg(f_j) = [L_j]$ *for some line bundles $L_1, \dots, L_s$. Then $F = \sum_{j=1}^{s} y_j f_j$ is homogeneous in R and its degree is the isomorphism class $[O_{\mathbf{P}(E)}(1)]$ of the canonical line bundle on $\mathbf{P}(E) = \mathbf{P}(L_1 \oplus \cdots \oplus L_s)$.*

Proof. To prove that F is a homogeneous polynomial we will repeat the arguments in the proof of Lemma 3.5 in [**CCD**]. Let $D_1, \dots, D_n$ be the torus-invariant divisors on $\mathbf{P} = \mathbf{P}_\Sigma$ corresponding to the 1-dimensional cones of the fan Σ. Then the pullback $\pi^* D_i$ is the torus-invariant divisor of $\mathbf{P}(E)$ corresponding to the cone generated by $\tilde{e}_i$. Also denote by D'_j the torus-invariant divisor corresponding to $\tilde{n}_j$. Let $\tilde{M} = M \oplus M'$ be the lattice dual to $\tilde{N} = N \oplus N'$ with $M' = \mathrm{Hom}(N', \mathbf{Z})$ having $\{n_2^*, \dots, n_s^*\}$ as a basis dual to $\{n_2, \dots, n_s\}$. The divisor corresponding to the character $\chi^{n_j^*}$ is

$$\begin{aligned}\mathrm{div}(\chi^{n_j^*}) &= \sum_{i=1}^{n} \langle n_j^*, \tilde{e}_i\rangle \pi^* D_i + \sum_{k=1}^{s} \langle n_j^*, \tilde{n}_k\rangle D'_k\\ &= \sum_{i=1}^{n} (h_1(e_i) - h_j(e_i))\pi^* D_i - D'_1 + D'_j.\end{aligned}$$

Therefore, $[D'_j] + [\pi^* L_j]$ all have the same degree in the Chow group $A_{d+s-2}(\mathbf{P}(E))$, and, consequently, F is a homogeneous polynomial.

Now consider the following exact sequence [**M**]:

$$0 \to O_{\mathbf{P}(E)} \to \pi^* E^* \otimes O_{\mathbf{P}(E)}(1) \to \mathrm{T}_{\mathbf{P}(E)} \to \pi^* \mathrm{T}_{\mathbf{P}} \to 0,$$

where T_X denotes the tangent bundle, E^* is the dual bundle. From here we can compute the Chern class

$$c_1(\mathrm{T}_{\mathbf{P}(E)}) = c_1(\pi^*\mathrm{T}_{\mathbf{P}}) + c_1(\pi^*E^* \otimes O_{\mathbf{P}(E)}(1)) \\ = \pi^*c_1(\mathrm{T}_{\mathbf{P}}) - \pi^*c_1(E) + s \cdot c_1(O_{\mathbf{P}(E)}(1)).$$

Hence, $s \cdot c_1(O_{\mathbf{P}(E)}(1)) = \pi^*c_1(L_1) + \cdots + \pi^*c_1(L_s) + c_1(\mathrm{T}_{\mathbf{P}(E)}) - \pi^*c_1(\mathrm{T}_{\mathbf{P}})$. On the other hand, from the generalized Euler exact sequence [**BC**, §12] we get

$$0 \to O_{\mathbf{P}}^{n-d} \to \oplus_{i=1}^n O_{\mathbf{P}}(D_i) \to \mathrm{T}_{\mathbf{P}} \to 0.$$

This implies that $c_1(\mathrm{T}_{\mathbf{P}}) = [D_1] + \cdots + [D_n]$. Similarly we have $c_1(\mathrm{T}_{\mathbf{P}(E)}) = [\pi^*D_1] + \cdots + [\pi^*D_n] + [D'_1] + \cdots + [D'_s]$. Under the identification $\mathrm{Pic}(\mathbf{P}(E)) \hookrightarrow A_{d+s-2}(\mathbf{P}(E))$ the first Chern class of a line bundle on $\mathbf{P}(E)$ is exactly its isomorphism class in the Picard group $\mathrm{Pic}(\mathbf{P}(E))$. Therefore

$$s \cdot [O_{\mathbf{P}(E)}(1)] = [\pi^*L_1] + \cdots + [\pi^*L_s] + [D'_1] + \cdots + [D'_s] = s \cdot ([\pi^*L_2] + [D'_2]).$$

