# Cohomology of Quotients in Symplectic and Algebraic Geometry

Frances Clare Kirwan

# Contents

# 1 Introduction

The aim of these notes is to develop a general procedure for computing the rational cohomology of quotients of group actions in algebraic geometry. The main results were announced in [Ki].

We shall consider linear actions of complex reductive groups on nonsingular complex projective varieties. To any such action there is an associated projective 'quotient' variety defined by Mumford in [M]. This quotient variety does not coincide with the ordinary topologiacl quotient of the action. For example, consider the action of SL(2) on complex projective space $\mathbb{P}_n$ where $\mathbb{P}_n$ is identified with the space of binary forms of degree $n$, or equivalently of unordered sets of $n$ points on the projective line $\mathbb{P}_1$. The orbit where all $n$ points coincide is contained in the closure of every ther orbit and hence the topological quotient cannot possibly be given the structure of a projective variety. To obtain a quotient which is a variety, such 'bad' orbits have to be left out.

The quotient varity can be described as follows. Suppose that $X$ is a projective variety embedded in some $\mathbb{P}_n$ and that $G$ is a complex reductive group acting on $X$ via the homomorphism $\phi : G \to \mathrm{GL}(n+1)$. If $A(X)$ denotes the graded coordinate ring of $X$ then the invariant subring $A(X)^G$ is a finitely generated graded ring: let $M$ be its associated projective variety. The inclusion of $A(X)^G$ into $A(X)$ induces a $G$-invariant surjective morphism $\psi$ from an open subset $X^{\mathrm{ss}}$ of $X$ to $M$. (The points of $X^{\mathrm{ss}}$ are called semistable for the action). There is an open subset $M'$ of $M$ which is an orbit space for the action of $G$ on its inverse image under $\psi$, in the sense that each fibre is a single orbit of $G$.

So we have two 'quotients' $M$ and $M'$ associated to the action of $G$ on $X$. Our main purpose here is to find a procedure for calculating the cohomology, or at least the Betti numbers, of these in the good caes when they coincide. This happens precisely when $M$ is topologically the ordinary quotient $X^{\mathrm{ss}}/G$. In fact, we make the slightly stronger requirement that the stabiliser in $G$ of every semistable point of $X$ should be finite; this is equivalent to requiring that every semistable point should be (properly) stable. Under these conditions, *an explicit formula is obtained for the Betti numbers of the quotient* $M$ (see theorem 8.12). This formula involves the cohomology of $X$ and certain linear sections of $X$, together with the classifying spaces for $G$ and certain reductive subgroups of $G$.

For example, consider again the action of SL(2) on binary forms of degree $n$. Then good cases occur when $n$ is odd, and one finds that the nonzero Betti number of the quotient $M$ are given by

$$\dim H^{2j}(M;\mathbb{Q}) = \left[1 + \frac{1}{2}\min\left(j, n-3-j\right)\right], \qquad 0 \le j \le n-3$$

Our approach to the problem follows the method used by Atiyah and Bott to calculate the cohomology of moduli spaces of vector bundles over Riemann surfaces [A & B]. It consists in finding a canonical stratification[1] of $X$ associated to the action of $G$ whose unique open stratum

[1]It has been pointed out by the referee that the term 'stratification' is usually reserved for the decomposition

coincides with the set $X^{ss}$ of semistable points provided $X^{ss} \neq \emptyset$. There are then Morse-type inequalities relating the Betti numbers of $X$ to those of $X^{ss}$ and the other strata; and since the stratification is $G$-invariant there are also exist *equivariant Morse inequalities* which turn out in fact to be *equalities*.

Recall that the rational equivariant cohomology $H^\star(Y;\mathbb{Q})$ of a space $Y$ acted on by $G$ is defined to be $H^\star(Y_G;\mathbb{Q})$ where $Y_G = Y \times_G EG$ and $EG \to BG$ is the universal classifying bundle for $G$. When the rational equivariant Morse inequalities of a stratification are equalities they can be stated in the form

1.1
$$\dim H^n_G(Y;\mathbb{Q}) = \sum_S \dim H_G^{n-\lambda(S)}(S;\mathbb{Q}), \qquad n \geq 0$$

where the sum runs over all the strata $S$ of the stratification and $\lambda(S)$ is the codimension of $S$ in $X$ (see [A&B] §1). Moreover, using the assumption that every point of $X^{ss}$ has finite stabilisr in $G$ we can show that

$$H^\star_G(X^{ss};\mathbb{Q}) = H^\star(X^{ss}/G;\mathbb{Q}) = H^\star(M;\mathbb{Q})$$

Hence the Morse equalities will give us formulae for the Betti numbers of $M$ in terms of the rational equivariant cohomology of $X$ and of the other strata.

The Morse inequalities are obtained by building up $X$ from the strata and using the Thom-Gysin sequences of rational equivariant cohomology that occur every time a stratum is added. Of course, any coefficient field may be used instead of $\mathbb{Q}$, but then the Morse inequalities are not necessarily equalities, and the cohomology of the quotient $M$ may not be isomorphic to the equivariant cohomology of $X^{ss}$. So information about the torsion of $M$ can only be obtained in special cases.

As in [A & B] there are two different approaches to the problem of defining a suitable stratification. One approach is purely algebraic, and leads to a definition of a stratification given a linear reductive group action on a projective variety defined over any algebraically closed field. This method will be developed in Part II. It is based on work of Kempf (see [K] and [Hess] and [K & N]). The paper [Ne] by Ness has very close links with much of what is covered here and in Part I, although our results were arrived at independently.

The alternative approach is based on Morse theory and symplectic geometry, and will be developed in Part I. The idea is to associate a certain function $f$ in a canonical way to the action of $G$ on $X$ and use it to define a 'Morse stratification' of $X$. The stratum to which any point of $X$ belongs is determined by the limit of its path of steepest descent for the Kähler metric under the function $f$.

which is topologically locally trivial in a neighbourhood of each stratum (Whitney stratifications, for example). The stratifications in these notes are not required to satisfy this property (see definition 2.11 below). Perhaps they should be more properly called 'manifold decompositions'.

The advantages of this approach are that it is conceptually simpler and that it can be applied to compact Kähler manifolds as well as to nonsingular projective varieties. More generally still it enables us to calculate the rational cohomology of the 'symplectic quotient', when it exists, of any symplectic manifold by the actio of a compact Lie group.

The function to which Atiyah and Bott apply the methods of Morse theory in their special case (where the group and the space are both infinite-dimensional) is the Yang-Mills functional. As pointed out in [A & B] the latter can be described in terms of symplectic geometry as the norm-square of the moment map. But in this form it makes sense in our situation.

Recall that a symplectic manifold $X$ is a smooth manifold equipped with a nondegenerate closed 2-form $\omega$; and a compact Lie group $K$ acts symplectically on the manifold if its acts smoothly and preserves $\omega$. Associated to such an action one has the concept of a moment map $\mu : X \to \mathfrak{k}^\star$ where $\mathfrak{k}^\star$ is the dual of the Lie algebra of $K$. For example, when SO(3) acts on the cotangent bundle $T^\star\mathbb{R}^3$ (phase space) the moment map can be identified with angular momentum. The existenc e of a moment map is guaranteed by conditions such as the semisimplicity of $G$ or the vanishing of $H^1(X;\mathbb{Q})$.

Consider again a reductive group $G$ acting on a nonsingular complex projectiv variety $X \subseteq \mathbb{P}_n$ via a homomorphism $\phi : G \to \mathrm{GL}(n+1)$. Since $G$ is reductive, it is the complexification of a maximal subgroup $K$. We may assume that $K$ acts unitarily on $\mathbb{C}^{n+1}$ and so preserves the standard Kähler structure on $\mathbb{P}_n$. This Kähler structure makes $X$ into a symplectic manifold on which $K$ acts symplectically. (It also gives a natural choice of Riemannian metric on $X$). There is a moment map $\mu : X \to \mathfrak{k}^\star$ assocaited to this action which can be described explicitly (see 2.7). If we fix an invariant inner product on the Lie algebra of $K$ then the norm-square of the moment map $\mu$ provides us with a $K$-invariant Morse function $f$ on $X$.

Unfortunately, the Morse function is not nondegenerate in the sense of Bott, so the results of Morse theory cannot be applied to it directly. To avoid this problem, one can use the approach of Part II to define the stratification algebraically and prove that it has all the properties one wants, showing later that it is is in fact in a natural sense the Morse stratification for $f$. On the other hand, of one is prepared to do a little local analysis, one can extend the arguments of Morse theory to degenerate functions which are reasonably well-behaved. It will be shown that the norm-square of the moment map is well-behaved in this sense: this is true when $X$ is any symplectic manifold ated on by a compact group.

More precisely, we shall see that the set of critical points for the function $f = ||\mu||^2$ is a finite disjoint union of closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$ along each of which $f$ is *minimally degenerate* in the following sense: A locally closed submanifold $\Sigma_\beta$ containing $C_\beta$ with orientable normal bundle in $X$ is a *minimising submanifold* for $f$ if

1. the restriction of $f$ to $\Sigma_\beta$ achieves its minimum value exactly on $C_\beta$ and

2. the tangent space to $\Sigma_\beta$ at any point $x \in C_\beta$ is maximal among subspaces of $T_xX$ on which

the Hessian $H_x(f)$ is positive definite.

If a minimising submanifold $\Sigma_\beta$ exists, then $f$ is called minimally degenerate along $C_\beta$.

In the appendix it is shown that these conditions imply that $f$ induces a smooth stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$ such that a point lies in the stratum $S_\beta$ if its path of steepest descent for $f$ has a limit point in the critical subset $C_\beta$. (For this $X$ must be given a suitable metric; when $X$ is Kähler and $f = ||\mu||^2$ the Kähler metric will do). The stratum $S_\beta$ then coincides with $\Sigma_\beta$ near $C_\beta$. The proof is not hard but involves some analysis of differential equations near critical points.

As has already been mentioned, it turns out that the unique open stratum of this stratification coincides with the set $X^{\mathrm{ss}}$; and that in good cases its $G$-equivariant rational cohomology is isomorphic to the ordinary rational cohomology of the quotient variety $M$, which is what we're after. Morever, the stratification is $G$-invariant and equivariantly perfect over $\mathbb{Q}$ in the sense that its equivariant Morse inequalities are in fact equalities. Thus, formula 1.1 can be used to calculate the Betti numbers of $M$ in terms of the equivariant cohomology of $X$ itself and of the nonsemistable strata.

In order that this formula should be useful, it is necessary to investigate the nonsemistable strata. It turns out that the equivariant cohomology of these can be calculated inductively. In fact, each stratum $S_\beta$ has the form

1.2
$$S_\beta \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$$

where $Y_\beta^{\mathrm{ss}}$ is a locally closed nonsingular subvariety of $X$ and $P_\beta$ is a parabolic subgroup of $G$ (see theorem 6.18). This implies that the $G$-equivariant cohomology of $S_\beta$ is isomorphic to the $P_\beta$-equivariant cohomology of $Y_\beta^{\mathrm{ss}}$. Moreover, there is a linear action of a maximal reductive subgroup of $P_\beta$ on a proper nonsingular closed subvariety $Z_\beta$ of $X$ such that $Y_\beta^{\mathrm{ss}}$ retracts equivariantly onto the subset $Z_\beta^{\mathrm{ss}}$ of semistable points for this action. It follows that $H^\star_{P_\beta}(Y_\beta^{\mathrm{ss}};\mathbb{Q})$ is isomorphic to the rational equivariant cohomology of $Z_\beta^{\mathrm{ss}}$ with respect to this reductive subgroup. By induction, we may assume that this is known.

It now remains to consider the equivariant cohomology $H^\star_G(X;\mathbb{Q})$ of $X$ itself. We shall assume for convenience that $G$ is connected; then one can show (see Proposition 5.8) that the spectral sequence of the fibratin

$$X_G = X \times_G EG \to BG$$

degenerates over $\mathbb{Q}$. This means that the equivariant cohomology of $X$ is isomorphic to the tensor product

$$H^\star(X;\mathbb{Q}) \otimes H^\star(BG;\mathbb{Q})$$

of the cohomology of $X$ and $BG$. (It is easy to deduce from this what happens for disconnected groups: For if $G$ has identity component $\Gamma$, then $H^\star_G(X;\mathbb{Q})$ is the invariant part of $H^\star_\Gamma(X;\mathbb{Q})$ under the action of the finite group $G/\Gamma$).

Thus the formula for $H^\star_G(X^{\text{ss}};\mathbb{Q})$ in terms of the equivariant cohomology of $X$ and of the nonsemistable strata give us an *inductive procedure* for calculating $H^\star_G(X^{\text{ss}};\mathbb{Q})$. This leads to an *explicit formula* for $H^\star_G(X^{\text{ss}};\mathbb{Q})$ and hence also in good cases for *the Betti numbers of the quotient variety* $M$. This fomrula involves the cohomology of $X$ and certain nonsingular subvarieties of $X$ together with the cohomology of the classifying spaces of $G$ and various reductive subgroups of $G$ (see theorem 8.12).

The stratification induced by the norm-square of the moment map has also been studies by Ness in [Ne]. Moreover, related research on Betti numbers of quotients by $\mathbb{C}^\star$ and $\mathrm{SL}(2,\mathbb{C})$ actions has been done independently by Bialynicki-Birula and Sommese. In fact, in their paper [B-B & S], they conside quotients of many different open subsets by $G$, not just $X^{\text{ss}}$, and completely classify those subsets for which quotients exist.

When $X$ is merely a compact symplectic manifold acted on by a compact group $K$, the function $f = ||\mu||^2$ still induces a smooth stratification of $X$, although most of the structure of the Morse strata is lost. The lost of structure is to be expected because the stratification depends on choosing a $K$-invariant Riemannian metric on $X$ and there is no longer a natural choice given by the real part of the $K$-invariant metric Kähler metric. So we concentrate on the *critical subsets* $C_\beta$ instead (which are not necessarily submanifolds of $X$).

In fact, the form in which the Morse inequalities are usually stated is that in which the cohomology of each Morse stratum $S_\beta$ is replaced by that of its critical subset $C_\beta$. This replacelemtn is allowable because the inclusion of $C_\beta$ in $S_\beta$ is an equivalence of both equivariant and ordinary Čech cohomology. These critical subsets are independent of the choice of metric. They have the following description in terms of minimal sets for small manifolds which is analogous to 1.2. For each $\beta$, there is a symplectic submanifold $Z_\beta$ of $X$ acted on by a compact subgroup $\mathrm{Stab}\,\beta$ of $G$ and a moment map $\mu_\beta$ for this action such that

$$C_\beta \cong K \times \times_{\mathrm{Stab}\,\beta} \mu_\beta^{-1}(0).$$

Since $f$ is equivariantly perfect and

$$H^\star_K(X;\mathbb{Q}) \cong H^\star(X;\mathbb{Q}) \otimes H^\star(BK;\mathbb{Q})$$

we obtain an inductive procedure for calculating the dimensions of the equivariant cohomology groups $H^n_K(\mu^{-1}(0);\mathbb{Q})$ of the minimum critical set $\mu^{-1}(0)$ for $f$.

The reason by $H^\star_K(\mu^{-1}(0);\mathbb{Q})$ is interesting is that when a symplectic quotient of the action of $K$ on $X$ exists, then its rational cohomology is isomorphic to $H^\star_K(\mu^{-1}(0);\mathbb{Q})$. In order that the symplectic quotient should exist in a reasonable sense one has to assume that there is a moment map $\mu : X \to \mathfrak{k}^\star$ and that the stabiliser in $K$ of every $x \in \mu^{-1}(0)$ is finite. Then one finds that $\mu^{-1}(0)$ is a submanifold of $X$, and that the Kähler form $\omega$ on $X$ induces a symplectic structure on the topological quotient $\mu^{-1}(0)/K$ which is a manifold except for singularities due to the presence of finite isotropy groups. With this structure $\mu^{-1}(0)/K$ is a natural symplectic quotient (or Marsden-Weinstein reduction) of $X$ by $K$. Because of the assumption on stabilisers, its rational cohomology is isomorphic to $H^\star_K(\mu^{-1}(0);\mathbb{Q})$.

As we have already seen, the link between algebraic and symplectic geometry is through Kähler geometry. Except for the connection with semistability and invariant theory, the results for projective varieties hold when $X$ is any compact Kähler manifold acted on by a complex group $G$, provided that $G$ is the complexification of a maximal compact subgroup $K$ which preserves the Kähler structure on $X$ and that there exists a moment map $\mu : X \to \mathfrak{k}^\star$. We obtain an equivariantly perfect stratification of $X$ such that each stratum is a locally closed complex submanifold of $X$ and can be decomposed in a form analogous to that described in 1.2. Moreover it turns out that if the symplectic quotient $\mu^{-1}(0)/K$ exists then it can be identified with the quotient of the minimum stratum $X^{\min}$ by the complex group $G$. Because of this it can be givne the structure of a compact Kähler manifold, except for singularities caused by finite isotropy groups. So

$$X^{\min}/G = \mu^{-1}(0)/K$$

is a natural Kähler quotient of $X$ by $G$, and its Betti numbers can be calculated by the method already described.

In particular, in the case of a linear action on a projective variety, the quotietn $M$ obtained from invariant theory coincides topologically with the quotient $\mu^{-1}(0)/K$; inf act, this is true in all cases, not only good ones.

The set $\mathcal{B}$ which indexes the critical subsets $C_\beta$ and also the stratification can be identified with a finite set of orbits of the adjoint representation of $K$ on its Lie algebra $\mathfrak{k}$. Each orbit in $\mathcal{B}$ is the image under the moment map $\mu : X \to \mathfrak{k}^\star \cong \mathfrak{k}$ of the critical subset which it indexes. If a choice is made of a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra of some maximal torus of $K$, then each adjoint orbit intersects $\mathfrak{t}_+$ in a unique point, so $\mathcal{B}$ can be regarded alteratively as a finite set of points in $\mathfrak{t}_+$. When $X \subseteq \mathbb{P}_n$ is a projective variety on which $K$ acts linearly via a homomorphism $\phi : K \to \mathrm{GL}(n+1)$, these points can be described in terms of the weights of the representation of $K$ given by $\phi$ as follows: A point of $\mathfrak{t}_+$ lies in $\mathcal{B}$ if it is the closest point to the origin of the convex hull of a nonempty set of these weights. (Recall that there is a fixed invariant inner product on $\mathfrak{k}$ which is used to identify $\mathfrak{k}$ with its dual). This is true also in the general symplectic case if the definition of weight is extende appropriately.

In terms of this last description, if $\beta \in \mathcal{B}$ then the submanifold $Z_\beta$ of $X$ which appeared in the inductive description of the critical subset $C_\beta$ and of the stratum $S_\beta$ is the union of certain components of the fixed points set of the subtorus of $K$ generated by $\beta$. This subgroup $\operatorname{Stab}\beta$ is the stabiliser of $\beta$ under the adjoint action of $K$ on its Lie algebra, and in the Kähler case, the complexification of Stab $\beta$ is a maximal reductive subgroup of the parabolic subgroup $P_\beta$.

The function $f = ||\mu||^2$ is not unique in possessing the properties described above. The same arguments work for any convex function of the moment map (cf. [A & B] §§8 and 12).

Finally, it should be noted that the assumption of the compactness is not essential (see §9). There are interesting examples of quasi-projective varieties and noncompact symplectic manifolds to which the same sort of analysis can be applied by taking a little extra care. These include the original examples of symplectic manifolds, viz. cotangent bundles.

The layout of the first part is as follows. §§2-5 are concerned with any symplectic action of a compact group $K$ on a compact symplectic manifold $X$. In §2 we introduce the moment map $\mu$, giving particular emphasis to the case when a compact group acts linearly on a nonsingular complex projective varity. We then describe the Morse stratification associated to the nondegenerate Morse function, and discuss how the ideas of Morse theory might be applied to $f = ||\mu||^2$ even though it is degenerate. In §3 we describe the set of critical points for $f$ as a finite disjoint union of closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$. It is then shown in §4 that $f$ is minimally degenerate along each critical subset $C_\beta$. This implies that there are Morse inequalitie relating the Betti numbers of the symplectic manifold $X$ to those of the subsets $C_\beta$; the proof of this fact is left to the appendix. In §5 these Morse inequalities are shown to be equalities for rational equivariant cohomology (see theorem 5.4). Inductive and explicit formulae are obtained for the dimensions of the cohomology groups $H^n_K(\mu^{-1}(0);\mathbb{Q})$ and it is shown that these coincide with the Betti numbers of the symplectic quotient $\mu^{-1}(0)/K$ when it exists.

The next two sections study the case when $X$ is a Kähler manifold so that there is a natural choice of metric on $X$. In §6 we see that the function $f = ||\mu||^2$ induces a Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ with respect to this metric such that the strata $S_\beta$ are localled closed complex submanifolds of $X$ and are invariant under the action of the complex group $G$. It is also known that the strata $S_\beta$ have the structure described in 1.2 above. The cohomological formulae of §5 are interpreted in the Kähler case, and there is a brief discussin of how the stratification is affected if the choices of moment map and of inner product on the Lie algebra $\mathfrak{k}$ are altered. In §7 we see that that if a symplectic quotient exists for the action of $K$ on $X$, then it has a natural Kähler structure and can be regarded as a Kähler quotient of the action of $G$ on $X$.

Then in §8 we consider the case when $G$ is a complex reductive group acting linearly on $X$ which is a nonsingular complex projective variety. It is showns that the open subset $X^{\mathrm{ss}}$ of semistable points for the action coincides with the minimum stratum of the Morse stratification, so that §5 gives us an inductive formula for its rational equivariant cohomology. In good cases, when the stabiliser of every semistable point is finite, we deduce that the projective quotient variety defined in geometric invariant theory coincides with the symplectic quotient $\mu^{-1}(0)/K$. Our original aim is then achieved by interpreting the formulae of §5 to give formulae for the Betti numbers of this quotient variety (see Theorem 8.12).

Section 9 contains some remarks on how to loosen the requirement of compactness. Examples are given of formulae obtained by looing at the symplectic actions on cotangent bundles induced by arbitrary actions of compact groups on manifolds.

