COHOMOLOGY OF q–CONVEX SPACES IN TOP DEGREES

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Abstract.— It is shown that every strongly q–complete subvariety of a complex analytic space has a fundamental system of strongly q–complete neighborhoods. As a consequence, we find a simple proof of Ohsawa's result that every non compact irreducible n–dimensional analytic space is strongly n–complete. Finally, it is shown that L^2–cohomology theory readily implies Ohsawa's Hodge decomposition and Lefschetz isomorphism theorems for absolutely q–convex manifolds.

1. Introduction.

Let $(X, \mathcal{O}_X)$ be a complex analytic space, possibly non reduced. Recall that a function φ on X is said to be strongly q–convex in the sense of Andreotti-Grauert [A-G] if there exists a covering of X by open patches A_λ isomorphic to closed analytic sets in open sets $\Omega_\lambda \subset \mathbb{C}^{N_\lambda}$, $\lambda \in I$, such that each restriction $\varphi_{\restriction A_\lambda}$ admits an extension $\widetilde{\varphi}_\lambda$ on Ω_λ which is strongly q–convex, *i.e.* such that $i\partial\overline{\partial}\widetilde{\varphi}_\lambda$ has at most $q-1$ negative or zero eigenvalues at each point of Ω_λ . The strong q–convexity property is easily shown not to depend on the covering nor on the embeddings $A_\lambda \subset \Omega_\lambda$.

The space X is said to be strongly q–complete, resp. strongly q–convex, if X has a smooth exhaustion function φ such that φ is strongly q–convex on X , resp. on the complement $X \setminus K$ of a compact set $K \subset X$. The main new result of this paper is :

Theorem 1.— *Let Y be an analytic subvariety in a complex space X . If Y is strongly q–complete, then Y has a fundamental family of strongly q–complete neighborhoods V in X .*

The special case of Stein neighborhoods ($q = 1$) has been proved long ago by Y.T. Siu [S3]. The special case when $q = \dim Y + 1$ is due to D. Barlet, who used it in the study of the convexity of spaces of cycles (cf. [Ba]). This case is also a consequence of the results of T. Ohsawa [Oh2], who obtained simultaneously a proof for $q = \dim Y$. Somewhat surprisingly, our proof of the general case is much simpler that the original proof of Siu for the Stein case, and also probably simpler than the partial proofs of Barlet and Ohsawa. The main idea is to extend an exhaustion of Y to a neighborhood by means of a patching procedure. However, up to our knowledge, the extension can be done only after the exhaustion of Y has been slightly modified in a neighborhood of the singular set (cf. theorem 4). Theorem 1 follows now rather easily from the fact that any subvariety Y is the set of $-\infty$ poles of an "almost plurisubharmonic" function (a function whose complex Hessian has locally bounded negative part). Theorem 1 can be used to obtain a short proof of Ohsawa's results on n–convexity of n–dimensional complex spaces :

Theorem 2 (Ohsawa [Oh2]).— *Let X be a complex space such that all irreducible components have dimension $\leq n$.*

(a) *X is always strongly $(n+1)$–complete.*

(b) *If X has no compact irreducible component of dimension n , then X is strongly n–complete.*

(c) *If X has only finitely many irreducible components of dimension n , then X is strongly n–convex.*

The main step consists in proving that a n–dimensional connected non compact manifold always has a strongly subharmonic exhaustion function with respect to any hermitian metric (a result due to Greene and Wu [G-W]). The proof is then completed by induction on n , using theorem 1.

These results will usually be applied in connection with the Andreotti-Grauert theorem [A-G]. Let $\mathcal{F}$ be a coherent sheaf over an analytic space X . The Andreotti-Grauert theorem asserts that $H^q(X,\mathcal{F})$ is finite dimensional if X is strongly q–convex and equal to zero if X is strongly q–complete. When $\dim X \leq n$, a combination with theorem 2 yields :

- $H^n(X,\mathcal{F}) = 0$ if X has no compact n–dimensional component;
- $\dim H^n(X,\mathcal{F}) < \infty$ if X has only finitely many ones.

The special case of these statements when $\mathcal{F}$ is a vector bundle over a manifold goes back to Malgrange [Ma]. The general case was first completed by Siu [S1,S2], with a direct but much more complicated method.

Finally, we show that Ohsawa's Hodge decomposition theorem for an absolutely q–convex Kähler manifold M is a direct consequence of Hodge decomposition for L^2 harmonic forms; the key fact is the observation that any smooth form of degree $k \geq n+q$ becomes L^2 for some suitably modified Kähler metric; thus $H^k(M,\mathbb{C})$ can be considered as a direct limit of L^2–cohomology groups. The Lefschetz isomorphism on L^2–cohomology groups then produces in the limit an isomorphism from the cohomology with compact supports onto the cohomology without supports.

Theorem 3 (Ohsawa [Oh1], [O-T]).— *Let (M,ω) be a Kähler n–dimensional manifold. Suppose that M is absolutely q–convex, i.e. admits a smooth plurisubharmonic exhaustion function that is strongly q–convex on $M \setminus K$ for some compact set K . Set $\Omega^r = \mathcal{O}(\Lambda^r T^\star M)$. Then the De Rham cohomology groups with arbitrary (resp. compact) supports have decompositions*

$$H^k(M,\mathbb{C}) \simeq \bigoplus_{r+s=k} H^s(M,\Omega^r) , \qquad H^r(M,\Omega^s) \simeq \overline{H^s(M,\Omega^r)} , \qquad k \geq n+q ,$$

$$H^k_c(M,\mathbb{C}) \simeq \bigoplus_{r+s=k} H^s_c(M,\Omega^r) , \qquad H^r_c(M,\Omega^s) \simeq \overline{H^s_c(M,\Omega^r)} , \qquad k \leq n-q ,$$

and these groups are finite dimensional. Moreover, there is a Lefschetz isomorphism

$$\omega^{n-r-s} \wedge \bullet \quad : \quad H^s_c(M,\Omega^r) \longrightarrow H^{n-r}(M,\Omega^{n-s}) , \qquad r+s \leq n-q .$$

Observe that the finiteness statement holds as soon as X is strongly q–convex (this is a consequence of Morse theory for the De Rham groups and a consequence of the Andreotti-Grauert theorem for the Dolbeault groups). By an example of Grauert and Riemenschneider [G-R] (cf. also [Oh1]), neither Hodge decomposition nor Hodge symmetry necessarily hold on a strongly q–convex manifold in degrees $\geq n+q$ or $\leq n-q$: if V is a positive rank q vector bundle over a projective m–fold Y , then the space X equal to $P(V \oplus \mathcal{O}) = V \cup V_\infty$ minus the unit ball bundle $\overline{B}(V)$ is q–convex, however with $n = q+m$ it can be checked that

$$H^2_c(X,\mathbb{C}) = \mathbb{C} , \qquad H^0_c(X,\Omega^2) = 0 , \qquad H^2_c(X,\mathcal{O}) \supset H^1(Y,V^\star) ,$$

and there are examples where $q = m \geq 2$ and $H^1(Y,V^\star)$ is arbitrarily large.