It can be easily checked that D'_2 is a Cartier divisor on $\mathbf{P}(E)$. Hence all classes $[O_{\mathbf{P}(E)}(1)]$, $[\pi^*L_2]$ and $[D'_2]$ lie in the Picard group $\mathrm{Pic}(\mathbf{P}(E))$. But this group is free abelian, because $\mathbf{P}(E)$ is complete. So the above equality is divisible by s: $[O_{\mathbf{P}(E)}(1)] = [\pi^*L_2] + [D'_2] = \deg(F)$. □

From now on we assume that $\mathbf{P} = \mathbf{P}_\Sigma$ is a complete simplicial toric variety and that $\deg(f_j) \in \mathrm{Pic}(\mathbf{P})$, $j = 1, \ldots, s$. Denote by Y the hypersurface in $\mathbf{P}(E)$ defined by $F = \sum_{j=1}^s y_j f_j$.

Lemma 2.2. *$X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection iff the hypersurface Y is quasi-smooth.*

Proof. $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection means that whenever $x \in \mathbf{V}(f_1, \ldots, f_s) \setminus Z(\Sigma)$, the $\mathrm{rank}\big(\frac{\partial f_j}{\partial x_i}(x)\big)_{i,j} = s$. And Y is quasi-smooth iff $z = (x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$ implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j = 1, \ldots, s$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \ldots, n$, is nonzero.

So let $(x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$, then there is a cone $\tilde{\sigma} + \sigma' \in \tilde{\Sigma}$ with $\sigma \in \Sigma$, $\sigma' \in \Sigma'$, such that $\prod_{\tilde{e}_i \notin \tilde{\sigma}} x_i \prod_{\tilde{n}_j \notin \sigma'} y_j \neq 0$ where x_i, y_j are the coordinates of (x, y). If $f_1(x) = \cdots = f_s(x) = 0$, then $x \in \mathbf{V}(f_1, \ldots, f_s) \setminus Z(\Sigma)$ because $\prod_{e_i \in \sigma} x_i \neq 0$. And if $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \ldots, n$, is nonzero.

Conversely, suppose Y is quasi-smooth. Pick any $x \in \mathbf{V}(f_1, \ldots, f_s) \setminus Z(\Sigma)$, then $(x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$ for each $y = (y_1, \ldots, y_s) \neq 0$. Therefore $\sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some i, which means the $\mathrm{rank}\big(\frac{\partial f_j}{\partial x_i}(x)\big)_{i,j}$ is maximal. □

3. Cohomology of quasi-smooth intersections.

Since a quasi-smooth intersection is a compact V-manifold (Proposition 1.3), the cohomology on it has a pure Hodge structure. Using Proposition 1.4 and the Poincaré duality, we can compute the cohomology of a quasi-smooth intersection except for the cohomology in the middle dimension $d-s$. So we introduce the following definition.

Definition 3.1. The *variable cohomology group* $H^{d-s}_{\rm var}(X)$ is $\operatorname{coker}(H^{d-s}(\mathbf{P}) \stackrel{i^*}{\to} H^{d-s}(X))$.

The variable cohomology group also has a pure Hodge structure.

Proposition 3.2. *Let $X = X_{f_1} \cap \ldots \cap X_{f_s}$ be a quasi-smooth intersection of ample hypersurfaces. Then there is an exact sequence of mixed Hodge structures*

$$0 \to H^{d-s-1}(\mathbf{P}) \stackrel{\cup[X]}{\to} H^{d+s-1}(\mathbf{P}) \to H^{d+s-1}(\mathbf{P}\setminus X) \to H^{d-s}_{\rm var}(X)\to 0,$$

where $[X] \in H^{2s}(\mathbf{P})$ is the cohomology class of X.

