COHOMOLOGY OF FACE RINGS, AND TORUS ACTIONS

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Abstract. In this survey article we present several new developments of 'toric topology' concerning the cohomology of face rings (also known as Stanley–Reisner algebras). We prove that the integral cohomology algebra of the *moment-angle complex* $\mathcal{Z}_K$ (equivalently, of the complement $U(K)$ of the coordinate subspace arrangement) determined by a simplicial complex K is isomorphic to the Tor-algebra of the face ring of K. Then we analyse Massey products and formality of this algebra by using a generalisation of Hochster's theorem. We also review several related combinatorial results and problems.

1. Introduction

This article centres on the cohomological aspects of 'toric topology', a new and actively developing field on the borders of equivariant topology, combinatorial geometry and commutative algebra. The algebro-geometric counterpart of toric topology, known as 'toric geometry' or algebraic geometry of *toric varieties*, is now a well established field in algebraic geometry, which is characterised by its strong links with combinatorial and convex geometry (see the classical survey paper [10] or more modern exposition [13]). Since the appearance of Davis and Januszkiewicz's work [11], where the concept of a *(quasi)toric manifold* was introduced as a topological generalisation of smooth compact toric variety, there has grown an understanding that most phenomena of smooth toric geometry may be modelled in the purely topological situation of smooth manifolds with a nicely behaved torus action.

One of the main results of [11] is that the equivariant cohomology of a toric manifold can be identified with the *face ring* of the quotient simple polytope, or, for more general classes of torus actions, with the face ring of a certain simplicial complex K. The ordinary cohomology of a quasitoric manifold can also be effectively identified as the quotient of the face ring by a *regular sequence* of degree-two elements, which provides a generalisation to the well-known Danilov–Jurkiewicz theorem of toric geometry. The notion of the face ring of a simplicial complex sits in the heart of Stanley's 'Combinatorial commutative algebra' [24], linking geometrical and combinatorial problems concerning simplicial complexes with commutative and homological algebra. Our concept of toric topology aims at extending these links and developing new applications by applying the full strength of the apparatus of equivariant topology of torus actions.

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The article surveys certain new developments of toric topology related to the cohomology of face rings. Introductory remarks can be found at the beginning of each section and most subsections. A more detailed description of the history of the subject, together with an extensive bibliography, can be found in [8] and its extended Russian version [9].

The current article represents the work of the algebraic topology and combinatorics group at the Department of Geometry and Topology, Moscow State University, and the author thanks all its members for the collaboration and insight gained from numerous discussions, particularly mentioning Victor Buchstaber, Ilia Baskakov, and Arseny Gadzhikurbanov. The author is also grateful to Nigel Ray for several valuable comments and suggestions that greatly improved this text and his hospitality during the visit to Manchester sponsored by an LMS grant.

2. Simplicial complexes and face rings

The notion of the face ring $\mathbf{k}[K]$ of a simplicial complex K is central to the algebraic study of triangulations. In this section we review its main properties, emphasising functoriality with respect to simplicial maps. Then we introduce the bigraded Tor-algebra $\mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$ through a finite free resolution of $\mathbf{k}[K]$ as a module over the polynomial ring. The corresponding bigraded Betti numbers are important combinatorial invariants of K.

2.1. **Definition and main properties.** Let $K = K^{n-1}$ be an arbitrary $(n-1)$-dimensional simplicial complex on an m-element vertex set V, which we usually identify with the set of ordinals $[m] = \{1,\ldots,m\}$. Those subsets $\sigma \subseteq V$ belonging to K are referred to as *simplices*; we also use the notation $\sigma \in K$. We count the empty set $\varnothing$ as a simplex of K. When it is necessary to distinguish between combinatorial and geometrical objects, we denote by $|K|$ a *geometrical realisation* of K, which is a triangulated topological space.

Choose a ground commutative ring $\mathbf{k}$ with unit (we are mostly interested in the cases $\mathbf{k} = \mathbb{Z}, \mathbb{Q}$ or finite field). Let $\mathbf{k}[v_1,\ldots,v_m]$ be the graded polynomial algebra over $\mathbf{k}$ with $\deg v_i = 2$. For an arbitrary subset $\omega = \{i_1,\ldots,i_k\} \subseteq [m]$, denote by v_ω the square-free monomial $v_{i_1}\ldots v_{i_k}$.

The *face ring* (or *Stanley–Reisner algebra*) of K is the quotient ring

$$\mathbf{k}[K] = \mathbf{k}[v_1,\ldots,v_m]/\mathcal{I}_K,$$

where $\mathcal{I}_K$ is the homogeneous ideal generated by all monomials v_σ such that σ is not a simplex of K. The ideal $\mathcal{I}_K$ is called the *Stanley–Reisner ideal* of K.

Example 2.1. Let K be a 2-dimensional simplicial complex shown on Figure 1. Then

$$\mathbf{k}[K] = \mathbf{k}[v_1,\ldots,v_5]/(v_1v_5, v_3v_4, v_1v_2v_3, v_2v_4v_5).$$

Despite its simple construction, the face ring appears to be a very powerful tool allowing us to translate the combinatorial properties of different particular classes of simplicial complexes into the language of commutative algebra. The resulting field of 'Combinatorial commutative algebra', whose

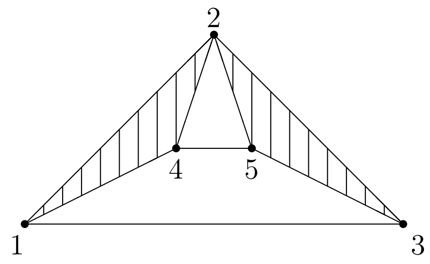


FIGURE 1

foundations were laid by Stanley in his monograph [24], has attracted a lot of interest from both combinatorialists and commutative algebraists.

Let K_1 and K_2 be two simplicial complexes on the vertex sets $[m_1]$ and $[m_2]$ respectively. A set map $\varphi\colon [m_1] \to [m_2]$ is called a *simplicial map* between K_1 and K_2 if $\varphi(\sigma) \in K_2$ for any $\sigma \in K_1$; we often identify such φ with its restriction to K_1 (regarded as a collection of subsets of $[m_1]$), and use the notation $\varphi\colon K_1 \to K_2$.

Proposition 2.2. *Let $\varphi\colon K_1 \to K_2$ be a simplicial map. Define a map $\varphi^*\colon \mathbf{k}[w_1, \dots, w_{m_2}] \to \mathbf{k}[v_1, \dots, v_{m_1}]$ by*

$$\varphi^*(w_j) := \sum_{i \in \varphi^{-1}(j)} v_i.$$

Then φ^* induces a homomorphism $\mathbf{k}[K_2] \to \mathbf{k}[K_1]$, which we will also denote by φ^*.

Proof. We have to check that $\varphi^*(\mathcal{I}_{K_2}) \subseteq \mathcal{I}_{K_1}$. Suppose $\tau = \{j_1, \dots, j_s\} \subseteq [m_2]$ is not a simplex of K_2. Then

$$\varphi^*(w_{j_1} \cdots w_{j_s}) = \sum_{i_1 \in \varphi^{-1}(j_1), \dots, i_s \in \varphi^{-1}(j_s)} v_{i_1} \cdots v_{i_s}. \tag{2.1}$$

We claim that $\sigma = \{i_1, \dots, i_s\}$ is not a simplex of K_1 for any monomial $v_{i_1} \cdots v_{i_s}$ in the right hand side of the above identity. Indeed, if $\sigma \in K_1$, then $\varphi(\sigma) = \tau \in K_2$ by the definition of simplicial map, which leads to a contradiction. Hence, the right hand side of (2.1) is in $\mathcal{I}_{K_1}$. □

2.2. **Cohen–Macaulay rings and complexes.** Cohen–Macaulay rings and modules play an important role in homological commutative algebra and algebraic geometry. A standard reference for the subject is [6], where the reader may find proofs of the basic facts about Cohen–Macaulay rings and regular sequences mentioned in this subsection. In the case of simplicial complexes, the Cohen–Macaulay property of the corresponding face rings leads to important combinatorial and topological consequences.

Let $A = \oplus_{i \geq 0} A^i$ be a finitely-generated commutative graded algebra over $\mathbf{k}$. We assume that A is connected ($A^0 = \mathbf{k}$) and has only even-degree graded components, so that we do not need to distinguish between graded and non-graded commutativity. We denote by A_+ the positive-degree part of A and by $\mathcal{H}(A_+)$ the set of homogeneous elements in A_+.

A sequence $t_1, \ldots, t_n$ of algebraically independent homogeneous elements of A is called an *hsop* (homogeneous system of parameters) if A is a finitely-generated $\mathbf{k}[t_1, \ldots, t_n]$-module (equivalently, $A/(t_1, \ldots, t_n)$ has finite dimension as a $\mathbf{k}$-vector space).

Lemma 2.3 (Nöther normalisation lemma)**.** *Any finitely-generated graded algebra A over a field $\mathbf{k}$ admits an hsop. If $\mathbf{k}$ has characteristic zero and A is generated by degree-two elements, then a degree-two hsop can be chosen.*

A degree-two hsop is called an *lsop* (linear system of parameters).

A sequence $\boldsymbol{t} = t_1, \ldots, t_k$ of elements of $\mathcal{H}(A_+)$ is called a *regular sequence* if t_{i+1} is not a zero divisor in $A/(t_1, \ldots, t_i)$ for $0 \le i < k$. A regular sequence consists of algebraically independent elements, so it generates a polynomial subring in A. It can be shown that $\boldsymbol{t}$ is a regular sequence if and only if A is a *free* $\mathbf{k}[t_1, \ldots, t_k]$-module.

An algebra A is called *Cohen–Macaulay* if it admits a regular hsop $\boldsymbol{t}$. It follows that A is Cohen–Macaulay if and only if it is a free and finitely generated module over its polynomial subring. If $\mathbf{k}$ is a field of zero characteristic and A is generated by degree-two elements, then one can choose $\boldsymbol{t}$ to be an lsop. A simplicial complex K is called *Cohen–Macaulay* (over $\mathbf{k}$) if its face ring $\mathbf{k}[K]$ is Cohen–Macaulay.

Example 2.4. Let $K = \partial\Delta^2$ be the boundary of a 2-simplex. Then

$$\mathbf{k}[K] = \mathbf{k}[v_1, v_2, v_3]/(v_1 v_2 v_3).$$

The elements $v_1, v_2 \in \mathbf{k}[K]$ are algebraically independent, but do not form an hsop, since $\mathbf{k}[K]/(v_1, v_2) \cong \mathbf{k}[v_3]$ is not finite-dimensional as a $\mathbf{k}$-space. On the other hand, the elements $t_1 = v_1 - v_3$, $t_2 = v_2 - v_3$ of $\mathbf{k}[K]$ form an hsop, since $\mathbf{k}[K]/(t_1, t_2) \cong \mathbf{k}[t]/t^3$. It is easy to see that $\mathbf{k}[K]$ is a free $\mathbf{k}[t_1, t_2]$-module with one 0-dimensional generator 1, one 1-dimensional generator v_1, and one 2-dimensional generator v_1^2. Thus, $\mathbf{k}[K]$ is Cohen–Macaulay and (t_1, t_2) is a regular sequence.

For an arbitrary simplex $\sigma \in K$ define its *link* and *star* as the subcomplexes

$$\begin{aligned} \operatorname{link}_K \sigma &= \{\tau \in K \colon \sigma \cup \tau \in K,\ \sigma \cap \tau = \varnothing\}; \\ \operatorname{star}_K \sigma &= \{\tau \in K \colon \sigma \cup \tau \in K\}. \end{aligned}$$

If $v \in K$ is a vertex, then $\operatorname{star}_K v$ is the subcomplex consisting of all simplices of K containing v, and all their subsimplices. Note also that $\operatorname{star}_K v$ is the cone over $\operatorname{link}_K v$.