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2. Existence of q–convex neighborhoods.

The first step in the proof of theorem 1 is the following approximate extension theorem for strongly q–convex functions.

Theorem 4.— *Let Y be an analytic set in a complex space X and ψ a strongly q–convex C^∞ function on Y . For every continuous function $\delta > 0$ on Y , there exists a strongly q–convex C^∞ function φ on a neighborhood V of Y such that $\psi \leq \varphi_{\restriction Y} < \psi + \delta$.*

Proof.— Without loss of generality, we may assume Y closed in X . Let Z_k be the union of all irreducible components of dimension $\leq k$ of one of the sets Y , $Y_{\rm sing}$, $(Y_{\rm sing})_{\rm sing}$, ... It is clear that $Z_k \setminus Z_{k-1}$ is a smooth k–dimensional submanifold of Y (possibly empty) and that $\bigcup Z_k = Y$. We shall prove by induction on k the following statement :

There exists a C^∞ function φ_k on X which is strongly q–convex along Y and on a closed neighborhood $\overline{V}_k$ of Z_k in X , such that $\psi \leq \varphi_{k\restriction Y} < \psi + \delta$.

We first observe that any smooth extension φ_{-1} of ψ to X satisfies the requirements with $Z_{-1} = V_{-1} = \emptyset$. Assume that V_{k-1} and φ_{k-1} have been constructed. Then $Z_k \setminus V_{k-1} \subset Z_k \setminus Z_{k-1}$ is contained in $Z_{k,\mathrm{reg}}$. The closed set $Z_k \setminus V_{k-1}$ has a locally finite covering (A_λ) in X by open coordinate patches $A_\lambda \subset \Omega_\lambda \subset \mathbb{C}^{N_\lambda}$ in which Z_k is given by equations $z'_\lambda = (z_{\lambda,k+1}, \ldots, z_{\lambda,N_\lambda}) = 0$. Let θ_λ be C^∞ functions with compact support in A_λ such that $0 \le \theta_\lambda \le 1$ and $\sum \theta_\lambda = 1$ on $Z_k \setminus V_{k-1}$. We set

$$\varphi_k(x) = \varphi_{k-1}(x) + \sum \ \theta_\lambda(x)\, \varepsilon_\lambda^3 \, \log(1 + \varepsilon_\lambda^{-4} |z'_\lambda|^2) \quad \text{ on } \ X \ .$$

For $\varepsilon_\lambda > 0$ small enough, we will have $\psi \le \varphi_{k-1\restriction Y} \le \varphi_{k\restriction Y} < \psi + \delta$. Now, we check that φ_k is still strongly q–convex along Y and near every point $x_0 \in \overline{V}_{k-1}$, and that φ_k becomes strongly q–convex near every point $x_0 \in Z_k \setminus V_{k-1}$. We may assume that $x_0 \in \operatorname{Supp} \theta_\mu$ for some μ , otherwise φ_k coincides with φ_{k-1} in a neighborhood of x_0 . Select μ and a small neighborhood $W \subset\subset \Omega_\mu$ of x_0 such that

(a) if $x_0 \in Z_k \setminus V_{k-1}$ then $\theta_\mu(x_0) > 0$ and $A_\mu \cap W \subset\subset \{\theta_\mu > 0\}$;

(b) if $x_0 \in A_\lambda$ for some λ (there is a finite set I of such λ's), then $A_\mu \cap W \subset\subset A_\lambda$ and $z_{\lambda\restriction A_\mu \cap W}$ has a holomorphic extension $\widetilde{z}_\lambda$ to $\overline{W}$;

(c) if $x_0 \in \overline{V}_{k-1}$ then $\varphi_{k-1\restriction A_\mu \cap W}$ has a strongly q–convex extension $\widetilde{\varphi}_{k-1}$ to $\overline{W}$;

(d) if $x_0 \in Y \setminus \overline{V}_{k-1}$, $\varphi_{k-1\restriction Y \cap W}$ has a strongly q–convex extension $\widetilde{\varphi}_{k-1}$ to $\overline{W}$.

Otherwise take an arbitrary smooth extension $\widetilde{\varphi}_{k-1}$ of $\varphi_{k-1\restriction A_\mu \cap W}$ to $\overline{W}$ and let $\widetilde{\theta}_\lambda$ be an extension of $\theta_{\lambda\restriction A_\mu \cap W}$ to $\overline{W}$. Then

$$\widetilde{\varphi}_k = \widetilde{\varphi}_{k-1} + \sum \widetilde{\theta}_\lambda \, \varepsilon_\lambda^3 \, \log(1 + \varepsilon_\lambda^{-4} |\widetilde{z}'_\lambda|^2)$$

is an extension of $\varphi_{k\restriction A_\mu \cap W}$ to $\overline{W}$, resp. of $\varphi_{k\restriction Y \cap W}$ to $\overline{W}$ in case (d). As the function $\log(1 + \varepsilon_\lambda^{-4} |\widetilde{z}'_\lambda|^2)$ is plurisubharmonic and as its first derivative $\langle \widetilde{z}'_\lambda, d\widetilde{z}'_\lambda \rangle \, (\varepsilon_\lambda^4 + |\widetilde{z}'_\lambda|^2)^{-1}$ is bounded by $\mathrm{O}(\varepsilon_\lambda^{-2})$, we see that

$$i\partial\overline{\partial}\widetilde{\varphi}_k \ge i\partial\overline{\partial}\widetilde{\varphi}_{k-1} - \mathrm{O}(\textstyle\sum \varepsilon_\lambda) \ .$$

