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# COHOMOLOGY OF FIBRE SPACES WITH GROUP BUNDLE COEFFICIENTS

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ABSTRACT. The purpose of this paper is to construct a resolution of a group bundle $\gamma$, a classifying spectrum for a cohomology functor $H^*( \ ;\gamma)$ (coefficients in $\gamma$) defined on a category of fibre spaces, and to clarify and note some implications of the close relationship between these two constructions.

1. **Introduction.** A group bundle, as used in this paper, consists of two Hausdorff $k$-spaces $E(\gamma)$ (the total space) and $B$ (the base space), a continuous projection $E(\gamma)\to B$, and a fibre preserving map from the fibre product (in the category of $k$-spaces) $\gamma\times\gamma$ to $\gamma$ that induces on each fibre of $\gamma$ an abelian group structure. Further, the 0-section and the inversion map are required to be continuous (see **[2]**, **[3]**, **[4]**). This notion of group bundle differs from the usual notion in that forming fibre products in the sense of $k$-spaces allows for an increase in the number of group bundle structures on $\gamma$. If, for example, $\gamma$ is locally compact the two notions coincide. Define $H^p(B;\gamma)=H^p(B;\tilde{\gamma})$ where $\tilde{\gamma}$ is the sheaf of germs of sections of $\gamma$ and call $\gamma$ acyclic if $\tilde{\gamma}$ is acyclic (see **[1]**).

2. **Definitions and remarks.** An exact sequence of group bundles $0\to\gamma_0\to\nu_0\to^{i_1}\nu_1\to\cdots$ (over a fixed base $B$) is a *resolution* of $\gamma_0$ if the associated sequence of sheaves $0\to\tilde{\gamma}_0\to\tilde{\nu}_0\to\cdots$ is exact. The resolution is *acyclic* if each $\nu_n$ is acyclic. By **[1**, p. 34], $H^n(B;\gamma_0)\simeq H^n(C(\tilde{\gamma}_0))$ where $C(\tilde{\gamma}_0)$ is the cochain complex of sheaf sections: $0\to\Gamma(\tilde{\nu}_0)\to\Gamma(\tilde{\nu}_1)\to\cdots$ associated to an acyclic resolution of $\gamma_0$. Define $\gamma_n$ to be the image of $i_n$. Since $C(\tilde{\gamma}_0)$ is isomorphic to the complex of bundle sections

$$C(\gamma_0):0\to\Gamma(\nu_0)\xrightarrow{\Gamma(i_1)}\Gamma(\nu_1)\to\cdots$$

and

$$\mathrm{Ker}(\Gamma(i_{n+1}))=\Gamma(\gamma_n),$$

there is a natural isomorphism $H^n(B;\gamma_0)\simeq\Gamma(\gamma_n)/\mathrm{Im}\,\Gamma(i_n)$. This result can be interpreted as

2.1. $H^n(B;\gamma_0)$ is naturally isomorphic to the group of equivalence

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classes of sections of $\gamma_n$, where two sections are identified if and only if their difference lifts (by $i_n$) to a section of $\nu_{n-1}$ ($i_0=0$).

Further, the sequence $0\to\gamma_n\to\nu_n\to\nu_{n+1}\to\cdots$ is clearly an acyclic resolution, hence

2.2. $H^p(B;\gamma_n)\simeq H^p(C(\gamma_n))\simeq H^{p+n}(B;\gamma_0)$ to ($n\geqq 0, p>0$).

3. **An acyclic resolution.** A group bundle is said to be an $(L)NDR$ group bundle if (locally) the 0-section is a vertical deformation retract of an open neighborhood; see [3], [4, 6.2]. In [3] the following is proved.

LEMMA. *If $\gamma_0$ is an $(L)NDR$ group bundle, then there is an exact sequence of group bundles (depending functorially on $\gamma_0$).*

3.1. $0\to\gamma_0\to^i \nu_0\to^j \gamma_1\to 0$ *for which*

3.2. $\gamma_1$ *is an $(L)NDR$ group bundle.*

3.3. $j:E(\nu_0)\to E(\gamma_1)$ *has local sections, i.e. there is an open cover $\{U_\alpha\}$ of $E(\gamma_1)$ and maps $s_\alpha$ such that $js_\alpha=\mathrm{id}$ on $U_\alpha$.*

3.4. $\nu_0$ *is shrinkable to the* 0-*section.*

3.5. $j$ *has the covering homotopy property for vertical homotopies $H_t:X\to E(\gamma_1)$ if $\gamma_0$ is NDR (LNDR if X is paracompact).*

Property 3.2 implies a long exact sequence $0\to\gamma_0\to\nu_0\to^{i_1}\nu_1\to\cdots$ can be constructed inductively by applying the lemma to $\gamma_1$ and splicing the resulting sequence 3.1 to the sequence 3.1 associated to $\gamma_0$, etc. Since "$\sim$" is left exact (by construction [3] $\gamma_0$ has topology induced by i) and $\tilde{j}$ is onto by 3.3 (lift germs of sections of $\gamma_1$ by an appropriate $s_\alpha$) the sequence so obtained is a resolution, hereafter called the *canonical resolution.*