The following fundamental theorem characterises Cohen–Macaulay complexes combinatorially.

Theorem 2.5 (Reisner)**.** *A simplicial complex K is Cohen–Macaulay over $\mathbf{k}$ if and only if for any simplex $\sigma \in K$ (including $\sigma = \varnothing$) and $i < \dim(\operatorname{link}_K \sigma)$, it holds that $\widetilde{H}_i(\operatorname{link}_K \sigma; \mathbf{k}) = 0$.*

Using standard techniques of PL topology the previous theorem may be reformulated in purely topological terms.

Proposition 2.6 (Munkres). *K^{n-1} is Cohen–Macaulay over $\mathbf{k}$ if and only if for an arbitrary point $x \in |K|$, it holds that*

$$\widetilde{H}_i(|K|;\mathbf{k}) = H_i(|K|, |K|\setminus x;\mathbf{k}) = 0 \quad \text{for } i < n-1.$$

Thus any triangulation of a sphere is a Cohen–Macaulay complex.

2.3. **Resolutions and Tor-algebras.** Let M be a finitely-generated graded $\mathbf{k}[v_1,\dots,v_m]$-module. A *free resolution* of M is an exact sequence

$$\dots \xrightarrow{d} R^{-i} \xrightarrow{d} \dots \xrightarrow{d} R^{-1} \xrightarrow{d} R^0 \longrightarrow M \to 0, \tag{2.2}$$

where the R^{-i} are finitely-generated graded free $\mathbf{k}[v_1,\dots,v_m]$-modules and the maps d are degree-preserving. By the Hilbert syzygy theorem, there is a free resolution of M with $R^{-i}=0$ for $i>m$. A resolution (2.2) determines a bigraded differential $\mathbf{k}$-module $[R,d]$, where $R=\bigoplus R^{-i,j}$, $R^{-i,j} := (R^{-i})^j$ and $d\colon R^{-i,j} \to R^{-i+1,j}$. The bigraded cohomology module $H[R,d]$ has $H^{-i,k}[R,d]=0$ for $i>0$ and $H^{0,k}[R,d]=M^k$. Let $[M,0]$ be the bigraded module with $M^{-i,k}=0$ for $i>0$, $M^{0,k}=M^k$, and zero differential. Then the resolution (2.2) determines a bigraded map $[R,d]\to[M,0]$ inducing an isomorphism in cohomology.

Let N be another module; then applying the functor $\otimes_{\mathbf{k}[v_1,\dots,v_m]}N$ to a resolution $[R,d]$ we get a homomorphism of differential modules

$$[R\otimes_{\mathbf{k}[v_1,\dots,v_m]} N, d] \to [M\otimes_{\mathbf{k}[v_1,\dots,v_m]} N, 0],$$

which in general does not induce an isomorphism in cohomology. The $(-i)$th cohomology module of the cochain complex

$$\dots \longrightarrow R^{-i}\otimes_{\mathbf{k}[v_1,\dots,v_m]} N \longrightarrow \dots \longrightarrow R^0\otimes_{\mathbf{k}[v_1,\dots,v_m]} N \longrightarrow 0$$

is denoted by $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$. Thus,

$$\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N) := \frac{\operatorname{Ker}\bigl[d\colon R^{-i}\otimes_{\mathbf{k}[v_1,\dots,v_m]} N \to R^{-i+1}\otimes_{\mathbf{k}[v_1,\dots,v_m]} N\bigr]}{d(R^{-i-1}\otimes_{\mathbf{k}[v_1,\dots,v_m]} N)}.$$

Since all the R^{-i} and N are graded modules, we actually have a *bigraded* $\mathbf{k}$-module

$$\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(M,N) = \bigoplus_{i,j} \mathrm{Tor}^{-i,j}_{\mathbf{k}[v_1,\dots,v_m]}(M,N).$$

The following properties of $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$ are well known.

Proposition 2.7. (a) *the module $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N)$ does not depend on a choice of resolution in* (2.2)*;*

(b) $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\,\cdot\,,N)$ *and* $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,\,\cdot\,)$ *are covariant functors;*

(c) $\mathrm{Tor}^{0}_{\mathbf{k}[v_1,\dots,v_m]}(M,N) \cong M\otimes_{\mathbf{k}[v_1,\dots,v_m]} N$*;*

(d) $\mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(M,N) \cong \mathrm{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(N,M)$.

Now put $M=\mathbf{k}[K]$ and $N=\mathbf{k}$. Since $\deg v_i = 2$, we have

$$\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}[K],\mathbf{k}\bigr) = \bigoplus_{i,j=0}^{m} \mathrm{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\bigl(\mathbf{k}[K],\mathbf{k}\bigr)$$

Define the *bigraded Betti numbers* of $\mathbf{k}[K]$ by

$$\beta^{-i,2j}\big(\mathbf{k}[K]\big) := \dim_{\mathbf{k}} \operatorname{Tor}^{-i,2j}_{\mathbf{k}[v_1,\dots,v_m]}\big(\mathbf{k}[K],\mathbf{k}\big), \qquad 0 \le i,j \le m. \tag{2.3}$$

We also set

$$\beta^{-i}(\mathbf{k}[K]) = \dim_{\mathbf{k}} \operatorname{Tor}^{-i}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}[K],\mathbf{k}) = \sum_j \beta^{-i,2j}(\mathbf{k}[K]).$$

Example 2.8. Let K be the boundary of a square. Then

$$\mathbf{k}[K] \cong \mathbf{k}[v_1,\dots,v_4]/(v_1v_3, v_2v_4).$$

Let us construct a resolution of $\mathbf{k}[K]$ and calculate the corresponding bigraded Betti numbers. The module R^0 has one generator 1 (of degree 0), and the map $R^0 \to \mathbf{k}[K]$ is the quotient projection. Its kernel is the ideal $\mathcal{I}_K$, generated by two monomials v_1v_3 and v_2v_4. Take R^{-1} to be a free module on two 4-dimensional generators, denoted v_{13} and v_{24}, and define $d\colon R^{-1} \to R^0$ by sending v_{13} to v_1v_3 and v_{24} to v_2v_4. Its kernel is generated by one element $v_2v_4v_{13} - v_1v_3v_{24}$. Hence, R^{-2} has one generator of degree 8, say a, and the map $d\colon R^{-2} \to R^{-1}$ is injective and sends a to $v_2v_4v_{13} - v_1v_3v_{24}$. Thus, we have a resolution

$$0 \longrightarrow R^{-2} \longrightarrow R^{-1} \longrightarrow R^0 \longrightarrow M \longrightarrow 0$$

where $\operatorname{rank} R^0 = \beta^{0,0}(\mathbf{k}[K]) = 1$, $\operatorname{rank} R^{-1} = \beta^{-1,4} = 2$ and $\operatorname{rank} R^{-2} = \beta^{-2,8} = 1$.

The Betti numbers $\beta^{-i,2j}(\mathbf{k}[K])$ are important combinatorial invariants of the simplicial complex K. The following result expresses them in terms of homology groups of subcomplexes of K.

Given a subset $\omega \subseteq [m]$, we may restrict K to ω and consider the *full subcomplex* $K_\omega = \{\sigma \in K : \sigma \subseteq \omega\}$.

Theorem 2.9 (Hochster)**.** *We have*

$$\beta^{-i,2j}\big(\mathbf{k}[K]\big) = \sum_{\omega\subseteq[m]\colon |\omega|=j} \dim_{\mathbf{k}} \widetilde{H}^{j-i-1}(K_\omega;\mathbf{k}),$$

where $\widetilde{H}^*(\cdot)$ *denotes the reduced cohomology groups and we assume that* $\widetilde{H}^{-1}(\varnothing) = \mathbf{k}$.

Hochster's original proof of this theorem uses rather complicated combinatorial and commutative algebra techniques. Later in subsection 5.1 we give a topological interpretation of the numbers $\beta^{-i,2j}(\mathbf{k}[K])$ as the bigraded Betti numbers of a topological space, and prove a generalisation of Hochster's theorem.

Example 2.10 (Koszul resolution)**.** Let $M = \mathbf{k}$ with the $\mathbf{k}[v_1,\dots,v_m]$-module structure defined via the map $\mathbf{k}[v_1,\dots,v_m] \to \mathbf{k}$ sending each v_i to 0. Let $\Lambda[u_1,\dots,u_m]$ denote the exterior $\mathbf{k}$-algebra on m generators. The tensor product $R = \Lambda[u_1,\dots,u_m] \otimes \mathbf{k}[v_1,\dots,v_m]$ (here and below we use $\otimes$ for $\otimes_{\mathbf{k}}$) may be turned to a differential bigraded algebra by setting

$$\begin{aligned} &\operatorname{bideg} u_i = (-1,2), \quad \operatorname{bideg} v_i = (0,2),\\ &du_i = v_i, \quad dv_i = 0, \end{aligned} \tag{2.4}$$

and requiring d to be a derivation of algebras. An explicit construction of a cochain homotopy shows that $H^{-i}[R,d]=0$ for $i>0$ and $H^0[R,d]=\mathbf{k}$. Since $\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[v_1,\ldots,v_m]$ is a free $\mathbf{k}[v_1,\ldots,v_m]$-module, it determines a free resolution of $\mathbf{k}$. It is known as the *Koszul resolution* and its expanded form (2.2) is as follows:

$$\begin{aligned}0\to\Lambda^m[u_1,\ldots,u_m]\otimes\mathbf{k}[v_1,\ldots,v_m]\longrightarrow\cdots\\ \longrightarrow\Lambda^1[u_1,\ldots,u_m]\otimes\mathbf{k}[v_1,\ldots,v_m]\longrightarrow\mathbf{k}[v_1,\ldots,v_m]\longrightarrow\mathbf{k}\to 0\end{aligned}$$

where $\Lambda^i[u_1,\ldots,u_m]$ is the subspace of $\Lambda[u_1,\ldots,u_m]$ spanned by monomials of length i.

Now let us consider the differential bigraded algebra $[\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[K],d]$ with d defined as in (2.4).

Lemma 2.11. *There is an isomorphism of bigraded modules:*

$$\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})\cong H\big[\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[K],d\big]$$

which endows $\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$ *with a bigraded algebra structure in a canonical way.*

Proof. Using the Koszul resolution in the definition of Tor, we calculate

$$\begin{aligned}\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})&\cong\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k},\mathbf{k}[K])\\ &=H\big[\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[v_1,\ldots,v_m]\otimes_{\mathbf{k}[v_1,\ldots,v_m]}\mathbf{k}[K]\big]\\ &\cong H\big[\Lambda[u_1,\ldots,u_m]\otimes\mathbf{k}[K]\big].\end{aligned}$$

The cohomology in the right hand side is a bigraded algebra, providing an algebra structure for $\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$. □

The bigraded algebra $\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_m]}(\mathbf{k}[K],\mathbf{k})$ is called the Tor-*algebra* of the simplicial complex K.

Lemma 2.12. *A simplicial map* $\varphi\colon K_1\to K_2$ *between two simplicial complexes on the vertex sets* $[m_1]$ *and* $[m_2]$ *respectively induces a homomorphism*

$$\varphi_t^*\colon\ \operatorname{Tor}_{\mathbf{k}[w_1,\ldots,w_{m_2}]}(\mathbf{k}[K_2],\mathbf{k})\to\operatorname{Tor}_{\mathbf{k}[v_1,\ldots,v_{m_1}]}(\mathbf{k}[K_1],\mathbf{k})\tag{2.5}$$

of the corresponding Tor-*algebras.*

Proof. This follows directly from Propositions 2.2 and 2.7 (b). □

3. Toric spaces

Moment-angle complexes provide a functor $K\mapsto\mathcal{Z}_K$ from the category of simplicial complexes and simplicial maps to the category of spaces with torus action and equivariant maps. This functor allows us to use the techniques of equivariant topology in the study of combinatorics of simplicial complexes and commutative algebra of their face rings; in a way, it breathes a geometrical life into Stanley's 'combinatorial commutative algebra'. In particular, the calculation of the cohomology of $\mathcal{Z}_K$ opens a way to a topological treatment of homological invariants of face rings.