Therefore, for ε_λ small enough, $\widetilde{\varphi}_k$ remains q–convex on $\overline{W}$ in cases (c) and (d). Since all functions $\widetilde{z}'_\lambda$ vanish along $Z_k \cap W$, we have

$$i\partial\overline{\partial}\widetilde{\varphi}_k \ge i\partial\overline{\partial}\widetilde{\varphi}_{k-1} + \sum_{\lambda \in I} \theta_\lambda \, \varepsilon_\lambda^{-1} \, i\partial\overline{\partial} |\widetilde{z}'_\lambda|^2 \ge i\partial\overline{\partial}\widetilde{\varphi}_{k-1} + \theta_\mu \, \varepsilon_\mu^{-1} \, i\partial\overline{\partial} |z'_\mu|^2$$

at every point of $Z_k \cap W$. Moreover $i\partial\overline{\partial}\widetilde{\varphi}_{k-1}$ has at most $(q-1)$–negative or zero eigenvalues on TZ_k since $Z_k \subset Y$, whereas $i\partial\overline{\partial}|z'_\mu|^2$ is positive definite in the normal directions to Z_k in Ω_μ . In case (a), we thus find that $\widetilde{\varphi}_k$ is strongly q–convex on $\overline{W}$ for ε_μ small enough; we also observe that only finitely many conditions are required on each ε_λ if we choose a locally finite covering of $\bigcup \operatorname{Supp} \theta_\lambda$ by neighborhoods W as above. Therefore, for ε_λ small enough, φ_k is strongly q–convex on a neighborhood $\overline{V}'_k$ of $Z_k \setminus V_{k-1}$. The function φ_k and the set $V_k = V_{k-1} \cup V'_k$ satisfy the requirements at order k . It is clear that we can choose the sequence φ_k stationary on every compact subset of X; the limit φ and the open set $V = \bigcup V_k$ fulfill theorem 4. $\square$

The second step is the existence of almost psh (plurisubharmonic) functions having poles along a prescribed analytic set. By an almost psh function on a manifold, we mean a function that is locally equal to the sum of a psh function and of a smooth function, or equivalently, a function whose complex Hessian has bounded negative part. On a complex space, we require that our function can be locally extended as an almost psh function in the ambient space of an embedding.

Lemma 5.— *Let Y be an analytic subvariety in a complex space X . There exists an almost plurisubharmonic function v on X such that $v \in C^\infty(X \setminus Y)$ and $v = -\infty$ on Y (with logarithmic poles along Y).*

Proof.— Since $\mathcal{I}_Y \subset \mathcal{O}_X$ is a coherent subsheaf, there is a locally finite covering of X by patches A_λ isomorphic to analytic sets in balls $B(0, r_\lambda) \subset \mathbb{C}^{N_\lambda}$, such that $\mathcal{I}_Y$ admits a system of generators $g_\lambda = (g_{\lambda,j})$ on a neighborhood of each set $\overline{A}_\lambda$. We set

$$\begin{aligned} v_\lambda(z) &= \log|g_\lambda(z)|^2 - \frac{1}{r_\lambda^2 - |z - z_\lambda|^2} \quad \text{ on } \ A_\lambda \ , \\ v(z) &= m\big(\ldots, v_\lambda(z), \ldots\big) \quad \text{ for } \ \lambda \ \text{ such that } \ A_\lambda \ni z \ , \end{aligned}$$

where m is a regularized max function defined as follows : select a smooth function ρ on $\mathbb{R}$ with support in $[-1/2, 1/2]$, such that $\rho \geq 0$, $\int_{\mathbb{R}} \rho(u) du = 1$, $\int_{\mathbb{R}} u\rho(u)\,du = 0$, and set

$$m(t_1, \ldots, t_p) = \int_{\mathbb{R}^p} \max\{t_1 + u_1, \ldots, t_p + u_p\} \prod_{1 \leq j \leq p} \rho(u_j)\, du_j \ .$$

It is clear that m is increasing in all variables and convex, thus m preserves plurisubharmonicity. Moreover, we have

$$m(t_1, \ldots, t_j, \ldots, t_p) = m(t_1, \ldots, \widehat{t_j}, \ldots, t_p)$$

as soon as $t_j < \max\{t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_p\} - 1$. As the generators $(g_{\lambda,j})$ can be expressed in terms of one another on a neighborhood of $\overline{A}_\lambda \cap \overline{A}_\mu$, we see that the quotient $|g_\lambda|/|g_\mu|$ remains bounded on this set. Therefore none of the values $v_\lambda(z)$ for $A_\lambda \ni z$ and z near ∂A_λ contributes to the value of $v(z)$, since $1/(r_\lambda^2 - |z - z_\lambda|^2)$ tends to $+\infty$ on ∂A_λ . It follows that v is smooth on $X \setminus Y$; as each v_λ is almost psh on A_λ , we also see that v is almost psh on X . $\square$

Proof of theorem 1.— By theorem 4 applied to a strongly q–convex exhaustion of Y and $\delta = 1$, there exists a strongly q–convex function φ on a neighborhood W_0 of Y such that $\varphi_{\restriction Y}$ is an exhaustion. Let W_1 be a neighborhood of Y such that $\overline{W}_1 \subset W_0$ and such that $\varphi_{\restriction \overline{W}_1}$ is an exhaustion. We are going to show that every neighborhood $W \subset W_1$ of Y contains a strongly q–complete neighborhood V . If v is the function given by lemma 5, we set

$$\widetilde{v} = v + \chi \circ \varphi \quad \text{ on } \ \overline{W}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth convex increasing function. If χ grows fast enough, we get $\widetilde{v} > 0$ on ∂W and the $(q-1)$–codimensional subspace on which $i\partial\overline{\partial}\varphi$ is positive definite (in some ambient space) is also positive definite for $i\partial\overline{\partial}\widetilde{v}$ provided that χ' be large enough to compensate the bounded negative part of $i\partial\overline{\partial}v$. Then $\widetilde{v}$ is strongly q–convex. Let θ be a smooth convex increasing function on $]-\infty, 0[$ such that $\theta(t) = 0$ for $t < -3$ and $\theta(t) = -1/t$ on $]-1, 0[$. The open set $V = \{z \in W \,;\, \widetilde{v}(z) < 0\}$ is a neighborhood of Y and $\widetilde{\psi} = \varphi + \theta \circ \widetilde{v}$ is a strongly q–convex exhaustion of V . $\square$

3. q–convexity properties in top degrees.