Proof. Consider the Gysin exact sequence:

(1)
$$\cdots \to H^{i-2s}(X) \stackrel{i_!}{\to} H^i(\mathbf{P}) \to H^i(\mathbf{P}\setminus X) \to H^{i-2s+1}(X) \stackrel{i_!}{\to} H^{i+1}(\mathbf{P}) \to \cdots .$$

Since i^* is Poincaré dual to the Gysin map $i_!$, it follows that $H^{d-s}_{\rm var}(X)$ is isomorphic to the kernel of $i_! : H^{d-s}(X) \to H^{d+s}(P)$. So we get an exact sequence

$$H^{d-s-1}(X) \stackrel{i_!}{\to} H^{d+s-1}(\mathbf{P}) \to H^{d+s-1}(\mathbf{P}\setminus X) \to H^{d-s}_{\rm var}(X)\to 0.$$

Now we use a commutative diagram

$$\begin{array}{ccc} H^{d-s-1}(X) & \stackrel{i_!}{\to} & H^{d+s-1}(\mathbf{P}) \\ {\scriptstyle i^*}\uparrow & \nearrow_{\cup[X]} & \\ H^{d-s-1}(\mathbf{P}). & & \end{array}$$

By Proposition 1.4 i^* is an isomorphism in this diagram, so it suffices to prove that the Gysin map $i_!$ is injective in the above diagram.

Lemma 3.3. *If $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection of ample hypersurfaces, then the Gysin map $H^{d-s-1}(X) \stackrel{i_!}{\to} H^{d+s-1}(\mathbf{P})$ is injective.*

Proof. Since the odd dimensional cohomology of a complete simplicial toric variety vanishes [**F1**, pp. 92-94] and $i^* : H^{d-s-1}(\mathbf{P}) \to H^{d-s-1}(X)$ is an isomorphism by Proposition 1.4, it follows that $H^{d-s-1}(X) = H^{d-s-1}(\mathbf{P}) = H^{d+s-1}(\mathbf{P}) = 0$ when $d+s-1$ is odd. So by the Gysin exact sequence (1) it is enough to show that $H^{d+s-2}(\mathbf{P}\setminus X) = 0$ when $d+s-2$ is odd. To prove this we use the Cayley trick again. Let Y be the hypersurface defined by

$F=\sum_{j=1}^{s} y_j f_j$. Then the natural map $\mathbf{P}(E)\setminus Y \to \mathbf{P}\setminus X$, induced by the projection $\pi:\mathbf{P}(E)\to\mathbf{P}$, is a $\mathbf{C}^{s-1}$ bundle in the Zariski topology. Notice that $\mathbf{P}\setminus X$ is simply connected, because $\mathbf{P}$ is simply connected [**F1**, p. 56] and X has codimension at least 2 in $\mathbf{P}$. Hence, the Leray-Serre spectral sequence implies that $H^i(\mathbf{P}(E)\setminus Y)=H^i(\mathbf{P}\setminus X)$ for $i\geq 0$. We have that $H^{d+s-2}(\mathbf{P}(E))=0$ for $d+s-2$ odd and Y is quasi-smooth by Lemma 2.2. So from the Gysin exact sequence

$$H^{d+s-2}(\mathbf{P}(E))\to H^{d+s-2}(\mathbf{P}(E)\setminus Y)\to H^{d+s-3}(Y)\xrightarrow{j_!} H^{d+s-1}(\mathbf{P}(E))$$

(here the Gysin map $j_!$ is induced by the inclusion $j: Y\hookrightarrow \mathbf{P}(E)$) it follows that we need to show injectivity of $j_!: H^{d+s-3}(Y)\to H^{d+s-1}(\mathbf{P}(E))$. Consider the commutative diagram

$$\begin{array}{ccc} H^{d+s-3}(Y) & \xrightarrow{j_!} & H^{d+s-1}(\mathbf{P}(E)) \\ {\scriptstyle j^*}\uparrow & \nearrow_{\cup[Y]} & \\ H^{d+s-3}(\mathbf{P}(E)) & & \end{array}$$

where $[Y]\in H^2(\mathbf{P}(E))$ is the cohomology class of Y. The canonical line bundle $O_{\mathbf{P}(E)}(1)$ is ample [**H**, III, §1], whence by Lemma 2.1, Y is ample. So by Proposition 10.8 [**BC**] $j^*: H^{d+s-3}(\mathbf{P}(E))\to H^{d+s-3}(Y)$ is an isomorphism and by Hard Lefschetz $\cup[Y]: H^{d+s-3}(\mathbf{P}(E))\to H^{d+s-1}(\mathbf{P}(E))$ is injective. Thus, from the above diagram the lemma follows. □