Part II gives an algebraic approach to the same problem. It is shown in §§12 and 13 that if $k$ is any *algebraically closed field* and $G$ is a reductive group acting linearly on a projective variety $X$ defined over $k$, then a stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$ can be defined which coincides with the stratification which coincides with the stratification defined in Part I when $k = \mathbb{C}$. The strata $S_\beta$ are all $G$-invariant subvarieties of $X$. Moreover, if $X$ is nonsingular then so are the strata $S_\beta$, and they have the structure described at 1.2 This algebraic definitio of the stratification relies heavily on the work of Kempf (as expounded in [He]).

The fact that such stratifications exist when $k$ is the algebraic closure of a finite field provides an alternative method for obtaining the formulae found in Part I for the Betti numbers of quotients of nonsingular complex projective varieties. For this one has t count points in quotients defined over finite fields, and then apply the Weil conjectures (see §15). This is the method used by Harder and Narasimhanin [H & N] to obtain formulae later rederived by Atiyah and Bott for the Betti numbers of the moduli spaces of vector bundles over a Riemann surface.

It is shown in §14 how the formulae for the Betti numbers can be refined to give Hodge numbers as well. As an immediate corollary we have that if the Hodge numbers $h^{p,q}$ of the variety $X$ vanish when $p \neq q$, then the same is true of the quotient variety.

In the final section, some detailed examples are given of the stratification and of calculating the rational cohomology of the quotients. One example studies is that of products of Grassmannians acted on by general linear groups. It will be shown in a future paper [Ki3] that this can be used to give an alternative derivation of the formulae of [A & B] for the cohomology of moduli spaces of vector bundles over Riemann surfaces. This alternative derivation uses finite-dimensional group actions whereas in [A&B] the groups and spaces are all infinite dimensional.

The formulae for the Betti numbers obtained in this monograph depend upon the restrictive assumption that the stabiliser of every semistable point is finite. This assumption implies in particular that the quotient variety has only the minor singularities due to the existence of finite isotropy groups, whereas in general the quotient has more serious singularities. However, provided that $X^{\mathrm{s}}$ is not empty, one can obtain interesting information even when there are semistable points which are not stable. In fact, there is a canonical way to blow up $X$ along a sequence of nonsingular subvarieties to obtain a projective variety $\tilde{X}$ with a linear action of $G$ for which every semistable point is stable. Then the geometric invariant theory quotient of $\tilde{X}$ (which has only minor singularities) can be regarded as an approximate desingularisation of the quotient of $X$, and there is a formula for its Betti numbers similar to that of Theorem 8.12 (see [Ki2]).

Finally, there are some differences of notation and also some inaccuracies in the announcement of these results in [Ki1]. One mistake is that the theorme as it stands is only valid when $G$ is connected, because remark (1) is only true in this case. Another is that in (d) of the proposition it is only the reductive part Stab $\beta$ of the parabolic subgroup $P_\beta$ which acts on $Z_\beta$, not the whole of $P_\beta$. Furthermore, the last sentence might be taken to imply that the geometric invariant theory quotient of a product of Grassmanians is torsion-free. This is not true since the projective inear groups $\mathrm{PGL}(m, \mathbb{C})$ have torsion.

I would like to thank all those who gave me help and advice, including Michael Pennington, Simon Donaldson, Michael Murray, John Roe, Graeme Segal and the referee, and to thank Linda Ness for sending me her results. I also thank Laura Schlesinger for her excellent typing, and the Science and Engineering Research Council of Great Britain for a grant which supported me during the course of my research. Above all, I wish to acknowledge my great debt to my supervisor Michael Atiyah, to whom most of the basic ideas of these notes are due.

# Part I. The Symplectic Approach

## 2 The moment map

This section introduces the concept of a moment map associated to a compact group action on a symplectic manifold. Special emphasis will be given to the examples of most interest to us, which are linear actions on nonsingular complex projective varieties. A precise formula is given in 2.7 for the moment map in these cases.

The moment map will be used to define a real valued function on the symplectic manifold concerned. We shall conclude this section by considering how the ideas of Morse theory might be applied to this function, in spite of the fact that it is not a nondegenerate Morse function.

A symplectic manifold is a smooth manifold $X$ equipped with a nondegenerate closed 2-form $\omega$. A compact Lie group $K$ is said to act symplectically on $X$ if $K$ acts smoothly and $k^\star\omega = \omega$ for all $k \in K$. We shall assume throughout that every compact group action on a symplectic manifold is symplectic unless specified otherwise.

Any Kähler manifold $X$ can be given the structure of a symplectic manifold by taking $\omega$ to be the Kähler form on $X$, which is the imaginary part of a hermitian metric $\eta$ on $X$. If $K$ is any compact Lie group acting on $X$ then the average

$$\int_K k^\star \eta$$

is a Kähler metric whose imaginary part is a $K$-invariant symplectic form on $X$.

The special case which will be of the most interest to us is the following.

**Example 2.1. Linear actions on complex projective spaces.**

Let $X$ be a nonsingular subvariety of some complex projective space $\mathbb{P}_n$ and suppose that a compact Lie group $K$ acts on $\mathbb{P}_n$ via a homomorphism $\varphi : K \to \mathrm{GL}(n+1)$. By conjugating $\varphi$ with a suitable element of $\mathrm{GL}(n+1)$ we may assume that $\varphi(K)$ is contained in the unitary group $\mathrm{U}(n+1)$. The restriction of the Fubini-Study metric on $\mathbb{P}_n$ then gives $X$ a Kähler structure which is preserved by $K$.

**Example 2.2. Configuration of points on the complex sphere.**

A particular case of 2.1 which will be used throughout to illustrate definitions and results is that of the diagonal action of $\mathrm{SU}(2)$ on the spaces $(\mathbb{P}_1)^n$ of sequences of points on the complex sphere. $(\mathbb{P}_1)^n$ is embedded in $\mathbb{P}_{2^n-1}$ by the Segre embedding. Alternatively one can consider the action of $\mathrm{SU}(2)$ on the space of unordered sets of $n$ points in $\mathbb{P}_1$ which can be identified with $\mathbb{P}_n$.

Let $\mathfrak{k}$ be the Lie algebra of $K$. Then the moment map for the action of $K$ on $X$ is a map $\mu : X \to \mathfrak{k}^\star$ which is $K$-equivariant with respect to the given action of $K$ on $X$ and the co-adjoint action $\mathrm{Ad}^\star$ of $K$ on $\mathfrak{k}^\star$ and satisfies the following condition.

**2.3**. For every $a \in \mathfrak{k}$ the composition of $d\mu : TX \to \mathfrak{k}^\star$ with evaluation at $a$ defines a 1-form $\omega$ on $X$. This 1-form is required to correspond under the duality defined by $\omega$ to the vector field $x \to a_x$ on $X$ induced by $a$. That is, for all $x \in X$ and $\xi \in T_xX$

$$d\mu(x)(\xi).a = \omega_x(\xi, a_x)$$

where . denotes the natural pairing of $\mathfrak{k}^\star$ and $\mathfrak{k}$. In other words the component of $\mu$ along $a$ is a Hamiltonian function for the vector field on $X$ defined by $a$.

$\mu$ is determined up to an additive constant by 2.3. When $K$ is semisimple $\mu$ is determined completely, since the only point of $\mathfrak{k}^\star$ fixed by the co-adjoint action is 0. If on the other hand $K$ is a torus the addition of a constant to $\mu$ does not affect its equivalence because $K$ acts trivially on $\mathfrak{k}^\star$. However if a moment map $\mu$ exists we can always make a canonical choice of $\mu$ by requiring that the integral of $\mu$ over $X$ (with the highest exterior power of $\omega$ as volume form) should vanish.

By a theorem of Marsden-Weinstein a moment map $\mu : X \to \mathfrak{k}^\star$ always exists (and is unique) when $K$ is semisimple. In addition if $H^1(X;\mathbb{Q}) = 0$ then a moment map always exists when $K$ is a torus. For the adjoint action of a torus on its Lie algebra is trivial, so by 2.3 we just have to solve the differential equations

$$d\mu(x)(\xi).a = \omega_x(a_x, \xi)$$

for each $a$ in some basis of the Lie algebra $\mathfrak{k}$. This is possible if $H^1(X, \mathbb{Q}) = 0$ since $d\omega = 0$.

A compact Lie group is the product of a semisimple group and a torus, at least modulo finite central extensions. Moreover if $K_1 \to K_2$ is a finite central extension then a moment map for $K_1$ is the same as a moment map for $K_2$. It follows that a moment map always exists when $H^1(X, \mathbb{Q}) = 0$.

It is easy to see that when $K$ acts on $X \subset \mathbb{P}_n$ via a homomorphism $\varphi : K \to \mathrm{U}(n+1)$ a moment map always exists. It is sufficient to prove existence when $\mathrm{U}(n+1)$ acts $\mathbb{P}_n$ since $\mu$ is a moment map for this action then the composition

2.4.

$$X \to \mathbb{P}_n \xrightarrow{\mu} \mathfrak{u}(n+1)^\star \xrightarrow{\varphi^\star} \mathfrak{k}^\star$$

is a moment map for the action of $K$ on $X$. But we have

**Lemma 2.5**. Let $x^\star = (x_0, \ldots, x_n)$ be any nonzero vector of $\mathbb{C}^{n+1}$ over $x = (x_0 : \ldots : x_n)$ in $\mathbb{P}^n$. Then the map $\mu : \mathbb{P}_n \to \mathfrak{u}(n+1)^\star$ defined by

$$\mu(x).a = \frac{\bar{x}^{\star\mathsf{t}} a x^\star}{2\pi ||x^\star||^2}$$

is a moment map for the action of $\mathrm{U}(n+1)$ on $\mathbb{P}_n$. Morever, $\mu$ is uniquely determined up to the addition of a scalar multiple of the trace.

*Proof.* Note first that

$$\mathfrak{u}(n+1) = \mathfrak{su}(n+1) \oplus i\mathbb{R}1_{n+1}$$

where 1 is the identity matrix. $i\mathbb{R}1_{n+1}$ is the Lie algebra of the central one-parameter compact subgroup of $\mathrm{U}(n+1)$ which acts trivially on $\mathbb{P}_n$,. The projection of $\mathfrak{u}(n+1)$ onto $i\mathbb{R}1_{n+1}$ is given by $a \to \mathrm{tr}(a)(n+1)^{-1}1_{n+1}$. Thus any moment map for $\mathrm{SU}(n+1)$ is unique and a moment for $\mathrm{U}(n+1)$ is unique up to the addition of a scalar mutliple of the trace.

Clearly the formula given for $\mu$ is independent of the choice of $x^\star$ and satisfies

$$\mu(kx).a = \frac{\bar{x}^{\star\mathsf{t}} \bar{k}^{\mathsf{t}} a k x^\star}{2\pi i ||x^\star||^2} = \mu(x).k^{-1}ak = \mathrm{Ad}^\star k \mu(x).a$$

for all $k, a \in \mathfrak{u}(n+1)$; so $\mu$ is $SU(n+1)$-equivariant.

In particular since $\mathrm{U}(n+1)$ acts transitively on $\mathbb{P}_n$, to prove that 2.3 holds it suffices to consider the point $o = (1 : 0 : \ldots : 0)$. The Kähler form at $p$ is given by

$$\omega_p = \frac{i}{2\pi} \sum_{j=1}^{n} dx_j \wedge d\bar{x}_j$$

with respect to local coordinates $(x_1, \ldots, x_n) \to (1 : x_1 : \ldots : x_n)$ near $p$. But in these coordinate the vector field induced by $a$ on $\mathbb{P}_n$ takes the values $(a_{10}, a_{20}, \ldots, a_{n0})$ at $p$. Also

$$\begin{aligned} d(2\pi i ||x^\star||^2)^{-1} \bar{x}^{\star\mathsf{t}} a x^\star) &= (2\pi i)^{-1} \sum_{j=1}^{n} (a_{0j} dx_j + a_{j0} d\bar{x}_j) \\ &= \frac{i}{2\pi} \sum (\bar{a}_{j0} dx_j - a_{j0} d\bar{x}_j) \end{aligned}$$

**Remark 2.6**. An alternative proof runs as follows. It is known that there is a natural homogeneous symplectic structure on any orbit in $\mathfrak{u}(n+1)^\star$ of the co-adjoint action of $\mathrm{U}(n+1)$ and that the corresponding moment map is the inclusion of the orbit in $\mathfrak{u}(n+1)^\star$. This is true for any compact group. The map from $\mathbb{P}_n$ to $\mathfrak{u}(n+1)$ given by

$$x \to \frac{x^\star \bar{x}^{\star\mathsf{t}}}{2\pi i ||x^\star||^2}$$

is a $\mathrm{U}(n+1)$-invariant symplectic isomorphism from $\mathbb{P}_n$ to the orbit of the skew-hermitian matrix $(2\pi i)^{-1}\mathrm{diag}(1,0,\ldots,0)$. For $x^\star\bar{x}^{\star\mathsf{t}}$ is hermitian of rank 1 with $x^\star$ as an eigenvector with eigenvalue $||x^\star||^2$. Lemma 2.5 follows from this because the inner product of $x^\star\bar{x}^{\star\mathsf{t}}$ with any $a \in \mathfrak{u}(n+1)$ is $\bar{x}^{\star\mathsf{t}}ax^\star$.

To sum up: by 2.4 and 2.5, given a nonsingular complex projective projective variety $X \subset \mathbb{P}_n$ and a compact group $K$ acting on $X$ by a homomorphism $\varphi : K \to \mathrm{U}(n+1)$ a moment map $\mu : X \to \mathfrak{k}^\star$ is defined by

2.7

$$\mu(x).a = \frac{\bar{x}^{\star\mathsf{t}}\varphi_\star(a)x^\star}{2\pi i||x^\star||^2}$$

for each $a \in \mathfrak{k}$ and $x \in X$. This moment map is functorial in $X, K$.

**2.8**. Consider the example 2.2 of configurations of points on the complex sphere $\mathbb{P}_1$ acted on by $\mathrm{SU}(2)$. Now $\mathfrak{su}(2)$ is isomorphic to $\mathbb{R}^3$ and $\mathbb{P}_1$ can be identified with $S^2$ in such a way that the moment map $\mu : (\mathbb{P}_1)^n \to \mathfrak{su}(2)$ sends a configuration of $n$ points on the sphere to its center of gravity in $\mathbb{R}^3$ (up to a scalar factor of $n$).

Henceforth we shall assume that a moment map $\mu$ exists for the action of $K$ on $X$.

Fix an inner product on $\mathfrak{k}$ which is invariant under the adjoint action of $K$ and denote the product of $a$ and $b$ by $a.b$; use it to identify $\mathfrak{k}$ with its dual.

For example if $K \subset \mathrm{U}(n+1)$ we can take the restriction to $\mathfrak{k}$ of the standard inner product given by $a.b = -\mathrm{tr}(ab)$ on $\mathfrak{u}(n+1)$. Then 2.6 implies that for each $x \in X$ the element $\mu(x)$ of $\mathfrak{k}^\star$ is identified with the orthogonal projection of the skew-hermitian matrix $(2\pi i||x^\star||^2)^{-1}x^\star\bar{x}^{\star\mathsf{t}}$ onto $\mathfrak{k}$.

Also choose a $K$-invariant Riemannian metric on $X$. If $X$ is Kähler (in particular, if $X$ is a projective variety) then the natural choice is the real part of the Kähler metric on $X$.

**Definition 2.9**. Let $f : X \to \mathbb{R}$ be the function defined by $f(x) = ||\mu(x)||^2$ where $||\,||$ is the norm on $\mathfrak{k}$ induced by the inner product.

We want to consider the function $f : X \to \mathbb{R}$ as a Morse function on $X$.

For any $x \in X$ let $\{x_t : t \geq 0\}$ be the trajectory of $-\mathrm{grad}f$ such that $x_0 = x$, i.e. the path of steepest descent of $f$ starting from $x$. Let

$$\omega(x) = \{y \in X : \text{every neighbourhood of } y \in X \text{ contains points } x_t \text{ for } t \text{ arbitrarily large}\}$$

be the set of limit points of the trajectory as $t \to \infty$. Then $\omega(x)$ is closed and nonempty (since $X$ is compact) and is connected. For suppose that there are disjoint open sets $U, V$ in $X$ such that $\omega(x) \subseteq U \cup V$. Then for every $y \notin U \cup V$ there is some $t_y \geq 0$ and a neighbourhood $W_y$ of $y$ such that $x_t \notin W_y$ for $t \geq t_y$. But $X \backslash U \cup V$ is compact so there is some $T > 0$ such that $t \geq T$ implies $x_t \in U \cup V$. Since the set $\{x_t : t \geq T\}$ is connected it is contained in $U$ or $V$, and thus $\omega(x)$ is also contained in either open. We conclude that

**2.10**. For every $x \in X$ the limit set $\omega(x)$ is connected. Also every point of $\omega(x)$ is critical for $f$.

If $f$ were a nondegenerate Morse function in the sense of Bott, then the set of critical points for $f$ on $X$ would be a finite disjoint union of connected submanifolds $\{C \in \mathcal{C}\}$ of $X$. Given such a function, 2.10 implies that for every $x$ there is a unique $C$ such that $\omega(x)$ is contained in $C$. The *Morse stratum* $S_C$ corresponding to any $C \in \mathcal{C}$ is then defined to consist of those $x \in X$ wth $\omega(x)$ contained in $C$. The strata $S_C$ retract onto the corresponding critical submanifolds $C$ and form a smooth stratification of $X$ in the following sense.

**Definition 2.11**. A finite collection $\{S_\beta : \beta \in \mathcal{B}\}$ of subsets form a stratification of $X$ if $X$ is the disjoint union of the strata $S_\beta$ and there is a strict partial order $>$ on the indexing set $\mathcal{B}$ such that

$$\bar{S}_\beta \subseteq \bigcup_{\gamma \geq \beta} S_\gamma$$

for every $\beta \in \mathcal{B}$. For the Morse stratification associated to a nondegenerate Morse function the partial order is given by $C > C'$ if $f(C) > f(C')$ where for $C \in \mathcal{C}$, $f(C)$ is the value taken by $f$ on $C$.

The stratification is smooth if every $S_\beta$ is a locally-closed submanifold of $X$ (possibly disconnected).

In fact the set of critical points for the function $f = ||\mu||^2$ has singularities in general so that $f$ cannot be a nondegenerate Morse function in the sense of Bott. Nevertheless we shall see that the critical set of $f$ is a finite disjoint union of closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$ on each of which $f$ takes a constant value. By 2.10 it follows that for every $x \in X$ there is a unique $\beta \in \mathcal{B}$ such that $\omega(x)$ is contained in $C_\beta$. So $X$ is the disjoint union of subsets $\{S_\beta\}$ where $x \in X$ lies in $S_\beta$ if the limit set $\omega(x)$ of its path of steepest descent for $f$ is contained in $C_\beta$. We shall find that for a suitable Riemannian metric the subsets $\{S_\beta : \beta \in \mathcal{B}\}$ form a smooth $K$-invariant stratification of $X$.

**Example 2.12**. The norm square of the moment map $\mu$ associated to the action of SU(2) on sequences of $n$ points in $\mathbb{P}_1$ identified with the unit sphere in $\mathbb{R}^3$ is given by

$$(x_1, \ldots, x_n) \to ||x_1 + x_2 + \ldots + x_n||^2$$

where $|| \; ||$ is the usual norm on $\mathbb{R}^3$. As is always the case $||\mu||^2$ takes its minimum value on $\mu^{-1}(0)$ which consists of all sequences with center of gravity at the origin. Note that if $n$ is even $\mu^{-1}(0)$ is singular near configurations containing two sets of $\frac{n}{2}$ coincident points. One can check that the critical configurations not contained in $\mu^{-1}(0)$ are those in which some number $r > \frac{n}{2}$ of the $n$ points coincide somewhere on the sphere and the other $n-r$ coincide at the antipodal point. The connected components of the set of non-minimal critical points are thus submanifolds and are indexed by subsets of $\{1, \ldots, n\}$ of cardinality greater than $\frac{n}{2}$. The union of the Morse strata corresponding to subsets of fixed cardinality $r$ consists of all sequences such that precisely $r$ of the points coincide somewhere on $\mathbb{P}_1$.

Given any smooth stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of the manifold $X$ one can build up the cohomology of $X$ inductively from the cohomology of the strata. This is done by using the Thom-Gysin sequences which for each $\beta \in \mathcal{B}$ relate the cohomology groups of the stratum $S_\beta$ and of the two open subsets

$$\bigcup_{\gamma<\beta} S_\gamma, \qquad \bigcup_{\gamma\leq\beta} S_\gamma$$

of $X$. These give us the famous Morse inequalities which can be expressed as follows. For any space $Y$ let $P_t(Y)$ be the Poincaré series given by

$$P_t(Y) = \sum_{i\geq 0} t^i \dim H^i(Y;\mathbb{Q})$$

Assume for convenience that if $\beta \in \mathcal{B}$ then each component of the stratum $S_\beta$ has the same codimension $d(\beta)$ in $X$. Then the Morse inequalities say that

2.13.
$$\sum_\beta t^{d(\beta)} P_t(S_\beta) - P_t(X) = (1+t)R(t)$$

where $R$ is a series with non-negative integer coefficients.

**2.14**. A smooth stratification of $X$ is called perfect if the Morse inequalities are equalities; that is, if

$$P_t(X) = \sum_\beta t^{d(\beta)} P_t(S_\beta)$$

When the stratification is induced by a nondegenerate Morse function $f$ one can replace $P_t(S_C)$ by $P_t(C)$ for each critical submanifold $C$ because the stratum $S_C$ retracts onto $C$: this is the form in which the Morse inequalities are usually seen. In this form the metric does not appear in the inequalities.

If a space $Y$ is acted on by a topological group then the equivariant cohomology $H^{\star}_G(Y,\mathbb{Q})$ is defined by
2.15

$$H^{\star}_G(Y,\mathbb{Q}) = H^{\star}(EG \times_G Y;\mathbb{Q})$$

where $EG \to BG$ is the universal classifying bundle for $G$ and $EG \times_G Y$ is the quotient of $EG \times Y$ by the diagonal action of $G$ acting on $EG$ on the right and on $Y$ on the left.