It is obvious by definition that a n–dimensional complex manifold M is strongly q–complete for $q \ge n+1$. If M is connected and non compact, this property also holds for $q = n$, i.e. there is a smooth exhaustion ψ on M such that $i\partial\overline{\partial}\psi$ has at least one positive eigenvalue everywhere. In fact, one can even show that M has strongly subharmonic exhaustion functions. Let ω be an arbitrary hermitian metric on M . We consider the Laplace operator Δ_ω defined by

$$\Delta_\omega v = \mathrm{Trace}_\omega(i\partial\overline{\partial}v) = \sum_{1\le j,k\le n} \omega^{jk}(z)\frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}$$

where (ω^{jk}) is the conjugate of the inverse matrix of (ω_{jk}) . Observe that Δ_ω coincides with the usual Laplace-Beltrami operator only if ω is Kähler. We will say that v is *strongly ω–subharmonic* if $\Delta_\omega v > 0$. Clearly, this property implies that $i\partial\overline{\partial}v$ has at least one positive eigenvalue at each point, i.e. that v is strongly n–convex. Moreover, since

$$\Delta_\omega\chi(v_1, \ldots, v_s) = \sum_j \frac{\partial\chi}{\partial t_j}(v_1, \ldots, v_s)\,\Delta_\omega v_j + \sum_{j,k}\frac{\partial^2\chi}{\partial t_j \partial t_k}(v_1, \ldots, v_s)\,\langle \partial v_j, \partial v_k\rangle_\omega,$$

subharmonicity has the advantage of being preserved by all convex increasing combinations, whereas a sum of strongly n–convex functions is not necessarily n–convex. We shall need the following partial converse.

Lemma 6.— *If ψ is strongly n–convex on M , there is a hermitian metric ω such that ψ is strongly subharmonic with respect to ω .*

Proof.— Let $U_\lambda \subset\subset U'_\lambda$, $\lambda \in \mathbb{N}$, be locally finite coverings of M by open balls equipped with coordinates such that $\partial^2\psi/\partial z_1\partial\overline{z}_1 > 0$ on $\overline{U}'_\lambda$. By induction on λ , we construct a hermitian metric ω_λ on M such that ψ is strongly ω_λ–subharmonic on $U_0 \cup \ldots \cup U_{\lambda-1}$. Starting from an arbitrary ω_0 , we obtain ω_λ from $\omega_{\lambda-1}$ by increasing the coefficient $\omega^{1\,1}_{\lambda-1}$ in $(\omega^{jk}_{\lambda-1}) = (\omega_{\lambda-1,kj})^{-1}$ on a neighborhood of $\overline{U}_\lambda$. Then $\omega = \lim\omega_\lambda$ is the required metric. □

Lemma 7.— *Let $U, W \subset M$ be open sets such that for every connected component U_s of U there is a connected component $W_{t(s)}$ of W such that $W_{t(s)} \cap U_s \ne \emptyset$ and $W_{t(s)} \setminus \overline{U}_s \ne \emptyset$. Then there exists a function $v \in C^\infty(M, \mathbb{R})$, $v \ge 0$, with support contained in $\overline{U} \cup \overline{W}$, such that v is strongly ω–subharmonic and > 0 on U .*

Proof.— We first prove that the result is true when U, W are small cylinders with the same radius and axis. Let $a_0 \in M$ be a given point and $z_1, \ldots, z_n$ holomorphic coordinates centered at a_0 . We set $\mathrm{Re}\, z_j = x_{2j-1}$, $\mathrm{Im}\, z_j = x_{2j}$, $x' = (x_2, \ldots, x_{2n})$ and $\omega = \sum \widetilde{\omega}_{jk}(x)dx_j \otimes dx_k$. Let U be the cylinder $|x_1| < r$, $|x'| < r$, and W the cylinder $r - \varepsilon < x_1 < r + \varepsilon$, $|x'| < r$. There are constants $c, C > 0$ such that

$$\sum \widetilde{\omega}^{jk}(x)\xi_j\xi_k \ge c|\xi|^2 \quad \text{and} \quad \sum |\widetilde{\omega}^{jk}(x)| \le C \quad \text{on } \overline{U} .$$

Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a nonnegative function equal to 0 on $]-\infty, -r] \cup [r+\varepsilon, +\infty[$ and strictly convex on $]-r, r]$. We take explicitly $\chi(x_1) = (x_1+r)\exp(-1/(x_1+r)^2)$

on $]-r,r]$ and

$$v(x)=\chi(x_1)\exp\big(1/(|x'|^2-r^2)\big)\quad\text{on}\ \ U\cup W\ ,\qquad v=0\quad\text{on}\ \ M\setminus(U\cup W)\ .$$

We have $v\in C^\infty(M,\mathbb{R})$, $v>0$ on U , and a simple computation gives

$$\begin{aligned}\frac{\Delta_\omega v(x)}{v(x)}&=\widetilde\omega^{11}(x)\big(4(x_1+r)^{-5}-2(x_1+r)^{-3}\big)\\&\quad+\sum_{j>1}\widetilde\omega^{1j}(x)\big(1+2(x_1+r)^{-2}\big)(-2x_j)(r^2-|x'|^2)^{-2}\\&\quad+\sum_{j,k>1}\widetilde\omega^{jk}(x)\Big(x_jx_k\big(4-8(r^2-|x'|^2)\big)-2(r^2-|x'|^2)^2\delta_{jk}\Big)(r^2-|x'|^2)^{-4}.\end{aligned}$$

For r small, we get

$$\frac{\Delta_\omega v(x)}{v(x)}\geq 2c(x_1+r)^{-5}-C_1(x_1+r)^{-2}|x'|(r^2-|x'|^2)^{-2}+(2c|x'|^2-C_2r^4)(r^2-|x'|^2)^{-4}$$

with constants C_1,C_2 independent of r . The negative term is bounded by $C_3(x_1+r)^{-4}+c|x'|^2(r^2-|x'|^2)^{-4}$, hence

$$\Delta_\omega v/v(x)\geq c(x_1+r)^{-5}+(c|x'|^2-C_2r^4)(r^2-|x'|^2)^{-4}\ .$$

The last term is negative only when $|x'|<C_4r^2$, in which case it is bounded by $C_5r^{-4}<c(x_1+r)^{-5}$. Hence v is strongly ω–subharmonic on U .