Definition 3.4. For a nonzero polynomial $F\in R=\mathbf{C}[x_1,\dots,x_n,y_1,\dots,y_s]$ the *Jacobian ring* $R(F)$ denotes the quotient of R by the ideal generated by the partial derivatives $\frac{\partial F}{\partial y_j}$, $j=1,\dots,s$, $\frac{\partial F}{\partial x_i}$, $i=1,\dots,n$.

Remark 3.5. If $F=y_1f_1+\cdots+y_sf_s$ is as in Lemma 2.1 with $f_j\in S_{\alpha_j}$, then $R(F)$ carries a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E))$. Moreover, there are canonical isomorphisms $A_{d+s-2}(\mathbf{P}(E))\cong A_{d-1}(\mathbf{P})\oplus A_d(\mathbf{P})\cong A_{d-1}(\mathbf{P})\oplus\mathbf{Z}$ ([**F2**]). With respect to this bigrading of the Chow group $A_{d+s-2}(\mathbf{P}(E))$ we have that $\deg(F)=(0,1)$, $\deg(f_j)=(\alpha_j,0)$, $\deg(y_j)=(-\alpha_j,1)$, which is very similar to the case when $\mathbf{P}$ is a projective space.

We now can state the main result.

Theorem 3.6. *Let $\mathbf{P}$ be a d-dimensional complete simplicial toric variety, and let $X\subset\mathbf{P}$ be a quasi-smooth intersection of ample hypersurfaces defined by $f_j\in S_{\alpha_j}$, $j=1,\dots,s$. If $F=y_1f_1+\cdots+y_sf_s$, then for $p\neq\frac{d+s-1}{2}$, we have a canonical isomorphism*

$$R(F)_{(d+s-p)\beta-\beta_0}\cong H^{p-s,d-p}_{\mathrm{var}}(X)$$

where $\beta_0=\deg(x_1\cdots x_n\cdot y_1\cdots y_s)$, $\beta=\deg(F)=\deg(f_j)+\deg(y_j)$. In the case $p=\frac{d+s-1}{2}$ there is an exact sequence

$$0\to H^{d-s-1}(\mathbf{P})\xrightarrow{\cup[X]} H^{d+s-1}(\mathbf{P})\to R(F)_{\frac{d+s+1}{2}\beta-\beta_0}\to H_{\mathrm{var}}^{\frac{d-s-1}{2},\frac{d-s+1}{2}}(X)\to 0.$$

Proof. Since $H^i(\mathbf{P})$ vanishes for i odd and has a pure Hodge structure of type (p,p) for i even, from Proposition 3.2 we get $\mathrm{Gr}_F^p H^{d+s-1}(\mathbf{P}\setminus X) \cong H_{var}^{p-s,d-p}(X)$ if $p \neq \frac{d+s-1}{2}$, and in case $p = \frac{d+s-1}{2}$ the following sequence

$$0 \to H^{d-s-1}(\mathbf{P}) \stackrel{\cup[X]}{\to} H^{d+s-1}(\mathbf{P}) \\ \to \mathrm{Gr}_F^{\frac{d+s-1}{2}} H^{d+s-1}(\mathbf{P}\setminus X) \to H_{var}^{\frac{d-s-1}{2},\frac{d-s+1}{2}}(X) \to 0$$

is exact.

Now use the isomorphism of mixed Hodge structures $H^i(\mathbf{P}\setminus X) \cong H^i(\mathbf{P}(E)\setminus Y)$ and by the Theorem 10.6 [**BC**] the desired result follows. □

4. Cohomology of nondegenerate intersections.

In this section we consider a special case of quasi-smooth intersections.