For any smooth stratification $\{S_\beta\}$ of $X$ whose strata are all invariant under the action of the group $K$ on $X$ we obtain equivariant Morse inequalities

2.16.
$$\sum_\beta t^{d(\beta)} P_t^K(S_\beta) - P_t^K(X) = (1+t)R(t)$$

where $R$ has nonnegative coefficients and $P_t^K$ denotes the equivariant Poincaré series.

The stratification is called equivariantly perfect if these are equalities.

It will be shown that the function $f = ||\mu||^2$ on $X$ is equivariantly perfect in the sense that

2.17.
$$P_t^K(X) = \sum_\beta t^{\lambda(\beta)} P_t^K(C_\beta)$$

where the sum is taken over the critical subsets $\{C_\beta\}$ and $\lambda(\beta)$ is the index of $f$ along $C_\beta$. This is done by showing that if $X$ is given a suitable metric then the stratification $\{S_\beta\}$ induced by $f$ is equivariantly perfect and each stratum $S_\beta$ retracts equivariantly onto the corresponding critical subsets $C_\beta$.

We shall finish this section with a criterion due to Atiyah and Bott for a stratification to be equivariantly perfect.

**Lemma 2.18**. Suppose $\{S_\beta : \beta \in \mathcal{B}\}$ is a smooth $K$-invariant stratification of $X$ such that for each $\beta$ the equivariant Euler class of the normal bundle to $S_\beta$ in $X$ is not a zero-divisor in $H^{\star}_K(S_\beta;\mathbb{Q})$. Then the stratification is equivariantly perfect over $\mathbb{Q}$.

*Proof.* We need to show that the equivariant Thom-Gysin sequences

$$\ldots \longrightarrow H_K^{n-d(\beta)}(S_\beta;\mathbb{Q}) \longrightarrow H_K^n\left(\bigcup_{\gamma\leq\beta} S_\gamma;\mathbb{Q}\right) \longrightarrow H_K^n\left(\bigcup_{\gamma<\beta} S_\gamma;\mathbb{Q}\right) \longrightarrow \ldots$$

split into short exact sequences for all $\beta$. It is enough to show that each map

$$H_K^{n-d(\beta)}(S_\beta;\mathbb{Q}) \longrightarrow H_K^n\left(\bigcup_{\gamma\leq\beta} S_\gamma;\mathbb{Q}\right)$$

is injective. This will certainly happen if the composition with the restriction map

$$H_K^n\left(\bigcup_{\gamma\leq\beta} S_\gamma;\mathbb{Q}\right) \longrightarrow H_K^n(S_\beta,\mathbb{Q})$$

is injective. But this composition is multiplication by the equivariant Euler class of the normal bundle to $S_\beta$ in $X$. The result follows. □

# 3 Critical points for the square of the moment map

Suppose that $K$ is a compact Lie group acting on a compact symplectic manifold $X$ and that $\mu : X \to \mathfrak{k}^\star$ is a moment map for this action. Our aim is to use the function $f = ||\mu||^2 : X \to \mathbb{R}$ as Morse function on $X$, where $||\,||$ is the norm associated to any inner product on $\mathfrak{k}$ which is invariant under the adjoint action of $K$. In this section we shall investigate the set of critical points for $f$.

As before if $a \in \mathfrak{k}$ let $x \to a_x$ be the vector field on $X$ generated by $a$.

**Lemma 3.1**. A point $x \in X$ is critical for $f$ iff $\mu(x)_x = 0$ where $\mu(x) \in \mathfrak{k}^\star$ is identified with an element of $\mathfrak{k}$ by using the fixed invariant inner product on $\mathfrak{k}$.

*Proof.* Let $\{a_i : 1 \le i \le \dim \mathfrak{k}\}$ be an orthonormal basis of $\mathfrak{k}$ and for $1 \le i \le \mathfrak{k}$ let $\mu_i : X \to \mathbb{R}$ be given by $\mu_i(x) = \mu(x).a_i$. Then

$$\mu(x) = \sum_i \mu_i(x) a_i$$

when $\mathfrak{k}^\star$ is identified with $\mathfrak{k}$ and

$$f(x) = ||\mu(x)||^2 = \sum_i (\mu_i(x))^2 \Rightarrow df(x) = \sum 2\mu_i(x) d\mu_i(x)$$

Now $df(x) \in T_x^\star X$ vanishes iff its $\omega$-dual in $T_x X$ does, where $\omega$ is the symplectic form on $X$. But by definition 2.3 of a moment map the $\omega$-dual of each $d\mu_i(x)$ is just the vector $(a_i)_x$. Hence the $\omega$-dual of $df(x)$ is

3.2.

$$2\left(\sum_i \mu_i(x) a_i\right)_x = 2(\mu(x))_x$$

and the result follows.

**3.3**. Now let $T$ be a maximal torus of $K$ and let $\mathfrak{t}$ be its Lie algebra. Then it is easy to check that the composition $\mu_T : X \to \mathfrak{k}^\star \to \mathfrak{t}^\star$ of $\mu$ with the restriction map $\mathfrak{k}^\star \to \mathfrak{t}^\star$ is a moment map for the action of $T$ on $X$. When the inner product of $\mathfrak{k}$ is used to identify $\mathfrak{k}^\star$ with $\mathfrak{k}$ and $\mathfrak{t}^\star$ with $\mathfrak{t}$ then $\mu_T$ becomes the orthogonal projection of $\mu$ onto $\mathfrak{t}$. Thus if $\mu(x) \in \mathfrak{t}$ then $\mu_T(x) = \mu(x)$ and hence $x$ is critical for the function $f = ||\mu||^2$ iff it is critical for the function $f_T = ||\mu_T||^2$ by 3.1.

Therefore we shall next investigate the critical points of $f_T$. The moment maps $\mu_T : X \to \mathfrak{t}^\star$ associated to torus actions on $X$ have been studied by Atiyah. Theorem 1 in [A2] tells us that

**3.4**. The image under $\mu_T$ of the fixed point set of $T$ on $X$ is a finite set $\mathbb{A}$ of points in $\mathfrak{t}^\star$ and $\mu_T(x)$ is the convex hull Conv($\mathbb{A}$) of $\mathbb{A}$ in $\mathfrak{t}^\star$.

The elements of $\mathbb{A}$ will be called the weights of the symplectic action of $T$ on $X$. This terminology is explained in the following example.

**Example 3.5**. Let $X \subset \mathbb{P}_n$ be a nonsingular complex projective variety and let $T$ act on $X$ via a homomorphism $\varphi : T \to \mathrm{U}(n+1)$. By conjugating $\varphi$ by an element of $\mathrm{U}(n+1)$ we may assume that

$$\varphi(t) = \mathrm{diag}(\alpha_0(t), \ldots, \alpha_n(t)), \ \ t \in T$$

where $\alpha_j : T \to S^1$, $0 \leq j \leq n$ are characters of $T$ whose derivatives at 1 are the weights of the representation of $T$ on $\mathbb{C}^{n+1}$. If the tangent space at 1 to $S^1$ is identified with the line $2\pi i\mathbb{R}$ in $\mathbb{C}$ and hence with $\mathbb{R}$ in the usual way then the derivative of each $\alpha_j$ at 1 can be identified with an element of $\mathfrak{t}^\star$. By abuse of notation this element of $\mathfrak{t}^\star$ will also be denoted by $\alpha_j$. Then the derivative $\varphi_\star$ of $\varphi$ at 1 is given by

$$\varphi_\star(\xi) = 2\pi i \,\mathrm{diag}(\xi.\alpha_0, \ldots, \xi.\alpha_n)$$

By 2.7 a moment map $\mu_T$ is given by

$$\mu_T(x).\xi = \frac{\bar{x}^{\star \mathrm{t}}\varphi_\star(\xi)x^\star}{2\pi i ||x^\star||^2} = \frac{1}{||x^\star||^2}\sum_j |x_j|^2 \alpha_j.\xi$$

for each $\xi \in \mathfrak{t}$ where $x^\star = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}\backslash\{0\}$ representing $x$. Thus

3.6.
$$\mu_T(x) = \frac{1}{||x^\star||^2}\sum_j |x_j|^2 \alpha_j$$

The point $x \in X$ is fixed by $T$ iff there is some $\alpha \in \mathfrak{t}^\star$ such that $\alpha_j = \alpha$ whenever $x_j \neq 0$; and then clearly $\mu_T(x) = \alpha$. So at least when $X$ is the whole projective space $\mathbb{P}_n$ the set $\mathbb{A}$ is just the set $\{\alpha_0, \ldots, \alpha_n\}$ os weights of the representation of $T$ on $\mathbb{C}^{n+1}$ and formula 3.6 shows immediately that $\mu_T(\mathbb{P}_n)$ is the convex hull of $\mathbb{A}$.

We need some definitions.

**Definition 3.7**. For any $\beta \in \mathfrak{t}$ let $T_\beta$ be the closure in $T$ of the real one-parameter subgroup $\exp \mathbb{R}\beta$. Thus $T_\beta$ is a subtorus of $T$. Let $\mu_\beta : X \to \mathbb{R}$ be given by $\mu_\beta(x) = \mu(x).\beta$. Then by definition of a moment map and cotangent field $x \to d\mu_\beta(x)$ on $X$ is $\omega$-dual to the vector field $x \to \beta_x$ induced by $\beta$ on $X$. If $x \in X$ then $\beta_x = 0$ iff $x$ is fixed by the subgroup $\exp \mathbb{R}\beta$ of $T$ and hence by its closure $T_\beta$ in $T$. Therefore the critical set of the function $\mu_\beta$ on $X$ is precisely the fixed point set of the subtorus $T_\beta$ of $T$. It is well known that

**3.8**. Every connected component of the fixed point set of a torus action on $X$ is a submanifold of $X$ and the induced action of the torus on its normal bundle in $X$ has no nonzero fixed vectors. (To see this one puts a $T$-invariant Riemannian metric on $X$ and uses normal coordinates).

Using this fact, Atiyah shows that

**3.9**. $\mu_\beta$ is a nondegenerate Morse function on $X$ in the sense of Bott.

**Definition 3.10**. Let $Z_\beta$ be the union of those connected components of the critical set of $\mu_\beta$ on which $\mu_\beta$ takes the value $||\beta||^2$. Thus if $x \in Z_\beta$ then $\mu(x)$ lies in the affine hyperplane in $\mathfrak{k}$ containing $\beta$ and perpendicular to the line from $\beta$ to the origin.

$Z_\beta$ is a submanifold of $X$ (possibly disconnected) fixed by $T_\beta$ and invariant under $T$. In fact, it is a symplectic submanifold of $X$.

**Example 3.11**. If $X$ is a smooth projective variety and $T$ acts on $X$ via $\varphi : T \to \mathrm{U}(n+1)$ then $Z_\beta$ is the intersection with $X$ of a linear subvariety of $\mathbb{P}_n$. If $\varphi(t) = \mathrm{diag}(\alpha_0(t), \ldots, \alpha_n(t))$ for $t \in T$ where $\alpha_j$ are the characters of $T$ identified with points of $\mathfrak{t}^\star$ then

$$Z_\beta = \{(x_0 : \ldots : x_n) \in X : x_j = 0 \text{ unless } \alpha_j.\beta = ||\beta||^2\}$$

Note that the inner product on $\mathfrak{t}$ gives $\mathfrak{t}$ the structure of a normed space. For any nonempty closed convex set $C \subset \mathfrak{t}$ there is then a unique of minimal norm in $C$. This point will be called the point of $C$ closest to the origin 0.

The point of these definitions is the following result.

**Lemma 3.12**. Let $x \in X$ and let $\beta = \mu_T(x) \in \mathfrak{t}$. Then $x$ is critical for $f_T = ||\mu_T||^2$ iff $x \in Z_\beta$; and if this is the case then $\beta$ is the closest point to 0 of the convex hull of some nonempty subset of the set $\mathbb{A}$ of weights defined in 3.4.

*Proof.* By 3.1 $x$ is critical iff $\beta_x = 0$, i.e. iff $x$ is fixed by $T_\beta$. Since $\mu_\beta = \mu_T(x).\beta = ||\beta||^2$ it follows that $x$ is fixed by $T_\beta$ iff it lies in $Z_\beta$ So $\beta$ is the closest point to 0 of $\mu_T(Z_\beta)$ if $\beta \in \mu_T(Z_\beta)$ and hence if $x \in Z_\beta$. But we can apply 3.4 to the action of $T$ on $Z_\beta$ to deduce that $\mu_T(Z_\beta)$ is the convex hull of the image under $\mu_T$ of the fixed point set of $T$ on $Z_\beta$ which is a subset of $\mathbb{A}$. The result now follows. □

This lemma can be used to describe the critical set of the function $f = ||\mu||^2$ associated to the action of the whole group $K$.

**Definition 3.13**. Let $\beta \in \mathfrak{t}$, the Lie algebra of a maximal torus $T$ of $K$. Then $\beta$ will be called the minimal combination of weights of the action of $T$ on $X$ if it is the closest point to the origin of the convex hull in $\mathfrak{t}$ of some nonempty subset of the set of weights $\mathbb{A}$ (defined in 3.4). Let $\mathfrak{t}_+$ be a fixed positive Weyl chamber in $\mathfrak{t}$ and denote by $\mathcal{B}$ the set of all minimal combinations of weights which lie in $\mathfrak{t}_+$.

$\mathcal{B}$ will be the indexing set for the stratification of $X$ which we shall associate to the function $f$.

**Definition 3.14**. For $\beta \in \mathcal{B}$ let $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$.

Then we have

**Lemma 3.15**. The critical set of $f$ on $X$ is the disjoint union of the closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$ of $X$.

*Proof.* For any $x \in X$ there is some $k \in K$ such that $\mathrm{Ad}\, k\mu(x) \in \mathfrak{t}_+$. By the defintion of moment map $\mathrm{Ad} k\mu(x) = \mu(kx)$. Since $f$ is a $K$-invariant map $x$ is critical for $f$ iff $kx$ is for any such $k$. But $\mu(kx) \in \mathfrak{t}$ so by 3.3 $kx$ is critical for $f = ||\mu||^2$ iff it is critical for $f_T = ||\mu_T||^2$. Let $\beta = \mu(kx) \in \mathfrak{t}_+$. By 3.12 $kx$ is critical for $f_T$ iff $kx \in Z_\beta$ and if this happens then $\beta \in \mathcal{B}$.

Therefore the critical set for $f$ is the union of the closed sets $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$ as $\beta$ runs over $\mathcal{B}$. Moreover for each $\beta \in \mathcal{B}$ the image of $C_\beta$ under $\mu$ is preisely the orbit of $\beta$ under the adjoint representations of $K$. Since any adjoint orbit in $\mathfrak{k}$ intersects the positive Weyl chamber in a unique point the subsets $\{C_\beta\}$ must be disjoint. The result follows.

The subsets $\{C_\beta : \beta \in \mathcal{B}\}$ will therefore be called the critical subsets for $f$.

**Corollary 3.16**. The image under $\mu$ of each connected component of the critical set for $f$ is a single adjoint orbit in $\mathfrak{k}^\star \cong \mathfrak{k}$. For each $\beta \in \mathcal{B}$, $C_\beta$ consists of those critical points for $f$ whose image under $\mu$ lies in the adjoint orbit of $\beta$. Thus the function $f = ||\mu||^2$ takes the value $||\beta||^2$ on $C_\beta$.

**Example 3.17**. If $X = (\mathbb{P}_1)^n$ is acted on by SU(2) as in 2.2 then $T_\beta$ is the maximal torus $T$ of SU(2) when $\beta \neq 0$. The fixed point set of $T$ consists of all configurations such that every point is either at 0 or $\infty$. Identify $\mathfrak{t}$ with $\mathbb{R}$ and give it the standard inner product so that the identity character of $T = S^1$ becomes 1 in $\mathbb{R}$. Take $\mathbb{R}^+$ as the positiv Weyl chamber. Then the moment map sends a configuration with $r$ points at 0 and the rest at $\infty$ to $2r - n \in \mathfrak{t}$. So

$$\mathcal{B} = \left\{ 2r - n : \frac{1}{2}n \leq r \leq n \right\} \cup \{0\}$$

and if $\beta = 2r - n$ then $Z_\beta$ consists of configurations with $r$ points at 0 and the rest at $\infty$. Thus the last lemma agrees with 2.12.

# 4 The square of the moment map as a Morse function

As in §§2-3 let $X$ be a compact symplectic manifold acted on by a compact Lie group $K$ and assume that a moment map $\mu : X \to \mathfrak{k}^\star$ exists. We want to apply Morse theory to the function $f = ||\mu||^2$ where $||\,||$ is the norm associated to any inner product on $\mathfrak{k}$ which is invariant under the adjoint action of $K$. Problems arise because the critical set of $f$ has singularities. However, we shall see in this section that $f$ is minimally degenerate. It is show in §10 that such functions are sufficiently well-behaved to have associated Morse inequalities.

To show that $f$ is minimally degenerate we need to find a minimising submanifold along each of the critical subsets $C_\beta$. That is, for each $\beta \in \mathcal{B}$ we require a submanifold $\Sigma_\beta$ of some neighbourhood of $C_\beta$ with orientable normal bundle in $X$ and such that the restriction of $f$ to $\Sigma_\beta$ takes its minimum value on $C_\beta$. We also require that for each $x \in C_\beta$ the tangent space $T_x\Sigma_\beta$ is maximal among subspaces of $T_xX$ on which the Hessian $H_x(f)$ is positive semidefinite.

First fix a $K$-invariant Riemannian metric on $X$.

**4.1**. Note that such a metric and the symplectic structure give $X$ a $K$-invariant almost-complex structure as follows. The metric can be used to identify the symplectic form with a skew-adjoint linear operator $A$ on the tangent bundle $TX$. Then $A^2 = -AA^\star$ and since $AA^\star$ is self-adjoint with positive eigenvalues it has a unique square root. If we rescale the metric by $(AA^\star)^{-\frac{1}{2}}$ then $A$ is replaced by $J = A(AA^\star)^{-\frac{1}{2}}$ so that $J^2 = -1$. Hence there is a complex structure on $TX$ such that $J$ is multiplication by $i$.

We can thus assume that the chosen $K$-invariant metric on $X$ has been suitably normalized so that

**4.2**. $X$ has a $K$-invariant almost-complex structure such that if $\xi \in T_xX$ then $i\xi$ is the dual with respect to the metric of the linear form $\zeta \to \omega_x(\zeta, \xi)$ on $T_xX$.

Note that this implies that

**4.3**. $\text{grad}\mu_\beta(x) = i\beta_x$ for all $x \in X$, since by the definition of a moment map the cotangent vector field $d\mu_\beta$ on $X$ is $\omega$-dual to the tangent vector field $x \to \beta_x$.

**Remark 4.4**. When $X$ is Kähler the real part of the Kähler metric is the obvious choice for a Riemannian metric on $X$. The induced almost-complex structure then coincides with the complex structure of $X$. In this section where $X$ is merely symplectic, the almost complex structure is used not only for convenience but also because it links up with the work of later sections on Kähler manifolds.

Recall from Lemma 3.15 that the set of critical points for $f$ on $X$ is the disjoint union of the closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$. The indexing set $\mathcal{B}$ is the set of minimal weight combinations in the

positive Weyl chamber as defined in 3.13. For each $\beta \in \mathcal{B}$ the critical subset $C_\beta$ is $K(Z_\beta \cap \mu^{-1}(\beta))$ where $Z_\beta$ is the symplectic submanifold of $X$ defined in 3.10. This submanifold is the union of certain components of the fixed point set of the subtorus $T_\beta$ generated by $\beta$ or, equivalently, of the critical set of the function $\mu_\beta$ on $X$ (which is nondegenerate as a Morse function in the sense of Bott). The components contained in $Z_\beta$ are those on which $\mu_\beta$ takes the value $||\beta||^2$.
4.6. For each $\beta$ there is a Morse stratum $Y_\beta$ associated to $Z_\beta$ which consists of all points of $X$ whose paths of steepest descent under $\mu_\beta$ have limit points in $Z_\beta$. This Morse stratum $Y_\beta$ (which depends on the chosen metric) is a locally-closed submanifold of $X$. (These facts are well known; a proof is given in the appendix, but this covers the more general case of minimally degenerate functions, which are harder to deal with than nondegenerate ones such as $\mu_\beta$).

**Example 4.7**. Consider again the projective variety $X = (\mathbb{P}_1)^n$ acted on diagonally by SU(2). We have seen in 3.17 that the nonzero elements of the indexing set $\mathcal{B}$ may be identified with integers $r$ such that $\frac{n}{2} \leq r \leq n$ and that $Z_r$ consists of sequences of points of which $r$ lie at 0 and the rest at $\infty$. It is not hard to see that $Y_r$ consists of all sequences of points precisely $r$ of which lie at 0.

Note that $KY_r$ thus consists of all sequences of points such that $r$ points and no more coincide somewhere on $\mathbb{P}_1$. By 2.12 this is exactly the Morse stratum indexed by $r$ for the function $||\mu||^2$ on $X$.

Recall that we need a minimising submanifold $\Sigma_\beta$ along each critical subset $C_\beta$. It will be shown that we can take $\Sigma_\beta$ to be an open subset of $KY_\beta$.

**Remark**. It will then follow from theorem 10.4 that the Morse stratum $S_\beta$ coincides with $KY_\beta$ in a neighbourhood of $C_\beta$. In fact, in the Kähler case we shall see that $S_\beta = KY_\beta^{\min}$ where $Y_\beta^{\min}$ is a certain open subset of $Y_\beta$. If one were onl interested in the Kähler case it would be possible to avoid minimising manifolds and simplify the appendix somewhat by using this fact. When $X$ is just a symplectic manifold the equality above does not hold for every invariant metric on $X$. For example consider $X = (\mathbb{P}_1)^n$ with symplectic form $\omega \oplus \ldots \oplus \omega$ and metric $2\rho \oplus \ldots \oplus \rho$ where $\omega$ and $\rho$ are the usual symplectic form and metric on $\mathbb{P}_1$. It may always be possible to choose a metric for which the equality holds, or at least when $\pi_1(X) = 0$ (that would follow immediately if it were shown that every simply connected compact symplectic manifold is Kähler) but this has not yet been proven.

First in order to show that $KY_\beta$ is smooth near $C_\beta$ we must investigate what elements of $K$ preserve $Y_\beta$.