Next, assume that U and W are connected. Then $U\cup W$ is connected. Fix a point $a\in W\setminus\overline{U}$. If $z_0\in U$ is given, we choose a path $\Gamma\subset U\cup W$ from z_0 to a which is piecewise linear with respect to holomorphic coordinate patches. Then we can find a finite sequence of cylinders (U_j,W_j) of the type described above, $1\leq j\leq N$, whose axes are segments contained in Γ , such that

$$U_j\cup W_j\subset U\cup W\ ,\quad \overline{W}_j\subset U_{j+1}\quad\text{and}\quad z_0\in U_0\ ,\quad a\in W_N\subset W\setminus\overline{U}\ .$$

For each such pair, we have a function $v_j\in C^\infty(M)$ with support in $\overline{U}_j\cup\overline{W}_j$, $v_j\geq 0$, strongly ω–subharmonic and >0 on U_j . By induction, we can find constants $C_j>0$ such that $v_0+C_1v_1+\cdots+C_jv_j$ is strongly ω–subharmonic on $U_0\cup\ldots\cup U_j$ and ω–subharmonic on $M\setminus\overline{W}_j$. Then

$$w_{z_0}=v_0+C_1v_1+\ldots+C_Nv_N\geq 0$$

is ω–subharmonic on U and strongly ω–subharmonic >0 on a neighborhood Ω_0 of the given point z_0 . Select a denumerable covering of U by such neighborhoods Ω_p and set $v(z)=\sum\varepsilon_pw_{z_p}(z)$ where ε_p is a sequence converging sufficiently fast to 0 so that $v\in C^\infty(M,\mathbb{R})$. Then v has the required properties.

In the general case, we find for each pair $(U_s,W_{t(s)})$ a function v_s with support in $\overline{U}_s\cup\overline{W}_{t(s)}$, strongly ω–subharmonic and >0 on U_s . Any convergent series $v=\sum\varepsilon_sv_s$ yields a function with the desired properties. □

Lemma 8.— *Let X be a connected, locally connected and locally compact topological space. If U is a relatively compact open subset of X , we let $\widetilde U$ be the union of U with all compact connected components of $X\setminus U$. Then $\widetilde U$ is open and relatively compact in X , and $X\setminus\widetilde U$ has only finitely many connected components, all non compact.*

Proof.— A rather easy exercise of general topology. Intuitively, $\widetilde{U}$ is obtained by "filling the holes" of U in X . $\square$

Theorem 9 (Greene-Wu [G-W]).— *Every n–dimensional connected non compact complex manifold M has a strongly subharmonic exhaustion function with respect to any hermitian metric ω . In particular, M is strongly n–complete.*

Proof.— Let $\varphi \in C^\infty(M,\mathbb{R})$ be an arbitrary exhaustion function. There exists a sequence of connected smoothly bounded open sets $\Omega'_\nu \subset\subset M$ with $\overline{\Omega}'_\nu \subset \Omega'_{\nu+1}$ and $M = \bigcup \Omega'_\nu$. Let $\Omega_\nu = \widetilde{\Omega}'_\nu$ be the relatively compact open set given by lemma 8. Then $\overline{\Omega}_\nu \subset \Omega_{\nu+1}$, $M = \bigcup \Omega_\nu$ and $M \setminus \Omega_\nu$ has no compact connected component. We set

$$U_1 = \Omega_2 \ , \qquad U_\nu = \Omega_{\nu+1} \setminus \overline{\Omega_{\nu-2}} \quad \text{for} \ \ \nu \geq 2 \ .$$

Then $\partial U_\nu = \partial\Omega_{\nu+1} \cup \partial\Omega_{\nu-2}$; any connected component $U_{\nu,s}$ of U_ν has its boundary $\partial U_{\nu,s} \not\subset \partial\Omega_{\nu-2}$, otherwise $\overline{U}_{\nu,s}$ would be open and closed in $M \setminus \Omega_{\nu-2}$, hence $\overline{U}_{\nu,s}$ would be a compact connected component of $M \setminus \Omega_{\nu-2}$. Therefore $\partial U_{\nu,s}$ intersects $\partial\Omega_{\nu+1} \subset U_{\nu+1}$. If $U_{\nu+1,t(s)}$ is a connected component of $U_{\nu+1}$ containing a point of $\partial U_{\nu,s}$, then $U_{\nu+1,t(s)} \cap U_{\nu,s} \neq \emptyset$ and $U_{\nu+1,t(s)} \setminus \overline{U}_{\nu,s} \neq \emptyset$. Lemma 7 implies that there is a nonnegative function $v_\nu \in C^\infty(M,\mathbb{R})$ with support in $U_\nu \cup U_{\nu+1}$, which is strongly ω–subharmonic on U_ν . An induction yields constants C_ν such that

$$\psi_\nu = \varphi + C_1 v_1 + \cdots + C_\nu v_\nu$$

is strongly ω–subharmonic on $\overline{\Omega_\nu} \subset U_0 \cup \ldots \cup U_\nu$, thus $\psi = \varphi + \sum C_\nu v_\nu$ is a strongly ω–subharmonic exhaustion function on M . $\square$

By an induction on the dimension, the above result can be generalized to an arbitrary complex space, as was first shown by T. Ohsawa [Oh2].

Proof of theorem 2 (a,b).— By induction on $n = \dim X$. For $n = 0$, property (b) is void and (a) is obvious (any function can then be considered as strongly 1–convex). Assume that (a) has been proved in dimension $\leq n-1$. Let X' be the union of $X_{\rm sing}$ and of the irreducible components of X of dimension at most $n-1$, and $M = X \setminus X'$ the n–dimensional part of $X_{\rm reg}$. As $\dim X' \leq n-1$, the induction hypothesis shows that X' is strongly n–complete. By theorem 1, there exists a strongly n–convex exhaustion function φ' on a neighborhood V' of X' . Take a closed neighborhood $\overline{V} \subset V'$ and an arbitrary exhaustion φ on X that extends $\varphi'_{\restriction \overline{V}}$. Since every function on a n–dimensional manifold is strongly $(n+1)$–convex , we conclude that X is at worst $(n+1)$–complete, as stated in (a).