Definition 4.1. A closed subset $X = X_{f_1}\cap\ldots\cap X_{f_s}$, defined by $\mathbf{D}$-homogeneous polynomials $f_1,\ldots,f_s$, is called a *nondegenerate intersection* if $X_{f_{j_1}} \cap \ldots \cap X_{f_{j_k}} \cap \mathbf{T}_\tau$ is a smooth subvariety of codimension k in $\mathbf{T}_\tau$ for any $\{j_1,\ldots,j_k\} \subset \{1,\ldots,s\}$ and $\tau \in \Sigma$. (Here $\mathbf{T}_\tau$ denotes the torus in $\mathbf{P}_\Sigma$ associated with a cone $\tau\in\Sigma$.)

We will show how to define a nondegenerate intersection in terms of the polynomials $f_1,\ldots,f_s$. For $\sigma\in\Sigma$, let $U_\sigma = \{x \in \mathbf{A}^n : \hat{x}_\sigma \neq 0\}$. We know that $\mathbf{P}_\Sigma$ has an affine toric open cover by $\mathbf{A}_\sigma = U_\sigma/\mathbf{D}(\Sigma)$, $\sigma\in\Sigma$ [**BC**]. Also $\mathbf{T}_\tau = (U_\tau \setminus \cup_{\gamma\prec\tau} U_\gamma)/\mathbf{D}(\Sigma)$. Notice that $U_\tau \setminus \cup_{\gamma\prec\tau} U_\gamma = \{x\in\mathbf{A}^n : \hat{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$ is a torus. So each $\mathbf{T}_\tau$ is a quotient of a torus by a D-subgroup, because $\mathbf{D}$ is diagonalizable [**BC**].

Lemma 4.2. *Let $T = (\mathbf{C}^*)^n/G$ be the quotient of a torus by a D-subgroup G. Suppose that $X \subset (\mathbf{C}^*)^n$ is an invariant subvariety with respect to the action of G. Then the geometric quotient X/G is smooth iff X is smooth.*

Proof. By the structure theorem of a D-group [**Hu**, §16.2] we can assume that $(\mathbf{C}^*)^n = G^\circ \times (\mathbf{C}^*)^k$, where $G^\circ \cong (\mathbf{C}^*)^{n-k}$ is the identity component of G, and $G = G^\circ \times H$ for some finite subgroup H in $(\mathbf{C}^*)^k$. Now it suffices to show the Lemma if G is a torus or a finite group. If $G = G^\circ$ then $X = (\mathbf{C}^*)^{n-k}\times p(X)$, where by $p(X)$ we mean the projection of X onto $(\mathbf{C}^*)^k$. Notice that $p(X)\cong X/G$, hence X is smooth iff X/G is smooth. In the case $G = H$ is a finite group it can be easily checked that $X \to X/G$ is an unramified cover [**Sh**, p. 346]. So X and X/G are smooth simultaneously. □

From this Lemma it follows that $X = X_{f_1}\cap\ldots\cap X_{f_s}$ is a nondegenerate intersection iff $\mathbf{V}(f_{j_1},\ldots,f_{j_k}) \cap V_\tau$ is a smooth subvariety of codimension k in the torus $V_\tau = \{x\in\mathbf{A}^n : \hat{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i\subset\tau\}$.

As in Section 2 we can consider the hypersurface $Y \subset \mathbf{P}(E)$ defined by $F = \sum_{j=1}^{s} y_j f_j$. $\square$

Lemma 4.3. *$X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection iff Y is a nondegenerate hypersurface.*

Proof. As shown above, $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection if the $\operatorname{rank}\big(\frac{\partial f_j}{\partial x_i}(x)\big)_{i\in\{i:\, e_i\notin\tau\}}^{j\in\{j_1,\ldots,j_k\}} = k$ for all $x \in \mathbf{V}(f_{j_1},\ldots,f_{j_k}) \cap V_\tau$, $\tau \in \Sigma$ and $\{j_1,\ldots,j_k\} \subset \{1,\ldots,s\}$. Similarly Y is nondegenerate iff $z = (x,y) \in \mathbf{V}(F) \cap V_{\tilde\tau+\tau'}$, $\tilde\tau + \tau' \in \tilde\Sigma$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$ (recall the definition of $\mathbf{P}(E)$ associated with $\tilde\Sigma$ in the Section 2) implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j \in \{j : \tilde n_j \notin \tau'\}$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^{s} y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde e_i \notin \tilde\tau\}$, is nonzero.