The space $\mathcal{Z}_K$ was introduced for arbitrary finite simplicial complex K by Davis and Januszkiewicz [11] as a technical tool in their study of (quasi)toric manifolds, a topological generalisation of smooth algebraic toric varieties.

Later this space turned out to be of great independent interest. For the subsequent study of $\mathcal{Z}_K$, its place within 'toric topology', and connections with combinatorial problems we refer to [8] and its extended Russian version [9]. Here we review the most important aspects of this study related to the cohomology of face rings.

3.1. **Moment-angle complexes.** The *m-torus* T^m is a product of m circles; we usually regard it as embedded in $\mathbb{C}^m$ in the standard way:

$$T^m = \bigl\{(z_1,\dots,z_m)\in\mathbb{C}^m\colon |z_i|=1,\quad i=1,\dots,m\bigr\}.$$

It is contained in the *unit polydisk*

$$(D^2)^m = \{(z_1,\dots,z_m)\in\mathbb{C}^m\colon |z_i|\le 1,\quad i=1,\dots,m\}.$$

For an arbitrary subset $\omega\subseteq V$, define

$$B_\omega := \{(z_1,\dots,z_m)\in(D^2)^m\colon |z_i|=1 \text{ for } i\notin\omega\}.$$

The subspace B_ω is homeomorphic to $(D^2)^{|\omega|}\times T^{m-|\omega|}$.

Given a simplicial complex K on $[m]=\{1,\dots,m\}$, we define the *moment-angle complex* $\mathcal{Z}_K$ by

$$\mathcal{Z}_K := \bigcup_{\sigma\in K} B_\sigma \subseteq (D^2)^m. \tag{3.1}$$

The torus T^m acts on $(D^2)^m$ coordinatewise and each subspace B_ω is invariant under this action. Therefore, the space $\mathcal{Z}_K$ inherits a torus action. The quotient $(D^2)^m/T^m$ can be identified with the *unit m-cube*:

$$I^m := \bigl\{(y_1,\dots,y_m)\in\mathbb{R}^m\colon 0\le y_i\le 1,\quad i=1,\dots,m\bigr\}.$$

The quotient B_ω/T^m is then the following $|\omega|$-dimensional face of I^m:

$$C_\omega := \bigl\{(y_1,\dots,y_m)\in I^m\colon y_i=1 \text{ if } i\notin\omega\bigr\}.$$

Thus the whole quotient $\mathcal{Z}_K/T^m$ is identified with a certain cubical subcomplex in I^m, which we denote by $\mathrm{cc}(K)$.

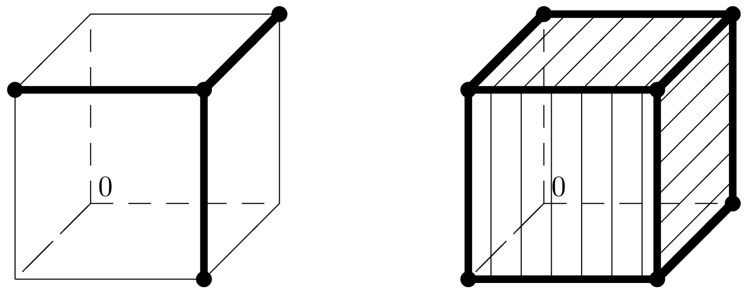
Lemma 3.1. *The cubical complex* $\mathrm{cc}(K)$ *is PL-homeomorphic to* $\operatorname{cone} K$*.*

Proof. Let K' denote the barycentric subdivision of K (the vertices of K' correspond to non-empty simplices σ of K). We define a *PL* embedding $i_c\colon \operatorname{cone}K'\hookrightarrow I^m$ by mapping each vertex σ to the vertex $(\varepsilon_1,\dots,\varepsilon_m)\in I^m$ where $\varepsilon_i=0$ if $i\in\sigma$ and $\varepsilon_i=1$ otherwise, the cone vertex to $(1,\dots,1)\in I^m$, and then extending linearly on the simplices of $\operatorname{cone}K'$. The barycentric subdivision of a face $\sigma\in K$ is a subcomplex in K', which we denote $K'|_\sigma$. Under the map i_c the subcomplex $\operatorname{cone}K'|_\sigma$ maps onto the face $C_\sigma\subset I^m$. Thus the whole complex $\operatorname{cone}K'$ maps homeomorphically onto $\mathrm{cc}(K)$, which concludes the proof. $\square$

It follows that the moment-angle complex $\mathcal{Z}_K$ can be defined by the pullback diagram

$$\begin{array}{ccc}
\mathcal{Z}_K & \longrightarrow & (D^2)^m \\
\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\
\operatorname{cone}K' & \xrightarrow{i_c} & I^m
\end{array}$$

where ρ is the projection onto the orbit space.



$K = 3$ points $\qquad\qquad$ $K = \partial\varDelta^2$

FIGURE 2. Embedding $i_c\colon \operatorname{cone} K' \hookrightarrow I^m$.

Example 3.2. The embedding i_c for two simple cases when K is a three point complex and the boundary of a triangle is shown on Figure 2. If $K = \Delta^{m-1}$ is the whole simplex on m vertices, then $\mathrm{cc}(K)$ is the whole cube I^m, and the above constructed PL-homeomorphism between $\operatorname{cone}(\Delta^{m-1})'$ and I^m defines the *standard triangulation* of I^m.

The next lemma shows that the space $\mathcal{Z}_K$ is particularly nice for certain geometrically important classes of triangulations.

Lemma 3.3. *Suppose that K is a triangulation of an $(n-1)$-dimensional sphere. Then $\mathcal{Z}_K$ is a closed $(m+n)$-dimensional manifold.*

In general, if K is a triangulated manifold then $\mathcal{Z}_K \setminus \rho^{-1}(1,\dots,1)$ is a noncompact manifold, where $(1,\dots,1) \in I^m$ is the cone vertex and $\rho^{-1}(1,\dots,1) \cong T^m$.

Proof. We only prove the first statement here; the proof of the second is similar and can be found in [9]. Each vertex v_i of K corresponds to a vertex of the barycentric subdivision K', which we continue to denote v_i. Let $\operatorname{star}_{K'} v_i$ be the star of v_i in K', that is, the subcomplex consisting of all simplices of K' containing v_i, and all their subsimplices. The space $\operatorname{cone} K'$ has a canonical *face structure* whose facets (codimension-one faces) are

$$F_i := \operatorname{star}_{K'} v_i, \quad i = 1,\dots,m, \tag{3.2}$$

and whose i-faces are non-empty intersections of i-tuples of facets. In particular, the vertices (0-faces) in this face structure are the barycentres of $(n-1)$-dimensional simplices of K.

For every such barycentre b we denote by U_b the subset of $\operatorname{cone} K'$ obtained by removing all faces not containing b. Since K is a triangulation of a sphere, $\operatorname{cone} K'$ is an n-ball, hence each U_b is homeomorphic to an open subset in I^n via a homeomorphism preserving the dimension of faces. Since each point of $\operatorname{cone} K'$ is contained in some U_b, this displays $\operatorname{cone} K'$ as a *manifold with corners*. Having identified $\operatorname{cone} K'$ with $\mathrm{cc}(K)$ and further $\mathrm{cc}(K)$ with $\mathcal{Z}_K/T^m$, we see that every point in $\mathcal{Z}_K$ lies in a neighbourhood homeomorphic to an open subset in $(D^2)^n \times T^{m-n}$ and thus in $\mathbb{R}^{m+n}$. $\square$

A particularly important class of examples of sphere triangulations arise from boundary triangulations of convex polytopes. Suppose P is a *simple* n-dimensional convex polytope, i.e. one where every vertex is contained in exactly n facets. Then the dual (or *polar*) polytope is *simplicial*, and we denote its boundary complex by K_P. K_P is then a triangulation of an $(n-1)$-sphere. The faces of $\operatorname{cone} K'_P$ introduced in the previous proof coincide with those of P.

Example 3.4. Let $K = \partial\Delta^{m-1}$. Then $\mathcal{Z}_K = \partial((D^2)^m) \cong S^{2m-1}$. In particular, for $m = 2$ from (3.1) we get the familiar decomposition

$$S^3 = D^2 \times S^1 \cup S^1 \times D^2 \subset D^2 \times D^2$$

of a 3-sphere into a union of two solid tori.

Using faces (3.2) we can identify the isotropy subgroups of the T^m-action on $\mathcal{Z}_K$. Namely, the isotropy subgroup of a point x in the orbit space $\operatorname{cone} K'$ is the coordinate subtorus

$$T(x) = \{(z_1, \ldots, z_m) \in T^m \colon z_i = 1 \text{ if } x \notin F_i\}.$$

In particular, the action is free over the interior (that is, near the cone point) of $\operatorname{cone} K'$.

It follows that the moment-angle complex can be identified with the quotient

$$\mathcal{Z}_K = \bigl(T^m \times |\operatorname{cone} K'|\bigr)/\!\sim,$$

where $(t_1, x) \sim (t_2, y)$ if and only if $x = y$ and $t_1 t_2^{-1} \in T(x)$. In the case when K is the dual triangulation of a simple polytope P^n we may write $(T^m \times P^n)/\!\sim$ instead. The latter T^m-manifold is the one introduced by Davis and Januszkiewicz [11], which thereby coincides with our moment-angle complex.

3.2. **Homotopy fibre construction.** The classifying space for the circle S^1 can be identified with the infinite-dimensional projective space $\mathbb{C}P^\infty$. The classifying space BT^m of the m-torus is a product of m copies of $\mathbb{C}P^\infty$. The cohomology of BT^m is the polynomial ring $\mathbb{Z}[v_1, \ldots, v_m]$, $\deg v_i = 2$ (the cohomology is taken with integer coefficients, unless another coefficient ring is explicitly specified). The total space ET^m of the universal principal T^m-bundle over BT^m can be identified with the product of m infinite-dimensional spheres.

In [11] Davis and Januszkiewicz considered the *homotopy quotient* of $\mathcal{Z}_K$ by the T^m-action (also known as the *Borel construction*). We refer to it as the *Davis–Januszkiewicz space*:

$$DJ(K) := ET^m \times_{T^m} \mathcal{Z}_K = ET^m \times \mathcal{Z}_K/\!\sim,$$

where $(e, z) \sim (et^{-1}, tz)$. There is a a fibration $p\colon DJ(K) \to BT^m$ with fibre $\mathcal{Z}_K$. The cohomology of the Borel construction of a T^m-space X is called the *equivariant cohomology* and denoted by $H^*_{T^m}(X)$.

A theorem of [11] states that the cohomology ring of $DJ(K)$ (or the equivariant cohomology of $\mathcal{Z}_K$) is isomorphic to $\mathbb{Z}[K]$. This result can be clarified by an alternative construction of $DJ(K)$ [8], which we review below.

The space BT^m has the canonical cell decomposition in which each factor $\mathbb{C}P^\infty$ has one cell in every even dimension. Given a subset $\omega \subseteq [m]$, define the subproduct

$$BT^\omega := \{(x_1, \ldots, x_m) \in BT^m \colon x_i = * \text{ if } i \notin \omega\}$$

where $*$ is the basepoint (zero-cell) of $\mathbb{C}P^\infty$. Now for a simplicial complex K on $[m]$ define the following cellular subcomplex:

$$BT^K := \bigcup_{\sigma\in K} BT^\sigma \subseteq BT^m. \tag{3.3}$$

Proposition 3.5. *The cohomology of BT^K is isomorphic to the Stanley–Reisner ring $\mathbb{Z}[K]$. Moreover, the inclusion of cellular complexes $i\colon BT^K \hookrightarrow BT^m$ induces the quotient epimorphism*

$$i^*\colon \mathbb{Z}[v_1,\dots,v_m] \to \mathbb{Z}[K] = \mathbb{Z}[v_1,\dots,v_m]/\mathcal{I}_K$$

in the cohomology.