**Definition 4.8**. For each $\beta \in \mathcal{B}$ let $\mathrm{Stab}\beta = \{k \in K : \mathrm{Ad}\, k(\beta) = \beta\}$ be the stabiliser of $\beta$ in $K$. $\mathrm{Stab}\beta$ is also the centralizer of the subtorus $T_\beta$ in $K$ so that it is connected if $K$ is connected and is a compact subgroup of $K$. Let $\mathrm{stab}\beta = \{a \in \mathfrak{k} : [a, \beta] = 0\}$ be the Lie algebra of $\mathrm{Stab}\beta$.

Stab $\beta$ acts on the symplectic submanifold $Z_\beta$ of $X$ and the composition of $\mu$ restricted to $Z_\beta$ with the orthogonal projection of $\mathfrak{k}$ onto stab $\beta$ is a moment map for this action; as usual $\mathfrak{k}$ and its dual are identified via the inner product. But if $x \in Z_\beta$ then $T_\beta$ fixes $x$ and hence also fixes $\mu(x)$ since $\mu$ is a $K$-equivariant map. Therefore $\mu(x) \in \mathrm{Stab}\,\beta$. It follows that
4.9. The restriction of $\mu$ onto $Z_\beta$ maps $Z_\beta$ to stab $\beta$ and can be regarded as a moment map for the action of Stab $\beta$ on $Z_\beta$.

In order to show that $KY_\beta$ is smooth in a neighbourhood of $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$ we need the following

**Lemma 4.10**. If $x \in Z_\beta \cap \mu^{-1}(\beta)$ then $\{k \in K : kx \in Y_\beta\} = \mathrm{Stab}\,\beta$ and $\{a \in \mathfrak{k} : a_x \in T_x Y_\beta\} = \mathrm{stab}\,\beta$.

*Proof.* It is clear from the definitions that $Z_\beta$ is invariant under Stab $\beta$ and that $Z_\beta \subseteq Y_\beta$. It follows that $\mathrm{Stab}\,\beta \subset \{k : kx \in Y_\beta\}$ and $\mathrm{stab}\,\beta \subseteq \{a : a_x \in T_x Y_\beta\}$. On the other hand suppose $k \in K$ is such that $kx \in Y_\beta$. Then the path of steepest descent from $kx$ for the function $\mu_\beta$ has limit point in $Z_\beta$ and by definition $\mu_\beta$ takes the value $||\beta||^2$ on $Z_\beta$. Thus as $\mu_\beta(kx) = \mu(kx).\beta$ we have $\mu(kx).\beta \geq ||\beta||^2$. But $||\mu(kx)||^2 = ||\mu(x)||^2 = ||\beta||^2$. Together these imply that $\beta = \mu(kx)$ and since $\mu(kx) = \mathrm{Ad}\,k\mu(x) = \mathrm{Ad}\,k\beta$ it follows that $k \in \mathrm{Stab}\,\beta$.

Now suppose that $a \in \mathfrak{k}$ is such that $a_x \in T_x Y$. For $t \in \mathbb{R}$ we have

$$\mu((\exp ta)x) = \beta + td\mu(x)(a_x) + e(t)$$

where $e(t) = O(t^2)$ as $t \to 0$; and

$$d\mu(x)(a_x) = [a, \mu(x)] = [a, \beta]$$

since $\mu$ is $K$-equivariant. As $[a, \beta].\beta = a.[\beta, \beta] = 0$, it follows that

$$\mu_\beta((\exp ta)x) = ||\beta||^2 + \beta.e(t)$$

But also

$$||\mu(\exp ta)x||^2 = ||\mu(x)|| = ||\beta||^2$$

for all $t$, so that

$$||\beta||^2 = ||\beta + t[a, \beta] + e(t)||^2 = ||\beta||^2 + t^2||[a, \beta]||^2 + 2\beta.e(t) + O(t^3)$$

as $t \to 0$. Thus

$$2\beta.e(t) = -t^2||[a, \beta]||^2 + O(t^3)$$

as $t \to 0$ and hence

$$\mu_\beta((\exp ta)x) = ||\beta||^2 - \frac{1}{2}t^2||[a,\beta]||^2 + O(t^3)$$

as $t \to 0$. But by assumption $a_x \in T_xY_\beta$ which is the sum of the nonnegative eigenspaces of the Hessian $H_x(\mu_\beta)$ of $\mu_\beta$ at $x$ since $\mu_\beta$ is nondegenerate in the sense of Bott. The last equation shows that this is impossible unless $[a,\beta] = 0$, i.e. unless $a \in \operatorname{stab}\beta$. This completes the proof. □

**Corollary 4.11**. The subset $KY_\beta$ of $X$ is a smooth submanifold when restricted to some $K$-invariant neighbourhood of $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$ in $X$.

*Proof.* Since $Y_\beta$ is invariant under $\operatorname{Stab}\beta$ the map $\sigma : K \times Y_\beta \to X$ given by $\sigma(k,x) = kx$ induces a map $\tilde{\sigma} : K \times_{\operatorname{Stab}\beta} Y_\beta \to X$ whose image is $KY_\beta$. It is easily checked from the definition of $Y_\beta$ that if $\epsilon > 0$ sufficiently small the subset $\{y \in Y_\beta : \mu_\beta(x) \leq ||\beta||^2 + \epsilon\}$ of $Y_\beta$ a is *compact* neighbourhood of $Z_\beta$ in $Y_\beta$. Moreover its complement in $Y_\beta$ is contained in the subset

$$\left\{ y \in X : ||\mu(y) \geq ||\beta|| + ||\frac{1}{||\beta||}\epsilon \right\}$$

of $X$ which is closed, $K$-invariant and does not meet $Z_\beta \cap \mu^{-1}(\beta)$. From this one can deduce easily that if $x \in Z_\beta \cap \mu^{-1}(\beta)$ then $\tilde{\sigma}$ maps each neighbourhood of the point in $K \times_{\operatorname{Stab}\beta} Y_\beta$ represented by $(1,x)$ onto a neighbourhood of $x$ in the image $KY_\beta$ of $\tilde{\sigma}$.

The derivative of $\sigma$ at any point of the form $(1,x)$ sends $(a,\xi) \in \mathfrak{k} \times T_xY_\beta$ to the tangent vector $a_x + \xi \in T_xX$. The tangent space of $K \times_{\operatorname{Stab}\beta} Y_\beta$ at a point represented by $(1,x)$ is the quotient of $\mathfrak{k} \times T_xY_\beta$ by the subspace consisting of all $(a,\xi)$ such that $a \in \operatorname{stab}\beta$ and $\xi = -a_x$. Thus 4.11 shows that the derivative of $\tilde{\sigma}$ is injective at a point rep. by $(1,x)$ with $x \in Z_\beta \cap \mu^{-1}(\beta)$ and hence also in some neighbourhood $V$ of this point. The preceding paragraph shows that the image $\tilde{\sigma}(V)$ is a neighbourhood of $x$ in $KY_\beta$. Therefore it follows from the inverse function theorem that the image $KY_\beta$ of $\tilde{\sigma}$ is smooth in some neighbourhood of $x$.

We have thus shown that $KY_\beta$ is smooth near $Z_\beta \cap \mu^{-1}(\beta)$. It follows that $KY_\beta$ is smooth in some $K$-invariant neighbourhood of $C_\beta$, as required. □

We are aiming to show that the intersection $\Sigma_\beta$ of $KY_\beta$ with a sufficiently small neighbourhood of $C_\beta$ is a minimising manifold for $f$ along $C_\beta$. The last corollary shows that the condition that $\Sigma_\beta$ be a locally-closed submanifold of $X$ can be satisfied. For the other conditions we need two technical lemmas.

**Lemma 4.12**. $Z_\beta$ is an almost complex submanifold of $X$. Moreoever $T_xY_\beta$ is a complex subspace of $T_xX$ for all $x \in Z_\beta$.

*Proof.* Suppose $x \in Z_\beta$. Then the compact torus $T_\beta$ generated by $\beta$ acts on $T_xX$ which decomposes into the sum,

$$V_0 \oplus \ldots \oplus V_p$$

of complex subspaces where $V_0$ is fixed by $T_\beta$ and is the tangent space to $Z_\beta$ while for each $j \geq 1$, $T_\beta$ acts on $V_j$ as scalar multiplication by some nontrivial character. Thus $\beta$ acts on each $V_j$ as multiplication by some $i\lambda_j$ with $\lambda_0 = 0$ and $\lambda_j$ real nonzero for $j \geq 1$. Also by 4.3 we have $\operatorname{grad} \mu_\beta(y) = i\beta_y$ for all $y \in Y$. Therefore the Hessian $H_x(\mu_\beta)$ of $\mu_\beta$ at $x$ acts on $V_j$ as multiplication by $\lambda_j$. Thus $T_xZ_\beta = V_0$ and $T_xY_\beta$ is the sum of those $V_j$ such that $\lambda_j \geq 0$ so both are complex subspaces of $T_xX$. The result follows. □

**Lemma 4.13**. Suppose $x \in C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$. Then the restriction of the symplectic form $\omega_x$ to $T_x(KY_\beta)$ is nondegenerate.

*Proof.* First note that by 4.11 $KY_\beta$ is smooth near $x$ so $T_x(KY_\beta)$ exists. Moreover since $\omega$ is invariant under $K$ we may assume that $x \in Z_\beta \cap \mu^{-1}(\beta)$ and then $T_x)(KY_\beta) = \mathfrak{k}_x + T_xY_\beta$. So any element of $T_x(KY_\beta)$ may be written in the form $a_x + \xi$ where $\xi \in T_xY_\beta$ and $a \in \mathfrak{k}$ is such that $a_x$ is orthogonal to $T_xY_\beta$ (with respect to the Riemannian metric on $X$). Suppose that $\omega_x(a_x + \xi, \zeta) = 0$ for all $\zeta \in T_x(KY_\beta)$. By 4.12, $i\xi \in T_xY_\beta$ so if $<,>$ denotes the metric then

$$0 = \omega_x(a_x + \xi, \imath\xi) = \langle a_x + \xi, \xi\rangle = \langle \xi, \xi\rangle$$

by 3.19 and the assumption on $a$.

Hence $\xi = 0$. But then as $\mathfrak{k}_x \subset T_x(KY_\beta)$

$$0 = \omega_x(a_x, b_x) = d\mu(x)(a_x).b$$

for every $b \in \mathfrak{k}$ (2.3), so

$$0 = d\mu(x)(a_x) = [a, \beta]$$

since $\mu(x) = \beta$. Thus $a \in \operatorname{stab} \beta$ and hence $a_x \in T_xY_\beta$ by 4.10. But by assumption $a_x$ is orthogonal to $T_xY_\beta$ so $a_x = 0$. This completes the proof. □

**Remark 4.14**. Lemma 4.13 implies that there is an open neighbourhood $\Sigma_\beta$ of the critical subset $C_\beta$ in $KY_\beta$ such that the restriction of the symplectic form $\omega$ to the tangent bundle $T\Sigma_\beta$ is nondegenerate. It follows that $\omega$ and the metric together induce a $K$-invariant almost complex structure on $\Sigma_\beta$ (4.1). It also follows that the normal bundle $\Sigma_\beta$ in $X$ can be identified with the $\omega$-orthogonal complement $T\Sigma_\beta^\perp$ in the restriction of $TX$ to $\Sigma_\beta$. Since $\omega$ is nondegenerate on $T\Sigma_\beta^\perp$ it gives a complex structure to this normal bundle as well.

At last we are in a position to prove

**Proposition 4.15**. There is a $K$-invariant open neighbourhood $\Sigma_\beta$ of $C_\beta$ in $KY_\beta$ which is a minimising manifold for $f$ along $C_\beta$.

So this goes to show that the function $f = ||\mu||^2$ is minimally degenerate along each critical subset $C_\beta$. By theorem 10.2 of the appendix this implies the existence of Morse inequalities for $f$ and also of equivariant Morse inequalities. Indeed, theorem 10.4 and lemma 10.3 imply the following

**Theorem 4.16**. Let $X$ be a compact symplectic manifold acted on by a compact Lie group $K$ and suppose $\mu : X \to \mathfrak{k}^\star$ is a moment map for this action. Fix an invariant inner product on $\mathfrak{k}$. Then the set of critical points for the function $f = ||\mu||^2$ is a finite disjoint union of closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$ on each of which $f$ takes a constant value. There is a smooth stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$ such that a point $x \in X$ lies in the stratum $S_\beta$ iff the limit set of the path of steepest descent for $f$ from $x$ (with respect to a suitable $K$-invariant metric) is contained in $C_\beta$. For each $\beta$ the inclusion $C_\beta \subset S_\beta$ is an equivalence of Cech cohomology and also $K$-equivariant cohomology.

Theorem 4.10 shows in addition that

4.17. If $\beta \in \mathcal{B}$ then the stratum $S_\beta$ coincides in a neighbourhood of $C_\beta$ with the minimising manifold $\Sigma_\beta$ (which is an open subset of $KY_\beta$ where $Y_\beta$ is defined as in 4.6). In particular if $x \in Z_\beta \cap \mu^{-1}(\beta)$ then

$$T_x S_\beta \supseteq T_x Z_\beta$$

From this together with remark 4.14 we deduce that

4.18. Both the tangent bundle and the normal bundle to each stratum $S_\beta$ have $K$-invariant complex structures in some neighbourhood of the critical set $C_\beta$.

Theorem 4.16 implies immediately the existence of equivariant Morse inequalities for $f = ||\mu||^2$. We shall not state these explicitly until the next section, where it will be shown that they are in fact equalities.

We shall conclude this section with some remarks about the codimensions of the components of the strata $S_\beta$ and the equivariant cohomology of the critical sets $C_\beta$.

Recall that when stating the Morse inequalities induced by a smooth stratification of $X$ in §2 we made the simplifying assumption that every stratum was connected and hence had a well defined codimension in $X$. In fact, the stratification $\{S_\beta\}$ defined in theorem 4.16 may contain disconnected strata. Therefore it is necessary to refine it so that the components of any stratum all have the same codimension.

For $\beta \in \mathcal{B}$ the critical subset $C_\beta$ was defined at 3.14 by

$$C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$$

where $Z_\beta$ is the union of certain components of the critical set of the nondegenerate Morse function $\mu_\beta$. Recall that the index of the Hessian $H_x(\mu_\beta)$ at any critical point $x$ for $\mu_\beta$ is the dimension of any subspace of the tangent space $T_xX$ to which the restriction of $H_x(\mu_\beta)$ is negative definite and which is maximal with this property. This is the same as the codimension of a maximal subspace of $T_xX$ on which $H_x$ is positive semi-definite. Since $\mu_\beta$ is a nondegenerate Morse function in the sense of Bott, the index of $H_x$ is constant along any component of the critical set of $\mu_\beta$. Its value is called the index of $\mu_\beta$ along this component. So we can make the following definition.

**Definition 4.19**. For any integer $m \geq 0$ let $Z_{\beta,m}$ be the union of those connected components of $Z_\beta$ along which the index of $\mu_\beta$ is $m$. Let

$$C_{\beta,m} = K(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

Then each $Z_{\beta,m}$ is a symplectic submanifold of $X$ and $C_\beta$ is the disjoint union of the closed subsets $\{C_{\beta,m} : 0 \leq m \leq \dim X\}$. The fact that these are disjoint comes from 4.10.

The point of this definition is the following

**Lemma 4.20**. The index of the Hessian $H_x(f)$ of $f = ||\mu||^2$ at any point $x \in C_{\beta,m}$ is

$$d(\beta, m) = m - \dim K + \dim \operatorname{Stab} \beta$$

This is the codimension of the component which contains $x$ of the stratum $S_\beta$.

*Proof.* By 4.17 the stratum $S_\beta$ coincides in a neighbourhood of $x$ with the minimising manifold $\Sigma_\beta$ for $f$ along $C_\beta$. It follows immediately from the definition of minimising manifold that the index of the Hessian $H_x(f)$ equals the codimension of the component of $\Sigma_\beta$ containing $x$. Thus it suffices to show that the component of $\Sigma_\beta$ containing $x$ has codimension $d(\beta, m)$ in $X$.

Since

$$C_{\beta,m} = K\left(Z_{\beta,m} \bigcap \mu^{-1}(\beta)\right)$$

and everything is invariant under $K$, we may assume $x \in Z_{\beta,m} \cap \mu^{-1}(\beta)$. By definition of the minimising manifold $\Sigma_\beta$ is an open subset of $KY_\beta$ where $Y_\beta$ is the Morse stratum consiting of all points in $X$ whose paths of steepest descent under the function $\mu_\beta$ have limit points in $Z_\beta$. Since $\mu_\beta$ is a nondegenerate Morse function, locally $Y_\beta$ is a submanifold of $X$ whose codimension is equal to the index of the Hessian $H_x(\mu_\beta)$. By definition of $Z_{\beta,m}$, this index is $m$.

In the proof of 4.11, we saw that $KY_\beta$ is locally diffeomorphic to $K \times_{\operatorname{Stab} \beta} Y_\beta$ near $x$. Therefore, its codimension is

$$d(\beta, m) = m - \dim K + \dim \operatorname{Stab} \beta$$

The result follows. □

It is easy to see that for each $\beta$

**4.21**. The critical subset $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$ is homeomorphic to $K \times_{\operatorname{Stab}\beta} (Z_\beta \cap \mu^{-1}(\beta))$

By 4.10, for each $x \in Z_\beta \cap \mu^{-1}(\beta)$ the set $\{k \in K : kx \in Z_\beta \cap \mu^{-1}(\beta))$ is just the subgroup $\operatorname{Stab} \beta$ of $K$. Thus there is a continuous bijection

$$K \times_{\operatorname{Stab}\beta} (Z_\beta \cap \mu^{-1}(\beta)) \to C_\beta$$

which must be a homeomorphism since both spaces are compact and Hausdorff.

As $Z_{\beta,m}$ is also preserved by $\operatorname{Stab}\beta$ we deduce that

**4.22**. Each $C_{\beta,m}$ is homeomorphic to $K \times_{\operatorname{Stab}\beta} (Z_{\beta,m} \cap \mu^{-1}(\beta))$.

It follows immediately (see [A & B] §13) that

**4.23**. The $K$-equivariant cohomology $H^\star_K(C_\beta; \mathbb{Q})$ is isomorphic to the $\operatorname{Stab}\beta$-equivariant rational cohomology of $Z_\beta \cap \mu^{-1}(\beta)$ and similarly that

$$H^\star_K(C_{\beta,m}; \mathbb{Q}) \cong H^\star_{\operatorname{Stab}\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta); \mathbb{Q})$$

for each $m$. Indeed, rational coefficients are not necessary here. Any field of coefficients will do.

We now have all the ingredients for writing down the equivariant Morse inequalities and proving that they are in fact equalities. This will be done in the next section.

# 5 Cohomological formulae

As in the previous section we suppose that $X$ is a compact symplectic manifold acted on by a compact Lie group $K$, that there is a fixed invariant inner product on $\mathfrak{k}$, and that $\mu : X \to \mathfrak{k}^\star \cong \mathfrak{k}$ is a moment map for the action. In the last section we saw that the function $f = ||\mu||^2$ is a minimally degenerate Morse function on $X$. This implies the existence of Morse inequalities for $f$. In this section we shall show that these inequalities calculated for rational equivariant cohomology are in fact equalities. Thus $f$ is equivariantly perfect for rational cohomology.

We shall see that this provides us with an inductive formula (from which an explicit formula will be derived) for the rational cohomology of the symplectic quotient of $X$ by $K$ when it exists.

At the end of the last section it was explained how the description of the critical set as the disjoint union of closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$ needs refining in order to state the Morse inequalities for $f$. The problem is that the subsets $C_\beta$ may be disconnected and hence the index of the Hessian of $f$ at points of $C_\beta$ may not be constant. Because of this we defined the closed subsets $\{C_{\beta,m} : \beta \in \mathcal{B}, 0 \leq m \leq \dim X\}$ such that each $C_\beta$ is the disjoint union of the subsets $\{C_{\beta,m} : 0 \leq m \leq \dim X\}$ and the index of the Hessian of $f$ at any point of $C_{\beta,m}$ is

$$d(\beta, m) = m - \dim K + \dim \operatorname{Stab} \beta$$

The statement that the function $f$ is equivariantly perfect for rational coefficients is now equivalent by 2.16 to the equality

5.1.

$$P_t^K(X) = \sum_{\beta,m} t^{d(\beta,m)} P_t^K(C_\beta, m)$$

For each $\beta$ and $m$ there is a symplectic submanifold $Z_{\beta,m}$ of $X$ acted on by Stab$\beta$ under the ajoint action of $K$ on $\mathfrak{k}$ such that

$$H_K^\star(C_{\beta,m}, \mathbb{Q}) \cong H_K^\star(Z_{\beta,m} \cap \mu^{-1}(\beta), \mathbb{Q})$$

Thus 5.1 is equivalent to the formula

5.2.

$$P_t^K(X) = \sum_{\beta,m} t^{d(\beta,m)} P_t^{\operatorname{Stab}\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

To show 5.1 and 5.2 hold, i.e. that $f$ is equivariantly perfect, we shall use criterion 2.18 together with the folowing result of Atiyah and Bott.

**5.3**. Suppose that $N$ is a complex vector bundle over a connected space $Y$ and that a compact group $K$ acts as a group of bundle automorphisms of $N$. Suppose that there is a subtorus $T_0$

of $K$ which acts trivially on $Y$ and that the representation of $T_0$ on the fibre of $N$ at any point of $Y$ has no nonzero fixed vectors. Then the equivariant Euler class of $N$ in $H_K^\star(Y, \mathbb{Q})$ is not a divisor of zero.

**Theorem 5.4**. Let $X$ be a symplectic manifold acted on by a compact group $K$ with moment map $\mu : X \to \mathfrak{k}^\star$ and give $\mathfrak{k}$ a fixed invariant inner product. Then the function $f = ||\mu||^2$ on $X$ is equivariantly perfect over the field of rational coefficients. Thus the equivariant Poincaré series of $X$ is given by

$$P_t^K(X) = \sum_{\beta,m} t^{d(\beta,m)} P_t^K(C_{\beta,m}) = \sum_{\beta,m} t^{d(\beta,m)} P_t^{\mathrm{Stab}\,\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

*Proof.* By theorem 4.16 there is a smooth $K$-invariant stratification $\{S_\beta\}$ of $X$ such that for each $\beta$ the stratum $S_\beta$ contains the critical subset $C_\beta$ and the inclusion of $C_\beta$ in $S_\beta$ is an equivalence of $K$-equivariant Cech cohomology.