In case (b), the hypothesis means that the connected components M_j of $M = X \setminus X'$ have non compact closure $\overline{M}_j$ in X . On the other hand, lemma 6 shows that there exists a hermitian metric ω on M such that $\varphi_{\restriction M\cap V}$ is strongly ω–subharmonic. Consider the open sets $U_{j,\nu} \subset M_j$ provided by lemma 10 below. By the arguments already used in theorem 9, one can find a strongly ω–subharmonic exhaustion $\psi = \varphi + \sum_{j,\nu} C_{j,\nu} v_{j,\nu}$ on X , with $v_{j,\nu}$ strongly ω–subharmonic on $U_{j,\nu}$, $\operatorname{Supp} v_{j,\nu} \subset U_{j,\nu} \cup U_{j,\nu+1}$ and $C_{j,\nu}$ large. Then ψ is strongly n–convex on X .

Lemma 10.— *For each j , there exists a sequence of open sets $U_{j,\nu} \subset\subset M_j$, $\nu \in \mathbb{N}$, such that*

(a) $M_j \setminus V' \subset \bigcup_\nu U_{j,\nu}$ *and $(U_{j,\nu})$ is locally finite in $\overline{M}_j$;*

(b) *for every connected component $U_{j,\nu,s}$ of $U_{j,\nu}$ there is a connected component $U_{j,\nu+1,t(s)}$ of $U_{j,\nu+1}$ such that $U_{j,\nu+1,t(s)} \cap U_{j,\nu,s} \neq \emptyset$ and $U_{j,\nu+1,t(s)} \setminus \overline{U}_{j,\nu,s} \neq \emptyset$.*

By lemma 8 applied to the space $\overline{M}_j$, there exists a sequence of relatively compact connected open sets $\Omega_{j,\nu}$ in $\overline{M}_j$ such that $\overline{M}_j \setminus \Omega_{j,\nu}$ has no compact connected component, $\overline{\Omega}_{j,\nu} \subset \Omega_{j,\nu+1}$ and $\overline{M}_j = \bigcup \Omega_{j,\nu}$. We define a compact set $K_{j,\nu} \subset M_j$ and an open set $W_{j,\nu} \subset \overline{M}_j$ containing $K_{j,\nu}$ by

$$K_{j,\nu} = (\overline{\Omega}_{j,\nu} \setminus \Omega_{j,\nu-1}) \setminus V' , \qquad W_{j,\nu} = \Omega_{j,\nu+1} \setminus \overline{\Omega}_{j,\nu-2} .$$

By induction on ν , we construct an open set $U_{j,\nu} \subset\subset W_{j,\nu} \setminus X' \subset M_j$ and a finite set $F_{j,\nu} \subset \partial U_{j,\nu} \setminus \overline{\Omega}_{j,\nu}$. We let $F_{j,-1} = \emptyset$. If these sets are already constructed for $\nu - 1$, the compact set $K_{j,\nu} \cup F_{j,\nu-1}$ is contained in the open set $W_{j,\nu}$, thus contained in a finite union of connected components $W_{j,\nu,s}$. We can write $K_{j,\nu} \cup F_{j,\nu-1} = \bigcup L_{j,\nu,s}$ where $L_{j,\nu,s}$ is contained in $W_{j,\nu,s} \setminus X' \subset M_j$. The open set $W_{j,\nu,s} \setminus X'$ is connected and non contained in $\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s}$, otherwise its closure $\overline{W}_{j,\nu,s}$ would have no boundary point $\in \partial\Omega_{j,\nu+1}$, thus would be open and compact in $\overline{M}_j \setminus \Omega_{j,\nu-2}$, contradiction. We select a point $a_s \in (W_{j,\nu,s} \setminus X') \setminus (\overline{\Omega}_{j,\nu} \cup L_{j,\nu,s})$ and a smoothly bounded connected open set $U_{j,\nu,s} \subset\subset W_{j,\nu,s} \setminus X'$ containing $L_{j,\nu,s}$ with $a_s \in \partial U_{j,\nu,s}$. Finally, we set $U_{j,\nu} = \bigcup_s U_{j,\nu,s}$ and let $F_{j,\nu}$ be the set of all points a_s . By construction, we have $U_{j,\nu} \supset K_{j,\nu} \cup F_{j,\nu-1}$, thus $\bigcup U_{j,\nu} \supset \bigcup K_{j,\nu} = M_j \setminus V'$, and $\partial U_{j,\nu,s} \ni a_s$ with $a_s \in F_{j,\nu} \subset U_{j,\nu+1}$. Property (b) follows. □

Proof of theorem 2 (c).— Let $Y \subset X$ be the union of X_{sing} with all irreducible components of X that are non compact or of dimension $< n$. Then $\dim Y \leq n-1$, so Y is n–convex and theorem 1 implies that there is an exhaustion function $\psi \in C^\infty(X, \mathbb{R})$ such that ψ is strongly n–convex on a neighborhood V of Y . Then the complement $K = X \setminus V$ is compact and ψ is strongly n–convex on $X \setminus K$. □

4. A simple proof of Ohsawa's Hodge decomposition theorem.

Let M be a complex n–dimensional manifold admitting a Kähler metric ω and a strongly q–convex plurisubharmonic exhaustion function ψ . For any convex increasing function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$, we consider the new Kähler metric

$$\omega_\chi = \omega + i\partial\overline{\partial}(\chi \circ \psi) = \omega + \chi'(\psi)\, i\partial\overline{\partial}\psi + \chi''(\psi)\, i\partial\psi \wedge \overline{\partial}\psi$$

and the associated geodesic distance δ_χ . Then the norm of $\chi''(\psi)^{1/2} d\psi$ with respect to ω_χ is less than 1 , thus if ρ is a primitive of $(\chi'')^{1/2}$ we have

$$\big|\rho(\psi(x)) - \rho(\psi(y))\big| \leq \delta_\chi(x, y) .$$

Hence ω_χ is complete as soon as $\lim_{+\infty} \rho(t) = +\infty$, that is $\int_0^{+\infty} \chi''(t)^{1/2} dt = +\infty$. In the sequel, we always assume that χ grows sufficiently fast at infinity so that this condition is fulfilled. We denote by $L^{2,(k)}_\chi(M) = \bigoplus_{r+s=k} L^{2,(r,s)}_\chi(M)$ the space of L^2 forms of degree k with respect to the metric ω_χ , by $\mathcal{H}^k_\chi(M)$ the subspace of L^2 harmonic forms of degree k with respect to the associated Laplace-Beltrami operator $\Delta_\chi = dd^\star_\chi + d^\star_\chi d$ and by $\mathcal{H}^{r,s}_\chi(M)$ the space of L^2–harmonic forms of bidegree (r,s) with respect to $\overline{\Box}_\chi = \overline{\partial}\,\overline{\partial}^\star_\chi + \overline{\partial}^\star_\chi\overline{\partial}$. As ω_χ is Kähler, we have $\Box_\chi = \overline{\Box}_\chi = \frac{1}{2}\Delta_\chi$, hence