Proof. Let B_i^{2k} denote the $2k$-dimensional cell in the ith factor of BT^m, and $C^*(BT^m)$ the cellular cochain module. A monomial $v_{i_1}^{k_1}\dots v_{i_p}^{k_p}$ represents the cellular cochain $(B_{i_1}^{2k_1}\dots B_{i_p}^{2k_p})^*$ in $C^*(BT^m)$. Under the cochain homomorphism induced by the inclusion $BT^K \subset BT^m$ the cochain $(B_{i_1}^{2k_1}\dots B_{i_p}^{2k_p})^*$ maps identically if $\{i_1,\dots,i_p\}\in K$ and to zero otherwise, whence the statement follows. $\square$

Theorem 3.6. *There is a deformation retraction $DJ(K) \to BT^K$ such that the diagram*

$$\begin{array}{ccc} DJ(K) & \xrightarrow{p} & BT^m \\ \downarrow & & \| \\ BT^K & \xrightarrow{i} & BT^m \end{array}$$

is commutative.

Proof. We have $\mathcal{Z}_K = \bigcup_{\sigma\in K} B_\sigma$, and each B_σ is T^m-invariant. Hence, there is the corresponding decomposition of the Borel construction:

$$DJ(K) = ET^m \times_{T^m} \mathcal{Z}_K = \bigcup_{\sigma\in K} ET^m \times_{T^m} B_\sigma.$$

Suppose $|\sigma| = s$. Then $B_\sigma \cong (D^2)^s \times T^{m-s}$, so we have

$$ET^m \times_{T^m} B_\sigma \cong (ET^s \times_{T^s} (D^2)^s) \times ET^{m-s}.$$

The space $ET^s \times_{T^s} (D^2)^s$ is the total space of a $(D^2)^s$-bundle over BT^s, and ET^{m-s} is contractible. It follows that there is a deformation retraction $ET^m \times_{T^m} B_\sigma \to BT^\sigma$. These homotopy equivalences corresponding to different simplices fit together to yield the required homotopy equivalence between $p\colon DJ(K) \to BT^m$ and $i\colon BT^K \hookrightarrow BT^m$. $\square$

Corollary 3.7. *The space $\mathcal{Z}_K$ is the homotopy fibre of the cellular inclusion $i\colon BT^K \hookrightarrow BT^m$. Hence* [11] *there are ring isomorphisms*

$$H^*(DJ(K)) = H^*_{T^m}(\mathcal{Z}_K) \cong \mathbb{Z}[K].$$

In view of the last two statements we shall also use the notation $DJ(K)$ for BT^K, and refer to the whole class of spaces homotopy equivalent to $DJ(K)$ as the *Davis–Januszkiewicz homotopy type.*

An important question arises: to what extent does the isomorphism of the cohomology ring of a space X with the face ring $\mathbb{Z}[K]$ determine the homotopy type of X? In other words, for given K, does there exist a 'fake' Davis–Januszkiewicz space, whose cohomology is isomorphic to $\mathbb{Z}[K]$, but which is not homotopy equivalent to $DJ(K)$? This question is addressed in [21]. It is shown there [21, Prop. 5.11] that if $\mathbb{Q}[K]$ is a *complete intersection ring* and X is a nilpotent cell complex of finite type whose rational cohomology is isomorphic to $\mathbb{Q}[K]$, then X is rationally homotopy equivalent to $DJ(K)$. Using the formality of $DJ(K)$, this can be rephrased by saying that the complete intersection face rings are *intrinsically formal* in the sense of Sullivan.

Note that the class of simplicial complexes K for which the face ring $\mathbb{Q}[K]$ is a complete intersection has a transparent geometrical interpretation: such K is a join of simplices and boundaries of simplices.

3.3. **Coordinate subspace arrangements.** Yet another interpretation of the moment-angle complex $\mathcal{Z}_K$ comes from its identification up to homotopy with the complement of the complex coordinate subspace arrangement corresponding to K. This leads to an application of toric topology in the theory of arrangements, and allows us to describe and effectively calculate the cohomology rings of coordinate subspace arrangement complements and in certain cases identify their homotopy types.

A *coordinate subspace* in $\mathbb{C}^m$ can be written as

$$L_\omega = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon z_{i_1} = \cdots = z_{i_k} = 0\} \tag{3.4}$$

for some subset $\omega = \{i_1, \ldots, i_k\} \subseteq [m]$. Given a simplicial complex K, we may define the corresponding *coordinate subspace arrangement* $\{L_\omega \colon \omega \notin K\}$ and its *complement*

$$U(K) = \mathbb{C}^m \setminus \bigcup_{\omega \notin K} L_\omega.$$

Note that if $K' \subset K$ is a subcomplex, then $U(K') \subset U(K)$. It is easy to see [8, Prop. 8.6] that the assignment $K \mapsto U(K)$ defines a one-to-one order preserving correspondence between the set of simplicial complexes on $[m]$ and the set of coordinate subspace arrangement complements in $\mathbb{C}^m$.

The subset $U(K) \subset \mathbb{C}^m$ is invariant with respect to the coordinatewise T^m-action. It follows from (3.1) that $\mathcal{Z}_K \subset U(K)$.

Proposition 3.8. *There is a T^m-equivariant deformation retraction*

$$U(K) \xrightarrow{\simeq} \mathcal{Z}_K.$$

Proof. In analogy with (3.3), we may write

$$U(K) = \bigcup_{\sigma \in K} U_\sigma, \tag{3.5}$$

where

$$U_\sigma := \{(z_1, \ldots, z_m) \in \mathbb{C}^m \colon z_i \neq 0 \text{ for } i \notin \sigma\}.$$

Then there are obvious homotopy equivalences (deformation retractions)

$$\mathbb{C}^\sigma \times (\mathbb{C} \setminus 0)^{[m]\setminus\sigma} \cong U_\sigma \xrightarrow{\simeq} B_\sigma \cong (D^2)^\sigma \times (S^1)^{[m]\setminus\sigma}.$$

These patch together to get the required map $U(K) \to \mathcal{Z}_K$. □

Example 3.9. 1. Let $K = \partial\Delta^{m-1}$. Then $U(K) = \mathbb{C}^m \setminus 0$ (recall that $\mathcal{Z}_K \cong S^{2m-1}$ in this case).

2. Let $K = \{v_1, \dots, v_m\}$ (m points). Then
$$U(K) = \mathbb{C}^m \setminus \bigcup_{1 \le i < j \le m} \{z_i = z_j = 0\},$$
the complement to the set of all codimension 2 coordinate planes.
3. More generally, if K is the i-skeleton of $\varDelta^{m-1}$, then $U(K)$ is the complement to the set of all coordinate planes of codimension $(i+2)$.

The reader may have noticed a similar pattern in several constructions of toric spaces appeared above; compare (3.1), (3.3) and (3.5). The following general framework was suggested to the author by Neil Strickland in a private communication.

Construction 3.10 (K-power)**.** Let X be a space and $W \subset X$ a subspace. For a simplicial complex K on $[m]$ and $\sigma \in K$, we set
$$(X, W)^\sigma := \bigl\{(x_1, \dots, x_m) \in X^m \colon x_j \in W \text{ for } j \notin \sigma\bigr\}$$
and
$$(X, W)^K := \bigcup_{\sigma \in K} (X, W)^\sigma = \bigcup_{\sigma \in K} \Bigl(\prod_{i \in \sigma} X \times \prod_{i \notin \sigma} W\Bigr).$$
We refer to the space $(X, W)^K \subseteq X^m$ as the *K-power* of (X, W). If X is a pointed space and $W = pt$ is the basepoint, then we shall use the abbreviated notation $X^K := (X, pt)^K$. Examples considered above include $\mathcal{Z}_K = (D^2, S^1)^K$, $\mathrm{cc}(K) = (I^1, S^0)^K$, $DJ(K) = (\mathbb{C}P^\infty)^K$ and $U(K) = (\mathbb{C}, \mathbb{C}^*)^K$.

Homotopy theorists would recognise the K-power as an example of the *colimit* of a diagram of topological spaces over the *face category* of K (objects are simplices and morphisms are inclusions). The diagram assigns the space $(X, W)^\sigma$ to a simplex σ; its colimit is $(X, W)^K$. These observations are further developed and used to construct models of loop spaces of toric spaces as well as for homotopy and homology calculations in [23] and [22].

3.4. **Toric varieties, quasitoric manifolds, and torus manifolds.** Several important classes of manifolds with torus action emerge as the quotients of moment-angle complexes by appropriate freely acting subtori.

First we give the following characterisation of lsops in the face ring. Let K^{n-1} be a simplicial complex and $t_1, \dots, t_n$ a sequence of degree-two elements in $\mathbf{k}[K]$. We may write

$$t_i = \lambda_{i1} v_1 + \dots + \lambda_{im} v_m, \quad i = 1, \dots, n. \tag{3.6}$$

For an arbitrary simplex $\sigma \in K$, we have $K_\sigma = \Delta^{|\sigma|-1}$ and $\mathbf{k}[K_\sigma]$ is the polynomial ring $\mathbf{k}[v_i \colon i \in \sigma]$ on $|\sigma|$ generators. The inclusion $K_\sigma \subset K$ induces the *restriction homomorphism* r_σ from $\mathbf{k}[K]$ to the polynomial ring, mapping v_i identically if $i \in \sigma$ and to zero otherwise.

Lemma 3.11. *A degree-two sequence $t_1, \dots, t_n$ is an lsop in $\mathbf{k}[K^{n-1}]$ if and only if for every $\sigma \in K$ the elements $r_\sigma(t_1), \dots, r_\sigma(t_n)$ generate the positive ideal $\mathbf{k}[v_i \colon i \in \sigma]_+$.*

Proof. Suppose (3.6) is an lsop. For simplicity we denote its image under any restriction homomorphism by the same letters. Then the restriction induces an epimorphism of the quotient rings:

$$\mathbf{k}[K]/(t_1,\dots,t_n)\to\mathbf{k}[v_i\colon i\in\sigma]/(t_1,\dots,t_n).$$

Since (3.6) is an lsop, $\mathbf{k}[K]/(t_1,\dots,t_n)$ is a finitely generated $\mathbf{k}$-module. Hence, so is $\mathbf{k}[v_i\colon i\in\sigma]/(t_1,\dots,t_n)$. But the latter can be finitely generated only if $t_1,\dots,t_n$ generates $\mathbf{k}[v_i\colon i\in\sigma]_+$.

The "if" part may be proved by considering the sum of restrictions:

$$\mathbf{k}[K]\to\bigoplus_{\sigma\in K}\mathbf{k}[v_i\colon i\in\sigma],$$

which turns out to be a monomorphism. See [6, Th. 5.1.16] for details. □

Obviously, it is enough to consider only restrictions to the maximal simplices in the previous lemma.

Suppose now that K is Cohen–Macaulay (e.g. K is a sphere triangulation). Then every lsop is a regular sequence (however, for $\mathbf{k}=\mathbb{Z}$ or a field of finite characteristic an lsop may fail to exist).

Now we restrict to the case $\mathbf{k}=\mathbb{Z}$ and organise the coefficients in (3.6) into an $n\times m$-matrix $\Lambda=(\lambda_{ij})$. For an arbitrary maximal simplex $\sigma\in K$ denote by Λ_σ the square submatrix formed by the elements λ_{ij} with $j\in\sigma$. The matrix Λ defines a linear map $\mathbb{Z}^m\to\mathbb{Z}^n$ and a homomorphism $T^m\to T^n$. We denote both by λ and denote the kernel of the latter map by T_Λ.