Let $(x,y) \in \mathbf{V}(F) \cap V_{\tilde\tau+\tau'}$, where $\tilde\tau + \tau' \in \tilde\Sigma$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$. Then $\prod_{\tilde e_i\notin\tilde\tau} x_i \prod_{\tilde n_j\notin\tau'} y_j \neq 0$ and $x_i = 0$ if $\tilde e_i \in \tilde\tau$, $y_j = 0$ if $\tilde n_j \in \tau'$. If $f_j(x) = 0$ for all $j \in \{j : \tilde n_j \notin \tau'\}$, then $x \in \mathbf{V}(f_{j_1},\ldots,f_{j_k}) \cap V_\tau$ where $\{j_1,\ldots,j_k\} = \{j : \tilde n_j \notin \tau'\}$. So if $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^{s} y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde e_i \notin \tilde\tau\}$, is nonzero.

Conversely, suppose Y is nondegenerate. Take any $x \in \mathbf{V}(f_{j_1},\ldots,f_{j_k}) \cap V_\tau$ with $\tau \in \Sigma$, $\{j_1,\ldots,j_k\} \subset \{1,\ldots,s\}$. Then $(x,y) \in \mathbf{V}(F) \cap V_{\tilde\tau+\tau'}$ for each $y \in V_{\tau'} = \{y \in \mathbf{A}^s : y_j \neq 0 \text{ if } \tilde n_j \notin \tau', y_j = 0 \text{ if } \tilde n_j \in \tau'\}$ where τ' is the cone generated by the complement of $\{\tilde n_{j_1},\ldots,\tilde n_{j_k}\}$ in the set $\{\tilde n_1,\ldots,\tilde n_s\}$. Therefore $\sum_{j=1}^{s} y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some i, which means the $\operatorname{rank}\big(\frac{\partial f_j}{\partial x_i}(x)\big)_{i\in\{i:\, e_i\notin\tau\}}^{j\in\{j_1,\ldots,j_k\}} = k$. $\square$

Since a nondegenerate hypersurface is quasi-smooth [**BC**], Lemma 2.2 shows that a nondegenerate intersection is quasi-smooth.

Definition 4.4 ([**BC**])**.** Given a polynomial $f \in S = \mathbf{C}[x_1,\ldots,x_n]$, we get the ideal quotient $J_1(f) = \langle x_1\partial f/\partial x_1, \ldots, x_n \partial f/\partial x_n\rangle : x_1 \cdots x_n$ (see [**CLO**, p. 193]) and the ring $R_1(f) = S/J_1(f)$.

Remark 4.5. If $F = \sum_{j=1}^{s} y_j f_j \in R$ is as in Lemma 2.1, then $R_1(F) = R/J_1(F)$ has a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E)) \cong A_{d-1}(\mathbf{P}) \oplus \mathbf{Z}$.

Theorem 4.6. *Let $X = X_{f_1} \cap \ldots \cap X_{f_s}$ be a nondegenerate intersection of ample hypersurfaces given by $f_j \in S_{\alpha_j}$, $j = 1,\ldots,s$. If $F = \sum_{j=1}^{s} y_j f_j \in R$, then there is a canonical isomorphism*

$$H^{p-s,d-p}_{\mathrm{var}}(X) = R_1(F)_{(d+s-p)\beta-\beta_0},$$

where $\beta_0 = \deg(x_1 \cdots x_n \cdot y_1 \cdots y_s)$, $\beta = \deg(F)$.