Let $S_{\beta,m}$ denote the union of those components of $S_\beta$ which have codimension $d(\beta, m)$. Then by 4.20 $\{S_{\beta,m} : \beta \in \mathcal{B}, 0 \leq m \leq \dim X\}$ is a smooth stratification of $X$ such that

$$H_K^\star(S_{\beta,m}, \mathbb{Q}) \cong H_K^\star(C_{\beta,m}, \mathbb{Q})$$

for all $\beta, m$. We must prove that this stratification is equivariantly perfect over $\mathbb{Q}$.

By 2.18 it is enough to show that the equivariant Euler class of the normal bundle to each stratum $S_{\beta,m}$ is not a zero divisor in $H_K^\star(S_{\beta,m})$. Under the composition of the isomorphisms

$$H_K^\star(S_{\beta,m}) \cong H_K^\star(C_{\beta,m}) \cong H_{\mathrm{Stab}\,\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

the equivariant Euler class of this normal bundle is identifed with the Stab$\beta$-equivariant Euler class of its restriction, $N$ say, to $Z_{\beta,m} \cap \mu^{-1}(\beta)$.

It follows from 4.18 that the bundle $N$ has a complex structure preserved by the action of Stab$\beta$. Also from 4.17 we see that $N$ is a quotient of the restriction to $Z_{\beta,m} \cap \mu^{-1}(\beta)$ of the normal bundle to $Z_{\beta,m}$. But by definition $Z_{\beta,m}$ is the union of certain components of the fixed point set of the subtorus $T_\beta$ of Stab $\beta$. So by 3.8 the action of $T_\beta$ on the normal bundle to $Z_{\beta,m}$ has no nonzero fixed vectors. The same is therefore true of the action of $T_\beta$ on $N$. Hence by 5.3 the equivariant Euler class of $N$ is not a divisor of zero in $H_{\mathrm{Stab}\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta)$. Note that we should really have considered each component of $Z_{\beta,m}$ separately. The result follows. □

**5.5**. The subset of $X$ on which the function $f = ||\mu||^2$ achieves its minimum is $\mu^{-1}(0)$ provided that $\mu^{-1}(0)$ is nonempty. This is a $K$-invariant subset of $X$. If we suppose that the stabiliser in $K$ of every point $x$ therein is finite then the quotient $\mu^{-1}(0)/K$ has a natural symplectic structure and is the symplectic quotient or Marsden-Weinstein reduction of $X$ by $K$.

To see why $\mu^{-1}(0)/K$ has a natural symplectic structure, note first that if every $x \in \mu^{-1}(0)$ has finite stabiliser then $d\mu(x)$ is surjective for each $x$. Otherwise, there is some $x \in \mu^{-1}(0)$ and some nonzero $a \in \mathfrak{k}$ such that

$$0 = d\mu(x)(\xi).a = \omega_x(\xi, a_x)$$

for all $\xi \in T_xX$. The second equality comes from the definition of a moment map. Then since $\omega$ is nondegenerate $a_x = 0$ so the one-parameter subgroup of $K$ generated by $a$ fixes $x$ which is impossible.

Thus $\mu^{-1}(0)$ is a submanifold of $X$ and $\mu^{-1}(0)/K$ is a rational homology manifold (it can be thought of as a manifold except for singularities caused by finite isotropy groups). Moreover from the fact that $\omega_x(\xi, a) = 0$ for $\xi \in T_x\mu^{-1}(0)$ and all $a \in \mathfrak{k}$ it is easy to deduce that $\omega$ induces a nondegenerate symplectic form on $\mu^{-1}(0)/K$.

In particular if $K$ acts freely on $\mu^{-1}(0)$ then $\mu^{-1}(0)/K$ is a symplectic manifold and moreover since $K$ is compact the natural map $\mu^{-1}(0) \to \mu^{-1}(0)/K$ is a locally trivial fibration with fiber $K$. It follows that the natural map

$$\mu^{-1}(0) \times_K EK \longrightarrow \mu^{-1}(0)/K$$

is a fibration with contractible fiber $EK$. Hence the equivariant cohomology of $\mu^{-1}(0)$ is isomorphic to the ordinary cohomology of the symplectic quotient of $X$ by $K$. Moreover for rational cohomology the same is true provided only that the stabiliser of every point $\mu^{-1}(0)$ is finite. Thus we have

**5.6**. If the stabiliser of every $x \in \mu^{-1}(0)$ is finite then the rational equivariant cohomology $H^\star_K(\mu^{-1}(0); \mathbb{Q})$ is isomorphic to the ordinary rational cohomology $H^\star(\mu^{-1}(0)/K, \mathbb{Q})$ of the symplectic quotient.

**5.7**. Since $\mu^{-1}(0)$ coincides with the critical subset $C_0$ of $X$ on which $f$ attains its minimum, theorem 5.4 provides a formula for the equivariant Poincaré series of $\mu^{-1}(0)$ in terms of the equivariant Poincaré series of $X$ itself and of all the series $P_t^{\mathrm{Stab}\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$ with $\beta \in \mathcal{B}$ and $0 \leq m \leq \dim X$. Moreover each $Z_{\beta,m}$ is a compact symplectic manifold on which the compact subgroup $\mathrm{Stab}\,\beta$ of $K$ act. We saw at 4.9 that the restriction of $\mu$ to $Z_\beta$ (which is the disjoint union of all the $Z_{\beta,m}$) can be regarded as a moment map for the action of $\mathrm{Stab}\,\beta$ on $Z_\beta$. As usual we use the fixed invariant inner product to identify $\mathfrak{k}^\star$ with $\mathfrak{k}$. Since $\mathrm{Ad}\,k(\beta) = \beta$ for all $k \in \mathrm{Stab}\,\beta$ by the definition of $\mathrm{Stab}\,\beta$, it follows immediately that the map $\mu - \beta$ sending $x \in Z_\beta$ to $\mu(x) - \beta$ is also a moment map for the action of $\mathrm{Stab}\,\beta$ on $Z_\beta$. The same is true when $Z_\beta$ is replaced by $Z_{\beta,m}$ for any $m$. As $Z_{\beta,m} \cap \mu^{-1}(\beta)$ is the inverse image of 0 under this moment map, theorem 5.4 will give us an inductive formula for the equivariant cohomology $H^\star_K(\mu^{-1}(0), \mathbb{Q})$

provided we can always calculate $P_t^K(X)$. But for connected groups we have

**Proposition 5.8**. Suppose that $X$ is a compact symplectic manifold acted on by a compact connected Lie group $K$ such that a moment map $\mu$ exists. Then the rational equivariant cohomology of $X$ is the tensor product of the ordinary ratioanl cohomology of $X$ and that of the classifying space $BK$. That is

$$P_t^K(X) = P_t(X)P_t(BK)$$

**Remark**. If $K$ is not connected, let $K_0$ be its identity component. Then it is not hard to show using 5.8 that $H_K^\star(X, \mathbb{Q})$ is the invariant part of

$$H^\star(X, \mathbb{Q}) \otimes H^\star(BK_0; \mathbb{Q})$$

under the action of the finite group $K/K_0$.

*Proof of 5.8.* By definition the equivariant cohomology of $X$ is the ordinary cohomology of $X \times_K EK$ where $EK \to BK$ is the classifying bundle for $K$. Write $X_K = X \times_K EK$.

There is a natural locally trivial fibration $X_K \to BK$ with $X$ as fibre. We need to show that this fibration is cohomologically trivial, i.e. that the associated spectral sequence degenerates.

First suppose that the group is a torus $T$. Let $\beta$ be a generic element of $\mathfrak{t}$ so that the subgroup $\exp \mathbb{R}\beta$ of $T$ is dense in $T$ and let $\mu_\beta : X \to \mathbb{R}$ be defined by $\mu_\beta = \mu(x).\beta$. Then by 3.9 $\mu_\beta$ is a nondegenerate Morse function on $X$ and its critical points are the fixed points of $T$ on $X$. Moreover the induced action of $T$ on the normal bundle to any of the components of the critical set has no nonzero fixed vectors so by 5.3 it follows that $\mu_\beta$ is equivariantly perfect for $T$. Thus

$$P_t^T(X) = \sum_C t^{d(C)} P_t(C_T)$$

where $C$ runs over the components of the fixed point set and $d(C)$ is the index of $\mu_\beta$ along $C$. But as $T$ acts trivially on each $C$ we have

$$C_T = C \times_T ET \cong C \times BT$$

so that $P_t(C_T) = P_t(C)P_t(BT)$. Thus

$$P_t^T(X) = P_t(BT) \sum_C t^{d(C)} P_t(C)$$

The ordinary Morse inequalities for $\mu_\beta$ imply that

$$\sum_C t^{d(C)} P_t(C) - P_t(X) = Q(t)(1+t)$$

where $Q(t) \geq 0$ in the sense that all its coefficients are nonnegative. In particular

$$P_t(X) \leq \sum_C t^{d(C)} P_t(C)$$

The Serre spectral sequence for the fibration $X_T \to BT$ starts with

$$E_2^{p,q} = H^p(X, \mathbb{Q}) \otimes H^q(BT, \mathbb{Q})$$

and $E_{r+1}^{p,q}$ is the quotient of a subgroup of $E_r^{p,q}$ for each $r \geq 2$. Thus $\dim E_r^{p,q}$ decreases as $r$ increases so that

$$\dim H^n(X_T, \mathbb{Q}) = \sum_{p+q=n} \dim E_\infty^{p,q} \leq \sum_{p+q=n} \dim E_2^{p,q}$$

which implies that $P_t^T(X) \leq P_t(X) P_t(BT)$. But

$$P_t(X) P_t(BT) \leq P_t(BT) \sum_C t^{d(C)} P_t(C) = P_t^T(X)$$

Therefore both these inequalities must be equalities

Now let $K$ be any compact connected group with maximal torus $T$. There are fibrations $BT \to BK$ $X_T \to X_K$ with fibre the flag manifold $K/T$. It is well known that the spectral sequences for these fibrations degenerate. To show this one must check that every cohomology class of the fibre $K/T$ extends to a cohomology class of $X_T$. But the $\mathbb{Q}$-cohomology of $K/T$ is multiplicatively generated by the Chern classes of the line bundles $L_\alpha$ on $K/T$ defined by characters $\alpha$ of $T$. Since $L_\alpha = \mathbb{C} \times_T K$ where the action of $T$ on $\mathbb{C}$ is multiplication by $\alpha$ the Chern class of the line bundle $\mathbb{C} \times_T (X \times ET)$ over $X_T = X \times_T ET$ restricts to $c_1(L_\alpha)$ on each fiber. Therefore

$$P_t^T(X) = P_t^K(X) P_t(K/T)$$

and $P_t(BT) = P_t(BT) P_t(K/T)$. The result now follows from the torus case.

**5.9**. This argument shows that every component $\mu_\beta$ of the moment map is both equivariantly perfect and perfect The function $f$ on the other hand is equivariantly perfect by theorem 5.4 but is not necessarily perfect. For example if $S^1$ acts on the complex sphere as rotation about some axis then $\mu$ is the projection on that axis and has a maximum and a minimum as its only critical points. Thus its Morse seris is $1 + t^2$ and its equivariant Morse series is $(1+t^2)(1-t^2)^{-1}$. On the other hand $f = ||\mu||^2$ has critical points at the poles and on the equator so its equivariant Morse series is

$$1 + 2t^2(1-t^2)^{-1} = (1+t^2)(1-t^2)^{-1}$$

but its ordinary Morse series is $(1+t)+2t^2$.

The fact that $\mu_\beta$ is perfect is used in the work of Carrell and Sommese on $\mathbb{C}^\star$-actions on Kähler manifolds (see [C & S] and also [B-B] and [C&G]).

By using the argument of 5.7 we obtain from theorem 5.4 and proposition 5.8 the inductive formula

5.10.

$$P_t^K(\mu^{-1}(0)) = P_t(X)P_t(BK) - \sum_{\substack{0 \neq \beta \\ 0 \leq m \leq \dim X}} t^{d(\beta,m)} P_t^{\mathrm{Stab}\,\beta}(Z_{\beta,m} \cap \mu^{-1}(\beta))$$

for the equivariant cohomology of $\mu^{-1}(0)$ when $K$ is connected.

Moreover if the symplectic quotient exists then by 5.6 its rational cohomology is the same as the rational equivariant cohomology of $\mu^{-1}(0)$ so 5.10 gives us a means of calculating it.

This inductive formula 5.10 was our first goal. It is not hard to deduce from it an explicit formula for $P_t^K(\mu^{-1}(0))$ in terms of the cohomology of certain symplectic submanifolds of $X$ and of the classifying spaces of certain subgroups of $K$. These submanifolds and subgroups are determined by the combinatorial geometry of the finite set of weights. The remainder of this section will be devoted to obtaining this explicit formula.

As in 3.4 let $\mathbb{A}$ be the set of weights of the action. That is, $\mathbb{A}$ is the image under $\mu_T$ of the fixed point set of the maximal torus $T$ of $K$ in $X$ which is a finite set. By definiton 3.13 the indexing set $\mathcal{B}$ of the stratification of the set of all minimal weight combinations in the positive Weyl chamber $\mathfrak{t}_+$. A minimal weight combination is the closest point to 0 of the convex hull of some nonempty set of weights. Thus any $\beta \in \mathcal{B}$ is the closest point to 0 of $\mathrm{Conv}\{\alpha \in \mathbb{A} : (\alpha-\beta).\beta = 0\}$. We have noted at 5.7 that $Z_\beta \cap \mu^{-1}(\beta)$ is the inverse image of 0 under the map $\mu - \beta : Z_\beta \to \mathrm{Stab}\,\beta$ and that this is a moment map for the action of $\mathrm{Stab}\,\beta$ on $Z_\beta$. By the definition of $Z_\beta$ as the union of those components of the fixed point set of $T_\beta$ on which $\mu_\beta$ takes the value $||\beta||^2$, the image under this moment map of the fixed points of $T$ (which is a maximal torus of $\mathrm{Stab}\,\beta$) on $Z_\beta$ is just the set

$$\{\alpha - \beta : \alpha \in \mathbb{A},\ (\alpha-\beta).\beta = 0\}$$

So we make the following

**Definition 5.11**. A sequence of points $\{\beta_1, \dots, \beta_q\}$ of nonzero elements of $\mathfrak{t}$ is called a $\beta$-sequence if for each integer $1 \leq j \leq q$

1. $\beta_j$ is the closest point to 0 of the convex hull

$$\mathrm{Conv}\{\alpha - \beta_1 - \beta_2 - \ldots - \beta_{j-1} : \alpha \in \mathbb{A},\ (\alpha - \beta_k).\beta_k = 0,\ 1 \leq k \leq j\}$$

2. $\beta_j$ lies in the unique Weyl chamber containing $\mathfrak{t}_+$ of the subgroup

$$\bigcap_{1\leq i\leq j} \mathrm{Stab}\,\beta_j$$

Note that $T$ is a maximal torus of $\cap_i \mathrm{Stab}\,\beta_j$ for each $j$ and its Weyl group is a subgroup of the Weyl group of $K$.

Thus a $\beta$-sequence of length one is just a nonzero element of the indexing set $\mathcal{B}$ while $(\beta_1, \beta_2)$ is a $\beta$-sequence of length 2 iff $\beta_1 \in \mathcal{B} - \{0\}$ and $\beta_2$ lies in the indexing set for the action of $\mathrm{Stab}\,\beta_1$ on $Z_{\beta_1}$ with moment map $\mu - \beta_1$.

**Definition 5.12**. For each $\beta$-sequence $\underline{\beta} = (\beta_1, \ldots, \beta_q)$ let $T_{\underline{\beta}}$ be the subtorus of $T$ generated by $\{\beta_1, \ldots, \beta_q\}$; that is, the closure in $T$ of the subgroup generated by the one-paramter subgroups $\{\exp \mathbb{R}\beta_j : 1 \leq j \leq q\}$. The fixed point set of $T_{\underline{\beta}}$ on $X$ is a (possibly disconnected) symplectic submanifold of $X$ and the projection $\mu_{\underline{\beta}}$ of $\mu$ onto the Lie algebra of $T_{\underline{\beta}}$ is constant along each of its components. Let $Z_{\underline{\beta}}$ be the union of those components on which $\mu_{\underline{\beta}} = \beta_q$.

**Lemma 5.13**. If $q \geq 2$ a sequence $\underline{\beta} = (\beta_1, \ldots, \beta_q)$ in $\mathfrak{t}$ is a $\beta$-sequence iff $\beta_1 \in \mathcal{B} - \{0\}$ and the sequence $\underline{\beta}' = (\beta_2, \ldots, \beta_q)$ is a $\beta$-sequence for the action of $\mathrm{Stab}\,\beta_1$ on $Z_{\beta_1}$ with moment map $\mu - \beta_1$. If this is so then $Z_{\underline{\beta}}$ is contained in $Z_{\beta_1}$ and coincides with $Z_{\underline{\beta}}$ where $Z_{\underline{\beta}'}$ is defined relative to the action of $\mathrm{Stab}\,\beta_1$ on $Z_{\beta_1}$.

*Proof.* This follows directly from the definitions. □

If $\beta$ lies in the Lie algebra $\mathfrak{t}_{\underline{\beta}}$ of $T_{\underline{\beta}}$ then every point of $Z_\beta$ is critical for the function $\mu_\beta$ (see 3.7). Since $\mu_\beta$ is nondegenerate the index $\mathrm{ind}\, H_x(\mu_\beta)$ of the Hessian $H_x(\mu_\beta)$ is constant along connected components of $Z_{\underline{\beta}}$; and so the index of its restriction to the tangent space of any $T_\beta$-invariant submanifold of $X$ containing $Z_{\underline{\beta}}$. So we make the following

**Definition 5.14**. Suppose $\underline{\beta} = (\beta_1, \ldots, \beta_q)$ is a $\beta$-sequence. For any integer $m$ let $Z_{\underline{\beta},m}$ be the union of those connected components $C$ of $Z_{\underline{\beta}}$ such that if $x$ is any point of $C$ then

$$m = \sum_{1\leq j\leq q} \mathrm{ind}\, H_x(\mu_{\beta_j}|T_j)$$

where $T_j = T_x(Z_{\beta_1} \cap \ldots \cap Z_{\beta_{j-1}})$. (So $Z_{\underline{\beta},m} = \emptyset$ unless $m$ lies between 0 and $\dim X$).

It follows immediately from definition 3.10 and 5.14 that if $m$ and $m_1$ are any integers then in the terminology of 5.13.

**5.15**. The intersection of $Z_{\underline{\beta},m}$ with $Z_{\beta_1,m_1}$ is $Z_{\underline{\beta}',m-m_1}$.

Now we can state the explicit formula for $P_t^K(\mu^{-1}(0))$.

**Theorem 5.16**. Let $X$ be a connected symplectic manifold acted on by a connected compact group $K$ with moment map $\mu : X \to \mathfrak{k}^\star$ and suppose that $\mathfrak{k}^\star$ is equipped with a fixed invariant inner product. Suppose that $\mu^{-1}(0) \neq \emptyset$. Then

$$P_t^K(\mu^{-1}(0)) = P_t(X)P_t(BK) + \sum_{\underline{\beta},m}(-1)^q t^{d(\underline{\beta},m)} P_t(Z_{\underline{\beta},m})P_t(B\text{Stab}\,\underline{\beta})$$

the sum being over all $\beta$-sequences $\underline{\beta} = (\beta_1, \ldots, \beta_q)$ and all integers $0 \leq m \leq \dim X$. Here $\beta$-sequences and the associated manifolds $Z_{\underline{\beta},m}$ are as defined in 5.11 and 5.14. Also for any $\beta$-sequence $\underline{\beta} = (\beta_1, \ldots, \beta_q)$

$$\text{Stab}\,\underline{\beta} = \bigcap_{1\leq j\leq q} \text{Stab}\,\beta_j$$

$B\text{Stab}\,\underline{\beta}$ is the classifying space for $\text{Stab}\underline{\beta}$ and $d(\underline{\beta}, m) = m - \dim K + \dim \text{Stab}\,\underline{\beta}$.

*Proof.* The proof is by induction on $\dim X$. By assumption $X$ is connected and $\mu^{-1}(0) \neq \emptyset$ so that if $\dim X = 0$ then $X$ consists of a single point $x$ and $\mu(x) = 0$. So there are no $\beta$-sequences and the result is trivial.

Now assume $\dim X > 0$. By 5.10

(a)
$$P_t^K(\mu^{-1}(0)) = P_t(X)P_t(BX) - \sum_{\beta_1,m_1} t^{d(\beta_1,m_1)} P_t^{\text{Stab}\,\beta_1}(Z_{\beta_1,m_1} \cap \mu^{-1}(\beta_1))$$

where the sum is over nonzero elements $\beta_1, m_1$ such that $\beta_1 \in \mathcal{B}$ and $0 \leq m \leq \dim X$ and $d(\beta_1, m_1) = m_1 - \dim K + \dim \text{Stab}\,\beta_1$. Moreover $Z_{\beta_1,m_1} \cap \mu^{-1}(\beta_1)$ is the inverse image of 0 under the moment map $\mu - \beta_1$ for the action of $\text{Stab}\,\beta_1$ on $Z_{\beta_1,m_1}$. Recall from 4.8 that since $K$ is connected so is $\text{Stab}\,\beta_1$. Without loss of generality we may assume that every component of $Z_{\beta_1,m_1}$ meets $\mu^{-1}(\beta_1)$. Since $\mu^{-1}(0)$ is nonempty and $\beta_1 \neq 0$ every component of $Z_{\beta_1,m_1}$ is a proper submanifold of $X$. Therefore by induction

(b)
$$P_t^{\text{Stab}\,\beta_1}(Z_{\beta_1,m_1} \cap \mu^{-1}(\beta_1)) = P_t(Z_{\beta_1,m_1})P_t(B\text{Stab}\,\beta_1) + \sum_{\underline{\beta}',m'}(-1)^{q-1} t^{(\underline{\beta}',m')} P_t(Z_{\underline{\beta}',m'})P_t(B\text{Stab}\underline{\beta}')$$

where the sum is over $\beta$-sequences $\underline{\beta}' = (\beta_2, \ldots, \beta_q)$ for the action of $\mathrm{Stab}\,\beta_1$ on $Z_{\beta_1}$ and integers $0 \leq m \leq \dim Z_{\beta_1}$. Moreover

$$\mathrm{Stab}\,\underline{\beta}' = \bigcap_{2\leq j\leq q} \mathrm{Stab}\,\beta_j \cap \mathrm{Stab}\,\beta_1 = \bigcap_{1\leq j\leq q} \mathrm{Stab}\,\beta_j$$

and $d(\underline{\beta}', m') = m' - \dim \mathrm{Stab}\,\beta_1 + \dim \mathrm{Stab}\,\underline{\beta}'$. Therefore the result follows immediately substituting (b) into (a) and using 5.13 and 5.15. □

**Corollary 5.17**. Under the same assumptions as the theorem, suppose that the symplectic quotient of $X$ by $K$ exists. Then its Betti numbers are the same as the equivariant Betti numbers of $\mu^{-1}(0)$ and are thus given by the formula 5.16.