Theorem 3.12. *The following conditions are equivalent:*
(a) *the sequence* (3.6) *is an lsop in* $\mathbb{Z}[K^{n-1}]$*;*
(b) $\det\Lambda_\sigma=\pm1$ *for every maximal simplex* $\sigma\in K$*;*
(c) $T_\Lambda\cong T^{m-n}$ *and* T_Λ *acts freely on* $\mathcal{Z}_K$*.*

Proof. The equivalence of (a) and (b) is a reformulation of Lemma 3.11. Let us prove the equivalence of (b) and (c). Every isotropy subgroup of the T^m-action on $\mathcal{Z}_K$ has the form

$$T^\sigma=\bigl\{(z_1,\dots,z_m)\in T^m\colon z_i=1\text{ if }i\notin\sigma\bigr\}$$

for some simplex $\sigma\in K$. Now, (b) is equivalent to the condition $T_\Lambda\cap T^\sigma=\{e\}$ for arbitrary maximal σ, whence the statement follows. □

We denote the quotient $\mathcal{Z}_K/T_\Lambda$ by $M_K^{2n}(\Lambda)$, and abbreviate it to M_K^{2n} or to M^{2n} when the context allows. If K is a triangulated sphere, then $\mathcal{Z}_K$ is a manifold, hence, so is M_K^{2n}. The n-torus $T^n=T^m/T_\Lambda$ acts on M_K^{2n}. This construction produces two important classes of T^n-manifolds as particular examples.

Let $K=K_P$ be a polytopal triangulation, dual to the boundary complex of a simple polytope P. Then the map λ determined by the matrix Λ may be regarded as an assignment of an integer vector to every facet of P. The map λ coming from a matrix satisfying the condition of Theorem 3.12(b) was called a *characteristic map* by Davis and Januszkiewicz [11]. We refer to the corresponding quotient $M_P^{2n}(\Lambda)=\mathcal{Z}_{K_P}/T_\Lambda$ as a *quasitoric manifold* (a toric manifold in the terminology of Davis–Januszkiewicz).

Let us assume further that P is realised in $\mathbb{R}^n$ with integer coordinates of vertices, so we can write

$$P^n = \big\{ \boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{l}_i, \boldsymbol{x} \rangle \geq -a_i,\ i = 1, \dots, m \big\}, \tag{3.7}$$

where $\boldsymbol{l}_i$ are inward pointing normals to the facets of P^n (we may further assume these vectors to be primitive), and $a_i \in \mathbb{Q}$. Let Λ be the matrix formed by the column vectors $\boldsymbol{l}_i$, $i = 1, \dots, m$. Then $\mathcal{Z}_{K_P}/T_\Lambda$ can be identified with the *projective toric variety* [10, 13] determined by the polytope P. The condition of Theorem 3.12(b) is equivalent to the requirement that the toric variety is non-singular. Thereby a non-singular projective toric variety is a quasitoric manifold (but there are many quasitoric manifolds which are not toric varieties).

We also note that smooth projective toric varieties provide examples of *symplectic* $2n$-dimensional manifolds with *Hamiltonian* T^n-action. These symplectic manifolds can be obtained via the process of *symplectic reduction* from the standard Hamiltonian T^m-action on $\mathbb{C}^m$. A choice of an $(m-n)$-dimensional toric subgroup provides a *moment map* $\mu\colon \mathbb{C}^m \to \mathbb{R}^{m-n}$, and the corresponding moment-angle complex $\mathcal{Z}_{K_P}$ can be identified with the level surface $\mu^{-1}(a)$ of the moment map for any of its regular values a. The details of this construction can be found in [8, p. 130].

Finally, we mention that if K is an arbitrary (not necessarily polytopal) triangulation of sphere, then the manifold $M_K^{2n}(\Lambda)$ is a *torus manifold* in the sense of Hattori–Masuda [19]. The corresponding multi-fan has K as the underlying simplicial complex. This particular class of torus manifolds has many interesting properties.

4. Cohomology of moment-angle complexes

The main result of this section (Theorem 4.7) identifies the *integral* cohomology algebra of the moment-angle complex $\mathcal{Z}_K$ with the Tor-algebra of the face ring of the simplicial complex K. Over the rationals this result was proved in [7] by studying the Eilenberg–Moore spectral sequence of the fibration $\mathcal{Z}_K \to DJ(K) \to BT^m$; a more detailed account of applications of the Eilenberg–Moore spectral sequence to toric topology can be found in [8]. The new proof, which works with integer coefficients as well, relies upon a construction of a special cellular decomposition of $\mathcal{Z}_K$ and subsequent analysis of the corresponding cellular cochains.

One of the key ingredients here is a specific cellular approximation of the diagonal map $\Delta\colon \mathcal{Z}_K \to \mathcal{Z}_K \times \mathcal{Z}_K$. Cellular cochains do not admit a functorial associative multiplication because a proper cellular diagonal approximation does not exist in general. The construction of moment-angle complexes is given by a functor from the category of simplicial complexes to the category of spaces with a torus action. We show that in this special case the cellular approximation of the diagonal is functorial with respect to those maps of moment-angle complexes which are induced by simplicial maps. The corresponding cellular cochain algebra is isomorphic to a quotient of the Koszul complex for $\mathbf{k}[K]$ by an acyclic ideal, and its cohomology is isomorphic to the Tor-algebra. The proofs have been sketched in [5]; here we follow the more detailed exposition of [9]. Another proof of Theorem 4.7 follows from a recent independent work of M. Franz [12, Th. 1.2].

4.1. **Cell decomposition.** The polydisc $(D^2)^m$ has a cell decomposition in which each D^2 is subdivided into cells 1, T and D of dimensions 0, 1 and 2 respectively, see Figure 3. Each cell of this complex is a product of cells of 3

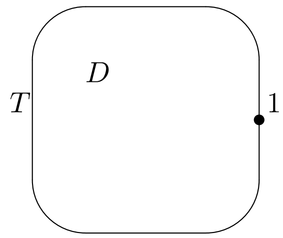


FIGURE 3

different types and we encode it by a word $\mathcal{T} \in \{D, T, 1\}^m$ in a three-letter alphabet. Assign to each pair of subsets $\sigma, \omega \subseteq [m]$, $\sigma \cap \omega = \varnothing$, the word $\mathcal{T}(\sigma, \omega)$ which has the letter D on the positions indexed by σ and letter T on the positions with indices from ω.

Lemma 4.1. *$\mathcal{Z}_K$ is a cellular subcomplex of $(D^2)^m$. A cell $\mathcal{T}(\sigma, \omega) \subset (D^2)^m$ belongs to $\mathcal{Z}_K$ if and only if $\sigma \in K$.*

Proof. We have $\mathcal{Z}_K = \cup_{\sigma \in K} B_\sigma$ and each B_σ is the closure of the cell $\mathcal{T}(\sigma, [m] \setminus \sigma)$. □

Therefore, we can consider the cellular cochain complex $C^*(\mathcal{Z}_K)$, which has an additive basis consisting of the cochains $\mathcal{T}(\sigma, \omega)^*$. It has a natural bigrading defined by

$$\operatorname{bideg} \mathcal{T}(\sigma, \omega)^* = (-|\omega|, 2|\sigma| + 2|\omega|),$$

so $\operatorname{bideg} D = (0, 2)$, $\operatorname{bideg} T = (-1, 2)$ and $\operatorname{bideg} 1 = (0, 0)$. Moreover, since the cellular differential does not change the second grading, $C^*(\mathcal{Z}_K)$ splits into the sum of its components having fixed second degree:

$$C^*(\mathcal{Z}_K) = \bigoplus_{j=1}^{m} C^{*,2j}(\mathcal{Z}_K).$$

The cohomology of $\mathcal{Z}_K$ thereby acquires an additional grading, and we may define the *bigraded Betti numbers* $b^{-i,2j}(\mathcal{Z}_K)$ by

$$b^{-i,2j}(\mathcal{Z}_K) := \operatorname{rank} H^{-i,2j}(\mathcal{Z}_K), \quad i, j = 1, \ldots, m.$$

For the ordinary Betti numbers we have $b^k(\mathcal{Z}_K) = \sum_{2j-i=k} b^{-i,2j}(\mathcal{Z}_K)$.

Lemma 4.2. *Let $\varphi \colon K_1 \to K_2$ be a simplicial map between simplicial complexes on the sets $[m_1]$ and $[m_2]$ respectively. Then there is an equivariant cellular map $\varphi_{\mathcal{Z}} \colon \mathcal{Z}_{K_1} \to \mathcal{Z}_{K_2}$ covering the induced map $|\operatorname{cone} K_1'| \to |\operatorname{cone} K_2'|$.*

Proof. Define a map of polydisks

$$\varphi_D\colon (D^2)^{m_1} \to (D^2)^{m_2}, \quad (z_1, \dots, z_{m_1}) \mapsto (w_1, \dots, w_{m_2}),$$

where

$$w_j := \prod_{i \in \varphi^{-1}(j)} z_i, \qquad j = 1, \dots, m_2$$

(we set $w_j = 1$ if $\varphi^{-1}(j) = \varnothing$). Assume $\tau \in K_1$. In the notation of (3.1), we have $\varphi_D(B_\tau) \subseteq B_{\varphi(\tau)}$. Since φ is a simplicial map, $\varphi(\tau) \in K_2$ and $B_{\varphi(\tau)} \subset \mathcal{Z}_{K_2}$, so the restriction of φ_D to $\mathcal{Z}_{K_1}$ is the required map. □

Corollary 4.3. *The correspondence $K \mapsto \mathcal{Z}_K$ gives rise to a functor from the category of simplicial complexes and simplicial maps to the category of spaces with torus actions and equivariant maps. It induces a natural transformation between the simplicial cochain functor of K and the cellular cochain functor of $\mathcal{Z}_K$.*

We also note that the maps respect the bigrading, so the bigraded Betti numbers are also functorial.

4.2. **Koszul algebras.** Our algebraic model for the cellular cochains of $\mathcal{Z}_K$ is obtained by taking the quotient of the Koszul algebra $[\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[K], d]$ from Lemma 2.11 by a certain acyclic ideal. Namely, we introduce a factor algebra

$$R^*(K) := \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K] / (v_i^2 = u_i v_i = 0,\ i = 1, \dots, m),$$

where the differential and bigrading are as in (2.4). Let

$$\varrho\colon \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K] \to R^*(K)$$

be the quotient projection. The algebra $R^*(K)$ has a finite additive basis consisting of the monomials of the form $u_\omega v_\sigma$ where $\omega \subseteq [m]$, $\sigma \in K$ and $\omega \cap \sigma = \varnothing$ (remember that we are using the notation $u_\omega = u_{i_1} \dots u_{i_k}$ for $\omega = \{i_1, \dots, i_k\}$). Therefore, we have an additive inclusion (a monomorphism of bigraded differential modules)

$$\iota\colon R^*(K) \to \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]$$

which satisfies $\varrho \cdot \iota = \mathrm{id}$.

The following statement shows that the finite-dimensional quotient $R^*(K)$ has the same cohomology as the Koszul algebra.