$$\mathcal{H}^k_\chi(M) = \bigoplus_{r+s=k} \mathcal{H}^{r,s}_\chi(M) \ , \qquad \mathcal{H}^{s,r}_\chi(M) = \overline{\mathcal{H}^{r,s}_\chi(M)} \tag{1}$$

for each $k = 0, 1, \ldots, 2n$. Since ω_χ is complete, we have orthogonal decompositions

$$L^{2,(r,s)}_\chi(M) = \mathcal{H}^{r,s}_\chi(M) \oplus \overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi} \oplus \overline{\mathrm{Im}^{r,s}\,\overline{\partial}^\star_\chi}$$

$$\mathrm{Ker}^{r,s}\,\overline{\partial}_\chi = \mathcal{H}^{r,s}_\chi(M) \oplus \overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi} \ , \tag{2}$$

where $\overline{\partial}_\chi$ is the unbounded $\overline{\partial}$ operator acting on L^2 forms with respect to ω_χ and where $\overline{\mathrm{Im}^{r,s}}$ means closure of the range (in the specified bidegree). In particular $\mathcal{H}^{r,s}_\chi(M)$ is isomorphic to the quotient $\mathrm{Ker}^{r,s}\,\overline{\partial}_\chi/\overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi}$. Of course, similar results also hold for Δ_χ–harmonic forms.

Lemma 11.— *Let u be a form of type (r,s) with L^2_{loc} coefficients on M . If $r+s \geq n+q$, then $u \in L^{2,(r,s)}_\chi(M)$ as soon as χ grows sufficiently fast at infinity.*

Proof.— At each point $x \in M$, there is an orthogonal basis $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ of T_xX in which

$$\omega = i\sum_{1\leq j\leq n} dz_j \wedge d\overline{z}_j \ , \qquad \omega_\chi = i\sum_{1\leq j\leq n} \lambda_j dz_j \wedge d\overline{z}_j \ ,$$

where $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of ω_χ with respect to ω . Then the volume elements $dV = \omega^n/2^n n!$ and $dV_\chi = \omega^n_\chi/2^n n!$ are related by

$$dV_\chi = \lambda_1 \ldots \lambda_n \, dV$$

and for a (r,s)–form $u = \sum_{I,J} u_{I,J} dz_I \wedge d\overline{z}_J$ we find

$$|u|^2_\chi = \sum_{|I|=r,|J|=s} \Big(\prod_{k\in I}\lambda_k \prod_{k\in J}\lambda_k\Big)^{-1} |u_{I,J}|^2 \ ,$$

in particular

$$|u|^2_\chi dV_\chi \leq \frac{\lambda_1 \ldots \lambda_n}{\lambda_1 \ldots \lambda_r \, \lambda_1 \ldots \lambda_s} |u|^2 dV = \frac{\lambda_{r+1} \ldots \lambda_n}{\lambda_1 \ldots \lambda_s} |u|^2 dV \ .$$

On the other hand, we have upper bounds

$$\lambda_j \leq 1 + C_1\chi'(\psi) \ , \quad 1 \leq j \leq n-1 \ , \qquad \lambda_n \leq 1 + C_1\chi'(\psi) + C_2\chi''(\psi)$$

where $C_1(x)$ is the largest eigenvalue of $i\partial\overline{\partial}\psi(x)$ and $C_2(x) = |\partial\psi(x)|^2$; to find the $n-1$ first inequalities, we need only apply the minimax principle on the kernel of $\partial\psi$. As $i\partial\overline{\partial}\psi$ has at most $q-1$ zero eigenvalues on $X \setminus K$, the minimax principle also gives lower bounds

$$\lambda_j \geq 1 \ , \quad 1 \leq j \leq q-1 \ , \qquad \lambda_j \geq 1 + c\,\chi'(\psi) \ , \quad q \leq j \leq n \ ,$$

where $c(x) \geq 0$ is the q–th eigenvalue of $i\partial\overline{\partial}\psi(x)$ and $c(x) > 0$ on $X \setminus K$. Assuming $\chi' \geq 1$, we infer easily

$$\begin{aligned} \frac{|u|^2_\chi dV_\chi}{|u|^2 dV} &\leq \frac{\big(1 + C_1\chi'(\psi)\big)^{n-r-1}\big(1 + C_1\chi'(\psi) + C_2\chi''(\psi)\big)}{\big(1 + c\chi'(\psi)\big)^{s-q+1}} \\ &\leq C_3\big(\chi'(\psi)^{n+q-r-s-1} + \chi''(\psi)\chi'(\psi)^{n+q-r-s-2}\big) \quad \text{on } X \setminus K \ . \end{aligned}$$

For $r+s\ge n+q$, this is less than

$$C_3\big(\chi'(\psi)^{-1}+\chi''(\psi)\chi'(\psi)^{-2}\big)\ ,$$

and it is easy to show that this quantity can be made arbitrarily small when χ grows sufficiently fast at infinity on M . $\square$

It is a well-known result of Andreotti-Grauert [A-G] that the natural topology on the cohomology groups $H^k(M,\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ over a strongly q–convex manifold is Hausdorff for $k\ge q$. If $\mathcal{F}=\mathcal{O}(E)$ is the sheaf of sections of a holomorphic vector bundle, this topology is given by the Fréchet topology on the Dolbeault complex of $L^2_{\rm loc}$ forms with $L^2_{\rm loc}$ $\overline\partial$–differential. In particular, the morphism

$$\mathrm{Ker}^{r,s}\overline\partial_\chi\longrightarrow H^s(M,\Omega^r)$$

is continuous and has a closed kernel, and therefore this kernel contains $\overline{\mathrm{Im}^{r,s}\,\overline\partial_\chi}$. We thus obtain a factorization

$$\mathcal{H}^{r,s}_\chi(M)\simeq\mathrm{Ker}^{r,s}\,\overline\partial_\chi/\overline{\mathrm{Im}^{r,s}\,\overline\partial_\chi}\longrightarrow H^s(M,\Omega^r)\ .$$