If $B$ is paracompact 3.4 implies $\tilde{\nu}_0$ is soft (thus acyclic, [1, p. 49], i.e., every section $s$ over a closed set $K$ extends to $B$. Indeed, there is an open set $U\supset K$ and $\sigma'\in\Gamma(\nu_0|U)$ such that $\tilde{\sigma}'|K=s$. Define $\sigma\in\Gamma(\nu_0)$ by

$$\begin{aligned}\sigma(b) &= 0 && \text{if } \tau(b)\leqq \tfrac{1}{2},\\ &= H_{2\tau(b)-1}(\sigma'(b)) && \text{if } \tau(b)\geqq \tfrac{1}{2},\end{aligned}$$

where $H_t$ is a vertical homotopy shrinking $\nu_0$ (3.4) ($H_0=0$, $H_1=\mathrm{id}$), $\tau:B\to I$ is such that $\tau^{-1}(1)\supset\bar{U}_1$, $\tau|(B-U)=0$ where $U_1$ is open and $U\supset\bar{U}_1\supset U_1\supset K$. Clearly $\sigma|U_1=\sigma'|U_1$, thus $\tilde{\sigma}$ extends $s$. This proves

3.6. THEOREM. *Any LNDR group bundle over a (paracompact) k-space has a (acyclic) resolution.*

Assume $B$ is paracompact for the rest of this section. The construction of this resolution implies that

$$0\to\gamma_{n-1}\to\nu_{n-1}\xrightarrow{i_n}\gamma_n\to 0$$

satisfies 3.4, and 3.5 if $\gamma_0$ is $LNDR$. This gives a "homotopic" description of the "algebraic" equivalence relation on sections of $\gamma_n$ (2.1); namely, for $s_0, s_1 \in \Gamma(\gamma_n)$ there is $s \in \Gamma(\nu_{n-1})$ such that $i_n s = s_1 - s_0$ if and only if $s_0$ and $s_1$ are vertically homotopic. Indeed, if $i_n s = s_1 - s_0$ define the homotopy by $H'_t(b) = i_n(H_t(s(b))) + s_0(b)$ where $H_t$ is a shrinking of $\nu_{n-1}$ ($H_0 = 0$, $H_1 = \mathrm{id}$). Conversely, given an $H'_t$ ($H'_0 = s_0$, $H'_1 = s_1$) then $H''_t = H'_t - s_0$ is a vertical homotopy between 0 and $s_1 - s_0$ such that $H''_0$ is covered by the 0-section of $\nu_{n-1}$. By 3.5, $H''_1$ is covered by a section $s$ and $i_n s = s_1 - s_0$, thus

3.7. If $\gamma_0$ is *LNDR* there is a natural isomorphism between $H^n(B; \gamma_0)$ and the group of equivalence classes of vertically homotopic sections of $\gamma_n$.

4. **A classifying spectrum.** For $B$ a $k$-space let $P_B$ be the category of paracompact $k$-fibre spaces ([**2**], [**3**]) over $B$ ($\xi \in P_B$ means $E(\xi)$ is a paracompact $k$-space). For $\gamma_0$ a group bundle on $B$ define $H^p(\xi; \gamma_0)$ ($p$th cohomology of $\xi$ with coefficients in $\gamma_0$) as $H^p(E(\xi); \xi(\gamma_0))$ where the group bundle $\xi(\gamma_0)$ is the pullback (in category of $k$-spaces) of $\gamma_0$ via the projection $P(\xi)$ of $\xi$. It is not hard to see that conditions 3.1–3.5 are invariant under pullback (the pullback by $P(\xi)$ of 3.1 is an exact sequence on $E(\xi)$ satisfying 3.2–3.5). Consequently the pullback of the canonical resolution of $\gamma_0$ is an acyclic resolution of $\xi(\gamma_0)$. Since pullback induces an isomorphism between $\Gamma(\xi(\nu_n))$ and $\mathrm{Hom}(\xi, \nu_n)$ (the set of fibre preserving maps $\xi \to \nu_n$), 2.1 implies

4.1. If $\gamma_0$ is an *LNDR* then $H^*(\xi; \gamma_0)$ is the cohomology of the complex $\{\mathrm{Hom}(\xi, \nu_n), \mathrm{Hom}(i_{n+1})\}$.

Result 3.7 (with $(E(\xi); \xi(\gamma_n))$ replacing $(B; \gamma_0)$) plus the fact that pull back induces an isomorphism between the group of equivalence classes of vertically homotopic sections of $\xi(\gamma_n)$ and the group of equivalence classes of vertically homotopic fibre maps $\xi \to \gamma_n$, denoted by $[\xi, \gamma_n]$, gives

4.2. THEOREM. *For $\gamma_0$ an LNDR the image sequence $\{\gamma_1, \gamma_2, \cdots\}$ of the canonical resolution of $\gamma_0$ is a classifying spectrum for the functor $H^*(\ ; \gamma_0)$ defined on $P_B$, i.e., $H^n(\xi; \gamma_0) \simeq [\xi, \gamma_n]$, $n > 0$.*

The following is proved in [**3**].

4.3. If $\gamma_0$ is *NDR* then $[\xi, \gamma_1]$ and the group of isomorphism classes of numerable principal $\gamma_0$ bundles on $\xi$ are naturally isomorphic. Normality of $E(\xi)$ is not needed, but if $E(\xi)$ is paracompact, numerable can be omitted and *NDR* replaced by *LNDR*. A similar interpretation holds for $[\xi, \gamma_n]$.

Combining 4.2, 2.2 and 4.3 gives

4.4. THEOREM. *If $\gamma_0$ is LNDR then $H^p(\xi; \gamma_n)$ is naturally isomorphic to the group of isomorphism classes of principal $\gamma_{p+n-1}$ bundles on $\xi$ for $\xi \in P_B$ ($n \geqq 0, p > 0$).*

Besides providing a geometric interpretation of $H^*(\ ;\gamma_0)$ this implies ($p=1$) that $\gamma_n$ is the classifying group bundle (see [3]) of $\gamma_{n-1}$ (actually

$$E(\nu_n) \xrightarrow{i_n} E(\gamma_n) \rightarrow B$$

is a universal bundle for $\gamma_{n-1}$).

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