Lemma 4.4. *The quotient map $\varrho\colon \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K] \to R^*(K)$ induces an isomorphism in cohomology.*

Proof. The argument is similar to that used in the proof of the acyclicity of the Koszul resolution. We construct a cochain homotopy between the maps id and $\iota \cdot \varrho$ from $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]$ to itself, that is, a map s satisfying

$$ds + sd = \mathrm{id} - \iota \cdot \varrho. \tag{4.1}$$

First assume that $K = \Delta^{m-1}$. We denote the corresponding bigraded algebra $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]$ by

$$E = E_m := \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[v_1, \dots, v_m], \tag{4.2}$$

while $R^*(K)$ is isomorphic to

$$(4.3)\qquad \big(\Lambda[u]\otimes\mathbb{Z}[v]/(v^2=uv=0)\big)^{\otimes m}=R^*(\varDelta^0)^{\otimes m}.$$

For $m=1$, the map $s_1\colon E^{0,*}=\mathbf{k}[v]\to E^{-1,*}$ given by

$$s_1(a_0+a_1v+\ldots+a_jv^j)=(a_2v+a_3v^2+\ldots+a_jv^{j-1})u$$

is a cochain homotopy. Indeed, we can write an element of E as either x or xu with $x=a_0+a_1v+\ldots+a_jv^j\in E^{0,2j}$. In the former case, $ds_1x=x-a_0-a_1v=x-\iota\varrho x$ and $s_1dx=0$. In the latter case, $xu\in E^{-1,2j}$, then $ds_1(xu)=0$ and $s_1d(xu)=xu-a_0u=xu-\iota\varrho(xu)$. In both cases (4.1) holds. Now we may assume by induction that for $m=k-1$ there is a cochain homotopy operator $s_{k-1}\colon E_{k-1}\to E_{k-1}$. Since $E_k=E_{k-1}\otimes E_1$, $\varrho_k=\varrho_{k-1}\otimes\varrho_1$ and $\iota_k=\iota_{k-1}\otimes\iota_1$, a direct check shows that the map

$$s_k=s_{k-1}\otimes\mathrm{id}+\iota_{k-1}\varrho_{k-1}\otimes s_1$$

is a cochain homotopy between id and $\iota_k\varrho_k$, which finishes the proof for $K=\varDelta^{m-1}$.

In the case of arbitrary K the algebras $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[K]$ are $R^*(K)$ are obtained from (4.2) and (4.3) respectively by factoring out the Stanley–Reisner ideal $\mathcal{I}_K$. This factorisation does not affect the properties of the constructed map s, which finishes the proof. $\square$

Now comparing the additive structure of $R^*(K)$ with that of the cellular cochains $C^*(K)$, we see that the two coincide:

Lemma 4.5. *The map*

$$g\colon R^*(K)\to C^*(\mathcal{Z}_K),$$
$$u_\omega v_\sigma\mapsto\mathcal{T}(\sigma,\omega)^*$$

is an isomorphism of bigraded differential modules. In particular, we have an additive isomorphism

$$H[R^*(K)]\cong H^*(\mathcal{Z}_K).$$

Having identified the algebra R^* with the cellular cochains of $\mathcal{Z}_K$, we can also interpret the cohomology isomorphism from Lemma 4.4 topologically. To do this we shall identify the Koszul algebra $\Lambda[u_1,\ldots,u_m]\otimes\mathbb{Z}[K]$ with the cellular cochains of a space homotopy equivalent to $\mathcal{Z}_K$.

Let S^∞ be an infinite-dimensional sphere obtained as a direct limit (union) of standardly embedded odd-dimensional spheres. The space S^∞ is contractible and has a cell decomposition with one cell in every dimension. The boundary of an even-dimensional cell is the closure of the appropriate odd-dimensional cell, while the boundary of an odd cell is zero. The 2-skeleton of this cell decomposition is a 2-disc decomposed as shown on Figure 3, while the 1-skeleton is the circle $S^1\subset S^\infty$. The cellular cochain complex of S^∞ can be identified with the algebra

$$\Lambda[u]\otimes\mathbb{Z}[v],\quad \deg u=1,\deg v=2,\quad du=v,dv=0.$$

From the obvious functorial properties of Construction 3.10 we obtain a deformation retraction

$$\mathcal{Z}_K=(D^2,S^1)^K\hookrightarrow(S^\infty,S^1)^K\longrightarrow(D^2,S^1)^K$$

onto a cellular subcomplex.

The cellular cochains of the K-power $(S^\infty, S^1)^K$ can be identified with the Koszul algebra $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K]$. Since $\mathcal{Z}_K \subset (S^\infty, S^1)^K$ is a deformation retract, the cellular cochain map

$$\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[K] = C^*\big((S^\infty, S^1)^K\big) \to C^*(\mathcal{Z}_K) = R^*(K),$$

induces an isomorphism in cohomology. In fact, the algebraic homotopy map s constructed in the proof of Lemma 4.4 is the map induced on the cochains by the topological homotopy.

4.3. **Cellular cochain algebras.** Here we introduce a multiplication for cellular cochains of $\mathcal{Z}_K$ and establish a ring isomorphism in Lemma 4.5. This task runs into a complication because cellular cochains in general do not carry a functorial associative multiplication; the classical definition of the cohomology multiplication involves a diagonal map, which is not cellular. However, in our case there is a way to construct a canonical cellular approximation of the diagonal map $\Delta\colon \mathcal{Z}_K \to \mathcal{Z}_K \times \mathcal{Z}_K$ in such a way that the resulting multiplication in cellular cochains coincides with that in $R^*(K)$.

The standard definition of the multiplication in cohomology of a cell complex X via cellular cochains is as follows. Consider a composite map of cellular cochain complexes:

$$C^*(X) \otimes C^*(X) \xrightarrow{\times} C^*(X \times X) \xrightarrow{\widetilde{\Delta}^*} C^*(X). \tag{4.4}$$

Here the map $\times$ assigns to a cellular cochain $c_1 \otimes c_2 \in C^{q_1}(X) \otimes C^{q_2}(X)$ the cochain $c_1 \times c_2 \in C^{q_1+q_2}(X \times X)$ whose value on a cell $e_1 \times e_2 \in X \times X$ is $(-1)^{q_1 q_2} c_1(e_1) c_2(e_2)$. The map $\widetilde{\Delta}^*$ is induced by a cellular approximation $\widetilde{\Delta}$ of the diagonal map $\Delta\colon X \to X \times X$. In cohomology, the map (4.4) induces a multiplication $H^*(X) \otimes H^*(X) \to H^*(X)$ which does not depend on a choice of cellular approximation and is functorial. However, the map (4.4) is not itself functorial because of the arbitrariness in the choice of a cellular approximation.

In the special case $X = \mathcal{Z}_K$ we may apply the following construction. Consider a map $\widetilde{\Delta}\colon D^2 \to D^2 \times D^2$, defined in polar coordinates $z = \rho e^{i\varphi} \in D^2$, $0 \le \rho \le 1$, $0 \le \varphi < 2\pi$ as follows:

$$\rho e^{i\varphi} \mapsto \begin{cases} (1 + \rho(e^{2i\varphi} - 1), 1) & \text{for } 0 \le \varphi \le \pi, \\ (1, 1 + \rho(e^{2i\varphi} - 1)) & \text{for } \pi \le \varphi < 2\pi. \end{cases}$$

This is a cellular map taking ∂D^2 to $\partial D^2 \times \partial D^2$ and homotopic to the diagonal $\Delta\colon D^2 \to D^2 \times D^2$ in the class of such maps. Taking an m-fold product, we obtain a cellular approximation

$$\widetilde{\Delta}\colon (D^2)^m \to (D^2)^m \times (D^2)^m$$

which restricts to a cellular approximation for the diagonal map of $\mathcal{Z}_K$ for arbitrary K, as described by the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Z}_K & \longrightarrow & (D^2)^m \\ {\scriptstyle\widetilde{\Delta}}\big\downarrow & & \big\downarrow{\scriptstyle\widetilde{\Delta}} \\ \mathcal{Z}_K \times \mathcal{Z}_K & \longrightarrow & (D^2)^m \times (D^2)^m \end{array}.$$

Note that this diagonal approximation is functorial with respect to those maps $\mathcal{Z}_{K_1} \to \mathcal{Z}_{K_2}$ of moment-angle complexes that are induced by simplicial maps $K_1 \to K_2$ (see Lemma 4.2).

Lemma 4.6. *The cellular cochain algebra $C^*(\mathcal{Z}_K)$ defined by the diagonal approximation $\widetilde{\Delta}\colon \mathcal{Z}_K \to \mathcal{Z}_K \times \mathcal{Z}_K$ and* (4.4) *is isomorphic to $R^*(K)$. Therefore, we get an isomorphism of the cohomology algebras:*

$$H[R^*(K)] \cong H^*(\mathcal{Z}_K; \mathbb{Z}).$$

Proof. We first consider the case $K = \Delta^0$, that is, $\mathcal{Z}_K = D^2$. The cellular cochain complex of D^2 is additively generated by the cochains $1 \in C^0(D^2)$, $T^* \in C^1(D^2)$ and $D^* \in C^2(D^2)$ dual to the corresponding cells, see Figure 3. The multiplication defined in $C^*(D^2)$ by (4.4) is trivial, so we get a multiplicative isomorphism

$$R^*(\Delta^0) = \Lambda[u] \otimes \mathbb{Z}[v]/(v^2 = uv = 0) \to C^*(D^2).$$

Now, for $K = \Delta^{m-1}$ we obtain a multiplicative isomorphism

$$f\colon R^*(\Delta^{m-1}) = \Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[v_1, \ldots, v_m]/(v_i^2 = u_i v_i = 0) \to C^*((D^2)^m)$$

by taking the tensor product. Since $\mathcal{Z}_K \subseteq (D^2)^m$ is a cell subcomplex for arbitrary K we obtain a multiplicative map $q\colon C^*((D^2)^m) \to C^*(\mathcal{Z}_K)$. Now consider the commutative diagram

$$\begin{tikzcd} R^*(\Delta^{m-1}) \arrow[r, "f"] \arrow[d, "p"'] & C^*((D^2)^m) \arrow[d, "q"] \\ R^*(K) \arrow[r, "g"] & C^*(\mathcal{Z}_K). \end{tikzcd}$$

Here the maps p, f and q are multiplicative, while g is an additive isomorphism by Lemma 4.5. Take $\alpha, \beta \in R^*(K)$. Since p is onto, we have $\alpha = p(\alpha')$ and $\beta = p(\beta')$. Then

$$g(\alpha\beta) = gp(\alpha'\beta') = qf(\alpha'\beta') = gp(\alpha')gp(\beta') = g(\alpha)g(\beta),$$

and g is also a multiplicative isomorphism, which finishes the proof. $\square$

Combining the results of Lemmas 2.11, 2.12, 4.4 and 4.6, we come to the main result of this section.

Theorem 4.7. *There is an isomorphism, functorial in K, of bigraded algebras*

$$H^{*,*}(\mathcal{Z}_K; \mathbb{Z}) \cong \operatorname{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}\bigl(\mathbb{Z}[K], \mathbb{Z}\bigr) \cong H\bigl[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K], d\bigr],$$

where the bigrading and the differential in the last algebra are defined by (2.4).

As an illustration, we give two examples of particular cohomology calculations, which have a transparent geometrical interpretation. More examples of calculations may be found in [8].

Example 4.8. 1. Let $K = \partial\Delta^{m-1}$. Then

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \ldots, v_m]/(v_1 \cdots v_m).$$

The fundamental class of $\mathcal{Z}_K \cong S^{2m-1}$ is represented by the bideg $(-1, 2m)$ cocycle $u_1 v_2 v_3 \cdots v_m \in \Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K]$.