Consider the direct limit

$$\lim_{\overrightarrow{\chi}}\ \mathcal{H}^{r,s}_\chi(M)\longrightarrow H^s(M,\Omega^r)\tag{3}$$

over the set of smooth convex increasing functions χ with the ordering

$$\chi_1\preccurlyeq\chi_2\iff\chi_1\le\chi_2\ \text{ and }\ L^{2,(k)}_{\chi_1}(M)\subset L^{2,(k)}_{\chi_2}(M)\ \text{ for }\ k=r+s\ ;$$

this ordering is filtering by the proof of lemma 13. It is well known that the De Rham cohomology groups are always Hausdorff, hence there is a similar morphism

$$\lim_{\overrightarrow{\chi}}\ \mathcal{H}^{k}_\chi(M)\longrightarrow H^k(M,\mathbb{C})\ .\tag{4}$$

The first decomposition in theorem 3 follows now from (1) and the following simple lemma.

Lemma 12.— *The morphisms* (3), (4) *are one-to-one for* $k=r+s\ge n+q$ *.*

Proof.— Let us treat for example the case of (3). Let u be a smooth $\overline\partial$–closed form of bidegree (r,s) , $r+s\ge n+q$. Then there is a choice of χ for which $u\in L^{2,(r,s)}_\chi(M)$, so $u\in\mathrm{Ker}^{r,s}\,\overline\partial_\chi$ and (3) is surjective. If a class $\{u\}\in\mathcal{H}^{r,s}_{\chi_0}(M)$ is mapped to zero in $H^s(M,\Omega^r)$, we can write $u=\overline\partial v$ for some smooth form v of bidegree $(r,s-1)$. In the case $r+s>n+q$, we have $v\in L^{2,(r,s-1)}_\chi(M)$ for some $\chi\succcurlyeq\chi_0$. Hence the class of $u=\overline\partial_\chi v$ in $\mathcal{H}^{r,s}_\chi(M)$ is zero and (3) is injective. When $r+s=n+q$, the form v need not lie in any space $L^{2,(r,s-1)}_\chi(M)$, but it suffices to show that $u=\overline\partial v$ is in the closure of $\mathrm{Im}^{r,s}\,\overline\partial_\chi$ for some χ . Let $\theta\in C^\infty(\mathbb{R},\mathbb{R})$ be a cut-off function such that $\theta(t)=1$ for $t\le 1/2$, $\theta(t)=0$ for $t\ge1$ and $|\theta'|\le3$. Then

$$\overline\partial\big(\theta(\varepsilon\psi)v\big)=\theta(\varepsilon\psi)\overline\partial v+\varepsilon\theta'(\varepsilon\psi)\overline\partial\psi\wedge v\ .$$

By the proof of lemma 11, there is a continuous function $C(x)>0$ such that $|v|^2_\chi dV_\chi\le C\big(1+\chi''(\psi)/\chi'(\psi)\big)|v|^2dV$, whereas $|\overline\partial\psi|^2_\chi\le1/\chi''(\psi)$ by the definition of ω_χ . Hence we see that

$$\int_M|\theta'(\varepsilon\psi)\overline\partial\psi\wedge v|^2_\chi dV_\chi\le 9\int_M C\big(1/\chi''(\psi)+1/\chi'(\psi)\big)|v|^2dV$$

is finite for χ large enough, and $\overline{\partial}\big(\theta(\varepsilon\psi)v\big)$ converges to $\overline{\partial}v = u$ in $L^{2,(r,s)}_\chi(M)$. $\square$

By Poincaré-Serre duality, the groups $H^k_c(M,\mathbb{C})$ and $H^s_c(M,\Omega^r)$ with compact supports are dual to $H^{2n-k}(M,\mathbb{C})$ and $H^{n-s}(M,\Omega^{n-r})$ as soon as the latter groups are Hausdorff and finite dimensional. This is certainly true for $k = r+s \le n-q$, thus we also obtain a Hodge decomposition

$$(5)\qquad H^k_c(M,\mathbb{C}) \simeq \bigoplus_{r+s=k} H^s_c(M,\Omega^r)\ ,\quad H^r_c(M,\Omega^s) \simeq \overline{H^s_c(M,\Omega^r)}\ ,\quad k \le n-q\ .$$

As in Ohsawa [Oh1], it is easy to prove that the Lefschetz isomorphism

$$(6)\qquad \omega_\chi^{n-r-s}\wedge\bullet\ \ :\ \ \mathcal{H}^{r,s}_\chi(M) \longrightarrow \mathcal{H}^{n-s,n-r}_\chi(M)$$

yields in the limit an isomorphism from the cohomology with compact support onto the cohomology without supports. Indeed, the natural morphism

$$(7)\qquad H^s_c(M,\Omega^r) \longrightarrow \mathrm{Ker}^{r,s}\,\overline{\partial}_\chi/\overline{\mathrm{Im}^{r,s}\,\overline{\partial}_\chi} \simeq \mathcal{H}^{r,s}_\chi(M)\ ,\qquad r+s \le n-q$$

is dual to $\mathcal{H}^{n-r,n-s}_\chi(M) \longrightarrow H^{n-s}(M,\Omega^{n-r})$, which is surjective for χ large by lemma 11 and the finite dimensionality of the target space. Hence (7) is injective for χ large and after composition with (6) we get an injection

$$H^s_c(M,\Omega^r) \longrightarrow \mathcal{H}^{n-s,n-r}_\chi(M)\ .$$

If we take the direct limit over all χ , combine with the isomorphism (3) and observe that ω_χ has the same cohomology class as ω , we obtain an injective map

$$(8)\qquad \omega^{n-r-s}\wedge\bullet\ \ :\ \ H^s_c(M,\Omega^r) \longrightarrow H^{n-r}(M,\Omega^{n-s})\ ,\qquad r+s \le n-q\ .$$

As both sides have the same dimension by Serre duality and Hodge symmetry, this map must be an isomorphism. Since (8) can be factorized through $H^s(M,\Omega^r)$ or through $H^{n-r}_c(M,\Omega^{n-s})$, we infer that the natural morphism

$$(9)\qquad H^s_c(M,\Omega^r) \longrightarrow H^s(M,\Omega^r)$$

is injective for $r+s \le n-q$ and surjective for $r+s \ge n+q$. Of course, similar properties hold for the De Rham cohomology groups.

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