*Proof.* This follows immediately from 5.6 and 5.16. □

**Remark**. These results can be extended to the case where $K$ is not connected by using the remark which follows 5.8.

We shall conclude this section with an example.

**Example 5.18**. As before consider the diagonal action of SU(2) on $(\mathbb{P}_1)^n$. The action of SU(2) on $\mu^{-1}(0)$ is free provided $n$ is odd, since any configuration with center of gravity at 0 must contain at least three distinct points. Since SU(1) has rank 1 and $\beta$-sequences consist of mutually orthogonal points, every $\beta$-sequence must be of length 1 and so can be identified with an element of $\mathcal{B} - \{0\}$. We have seen in 3.17 that any $\beta \in \mathcal{B} - \{0\}$ corresponds to an integer $r$ such that $\frac{n}{2} < r \leq n$ and that $Z_r$ consists of sequences containing $r$ copies of 0 and $n-r$ copies of $\infty$. Thus $Z_{r,m} = \emptyset$ unless $m = 2(r-1)$ and so the rational cohomology of the Marsden-Weinstein reduction is

$$P_t(\mathbb{P}_1^n)P_t(B\mathrm{SU}(2)) - \sum_{\frac{n}{2}<r\leq n} \binom{n}{r} t^{2(r-1)} P_t(BS^1)$$

$$= (1+t^2)(1-t^4)^{-1} - \sum \binom{n}{r} t^{2(r-1)}(1-t^2)^{-1}$$

When $n$ is odd this is a polynomial in $t^2$ of degree $n-3$ such that the coefficient of $t^{2j}$ is

$$1 + (n-1) + \binom{n-1}{2} + \ldots + \binom{n-1}{\min(j, n-3-j)}$$

It is *not* a polynomial when $n$ is even.

Further examples will be given in Part II.

# 6 Complex group actions on Kähler manifolds

Suppose now that $Y$ is a compact Kähler manifold acted on by a complex Lie group $G$ and that $G$ is the complexifcation of a maximal compact subgroup $K$. Thus if $\mathfrak{k}$, $\mathfrak{g}$ are the Lie algebras of $K, G$ then $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Suppose also that $K$ preserves the Kähler structure on $X$. This condition is always satisfied if the Kähler metric is replaced by its average over $K$. In particular, $X$ might be a nonsingular complex projective variety acted on linearly by a complex reductive group.

The Kähler structure makes $X$ into a symplectic manifold acted on by $K$ and in addition gives $X$ a $K$-invariant Riemannian metric. Assume that a moment map $\mu : X \to \mathfrak{k}^\star$ exists for the action of $K$ on $X$. This always happens if for example $K$ is semisimple or $X$ is a projective variety or if $H^1(X, \mathbb{Q}) = 0$. Let $f : X \to \mathbb{R}$ be the norm-square of the moment map with respect to some fixed invariant inner product on $\mathfrak{k}$.

When applying Morse theory to the function $f$ on a general symplectic manifold we concentrated on the set of critical points for $f$. We showed that there are Morse inequalities (in fact, equalities) relating the equivariant Betti numbers of $X$ to those of certain critical subsets of $f$. In order to establish these inequalities a metric was introduced on $X$. Then $f$ induced a stratification on $X$ such that the stratum containing any point was determined by the limit of its trajectory under $-\mathrm{grad}\, f$. This stratification was no more than a technical device: it was not canonically determined by the symplectic group action since it depended on the metric. However in the case of a Kähler manifold there is a canonical choice of metric. We shall see in this section that the stratification induced by $f$ and the Kähler metric has many elegant properties; in particular, the strata are all complex locally-closed submanifolds of $X$ and are invariant under the action of the complex group $G$.

**Definition 6.1**. For $\beta \in \mathcal{B}$ let $S_\beta$ consist of all points of $X$ whose paths of steepest descent for the Kähler metric have limit points in the critical subset $C_\beta$ defined at 3.14.

The subsets $\{S_\beta\}$ form a stratification of $X$ by lemma 10.7 of the appendix. We shall see that they have the following alternative description in terms of the moment map and the orbits of $G$.

**6.2**. A point $x \in X$ lies in $S_\beta$ iff $\beta$ is the unique closest point to 0 of $\mu(\overline{Gx}) \cap \mathfrak{t}^\star$.

Each stratum $S_\beta$ also has a decomposition analogous to the decomposition of $C_\beta$ as $K \times_{\mathrm{Stab}\beta} (Z_\beta \cap \mu^{-1}(\beta))$. It is described as follows. Recall from 4.9 that for each $\beta$ the symplectic manifold $Z_\beta$ of $X$ is acted on by the stabiliser of $\beta$ in $K$ and the restriction of $\mu - \beta$ to $Z_\beta$ is a moment map for the action of $\mathrm{Stab}\,\beta$ on $Z_\beta$.

**6.3**. Let $Z_\beta^{\min}$ be the subset of $Z_\beta$ consisting of those points $x \in Z_\beta$ such that the limit points of the path of steepest descent from $x$ for $||\mu - \beta||^2$ on $Z_\beta$ lie in $Z_\beta \cap \mu^{-1}(\beta)$.

So $Z_\beta^{\min}$ is the minimum Morse stratum of $Z_\beta$ associated to the square of the moment map $\mu - \beta$ and is an open subset of $Z_\beta$.

Recall also that $Y_\beta$ is the subset of $X$ consisting of all those points in $X$ whose paths of steepest descent under $\mu_\beta$ converge to points of $Z_\beta$. This subset is a locally closed submanifold of $X$ and the inclusion of $Z_\beta$ in $Y_\beta$ is a cohomology equivalence. In fact since $\mu_\beta$ is nondegenerate in the sense of Bott, it is straightforward to check that the path of steepest descent of any $y \in Y_\beta$ has a unique limit point $p_\beta(y)$ say, in $Z_\beta$ and that the function $p_\beta : Y_\beta \to Z_\beta$ defined thus is a retraction of $Y_\beta$ onto $Z_\beta$.

**Definition 6.4**. Let $Y_\beta^{\min}$ be the inverse image of $Z_\beta^{\min}$ under the retraction $p_\beta : Y_\beta \to Z_\beta$. Then $Y_\beta^{\min}$ is an open subset of $Y_\beta$ and retracts on $Z_\beta^{\min}$.

We shall see that $S_\beta = GY_\beta^{\min}$ for each $\beta$ and that there is a parabolic subgroup $P_\beta$ of $G$ which preserves $Y_\beta^{\min}$ such that $S_\beta$ is isomorphic to $G \times_{P_\beta} Y_\beta^{\min}$.

**Example 6.5**. Suppose that $X \subset \mathbb{P}_n$ is a complex projective variety acted on linearly by a complex reductive group $G$, and that $\alpha_0, \ldots, \alpha_n$ are the weights of the representation of $G$. Then we have seen at 3.11 that

$$Z_\beta = \{(x_0 : \ldots : x_n) \in X : x_j \neq 0 \,\text{unless}\, \alpha_j.\beta = ||\beta||^2\}$$

It is easily checked that $Y_\beta$ consists of all $(x_0 : \ldots : x_n) \in X$ such that $x_j = 0$ unless $\alpha_j.\beta = ||\beta||^2$ and $x_j \neq 0$ for some $j$ with $\alpha_j.\beta = ||\beta||^2$. In particular, suppose $X = \mathbb{P}_1^n$ and $G = \mathrm{SL}(2)$ acts diagonally on $X$ as in 2.2. By example 3.17, $\mathcal{B} - \{0\}$ can be identified with the set of integers $r$ such that $\frac{n}{2} < r \leq n$. It is easy to see from 3.17 that $Y_r^{\min} = Y_r$ consists of all sequences which contain precisely $r$ copies of 0. But the stratum $S_r$ consists of sequences which contain $r$ coincident points. So $S_r \cong G \times_B Y_r^{\min}$ where $B$ is the Borel subgroup of SL(2) fixing 0.

The basic lemma needed is the following.

**Lemma 6.6**. If $\beta \in \mathcal{B}$ then for any $x \in X$

$$\mathrm{grad}\, \mu_\beta(x) = i\beta_x$$

and

$$\mathrm{grad}\, f(x) = 2i\mu(x)_x$$

where $\mu(x) \in \mathfrak{k}^\star$ is identified with a point of $\mathfrak{k}$ by using the fixed inner product.

*Proof.* For any $x$ and $\xi \in T_xX$ we have

$$\langle \xi, \mathrm{grad}\, \mu_\beta(x) \rangle = d\mu_\beta(x)(\xi) = d\mu(x)(\xi).\beta = \omega_x(\xi, \beta_x) = \langle \xi, i\beta_x \rangle$$

Hence grad $\mu_\beta(x) = i\beta_x$. The argument used in 3.1 completes the proof.

**6.7**. Thus the trajectory from any $x \in X$ of $-\text{grad}\,\mu_\beta$ is $\{(\exp(it\beta)x : t \geq 0\}$ Moreover since $2i\mu(x)$ lies in $\mathfrak{g}$ all paths of steepest descent for $f = ||\mu||^2$ on $X$ are contained in orbits of the complex group $G$.

The complexification $T_{\mathbb{C}}$ of the maximal torus $T$ of $K$ is a maximal complex torus of $G$.

**Definition 6.8**. Let $B$ be the Borel subgroup of $G$ associated to the positive Weyl chamber $\mathfrak{t}_+$. That is, if

$$\mathfrak{g} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha} \mathfrak{g}^\alpha$$

is the root space decomposition of $\mathfrak{g}$ with respect to $T_{\mathbb{C}}$ then $B = \exp \mathfrak{b}$ where

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} + \sum_{\alpha+} \mathfrak{g}^\alpha$$

(see [A1], p. 146).

**Lemma 6.9**. For any $\beta \in \mathfrak{t}_+$ let $P_\beta \subset G$ consist of all $g \in G$ such that

$$(\exp it\beta) g (\exp it\beta)^{-1}$$

tends to a limit in $G$ as $t \to \infty$. Then $P_\beta$ is a parabolic subgroup of $G$ and is the product $B\text{Stab}\,\beta$ of the Borel subgroup with the stabiliser of $\beta$ in $K$.

*Proof.* It follows from the Peter-Weyl theorem that the compact group $K$ may be embedded in some unitary group $\mathrm{U}(n)$. Then, as $G$ is the complexification of $K$, it is isomorphic to a subgroup of the complex general linear group $\mathrm{GL}(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k} \subseteq \mathfrak{gl}(n)$. We may assume that the maximal torus $T$ is embedded in the diagonal matrices via $t \mapsto \text{diag}(\alpha_1(t), \ldots, \alpha_n(t))$ where $\alpha_1, \ldots, \alpha_n$ are characters of $T$. If we identify the $\alpha_i$ with elements of $\mathfrak{t}^\star$ by looking at their derivatives at the identity, then $\beta$ becomes the diagonal matrix with entries $2\pi i(\alpha_j.\beta)$. Without loss of generality, suppose that

$$\alpha_1.\beta \geq \ldots \geq \alpha_n.\beta;$$

then $P_\beta$ is the subgroup of $G$ which consists of all upper triangular block matrices where the blocks are determined by the different values of $\alpha_i.\beta$.

Given any root $\alpha$, the a matrix $x \in \mathfrak{g}$ lies in the root space $\mathfrak{g}^\alpha$ iff

$$[h, x] = (\alpha.h)x \qquad \forall h \in \mathfrak{t}_{\mathbb{C}}$$

Thus, if the $(i,j)$ component of $x$ is nonzero, then $\alpha = \alpha_j - \alpha_i$. If moreover $\alpha$ is a positive root then $\alpha.\beta \geq 0$ since $\beta \in \mathfrak{t}_+$ so $\alpha_j.\beta \geq \alpha_i.\beta$. This implies that every element of the Lie algebra $\mathfrak{b}$ has an upper triangular block decomposition. Hence the same is true of $B = \exp \mathfrak{b}$. Thus $P_\beta$ contains $B$ and so is a parabolic subgroup of $G$.

Since $G = BK$ (by [A1], p. 147), we deduce that $P_\beta = B(K \cap P_\beta)$. But as $K \subseteq \mathrm{U}(n)$, an element of $K$ lies in $P_\beta$ iff it is of block diagonal form, i.e. iff it ies in $\mathrm{Stab}\,\beta$. Therefore, $P_\beta = B\mathrm{Stab}\,\beta$ and the proof is complete. □

**Lemma 6.10**. The subsets $Y_\beta$ and $Y_\beta^{\min}$ of $X$ are invariant under $P_\beta$.

*Proof.* Suppose $p \in P_\beta$ so that $(\exp it\beta)p(\exp it\beta)^{-1}$ tends to some $s \in G$ as $t \to -\infty$. By 6.3 and 6.7 an element $y$ of $X$ lies in $Y_\beta$ iff $(\exp it\beta)y$ converges to an element $x$ of $Z_\beta$ as $t \to -\infty$. But $(\exp it\beta)y \to x$ as $t \to -\infty$ iff $(\exp it\beta)py \to sx$. Clearly, $s$ lies in the stabiliser of $\beta$ in $G$ and hence preserves both $Z_\beta$ and $Z_\beta^{\min}$. The result follows.

**Corollary 6.11**. If $x \in GY_\beta$ then $||\mu(x)||^2 \geq ||\beta||^2$. Equality occurs iff $\mu(x)$ lies in the adjoint orbit of $\beta$ in $\mathfrak{k}$.

*Proof.* Since $G = BK$ and $B \subseteq P_\beta$, it follows from 6.10 that $GY_\beta = KY_\beta$. As $||\mu(kx)||^2 = ||\mu(x)||^2$ for all $k \in K$, we can therefore assume that $x \in Y_\beta$. But then the path of steepest descent for the function $\mu_\beta$ from $x$ converges to a point $y \in Z_\beta$ and $\mu_\beta(y) = ||\beta||^2$ by definition of $Z_\beta$. So

$$\mu(x).\beta = \mu_\beta(x) \geq \mu_\beta(y) = ||\beta||^2$$

from which the result follows.

**Corollary 6.12**. If $x \in GY_\beta^{\min}$ then $\beta$ is the unique closest point to 0 of $\mu(\overline{Gx}) \cap \mathfrak{t}_+$.

*Proof.* Since the adjoint orbit of $\beta$ in $\mathfrak{k}$ intersects $\mathfrak{t}_+$ only at $\beta$, by 6.11 it suffices to show that $\beta$ lies in $\mu(\overline{Gx})$. Without loss of generality, $x \in Y_\beta^{\min}$ o that $(\exp it\beta)x$ converges to some $y \in Z_\beta^{\min}$ as $t \to -\infty$. Then $\overline{Gy} \subseteq \overline{Gx}$, so it is enough to show that $\beta \in \mu(\overline{Gy})$.

By definition of $Z_\beta^{\min}$ the path of steepest descent from $y$ of the function $||\mu - \beta||^2$ restricted to $Z_\beta$ has a limit point in $Z_\beta \cap \mu^{-1}(\beta)$. By 4.9, $\mu - \beta$ is a moment map for the action of $\mathrm{Stab}\,\beta$ on $Z_\beta$, so 6.7 implies that this path is contained in the orbit of $y$ under the complexification of $\mathrm{Stab}\,\beta$. Hence $\beta \in \mu(\overline{Gy})$. This completes the proof. □

What we are aiming to do is to show that $GY_\beta^{\min} = S_\beta$ for each $\beta$. Then 6.12 will show that 6.2 is as claimed an alternative definition of the stratification $\{S_\beta\}$.

**Remark 6.13**. The definition 6.2 is neater but less useful, and moreover cannot be given directly without some guarantee that $\mu(\overline{Gx}) \cap \mathfrak{t}_+$ contains a unique point which is closest to 0. In fact it has been proved recently by Mumford that if $X$ is a complex projective variety acted on linearly by $G$ then $\mu(\overline{Gx}) \cap \mathfrak{t}_+$ is convex for all $x \in X$ which implies that it contains a unique point closest to 0. Indeed the same is true when $\overline{Gx}$ is replaced by any closed $G$-invariant subvariety of $X$. This generalizes a similar result of Guillemin and Sternberg which requires the subvariety to be nonsingular and hence does not apply to $\overline{Gx}$ in general. Mumford's proof is algebraic but it can be adapted to the more general Kähler case by using lemma 8.8 and remark 8.9 below.

The next result we need is that $GY_\beta^{\min}$ is diffeomorphic to $G \times_{P_\beta} Y_\beta^{\min}$. This will imply in particular that $GY_\beta^{\min}$ is smooth.

We shall first make the following

**Assumption 6.14**. The minimum stratum $X^{\min}$ for the function $||\mu||^2$ on $X$ is contained in the minimum stratum denoted by $X_T^{\min}$ for the function $||\mu_T||^2$. Here as before $\mu_T$ is the composition $X \xrightarrow{\mu} \mathfrak{k}^\star \to \mathfrak{t}^\star$ and is a moment map for the action of the maximal torus $T$ on $X$.

The proofs of the following lemma and theorem will depend on this assumption holding for all actions of closed subgroups of $K$. But clearly the assumption is valid for all tori so that theorem 6.18 will hold for $T$ at least; and from this we will be able to deduce that the assumption is always valid.

**Lemma 6.15**. If $x \in Y_\beta^{\min}$ then

$$\{g \in G : gx \in Y_\beta^{\min}\} = P_\beta; \quad \{a \in \mathfrak{g} : a_x \in T_x Y_\beta^{\min}\} = \mathfrak{p}_\beta$$

*Proof.* Lemma 6.10 shows that

$$P_\beta \subseteq \{g \in G : gx \in Y_\beta^{\min}\}$$

For the reverse inclusion suppose that $g \in G$ is such that $gx \in Y_\beta^{\min}$. Let $N_K(T)$ be the normaliser of $T$ in $K$; then $G = BN_K(T)B$ by the Bruhat decomposition (see e.g. [A1], p. 135), so that

$$g = b_1 k b_2, \qquad k \in N_K(T), \qquad b_1, b_2 \in B$$

Since $B$ is contained in $P_\beta$ and both $x$ and $gx$ lie in $Y_\beta^{\min}$, so do $b_2x$ and $kb_2x = b_1^{-1}(gx)$. By assumption 6.14 applied to the moment map $\mu - \beta$ for the action of $\operatorname{Stab}\beta$ on $Z_\beta$ we have $Z_\beta^{\min} \subseteq Z_{\beta,T}^{\min}$. Therefore by applying 6.12 to the action of the complex torus $T_\mathbb{C}$ on $X$ it follows that $\beta$ is the closest point to 0 in $\mathfrak{t}$ of both $\mu_T(\overline{T_\mathbb{C} b_2 x})$ and $\mu_T(\overline{T_\mathbb{C} k b_2 x})$. Since $k$ normalises $T$ and $T_\mathbb{C}$ we have that

$$\mu_T(\overline{T_\mathbb{C} k b_2 x}) = \mu_T(\overline{k T_\mathbb{C} b_2 x}) = \operatorname{Ad} \mu_T(\overline{T_\mathbb{C} b_2 x})$$

As the inner product on $\mathfrak{k}$ is invariant under the adjoint action, this implies that $\mathrm{Ad}\,k(\beta) = \beta$, i.e. $k \in \mathrm{Stab}\,\beta$. Since $P_\beta = B\mathrm{Stab}\,\beta$ by 6.9, the element $g = b_1 k b_2$ lies in $P_\beta$.

It remains to show that $\{a \in \mathfrak{g} : a_x \in T_x Y_\beta^{\min}\} \subseteq P_\beta$ since we know that the reverse inclusion holds. By 6.9, $\mathfrak{p}_\beta = \mathfrak{b} + \mathrm{stab}\,\beta$ so it suffices to show that (a) $\{a \in \mathfrak{k}La_x \in T_x Y_\beta^{\min}\} \subseteq \mathrm{stab}\,\beta$. This has already been proved (see Lemma 4.10) in the particular case when $x \in Z_\beta \cap \mu^{-1}(\beta)$. Moreover (a) is a linear independence condition on $x$, and hence the subset of $Y_\beta^{\min}$ where it holds is an open neighbourhood of $Z_\beta \cap \mu^{-1}(\beta)$. It is also clearly invariant under $P_\beta$.

But by the proof of 6.12, given any $x \in Y_\beta^{\min}$ there is some $y \in Z_\beta \cap \mu^{-1}(\beta)$ which lies in the closure of the orbit of $x$ under the complexification $\mathrm{Stab}_{\mathbb{C}}(\beta)$ of $\mathrm{Stab}\,\beta$. Since $\mathrm{Stab}_{\mathbb{C}}\,\beta \subseteq P_\beta$ it follows that the only $P_\beta$-invariant neighbourhood of $Z_\beta \cap \mu^{-1}(\beta)$ is $Y_\beta^{\min}$ itself. Thus (a) must hold for every $x \in Y_\beta^{\min}$. This completes the proof. □

It follows from this lemma, by adapting the argument of Corollary 4.11, that

**6.16**. $GY_\beta^{\min}$ is smooth and diffeomorphic to $G \times_{P_\beta} Y_\beta^{\min}$.

**Remark 6.17**. In fact $Y_\beta$ is a locally-closed complex submanifold of $X$ from which it follows immediately that $GY_\beta^{\min}$ is also complex and is isomorphic as a complex manifold to $G \times_{P_\beta} Y_\beta^{\min}$.