2. Let $K = \{v_1, \dots, v_m\}$ (m points). Then $\mathcal{Z}_K$ is homotopy equivalent to the complement in $\mathbb{C}^m$ to the set of all codimension-two coordinate planes, see Example 3.9. Then

$$\mathbb{Z}[K] = \mathbb{Z}[v_1, \dots, v_m]/(v_i v_j,\ i \neq j).$$

The subspace of cocycles in $R^*(K)$ is generated by

$$v_{i_1} u_{i_2} u_{i_3} \cdots u_{i_k}, \quad k \geq 2 \text{ and } i_p \neq i_q \text{ for } p \neq q,$$

and has dimension $m\binom{m-1}{k-1}$. The subspace of coboundaries is generated by the elements of the form

$$d(u_{i_1} \cdots u_{i_k})$$

and is $\binom{m}{k}$-dimensional. Therefore

$$\begin{aligned} &\dim H^0(\mathcal{Z}_K) = 1,\\ &\dim H^1(\mathcal{Z}_K) = H^2(U(K)) = 0,\\ &\dim H^{k+1}(\mathcal{Z}_K) = m\tbinom{m-1}{k-1} - \tbinom{m}{k} = (k-1)\tbinom{m}{k}, \quad 2 \leq k \leq m, \end{aligned}$$

and multiplication in the cohomology of $\mathcal{Z}_K$ is trivial. Note that in general multiplication in the cohomology of $\mathcal{Z}_K$ is far from being trivial; for example if K is a sphere triangulation then $\mathcal{Z}_K$ is a manifold by Lemma 3.3.

The above cohomology calculation suggests that the complement of the subspace arrangement from the previous example is homotopy equivalent to a wedge of spheres. This is indeed the case, as the following theorem shows.

Theorem 4.9 (Grbić–Theriault [16]). *The complement of the set of all codimension-two coordinate subspaces in $\mathbb{C}^m$ has the homotopy type of the wedge of spheres*

$$\bigvee_{k=2}^{m} (k-1)\binom{m}{k} S^{k+1}.$$

The proof is based on an analysis of the homotopy fibre of the inclusion $DJ(K) \hookrightarrow BT^m$, which is homotopy equivalent to $\mathcal{Z}_K$ (or $U(K)$) by Corollary 3.7. We shall return to coordinate subspace arrangements once again in the next section.

5. Applications to combinatorial commutative algebra

5.1. **A multiplicative version of Hochster's theorem.** As a first application we give a proof of a generalisation of Hochster's theorem (Theorem 2.9) obtained by Baskakov in [3].

The bigraded structure in the cellular cochains of $\mathcal{Z}_K$ can be further refined as

$$C^*(\mathcal{Z}_K) = \bigoplus_{\omega \subseteq [m]} C^{*,2\omega}(\mathcal{Z}_K)$$

where $C^{*,2\omega}(\mathcal{Z}_K)$ is the subcomplex generated by the cochains $\mathcal{T}(\sigma, \omega \setminus \sigma)^*$ with $\sigma \subseteq \omega$ and $\sigma \in K$. Thus, $C^*(\mathcal{Z}_K)$ now becomes a $\mathbb{Z} \oplus \mathbb{Z}^m$-graded module, and the bigraded cohomology groups decompose accordingly as

$$H^{-i,2j}(\mathcal{Z}_K) = \bigoplus_{\omega \subseteq [m]\colon |\omega| = j} H^{-i,2\omega}(\mathcal{Z}_K) \tag{5.1}$$

where $H^{-i,2\omega}(\mathcal{Z}_K) := H^{-i}[C^{*,2\omega}(\mathcal{Z}_K)]$.

Given two simplicial complexes K_1 and K_2 with vertex sets V_1 and V_2 respectively, their *join* is the following complex on $V_1 \sqcup V_2$:

$$K_1 * K_2 := \{\sigma \subseteq V_1 \sqcup V_2 \colon \sigma = \sigma_1 \cup \sigma_2,\ \sigma_1 \in K_1,\ \sigma_2 \in K_2\}.$$

Now we introduce a multiplication in the sum

$$\bigoplus_{\substack{p \geq -1,\\ \omega \subseteq [m]}} \widetilde{H}^p(K_\omega)$$

where K_ω is the full subcomplex and $\widetilde{H}^{-1}(\varnothing) = \mathbb{Z}$, as follows. Take two elements $\alpha \in \widetilde{H}^p(K_{\omega_1})$ and $\beta \in \widetilde{H}^q(K_{\omega_2})$. Assume that $\omega_1 \cap \omega_2 = \varnothing$. Then we have an inclusion of subcomplexes

$$i \colon K_{\omega_1 \cup \omega_2} = K_{\omega_1} \sqcup K_{\omega_2} \hookrightarrow K_{\omega_1} * K_{\omega_2}$$

and an isomorphism of reduced simplicial cochains

$$f \colon \widetilde{C}^p(K_{\omega_1}) \otimes \widetilde{C}^q(K_{\omega_2}) \xrightarrow{\cong} \widetilde{C}^{p+q+1}(K_{\omega_1} * K_{\omega_2}).$$

Now set

$$\alpha \cdot \beta := \begin{cases} 0, & \omega_1 \cap \omega_2 \neq \varnothing, \\ i^* f(a \otimes b) \in \widetilde{H}^{p+q+1}(K_{\omega_1 \sqcup \omega_2}), & \omega_1 \cap \omega_2 = \varnothing. \end{cases}$$

Theorem 5.1 (Baskakov [3, Th. 1])**.** *There are isomorphisms*

$$\widetilde{H}^p(K_\omega) \xrightarrow{\cong} H^{p+1-|\omega|,2\omega}(\mathcal{Z}_K)$$

which are functorial with respect to simplicial maps and induce a ring isomorphism

$$\gamma \colon \bigoplus_{\substack{p \geq -1,\\ \omega \subseteq [m]}} \widetilde{H}^p(K_\omega) \xrightarrow{\cong} H^*(\mathcal{Z}_K).$$

Proof. Define a map of cochain complexes

$$\widetilde{C}^*(K_\omega) \to C^{*+1-|\omega|,2\omega}(\mathcal{Z}_K), \quad \sigma^* \mapsto \mathcal{T}(\sigma, \omega \setminus \sigma)^*.$$

It is a functorial isomorphism by observation, whence the isomorphism of the cohomology groups follows.

The statement about the ring isomorphism follows from the isomorphism $H^*(\mathcal{Z}_K) \cong H[R^*(K)]$ established in Lemma 4.5 and analysing the ring structure in $R^*(K)$. □

Corollary 5.2. *There is an isomorphism*

$$H^{-i,2j}(\mathcal{Z}_K) \cong \bigoplus_{\omega \subseteq [m] \colon |\omega| = j} \widetilde{H}^{j-i-1}(K_\omega).$$

As a further corollary we obtain Hochster's theorem (Theorem 2.9):

$$\mathrm{Tor}^{-i,*}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[K], \mathbb{Z}) \cong \bigoplus_{\omega \subseteq [m]} \widetilde{H}^{|\omega|-i-1}(K_\omega).$$

5.2. **Alexander duality and coordinate subspace arrangements revisited.** The multiplicative version of Hochster's can also be applied to cohomology calculations of subspace arrangement complements.

A coordinate subspace can be defined either by setting some coordinates to zero as in (3.4), or as the linear span of a subset of the standard basis in $\mathbb{C}^m$. This gives an alternative way to parametrise coordinate subspace arrangements by simplicial complexes. Namely, we can write

$$\{L_\omega\colon \omega\notin K\}=\{\mathrm{span}\langle e_{i_1},\dots,e_{i_k}\rangle\colon \{i_1,\dots,i_k\}\in\widehat{K}\}$$

where $\widehat{K}$ is the simplicial complex given by

$$\widehat{K}:=\{\omega\subseteq[m]\colon [m]\setminus\omega\notin K\}.$$

It is called the *dual complex* of K. The cohomology of full subcomplexes in K is related to the homology of links in $\widehat{K}$ by means of the following combinatorial version of the Alexander duality theorem.

Theorem 5.3 (Alexander duality)**.** *Let $K\neq\Delta^{m-1}$ be a simplicial complex on the set $[m]$ and $\sigma\notin K$, that is, $\widehat{\sigma}=[m]\setminus\sigma\in\widehat{K}$. Then there are isomorphisms*

$$\widetilde{H}_j(K_\sigma)\cong\widetilde{H}^{|\sigma|-3-j}(\mathrm{link}_{\widehat{K}}\,\widehat{\sigma}).$$

In particular, for $\sigma=[m]$ we get

$$\widetilde{H}_j(K)\cong\widetilde{H}^{m-3-j}(\widehat{K}),\quad -1\le j\le m-2.$$

A proof can be found in [9, §2.2]. Using the duality between the full subcomplexes of K and links of $\widehat{K}$ we can reformulate the cohomology calculation of $U(K)$ as follows.

Proposition 5.4. *We have*

$$\widetilde{H}_i(U(K))\cong\bigoplus_{\sigma\in\widehat{K}}\widetilde{H}^{2m-2|\sigma|-i-2}(\mathrm{link}_{\widehat{K}}\,\sigma).$$

Proof. From Proposition 3.8 and Corollary 5.2 we obtain

$$H_p(U(K))=\bigoplus_{\tau\subseteq[m]}\widetilde{H}_{p-|\tau|-1}(K_\tau).$$

Nonempty simplices $\tau\in K$ do not contribute to the above sum, since the corresponding subcomplexes K_τ are contractible. Since $\widetilde{H}^{-1}(\varnothing)=\mathbf{k}$ the empty subset of $[m]$ only contributes $\mathbf{k}$ to $H^0(U(K))$. Hence we may rewrite the above formula as

$$\widetilde{H}_p(U(K))=\bigoplus_{\tau\notin K}\widetilde{H}_{p-|\tau|-1}(K_\tau).$$

Using Theorem 5.3, we calculate

$$\widetilde{H}_{p-|\tau|-1}(K_\tau)=\widetilde{H}^{|\tau|-3-p+|\tau|+1}(\mathrm{link}_{\widehat{K}}\,\widehat{\tau})=\widetilde{H}^{2m-2|\widehat{\tau}|-p-2}(\mathrm{link}_{\widehat{K}}\,\widehat{\tau}),$$

where $\widehat{\tau}=[m]\setminus\tau$ is a simplex in $\widehat{K}$, as required. $\square$

Proposition 5.4 is a particular case of the well-known *Goresky–Macpherson formula* [15, Part III], which calculates the dimensions of the (co)homology groups of an arbitrary subspace arrangement in terms of its *intersection poset* (which coincides with the poset of faces of $\widehat{K}$ in the case of coordinate arrangements). We see that the study of moment-angle complexes not only allows us to retrieve the multiplicative structure of the cohomology of complex coordinate subspace arrangement complements, but also connects two seemingly unrelated results, the Goresky–Macpherson formula from the theory of arrangements and Hochester's formula from combinatorial commutative algebra.

5.3. **Massey products in the cohomology of $\mathcal{Z}_K$.** Here we address the question of existence of nontrivial Massey products in the Koszul complex

$$[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{Z}[K], d]$$

of the face ring. Massey products constitute a series of higher-order operations (or *brackets*) in the cohomology of a differential graded algebra, with the second-order operation coinciding with the cohomology multiplication, while the higher-order brackets are only defined for certain tuples of cohomology classes. A geometrical approach to constructing nontrivial triple Massey products in the Koszul complex of the face ring has been developed by Baskakov in [4] as an extension of the cohomology calculation in Theorem 5.1. It is well-known that non-trivial higher Massey products obstruct the *formality* of a differential graded algebra, which in our case leads to a family on nonformal moment-angle manifolds $\mathcal{Z}_K$.

Massey products in the cohomology of the Koszul complex of a local ring R were studied by Golod [14] in connection with the calculation of the Poincaré series of $\operatorname{Tor}_R(\mathbf{k}, \mathbf{k})$. The main result of Golod is a calculation of the Poincaré series for the class of rings with vanishing Massey products in the Koszul complex (including the cohomology multiplication). Such rings were called *Golod* in [17], where the reader can find a detailed exposition of Golod's theorem together with several further applications. The Golod property of face rings was studied in [20], where several combinatorial criteria for Golodness were given.