Proof. First we will show that there is an isomorphism of Hodge structures $H^{d-s}_{\rm var}(X)(1-s)\cong H^{d+s-2}_{\rm var}(Y)$. Let $\varphi: Y\to \mathbf{P}$ be the composition of the inclusion $j: Y\hookrightarrow \mathbf{P}(E)$ and the projection $\pi:\mathbf{P}(E)\to\mathbf{P}$. As in [**Te**], consider the following morphism of the Leray spectral sequences

$$\begin{array}{rccc} E_2^{p,q} = & H^p(\mathbf{P}, R^q\pi_*\mathbf{C}) & \Rightarrow & H^{p+q}(\mathbf{P}(E)) \\ & \downarrow & & \downarrow \\ {}'E_2^{p,q} = & H^p(\mathbf{P}, R^q\varphi_*\mathbf{C}) & \Rightarrow & H^{p+q}(Y). \end{array}$$

Since

$$\varphi^{-1}(X)=\begin{cases}\mathbb{P}^{s-1} & \text{if } x\in X,\\ \mathbb{P}^{s-2} & \text{if } x\notin X,\end{cases}$$

we have that (see [**Go**, p. 202], [**De**])

$$R^q\varphi_*\mathbf{C}=\begin{cases}\mathbf{C}_{\mathbf{P}}(-\frac{q}{2}) & \text{if } q \text{ is even and } 0\le q<2s-2,\\ \mathbf{C}_X(1-s) & \text{if } q=2s-2,\\ 0 & \text{otherwise.}\end{cases}$$

Also we have

$$R^q\pi_*\mathbf{C}=\begin{cases}\mathbf{C}_{\mathbf{P}}(-\frac{q}{2}) & \text{if } q \text{ is even and } 0\le q\le 2s-2,\\ 0 & \text{otherwise.}\end{cases}$$

The first spectral sequence degenerates at E_2, because for p or q odd $E_r^{p,q}$ vanishes. The second spectral sequence also degenerates at E_2:

$$h^{l-2s-2}(X)+\sum_{q=0}^{2s-4}h^{l-q}(\mathbf{P})=\sum_{p+q=l}\dim {}'E_2^{p,q}\ge\sum_{p+q=l}\dim{}'E_\infty^{p,q}=h^l(Y).$$

To show the degeneracy of ${}'E_2^{p,q}$ it suffices to show that the above inequality is an equality. From Proposition 10.8 [**BC**] and Proposition 3.2 we get

$$\begin{aligned} h^{d+s-2}(Y)=h^{d+s-2}(\mathbf{P}(E))+h^{d+s-1}(\mathbf{P}(E)\setminus Y)\\ -h^{d+s-1}(\mathbf{P}(E))+h^{d+s-3}(\mathbf{P}(E)),\\ h^{d-s}(X)=h^{d-s}(\mathbf{P})+h^{d+s-1}(\mathbf{P}\setminus X)-h^{d+s-1}(\mathbf{P})+h^{d-s-1}(\mathbf{P}).\end{aligned}$$

Hence, using the spectral sequence $E_2^{p,q}$, we can easily compute the Hodge numbers of $\mathbf{P}(E)$ and check that $h^{l-2s-2}(X)+\sum_{q=0}^{2s-4}h^{l-q}(\mathbf{P})=h^l(Y)$ for $l=d+s-2$. Using Proposition 1.4, we can similarly show the above equality for $l\ne d+s-2$ as well. So the spectral sequence ${}'E_2^{p,q}$ degenerates at E_2. Since $E_2^{d+s-2-q,q}={}'E_2^{d+s-2-q,q}$ for $q\ne 2s-2$ and, by Proposition 1.4, $E_2^{d-s,2s-2}\hookrightarrow{}'E_2^{d-s,2s-2}$, we get an isomorphism of Hodge structures (for details see [**Te**]):

$$H^{d+s-2}_{var}(Y)\cong{}'E_2^{d-s,2s-s}/E_2^{d-s,2s-2}\cong H^{d-s}_{var}(X)(1-s).$$

Now we only need to apply Theorem 11.8 [**BC**] to finish the proof. □

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