To see that $Y_\beta$ is complex recall that by definition $Y_\beta$ consists of those points $y \in X$ whose trajectories under $-\mathrm{grad}\,\mu_\beta$ converge to points of $Z_\beta$. Since $\mathrm{grad}\,\mu_\beta(x) = i\beta_x$ for all $x$ the vector field $-\mathrm{grad}\,\mu_\beta$ on $X$ is holomorphic. Moreover, by 4.12 (and its proof) if $x \in Z_\beta$ then the Hessian $H_x(\mu_\beta)$ acts as a complex linear transformation of the tangent space $T_x X$ and depends holomorphically on $x$ (again because the action of the group is complex analytic). Since $\mu_\beta$ is nondegenerate the local theory of ordinary differential equations tells us that $Y_\beta$ is a complex submanifold of $X$ in some neighbourhood of $Z_\beta$. But for every $y \in Y_\beta$ there is some $t \in \mathbb{R}$ such that the point $(\exp(it\beta))y$ of the path of steepest descent for $\mu_\beta$ from $y$ lies in the neighbourhood of $Z_\beta$. Since $\exp(it\beta)$ acts as a complex analytic isomorphism of $X$ which preserves $Y_\beta$ it follows that $Y_\beta$ is a complex submanifold of $X$ as required.

We can now prove the result we want, on the assumption that 6.14 and hence also 6.16 hold.

**Theorem 6.18**. Suppose that $X$ is a compact Kähler manifold acted on by a complex Lie group $G$ and that $G$ is the complexification of a maximal compact subgroup $K$ which preserves the Kähler structure. Suppose that $\mu : X \to \mathfrak{k}^*$ is a moment map for the action of $K$ on $X$ and let $\{S_\beta : \beta \in \mathcal{B}\}$ be the Morse stratification for the function $f = ||\mu||^2$ on $X$. Then for each $\beta$ we have $S_\beta = GY_\beta^{\min}$ and $x \in S_\beta$ iff $\beta$ is the unique closest point to 0 of $\mu(\overline{Gx}) \cap \mathfrak{t}_+$. The subsets $\{S_\beta\}$ form a smooth stratification of $X$ which is $G$-equivariantly perfect. Moreover $S_\beta \cong G \times_{P_\beta} Y_\beta^{\min}$ so that

$$H^{\star}_G(S_\beta;\mathbb{Q}) \cong H_{P_\beta}(Y_\beta^{\min};\mathbb{Q})$$

for each $\beta \in \mathcal{B}$.

*Proof.* First we shall use the results of the appendix to show that $S_\beta = GY_\beta^{\min}$.

Since $Y_\beta^{\min}$ is invariant under the action of the parabolic subgroup $P_\beta$ we have that $GY_\beta^{\min} = KY_\beta^{\min}$. So Proposition 4.15 implies that some open subset of $GY_\beta^{\min}$ is a minimising manifold for $f = ||\mu||^2$ along $C_\beta$. Moreover, by 6.7, the trajectory under $-\mathrm{grad}\, f$ of any $x \in GY_\beta^{\min}$ is contained in the orbit $Gx$ and hence in $GY_\beta^{\min}$. In particular, the gradient flow of $f$ is tangential to $GY_\beta^{\min}$, so by Theorem 10.4 of the appendix, $GY_\beta^{\min}$ coincides with the stratum $S_\beta$ in some neighbourhood $U$ of the critical subset $C_\beta$.

Suppose $x \in S_\beta$; then the path of steepest descent for $f$ from $x$ has a limit point in $C_\beta$ and therefore intersects $U \cap S_\beta = U \cap GY_\beta^{\min}$. But by 6.7 this path is contained in the orbit $Gx$ so $x$ must lie in $GY_\beta^{\min}$. Thus $S_\beta \subseteq GY_\beta^{\min}$ for each $\beta \in \mathcal{B}$. But 6.12 implies that the subsets $\{GY_\beta^{\min} : \beta \in \mathcal{B}\}$ are disjoint. Since

$$X = \bigcup_{\beta \in \mathcal{B}} S_\beta$$

it follows that $S_\beta = GY_\beta^{\min}$ for each $\beta \in \mathcal{B}$.

We have already seen at 6.16 that

$$GY_\beta^{\min} \cong G \times_{P_\beta} Y_\beta^{\min}$$

for each $\beta$ and hence

$$H^{\star}_G(GY_\beta^{\min};\mathbb{Q}) \cong H^{\star}_{P_\beta}(Y_\beta^{\min};\mathbb{Q})$$

by [A & B], §13. Finally, theorem 5.4 shows that the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is equivariantly perfect for $K$, and this implies immediately that it is equivariantly perfect for $G$ since $K$ and $G$ are homotopically equivalent. (Alternatively, the proof of 5.4 can be adapted easily to work for the complex group).

It remains to check that the assumption 6.14 is valid.

**Notation**. If it is necessary to make clear what group is involved, the stratum $S_\beta$ will be written $S_{\beta,K}$.

**Lemma 6.20**. Assumption 6.14 is always valid. That is, the minimum stratum $X^{\min} = S_{0,K}$ is contained in the maximum stratum $X_T^{\min} = S_{0,T}$ associated to the action of the maximal torus

$T$ of $K$.

*Proof.* Since 6.14 holds trivially for tori, Proposition 6.18 is valid for the maximal torus $T$ of $K$. Thus, if $x$ does not lie in the minimum stratum $X_T^{\min} = S_{0,T}$ for the torus then there exists some nonzero $\beta \in \mathcal{B}$ such that $x \in S_{\beta,T} \subseteq T_{\mathbb{C}} Y_\beta$. (Note that $Y_\beta$ is the same whether the group is $K$ or $T$; also $T_{\mathbb{C}} Y_\beta = Y_\beta$). Thus by Corollary 6.11 if $y \in \overline{Gx}$ then $||\mu(y)||^2 \geq ||\beta||^2 > 0$. Since the path of steepest descent for the function $||\mu||^2$ from $x$ is contained in $Gx$, we deduce that $x$ cannot lie in $X^{\min}$.

The proof of Theorem 6.18 for any group (torus or not) is now complete. □

**Remark 6.20**. By theorem 4.16 the inclusion of the minimum set $\mu^{-1}(0)$ for $f$ is the minimum stratum $X^{\min}$ is an equivalence of equivariant cohomology. So 5.10 and 5.16 may be interpreted as formulae for the equivariant Poincaré series $P_t^G(X^{\min})$. These formulae can also be derived directly from theorem 6.18.

If $G$ acts freely on the open subset $X^{\min}$ of $X$ then the quotient $X^{\min}/G$ is a complex manifold and it would be natural to hope that the rational cohomology of this is isomorphic to $H_G^\star(X^{\min}; \mathbb{Q})$. This could be proved by showing that the quotient map $X^{\min} \to X^{\min}/G$ is a locally trivial fibration. However this is unnecessary because in the next section we shall see that $X^{\min}/G$ is homeomorphic to the symplectic quotient $\mu^{-1}(0)/K$. This reduces the problem to the action of a compact group.

Let us conclude this section by considering how the stratification is affected if we alter the choice of a moment map or of the invariant inner product on $\mathfrak{k}$. From the algebraic point of view changing the moment map on a complex projective variety $X$ corresponds to changing the projective embedding of $X$.

First consider the inner product. Clearly if the group is a torus then any inner product is invariant and different choices give different stratifications. For example, take $(\mathbb{C}^\star)^2$ acting on $\mathbb{P}_1$ via $\varphi : (\mathbb{C}^\star)^2 \to \mathrm{GL}(2)$ given by

$$\varphi(h) = \begin{pmatrix} \alpha_0(h) & 0 \\ 0 & \alpha_1(h) \end{pmatrix}$$

where $\alpha_j : (\mathbb{C}^\star)^2 \to \mathbb{C}$ is the projection onto the $(j+1)$st factor. Then the stratum to which an element $(x_0 : x_1) \in \mathbb{P}_1$ belongs is determined by the closest point to 0 in the convex hull of $\{\alpha_j : x_j \neq 0\}$. But $\alpha_0$ and $\alpha_1$ are linearly independent so there exist inner products on the Lie algebra of the torus for which the closest point to 0 of their convex hull is respectively $\alpha_0, \alpha_1$ and neither of these. These give different stratifications of $\mathbb{P}_1$.

On the other hand if $G$ is semisimple then the stratification is independent of the choice of inner product. For then $G$ is, up to finite central extentions, the product $G_1 \times \ldots \times G_k$ of simple groups $G_i$ with maximal compact subgroup $K_i$ and maximal compact subtori $T_i$, say. Then $\mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_k$ and $\mathfrak{k}_\alpha$ must be mutually orthogonal under any invariant inner product on $\mathfrak{k}$. For each $i$ the projection $\mu_i$ of $\mu$ onto $\mathfrak{k}_i$ is a moment map for the action of $K_i$ on $X$. It is not hard to see that for any $x \in X$ the closest point to 0 of $\mu(\overline{Gx}) \cap \mathfrak{t}_+$ is

$$\beta = \beta_1 + \ldots + \beta_k$$

where $\beta_i$ is the closest point to 0 of $\mu_i(\overline{G_i x}) \cap (\mathfrak{t}_i)_+$. But since $\mathfrak{k}_i$ is simple the invariant inner product on $\mathfrak{k}_i$ is unique up to scalar multiplication and therefore each $\beta_i$ is independent of the choice of inner product.

Now consider the effect of changing the choice of moment map $\mu$. The only possible way to to do this is to add to $\mu$ a constant $\xi \in \mathfrak{k}^\star$ which is invariant under the adjoint action. Thus as has already been noted when $G$ is semisimple the moment map is unique. On the other hand, if $G$ is a torus $T_c$ an arbitrary constant may be added to the moment map. We know that the stratum containing any point $x$ is labelled by the closest point to 0 of $\mu(\overline{T_c x})$ which is the convex hull of some subset of $\{\alpha_0, \ldots, \alpha_n\}$. Thus by adding different constants to $\mu$ one can obtain a finite number of distinct stratifications of $X$.

Since any compact group is, up to finite central extensions, the product of a torus by a semisimple group, it is now easy to deduce what happens in general.

# 7 Quotients of Kähler manifolds

Suppose as in §6 that $X$ is a compact Kähler manifold acted on by a complex Lie group $G$, and that $G$ is the complexification of a maximal compact subgroup $K$. Assume that $K$ preseves the Kähler form $\omega$ on $X$ (if necessary replace $\omega$ by its average over $K$) and that a moment map $\mu : X \to \mathfrak{k}^\star$ exists for the symplectic action of $K$ on $X$.

Any torus in $G$ will always have fixed points in $X$ so we cannot hope ti give the topological quotient $X/G$ the structure of a Kähler manifold. However in good cases there is a compat Kähler manifold which it is natural to regard as the Kähler quotient of the action of $G$ on $X$. When $X$ is a complex projective variety on which $G$ acts linearly, this quotient coincides with the projective quotient defined by Mumford using geometric invariant theory. The good cases occur when the stabiliser in $K$ of every $x \in \mu^{-1}(0)$ is finite. Recall that this is the condition needed for there to be a symplectic quotient associated to the action.

As before let $X^{\min}$ be the subset of $X$ consisting of points whose paths of steepest descent under the function $f = ||\mu||^2$ have limit points in $\mu^{-1}(0)$. By 6.18 $X^{\min}$ is a $G$-invariant open subset of $X$. We shall see that when $K$ acts with finite stabilisers on $\mu^{-1}(0)$ then the symplectic quotient $\mu^{-1}(0)/K$ can be identified with $X^{\min}/G$ and thus has a complex structure. The symplectic form induced on $\mu^{-1}(0)/K$ is then holomorphic and makes $\mu^{-1}(0)/K$ into a compact Kähler manifold except for the singularities caused by finite isotropy groups. Manifolds with such singularities have been well studied; they are sometimes called $V$-manifolds). This is the natural Kähler quotient of $X$ by $G$.

The rational cohomology of this quotient can be calculated by using 5.10 or 5.17.

Recall from 5.5 that the condition that $K$ acts with finite stabilisers on $\mu^{-1}(0)$ implies that $\mu^{-1}(0)$ is smooth. The inclusion of $\mu^{-1}(0)$ in $X^{\min}$ induces a natural continuous map

$$\mu^{-1}(0)/K \to X^{\min}/G$$

In order to show that this map is a homeomorphism we need some lemmas.

**Lemma 7.1**. $G = K \exp i\mathfrak{k}$.

*Proof.* The left coset space $G/K$ is a complete Riemannian manifold (see [He]) so that the associated exponential map $\mathrm{Exp} : T_K(G/K) \to G/K$ is onto. Moreover $T_K(G/K) = \mathfrak{g}/\mathfrak{k}$ and by [He], p. 169 (4), we have

$$\mathrm{Exp}(a + \mathfrak{k}) = (\exp a)K, \qquad \forall a \in \mathfrak{g}$$

Since $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ the result follows. □

**Lemma 7.2**. If $x \in \mu^{-1}(0)$ then $Gx \cap \mu^{-1}(0) = Kx$.

*Proof.* Suppose $g \in G$ is such that $gx \in \mu^{-1}(0)$. We wish to show that there exists a $k \in K$ such that $gx = kx$. Since $\mu^{-1}(0)$ is $K$-invariant, by 7.1 it suffices to consider the case $g = \exp\, ia$ where $a \in \mathfrak{k}$.

Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(t) = \mu((\exp iat)x).a$. Then $h$ vanishes at 0 and 1 because $x$ and $(\exp ia)x$ both lie in $\mu^{-1}(0)$. Therefore, there is some $t \in (0, 1)$ such that

$$0 = h'(t) = d\mu(y)(ia_y).a = \omega_y(ia_y, a_y) = \langle a_y, a_y\rangle$$

where $y = (\exp iat)x$ and $\langle\,\rangle$ denotes the metric induces by the Kähler structure. Thus $a_y = 0$ so that $\exp ia\mathbb{R}$ fixes $y$ and hence also $x$. But then $(\exp ia)x = x \in Kx$, and the proof is complete. □

It is necessary to strengthen this result.

**Lemma 7.3**. Suppose that $x$ and $y$ lie in $\mu^{-1}(0)$ and $x \notin Ky$. Then there exist disjoint $G$-invariant neighbourhoods of $x$ and $y$ in $X$.

*Proof.* Since $K$ is compact and $x \notin Ky$ there is a compact $K$-invariant neighbourhood $V$ of $x \in \mu^{-1}(0)$ not containing $y$. Since $G = (\exp i\mathfrak{k})K$ by 7.1 it suffices to show that $(\exp i\mathfrak{k})V$ is a neighbourhood of $x \in X$ and that $y \notin \overline{(\exp i\mathfrak{k})V}$.

To see that $(\exp i\mathfrak{k})V$ is a neighbourhood of $x$ in $X$ consider the map $\sigma : \mathfrak{k} \times \mu^{-1}(0) \to X$ which sends $(a, w)$ to $(\exp ia)w$. This is a smooth map of smooth manifolds so it is enough to show that its derivative at $(1, x)$ is surjective. If not, there is some nonzero $\xi \in T_xX$ such that $\langle\xi, \zeta\rangle = 0$ for all $\zeta$ in the image of $d\sigma(1, x)$. In particular, $(\xi, ia_x\rangle = 0$ for all $a \in \mathfrak{k}$. But then

$$0 = \omega_x(\xi, a_x) = d\mu(x)(\xi).a$$

for all $a \in \mathfrak{k}$. Thus

$$\xi \in \ker d\mu(x) = T_x(\mu^{-1}(0))$$

and hence $\xi$ is in the image of $d\sigma(1, x)$ which is a contradiction.

Therefore, if

$$W = \exp\{ia : a \in \mathfrak{k}, ||a|| \le 1\}V$$

then $W$ is a compact neighbourhood of $x$ in $X$. Let

$$\epsilon = \inf\{\langle a_w, a_w\rangle : w \in W, a \in \mathfrak{k}, ||a|| = 1\}$$

If $w \in W$ then $w$ lies in the $G$-orbit of some $z \in \mu^{-1}(0)$ and it follows easily from 7.2 that the stabiliser of $w$ in $G$ is finite. Therefore $a_w \neq 0$ whenever $0 \neq a \in \mathfrak{k}$ and so $\epsilon > 0$.

Now suppose $z \in V$ and $a \in \mathfrak{k}$ is such that $||a|| = 1$. Consider the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(t) = \mu((\exp ita)z).a$. As in (the proof of) 7.2, it $t \in \mathbb{R}$ then $h'(t) = \langle a_w, a_w \rangle$ where $w = (\exp(ita)z$. Therefore $h'(t) \geq 0$ for all $t$ and $h'(t) \geq \epsilon$ when $t \in [0,1]$ by the choice of $\epsilon$. Since $h(0) = 0$ the mean value theorem implies that $h(t) \geq 0$ when $t \geq 1$. As $||a|| = 1$ it follows that

$$||\mu(\exp ita)z|| \geq \epsilon$$

when $t \geq 1$.

We deduce that if $z \in V$ then $||\mu(\exp iaz)|| \geq \epsilon$ whenever $a \in \mathfrak{k}$ and $||a|| \geq 1$. Hence as $V$ is compact $(\exp i\mathfrak{k})V$ is closed in a neighbourhood of $\mu^{-1}(0)$. Since $y \in \mu^{-1}(0)$ and $y \notin (\exp i\mathfrak{k})V$ by 7.2 it follows that $y \in \overline{(\exp i\mathfrak{k})V}$. □

Now we can prove the result we're aiming for.

**Theorem 7.4**. Let $X$ be a Kähler manifold acted on by a group $G$ which is the complexification of a maximal compact subgroup $K$ that preserves the Kähler structure on $X$. Suppose that a moment map $\mu : X \to \mathfrak{k}^\star$ exists for this action of $K$ and suppose that the stabiliser of every $x \in \mu^{-1}(0)$ is finite. Then $X^{\min} = G\mu^{-1}(0)$ and the natural map $\mu^{-1}(0)/K \to X^{\min}/G$ is a homeomorphism.

*Proof.* $G\mu^{-1}(0) \subset X^{\min}$ since $X^{\min}$ is $G$-invariant by 6.18 and contains $\mu^{-1}(0)$. Conversely if $x \in X^{\min}$ then there is some $y \in \mu^{-1}(0)$ lying in the closure of the path of steepest descent for $||\mu||^2$ from $x$. By 6.7 this path is contained in the orbit $Gx$ so that $y \in \overline{Gx}$. Then $Gy \subset \overline{Gx}$ so that either $y \in Gx$ or $\dim Gy < \dim Gx$. But by assumption the stabiliser of $y$ in $K$ is finite and this implies that $\dim Gy = \dim G \geq \dim Gx$ (7.2). We conclude that $y \in Gx$ so that $x \in G\mu^{-1}(0)$.

Thus $X^{\min} = G\mu^{-1}(0)$ so the natural map $\mu^{-1}(0)/K \to X^{\min}/G$ is surjective. Lemma 7.2 implies that it is injective while Lemma 7.3 shows that $X^{\min}/G$ is Hausdorff. Thus the map is a continuous bijection from a compact space to a Hausdorff space and therefore it is a homeomorphism

It follows (from the proof of 7.2) that if $K$ acts freely on $\mu^{-1}(0)$ then $G$ acts freely on the open $X^{\min}$ of $X$, so that the complex structure on $X^{\min}$ induces a complex structure on the topological quotient $X^{\min}/G$. The symplectic form on $\mu^{-1}(0)/K$ induced by $\omega$ is holomorphic with respect to this complex structure because $\omega$ is holomorphic on $X$ and indeed is a Kähler form since $\omega$ is Kähler. Hence the quotient $X^{\min}/G = \mu^{-1}(0)/K$ is a compact Kähler manifold. More generally when the stabiliser of every point in $\mu^{-1}(0)$ is finite the quotient $X^{\min}/G$ can be thought of as a Kähler manifold with singularities caused by the finite isotropy groups.

**Remark 7.5**. The proof of lemma 7.2 is independent of the assumption that the stabiliser of every point in $\mu^{-1}(0)$ is finite and it is also possible to prove lemma 7.3 without using this assumption. One uses the fact that if $a \in \mathfrak{k}$ then the function $\mu_a$ defined by $\mu_a = \mu(x).a$ is a nondegenerate Morse function on $X$. This implies that given any $y \in \mu_a^{-1}(0)$ and any neighbourhood $U$ of $y$ in $X$ there is a smaller neighbourhood $V$ of $y$ and $\epsilon > 0$ such that the intersection with $\mu_a^{-1}[-\epsilon, \epsilon]$ of any trajectory of $\text{grad}\, \mu_a$ which passes through a point of $V$ is contained in $U$. The proof of this when $y$ is not critical for $\mu_a$ is easy: see a proof of 7.3. Using this argument of 7.3 gives the result when $G = \mathbb{C}^\star$ and the torus case also follows without difficulty. The general case can be then be deduced from the facts that $G = KT_cK$ and that $K$ is compact.

From this it follows without the assumption of finite stabilisers that any $x \in X$ lies in $G\mu^{-1}(0)$ iff $x$ lies in $X^{\min}$ and its orbit $Gx$ is closed in $X^{\min}$; and also that the natural map $\mu^{-1}(0)/K \to G\mu^{-1}(0)/G$ is a homeomorphism. In particular when $X$ is a projective variety on which $G$ acts linearly one finds that $\mu^{-1}(0)/K$ is naturally homeomorphic to the geometric invariant theory quotient of $X$ by $G$.

# 8 The relationship with geometric invariant theory

From now on we'll assume that our Kähler manifold $X$ is in fact a nonsingular complex projective variety and that $G$ is a connected reductive complex group acting on $X$ linearly (as in example 2.1). Then geometric invariant theory associates to the action of $G$ on $X$ a projective 'quotient' variety $M$. In fact $M$ is the projective variety $\operatorname{Proj} A(X)^G$ where $A(X)^G$ is the invariant subring of the coordinate ring of $X$. In general $M$ has bad singularities even though $X$ is nonsingular. However in good cases $M$ coincides with the quotient in the usual sense of an open subset $X^{\mathrm{ss}}$ of $X$ by $G$ and the stabiliser in $G$ of every $x \in X^{\mathrm{ss}}$ is finite. This implies that $M$ behaves like a manifold for rational cohomology.