The difference between our situation and that of Golod is that we are mainly interested in the cohomology of the Koszul complex for the face ring of a sphere triangulation K. The corresponding face ring $\mathbf{k}[K]$ does not qualify for Golodness, as the corresponding moment-angle complex $\mathcal{Z}_K$ is a manifold, and therefore, the cohomology of the Koszul complex of $\mathbf{k}[K]$ must possess many non-trivial products. Our approach aims at identifying a class of simplicial complexes with non-trivial cohomology product but vanishing higher-order Massey operations in the cohomology of the Koszul complex.

Let K_i be a triangulation of a sphere S^{n_i-1} with $|V_i| = m_i$ vertices, $i = 1, 2, 3$. Set $m := m_1 + m_2 + m_3$, $n := n_1 + n_2 + n_3$, and

$$K := K_1 * K_2 * K_3, \quad \mathcal{Z}_K = \mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2} \times \mathcal{Z}_{K_3}.$$

Note that K is a triangulation of S^{n-1} and $\mathcal{Z}_K$ is an $(m+n)$-manifold.

Given $\sigma \in K$, the *stellar subdivision* of K at σ is obtained by replacing the star of σ by the cone over its boundary:

$$\zeta_\sigma(K) = (K \setminus \operatorname{star}_K \sigma) \cup (\operatorname{cone} \partial \operatorname{star}_K \sigma).$$

Now choose maximal simplices $\sigma_1 \in K_1$, $\sigma_2', \sigma_2'' \in K_2$ such that $\sigma_2' \cap \sigma_2'' = \varnothing$, and $\sigma_3 \in K_3$. Set

$$\widetilde{K} := \zeta_{\sigma_1 \cup \sigma_2'}(\zeta_{\sigma_2'' \cup \sigma_3}(K)).$$

Then $\widetilde{K}$ is a triangulation of S^{n-1} with $m+2$ vertices. Take generators

$$\beta_i \in \widetilde{H}^{n_i-1}(\widetilde{K}_{V_i}) \cong \widetilde{H}^{n_i-1}(S^{n_i-1}), \quad i = 1, 2, 3,$$

where $\widetilde{K}_{V_i}$ is the restriction of $\widetilde{K}$ to the vertex set of K_i, and set

$$\alpha_i := \gamma(\beta_i) \in H^{n_i - m_i, 2m_i}(\mathcal{Z}_{\widetilde{K}}) \subseteq H^{m_i+n_i}(\mathcal{Z}_{\widetilde{K}}),$$

where γ is the isomorphism from Theorem 5.1. Then

$$\beta_1\beta_2 \in \widetilde{H}^{n_1+n_2-1}(\widetilde{K}_{V_1 \sqcup V_2}) \cong \widetilde{H}^{n_1+n_2-1}(S^{n_1+n_2-1} \setminus \mathrm{pt}) = 0,$$

and therefore, $\alpha_1\alpha_2 = \gamma(\beta_1\beta_2) = 0$, and similarly $\alpha_2\alpha_3 = 0$. In these circumstances the triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^{m+n-1}(\mathcal{Z}_{\widetilde{K}})$ is defined. Recall that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is the set of cohomology classes represented by the cocycles $(-1)^{\deg a_1 + 1} a_1 f + e a_3$ where a_i is a cocycle representing α_i, $i = 1, 2, 3$, while e and f are cochains satisfying $de = a_1a_2$, $df = a_2a_3$. A Massey product is called *trivial* if it contains zero.

Theorem 5.5. *The triple Massey product*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^{m+n-1}(\mathcal{Z}_{\widetilde{K}})$$

in the cohomology of $(m+n+2)$-manifold $\mathcal{Z}_{\widetilde{K}}$ is non-trivial.

Proof. Consider the subcomplex of $\widetilde{K}$ consisting of those two new vertices added to K in the process of stellar subdivision. By Lemma 4.2, the inclusion of this subcomplex induces an embedding of a 3-dimensional sphere $S^3 \subset \mathcal{Z}_{\widetilde{K}}$. Since the two new vertices are not joined by an edge in $\mathcal{Z}_{\widetilde{K}}$, the embedded 3-sphere defines a non-trivial class in $H^3(\mathcal{Z}_{\widetilde{K}})$. By construction the dual cohomology class is contained in the Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. On the other hand, this Massey product is defined up to elements from the subspace

$$\alpha_1 \cdot H^{m_2+m_3+n_2+n_3-1}(\mathcal{Z}_{\widetilde{K}}) + \alpha_3 \cdot H^{m_1+m_2+n_1+n_2-1}(\mathcal{Z}_{\widetilde{K}}).$$

The multigraded components of the group $H^{m_2+m_3+n_2+n_3-1}(\mathcal{Z}_{\widetilde{K}})$ different from that determined by the full subcomplex $\widetilde{K}_{V_2 \sqcup V_3}$ do not affect the non-triviality of the Massey product, while the multigraded component corresponding to $\widetilde{K}_{V_2 \sqcup V_3}$ is zero since this subcomplex is contractible. The group $H^{m_1+m_2+n_1+n_2-1}(\mathcal{Z}_{\widetilde{K}})$ is treated similarly. It follows that the Massey product contains a unique nonzero element in its multigraded component and so is nontrivial. □

As is well known, the nontriviality of Massey products obstructs formality of manifolds, see e.g. [2].

Corollary 5.6. *For every sphere triangulation $\widetilde{K}$ obtained from another triangulation by applying two stellar subdivisions as described above, the 2-connected moment-angle manifold $\mathcal{Z}_{\widetilde{K}}$ is nonformal.*

In the proof of Theorem 5.5 the nontriviality of the Massey product is established geometrically. A parallel argument may be carried out algebraically in terms of the algebra $R^*(K)$, as illustrated in the following example.

Example 5.7. Consider the simple polytope P^3 shown on Figure 4. This

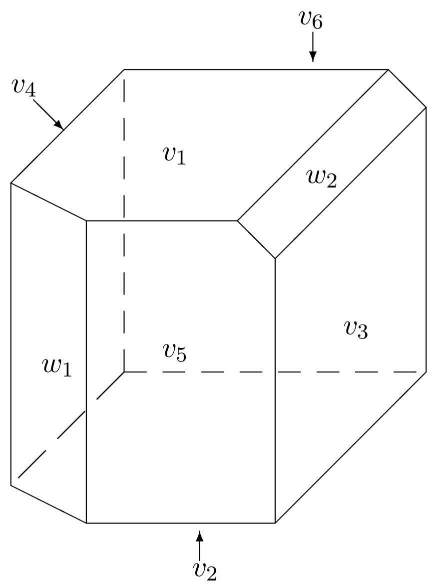


FIGURE 4

polytope is obtained by cutting two non-adjacent edges off a cube and has 8 facets. The dual triangulation K_P is obtained from an octahedron by applying stellar subdivisions at two non-adjacent edges. The face ring is

$$\mathbb{Z}[K_P] = \mathbb{Z}[v_1, \dots, v_6, w_1, w_2]/\mathcal{I}_{K_P},$$

where v_i, $i = 1, \dots, 6$, are the generators coming from the facets of the cube and w_1, w_2 are the generators corresponding to the two new facets, see Figure 4, and

$$\mathcal{I}_P = (v_1v_2, v_3v_4, v_5v_6, w_1w_2, v_1v_3, v_4v_5, w_1v_3, w_1v_6, w_2v_2, w_2v_4).$$

The corresponding algebra $R^*(K_P)$ has additional generators $u_1, \dots, u_6, t_1, t_2$ of total degree 1 satisfying $du_i = v_i$ and $dt_i = w_i$. Consider the cocycles

$$a = v_1u_2, \quad b = v_3u_4, \quad c = v_5u_6$$

and the corresponding cohomology classes $\alpha, \beta, \gamma \in H^{-1,4}[R^*(K)]$. The equations

$$ab = de, \quad bc = df$$

have a solution $e = 0$, $f = v_5u_3u_4u_6$, so the triple Massey product $\langle \alpha, \beta, \gamma \rangle \in H^{-4,12}[R^*(K)]$ is defined. This Massey product is nontrivial by Theorem 5.5.

The cocycle

$$af + ec = v_1v_5u_2u_3u_4u_6$$

represents a nontrivial cohomology class $[v_1v_5u_2u_3u_4u_6] \in \langle\alpha,\beta,\gamma\rangle$ and so the algebra $R^*(K_P)$ and the manifold $\mathcal{Z}_{K_P}$ are not formal.

In view of Theorem 5.5, the question arises of describing the class of simplicial complexes K for which the algebra $R^*(K)$ (equivalently, the Koszul algebra $[\Lambda[u_1,\dots,u_m]\otimes\mathbb{Z}[K],d]$ or the space $\mathcal{Z}_K$) is formal (in particular, does not contain nontrivial Massey products). For example, a direct calculation shows that this is the case if K is the boundary of a polygon.

5.4. **Toral rank conjecture.** Here we relate our cohomological calculations with moment-angle complexes to an interesting conjecture in the theory of transformation groups. This 'toral rank conjecture' has strong links with rational homotopy theory, as described in [1]. Therefore this last subsection, although not containing new results, aims at encouraging rational homotopy theorists to turn their attention to combinatorial commutative algebra of simplicial complexes.

A torus action on a space X is called *almost free* if all isotropy subgroups are finite. The *toral rank* of X, denoted $\mathrm{trk}(X)$, is the largest k for which there exists an almost free T^k-action on X.

The *toral rank conjecture* of Halperin [18] suggests that

$$\dim H^*(X;\mathbb{Q}) \geq 2^{\mathrm{trk}(X)}$$

for any finite dimensional space X. Equality is achieved, for example, if $X = T^k$.

Moment angle complexes provide a wide class of almost free torus actions:

Theorem 5.8 (Davis–Januszkiewicz [11, 7.1])**.** *Let K be an $(n-1)$-dimensional simplicial complex with m vertices. Then $\mathrm{trk}\,\mathcal{Z}_K \geq m-n$.*

Proof. Choose an lsop in $t_1,\dots,t_n$ in $\mathbb{Q}[K]$ according to Lemma 2.3 and write

$$t_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m, \quad i = 1,\dots,n.$$

Then the matrix $\Lambda = (\lambda_{ij})$ defines a linear map $\lambda\colon \mathbb{Q}^m \to \mathbb{Q}^n$. Changing λ to $k\lambda$ for a sufficiently large k if necessary, we may assume that λ is induced by a map $\mathbb{Z}^m \to \mathbb{Z}^n$, which we also denote by λ. It follows from Lemma 3.11 that for every simplex $\sigma \in K$ the restriction $\lambda|_{\mathbb{Z}^\sigma}\colon \mathbb{Z}^\sigma \to \mathbb{Z}^n$ of the map λ to the coordinate subspace $\mathbb{Z}^\sigma \subseteq \mathbb{Z}^m$ is injective.

Denote by T_Λ the subgroup in T^m corresponding to the kernel of the map $\lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$. Then T_Λ is a product of an $(m-n)$-dimensional torus N and a finite group. The intersection of the torus N with the coordinate subgroup $T^\sigma \subseteq T^m$ is a finite subgroup. Since the isotropy subgroups of the T^m-action on $\mathcal{Z}_K$ are of the form T^σ (see the proof of Theorem 3.12), the torus N acts on $\mathcal{Z}_K$ almost freely. □

Note that by construction the space $\mathcal{Z}_K$ is 2-connected.

In view of Theorem 5.1, we get the following reformulation of the toral rank conjecture for $\mathcal{Z}_K$:

$$\dim \bigoplus_{\omega\subseteq[m]} \widetilde{H}^*(K_\omega;\mathbb{Q}) \geq 2^{m-n}$$

for any simplicial complex K^{n-1} on m vertices.

Example 5.9. Let K the boundary of an m-gon. Then the calculation of [8, Exam. 7.22] shows that

$$\dim H^*(\mathcal{Z}_K) = (m-4)2^{m-2} + 4 \geq 2^{m-2}.$$

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