It turns out that the geometric invariant theory quotient $M$ coincides with the symplectic quotient $\mu^{-1}(0)/K$ and that the good cases occur precisely when the stabiliser in $K$ of every $x \in \mu^{-1}(0)$ is finite. So the work of the preceding sections can be used to obtain formulae for the Betti numbers of $M$ in these cases. The formulae involve the cohomology of $X$ and various subvarieties together with that of the classifying space of $G$ and certain reductive subgroups.

**Remark 8.1**. The example of $\mathrm{PGL}(n+1)$ shows that the assumption that $G$ acts on $X$ linearly via a homomorphism $\varphi : G \to \mathrm{GL}(n+1)$ involves some loss of generality. However the finite cover of $\mathrm{SL}(n+1)$ of $\mathrm{PGL}(n+1)$ has the same Lie algebra, moment map and orbits on $X$ as $\mathrm{PGL}(n+1)$. Moreover if $G$ is a connected reductive linear algebraic group acting algebraically on a smooth projective variety $X \subseteq \mathbb{P}_n$ then the action is given by a homomorphism $\varphi : G \to \mathrm{PGL}(n+1)$ provided we assume that $X$ is not contained in any hyperplane. The argument for this runs as follows. Firt we note that the induced action of $G$ on the Picard variety $\mathrm{Pic}(X)$ of $X$ is trivial. For it is enough to show that every Borel subgroup $B$ of $G$ acts trivially. But by [B] theorem 10.4, $B$ has a fixed point on each component of $\mathrm{Pic}(X)$. Applying this with $X$ replaced by $\mathrm{Pic}(X)$ we see that there is an ample bundle on $\mathrm{Pic}(X)$ fixed by $B$. By the theorem of [G&H] p. 326 it follows that the image of $B$ in the group of automorphisms of $\mathrm{Pic}(X)$ is discrete. Thus as $B$ is connected it must act trivially. (Alternatively, see [M] corollary 1.6). Now let $L$ be the hyperplane bundle on $X \subset \mathbb{P}_n$ which has automorphism group $\mathrm{GL}(n+1)$. Then $g^{\star}L \cong L$ for all $g \in G$ so that the action of $G$ on $X$ is covered by an automorphism of $L$ and hence is not contained in a hyperplane. So we get a well defined homomorphism $\varphi : G \to \mathrm{PGL}(n+1)$ which induces the action of $G$ on $X$.

We may now replace $G$ by its image in $\mathrm{PGL}(n+1)$ and then by the inverse image of this in $\mathrm{SL}(n+1)$ to obtain a linear action on $X$ with essentially the same properties as the original action.

The inclusion of $A(X)^G$ in $A(X)$ induces a surjective $G$-invariant morphism $\psi : X^{\mathrm{ss}} \to M$ from an open subset $X^{\mathrm{ss}}$ of $X$ to the quotient $M$. We shall see that $X^{\mathrm{ss}}$ always coincides with the minimum Morse stratum $X^{\min}$ associated to the function $f = ||\mu||^2$ on $X$. Therefore §5 and

§6 give us formulae for the equivariant Betti numbers of $X^{\mathrm{ss}}$. It may happen that a fibrr of $\psi$ contains more than one orbit of $G$ so that $M \neq X^{\mathrm{ss}}/G$. However there is an open subset $X^{\mathrm{s}}$ of $X^{\mathrm{ss}}$ such that every fiber which meets $X^{\mathrm{s}}$ is a single $G$-orbit (see [M] theorem 1.10). The image of $X^{\mathrm{s}}$ in $M$ is an open subset $M'$ of $M$ and $M' = X^{\mathrm{s}}/G$.

**Definition 8.2**. (see [M], definitions 1.7 and 1.8, noting that Mumford calls stable points 'properly stable': this seems to be no longer the accepted terminology). A point $x \in X$ is called semistable if there is a homogeneous nonconstant polynomial $F \in \mathbb{C}[X_0, \ldots, X_n]$ which is invariant under the natural action of $G$ on $\mathbb{C}[X_0, \ldots, X_n]$ and is such that $F(x) \neq 0$. $x$ is stable if there is an invariant $F$ with $F(x) \neq 0$ such that all orbits of $G$ in the affine set

$$X_F = \{y \in X : F(y) \neq 0\}$$

are closed in $X_F$ and in addition the stabiliser of $x$ in $G$ is finite.

$X^{\mathrm{ss}}$ is the set of semistable points and $X^{\mathrm{s}}$ is the set of stable points of $X$.

**Remark 8.3**. Suppose that the stabiliser in $G$ of every semistable point in $X$ is finite. Then if $x \in X^{\mathrm{ss}}$ there exists some homogeneous non-constant $G$-invariant polynomial $F$ such that $F(x) \neq 0$. Every point in $X_F$ is semistable so every $G$-orbit in $X_F$ has the same dimension as $G$. This implies that every orbit is closed in $X_F$ and thus that $x$ is stable. Hence $X^{\mathrm{s}} = X^{\mathrm{ss}}$.

We shall use the following facts which follow from [M], theorem 2.1 and proposition 2.2.

**8.4**. A point $x \in X$ is semistable for the action of $G$ on $X$ iff it is semistable for the action of every one-parameter subgroup $\lambda : \mathbb{C}^\star \to G$ on $X$.

**8.5**. If $\lambda : \mathbb{C}^\star \to \mathrm{GL}(n+1)$ is given by $z \mapsto \mathrm{diag}(z^{r_0}, \ldots, z^{r_n})$ with $r_i \in \mathbb{Z}$ then a point $x = (x_0 : \ldots : x_n) \in \mathbb{P}_n$ is semistable for the action of $\mathbb{C}^\star$ via $\lambda$ iff

$$\min\{r_j : x_j \neq 0\} \leq 0 \leq \max\{r_j : x_j \neq 0\}$$

Using this last fact we obtain

**Lemma 8.6**. When $G = \mathbb{C}^\star$ the set $X^{\mathrm{ss}}$ of semistable points coincides with the minimum Morse stratum $X^{\min}$ associated to the function $||\mu||^2$.

*Proof.* There are coordinates in $\mathbb{P}_n$ such that $G = \mathbb{C}^\star$ acts diagonally by $z \mapsto \mathrm{diag}(z^{r_0}, \ldots, z^{r_n})$, say. We have $K = \{e^{2\pi i t} : t \in \mathbb{R}\}$ so that $\phi_\star(\mathfrak{k})$ is the subspace $\mathfrak{u}(n+1)$ spanned by $2\pi i \mathrm{diag}(r_0, \ldots, r_n)$. Let $a \in \mathfrak{k}^\star$ be a basis element of norm 1. By 2.7, if $x = (x_0 : \ldots : x_n) \in X$ then

$$\mu(x) \left( \sum_{0 \leq j \leq n} r_j |x_j|^2 \right) \left( \sum_{0 \leq j \leq n} |x_j|^2 \right)^{-1} a$$

Now, by Theorem 6.18, $x \in X^{\min}$ iff $0 \in \mu(\overline{Gx})$. The map from $G$ to $X$ is given by $z \mapsto (z^{r_0}x_0 : \ldots : z^{r_n}x_n)$ extends uniquely to a map $\theta : \mathbb{P}_1 \to X$ with $\theta(0) = (y_0 : \ldots : y_n)$ where $y_j = x_j$ if $r_j = \min\{r_i : x_i \neq 0\}$ and $y_j = 0$ otherwise, and $\theta(\infty)(y'_0 : \ldots : y'_n)$ where $y'_j = x_j$ if $r_j = \max\{r_i : x_i \neq 0\}$ and $y'_j = 0$ otherwise. Then $\overline{Gx}$ is the image of $\mathbb{P}_1$ under $\theta$ and $\mu(\theta(0)) = \min\{r_j : x_j \neq 0\}$ while $\mu(\theta(\infty)) = \max\{r_j : x_j \neq 0\}$. On the other hand $0 \in \mu(Gx)$ iff either $r_j = 0$ whenever $x_j \neq 0$ or

$$\min\{r_j : x_j \neq 0\} < 0 < \max\{r_j : x_j \neq 0\}$$

This is because if

$$\min\{r_j : x_j \neq 0\} < 0 < \max\{r_j : x_j \neq 0\}$$

then

$$\sum_j r_j |x_j|^2 |z|^{2r_j}$$

tends to $\infty$ as $|z| \to \infty$ and tends to $-\infty$ as $|z| \to 0$. It follows that $0 \in \mu(\overline{Gx})$ iff

$$\min\{r_j : x_j \neq 0\} \leq 0 \leq \max\{r_j : x_j \neq 0\}$$

and thus by 8.5, $x \in X^{\min}$ iff $x \in X^{ss}$.

□

In order to deduce from the lemma that $X^{ss} = X^{\min}$ in the general case, we investigate next the relationship between the minimum strata associated to the action of the whole maximal compact subgroup $K$ and of its closed real one-parameter subgroup $\lambda : S^1 \to K$.

**Definition 8.7**. A complex one-parameter subgroup $\lambda : \mathbb{C}^\star \to G$ of $G$ will be callsed compatible with $K$ if it is the complexification of a closed real 1-PS $\lambda : S^1 \to K$ of $K$. When $\lambda$ is compatible with $K$ let $\mu_\lambda$ be the composition of $\mu$ with $\lambda^\star : \mathfrak{k}^\star \to \mathbb{R}$. Then $\mu_\lambda$ is a moment map for the action of $S^1$ on $X$ via $\lambda$.

**Lemma 8.8**. If $x \in X$ then $0 \in \mu(\overline{Gx})$ iff $0 \in \mu_\lambda(\overline{\lambda(\mathbb{C}^\star)x})$ for every 1-PS $\lambda : \mathbb{C}^\star \to G$ compatible with $K$. Equivalently the minimum stratum $X^{\min}$ is the intersection of the minimum strata $X^{\min}_\lambda$ associated to the action on $X$ of all the 1-PSs $\lambda$ compatible with $K$.

**Remark 8.9**. The proof of this lemma is valid when $X$ is any Kähler manifold and $G$ is the complexification of a compact subgroup $K$ which preserves the Kähler structure on $X$. We are going to see that when $X$ is a complex projective variety then $X^{\min} = X^{ss}$. Therefore this result can be regarded as a generalization to Kähler manifolds of the fundamental fact of geometric invariant theory which says that a point is semistable for the action of a group iff it is semistable for the action of every 1-PS.

*Proof.* First note that the proof given at 6.19 shows that $X^{\min} \subseteq X_\lambda^{\min}$ for every 1-PS $\lambda$ of $G$ compatible with $K$.

Now suppose $x$ does not lie in $X^{\min}$; then there exists a nonzero $\beta \in \mathcal{B}$ such that $x \in S_\beta$. By 6.18, $S_\beta = GY_\beta^{\min}$ and this is the same as $KY_\beta^{\min}$ since $Y_\beta^{\min}$ is invariant under the parabolic subgroup $P_\beta$ and $G = KP_\beta$. Therefore, $kx \in Y_\beta^{\min}$ for some $k \in K$.

Now $\mu(X)$ is compact in $\mathfrak{k}^\star \cong \mathfrak{k}$ and the rational points are dense in $\mathfrak{t}$. Therefore, there exists $\delta > 0$ and a rational point $\gamma \in \mathfrak{t}$ such that

$$\{\xi : \mu(X) : \xi.\beta \geq ||\beta||^2\} \subseteq \{\xi \in \mathfrak{k} : \xi.\gamma \geq \delta\}$$

By replacing $\gamma$ by $m\gamma$ for a suitable $m \in \mathbb{Z}$, we may assume that $\gamma$ is a lattice point of $\mathfrak{t}$ and hence corresponds to a complex 1-PS of $T_\mathbb{C}$ compatible with $T$. Since $kx \in Y_\beta^{\min}$, by 6.11 we have

$$\mu(\overline{\gamma(\mathbb{C}^\star)kx}) \subset \{\xi \in \mu(X) : \xi.\beta \geq ||\beta||^2\} \subseteq \{\xi \in \mathfrak{k} : \xi.\gamma \geq \delta\}$$

In particular $\mu_\gamma(\overline{\gamma(\mathbb{C}^\star)kx})$, which is the projection along $\gamma$ of $\mu(\overline{\gamma(\mathbb{C}^\star)kx})$, does not contain 0. Let $\lambda = \mathrm{Ad}(k)\gamma$; then $\lambda$ is a 1-PS of $G$ compatible with $K$ such that $0 \notin \mu_\lambda(\overline{\lambda(\mathbb{C}^\star)x})$ and hence $x \notin X_\lambda^{\min}$. Therefore,

$$\bigcap_\lambda X_\lambda^{\min} \subseteq X^{\min}$$

and the proof is complete. □

Any 1-PS $\lambda : \mathbb{C}^\star \to G$ has a conjugate $\mathrm{Ad}(g)\lambda = g\lambda g^{-1} : \mathbb{C}^\star \to G$ which is compatible with $K$. Therefore from 8.4, 8.6 and 8.8 and the fact that $X^{\min}$ is $G$-invariant we can deduce the following

**Theorem 8.10**. Let $X \subseteq \mathbb{P}_n$ be a nonsingular complex projective variety and let $G$ be a complex reductive algebraic group acting on $X$ via a homomorphism $\varphi : G \to \mathrm{GL}(n+1)$. Suppose that $G$ has a maximal compact subgroup $K$ such that $\varphi(K) \subseteq \mathrm{U}(n+1)$. Then the set $X^{\mathrm{ss}}$ of semistable points of $X$ coincides with the minimum Morse stratum $X^{\min}$ of the function $||\mu||^2$ on $X$ where $\mu : X \to \mathfrak{k}^\star$ is the moment map and $||\,||$ is the norm associated to any $K$-invariant inner product on $\mathfrak{k}$.

Suppose now that the stabiliser in $G$ of every semistable point is finite. Then by remark 8.3 we have $X^{\mathrm{ss}} = X^{\mathrm{s}}$. But we know that there is a morphism $\psi : X^{\mathrm{ss}} \to M$ from $X^{\mathrm{ss}}$ to the projective quotient $M$ such that each fiber which meets $X^{\mathrm{s}}$ is a single orbit under the action of $G$. Therefore $\psi$ induces a continuous bijection $\tilde{\psi} : X^{\mathrm{ss}}/G \to M$.

We saw in §7 that $X^{\mathrm{ss}}/G$ is a compact Hausdorff space and so is the projective variety $M$. Hence $\tilde{\psi}$ is a homeomorphism.

Thus we obtain formulae for the rational cohomology of the quotient variety $M$. Before stating these formulae in a theorem let us review the definitions of the terms involved and interpret them in the case of a linear reductive group action on a projective variety.

First recall from 3.5 that the moment map $\mu_T$ for the action of the compact maximal torus $T$ on $X$ is given by

$$\mu_T(x) = \frac{\sum_j |x_j|^2 \alpha_j}{\sum_j |x_j|^2}$$

where $\alpha_k$ are the weights of the action.

Choose an inner product which is invariant under the Weyl group action on the Lie algebra $\mathfrak{t}$ of $T$ and use it to identify $\mathfrak{t}^\star$ with $\mathfrak{t}$. Then a minimal combination of weights is by definition the closest point to 0 of the convex hull of some nonempty subset of $\{\alpha_0, \ldots, \alpha_n\}$. The indexing set $\mathcal{B}$ consists of all minimal weight combinations lying in the positive Weyl chamber $\mathfrak{t}_+$.

Note that if we assume the inner product to be rational (i.e. to take rational values on lattice points) then *each* $\beta \in \mathcal{B}$ is a rational point of $\mathfrak{t}_+$. Thus each subgroup $\exp \mathbb{R}\beta$ of $T$ is closed and hence the subtorus $T_\beta$ of $T$ generated by $\beta$ is 1-dimensional.

We saw in 3.11 that for each $\beta \in \mathbb{B}$ the submanifold $Z_\beta$ of $X$ is the intersection of $X$ with the linear subspace

$$\{x \in \mathbb{P}_n : x_j = 0 \text{ unless } \alpha_j.\beta = ||\beta||^2\}$$

of $\mathbb{P}_n$. Recall that $Z_\beta^{\min}$ was defined as the set of points in $Z_\beta$ whose paths of steepest descent for the function $|\mu - \beta|^2$ on $Z_\beta$ have limit points $Z_\beta \cap \mu^{-1}(\beta)$. Let $\mathrm{Stab}(\beta)$ be the stabiliser of $\beta$ under the adjoint action of $G$ and let $\mathrm{Stab}_K\beta$ be its intersection with $K$. By 4.9 $\mu - \beta$ is a moment map for the action of $\mathrm{Stab}_K\beta$ on $Z_\beta$.

**8.11**. In order to interpret the inductive formula of 5.10 we want to define a subset $Z_\beta^{\mathrm{ss}}$ of $Z_\beta$ somehow in terms of semistability so that $Z_\beta^{\mathrm{ss}}$ will coincide with $Z_\beta^{\min}$. There are at least two alternative ways to do this. One way is to let $G_\beta$ be the complexication of the connected closed subgroup of $\mathrm{Stab}_K(\beta)$ whose Lie algebra is the orthogonal complement to $\beta$ and to let $Z_\beta^{\mathrm{ss}}$ be the set of points of $Z_\beta$ which are semistable which are semistable for the linear action of $G_\beta$ on $Z_\beta$ defined by the homomorphism $\varphi$. Then $Z_\beta^{\mathrm{ss}} = Z_\beta^{\min}$ by theorem 8.10 because the projection onto the Lie algebra of $K \cap G_\beta$ of $\mu$ restricted to $Z_\beta$ is $\mu - \beta$. Another way is to note that since $\beta$ is a rational point of the center of stab$\beta$ there is a character $\chi : \mathrm{Stab}\,\beta \to \mathbb{C}^\star$ whose derivative is a positive integer multiple $r\beta$ of $\beta$. One can define $Z_\beta^{\mathrm{ss}}$ to be the set of semistable points of $Z_\beta$ under the action of $\mathrm{Stab}\,\beta$ where the action is linearised with respect to the $r$th tensor of the hyperplane bundle by the product of $\varphi^{\otimes r}$ with the inverse of the character $\chi$. The corresponding moment map $Z_\beta$ is then $r\mu - r\beta$ so that again $Z_\beta^{\rm ss} = Z_\beta^{\rm min}$. However the details are unimportant.

$Z_{\beta,m}$ can be reinterpreted as the union of those components of $Z_\beta$ which are contained in components of $Y_\beta$ of real codimension $m$. Finally, $\beta$-sequences $\underline{\beta} = (\beta_1, \ldots, \beta_q)$ are the corresponding linear sections $Z_{\underline{\beta}}$ and $Z_{\underline{\beta},m}$ of $X$ and subgroups Stab $\underline{\beta}$ can be defined as in §5.

**Theorem 8.12**. Let $X \subset \mathbb{P}_n$ be a complex projective variety acted on linearly by a connected complex reductive algebraic group $G$. The equivariant Poincaré series for $X^{\rm ss}$ is given by the inductive formula

$$P_t^G(X^{\rm ss}) = P_t(X)P_t(BG) - \sum_{\beta,m} t^{d(\beta,m)} P_t^{{\rm Stab}\,\beta}(Z_{\beta,m}^{\rm ss})$$

where the sum is over nonzero $\beta \in \mathcal{B}$ and integers $0 \le m \le \dim X$. Stab $\beta$ is a reductive subgroup of $G$ acting on $Z_{\beta,m}$ which is a smooth subvariety of $X$ for each $\beta, m$ and

$$d(\beta, m) = m - \dim G + \dim\ {\rm Stab}\,\beta$$

Suppose that the stabiliser of every semistable point in $X$ is fintie so that the projective quotient variety $M$ associated to the action in geometric invariant theory is topologically the quotient $X^{\rm ss}/G$. THen the rational cohomology of $M$ is isomorphic to the $G$-equivariant rational cohomology of $X^{\rm ss}$. It is given by the explicit formula

$$P_t(M) = P_t(X)P_t(BG) + \sum_{\underline{\beta},m} (-1)^{q_t} t^{d(\underline{\beta},m)} P_t(Z_{\underline{\beta},m}) P_t(B{\rm Stab}\underline{\beta})$$

Each $Z_{\underline{\beta},m}$ is a smooth subvariety of $X$ acted on by a reductive subgroup Stab$\underline{\beta}$ of $G$ and

$$d(\underline{\beta}, m) = m - \dim G + \dim\ {\rm Stab}\underline{\beta}$$

**Remark 8.13**. Note that the equivariant cohomology must be used in the inductive formula because the condition of finite isotropy groups may not be satisfied for all the subgroups Stab $\beta$ acting on the subvarieties $Z_{\beta,m}$.

*Proof.* This follows from 5.10, 5.16, 6.20, 8.10 and the remarks of the last few paragraphs.

**Remark 8.14**. When the stabiliser of every semistable point is finite then the geometric invariant theory quotient $X^{\rm ss}/G$ *is homeomorphic to* the symplectic quotient $\mu^{-1}(0)/K$ by theorem 7.5. In fact we can show that $M$ is homeomorphic to $\mu^{-1}(0)/K$ without any assumption on stabilisers as follows.

The inclusions $\mu^{-1}(0) \hookrightarrow X^{\rm min} = X^{\rm ss}$ together with the surjective $G$-invariant morphism $\psi : X^{\rm ss} \to M$ induces a continuous map $h : \mu^{-1}(0)/K \to M$. By the proofs of [M] theorem 1.10

and amplification 1.3 two points $x, y \in X^{\mathrm{ss}}$ are identified by $\psi$ iff the closures in $X^{\mathrm{ss}}$ of $Gx$ and $Gy$ meet each other. But by remark 7.8 $G\mu^{-1}(0)$ consists of those $x \in X^{\mathrm{ss}}$ such that $Gx$ is closed in $X^{\mathrm{ss}}$ so the map $h$ is injective. Moreover if $x \in X^{\mathrm{ss}} = X^{\mathrm{min}}$ then the closure of the path of steepest descent for the function $||\mu||^2$ from $x$ contains a point of $\mu^{-1}(0)$ and by 6.7 this path is contained in the orbit $Gx$. Thus $h$ is surjective. It follows that $h$ is a bijection from a compact space to a Hausdorff space and hence is a homeomorphism.

# 9 Some remarks on non-compact manifolds

So far we have considered only compact symplectic manifolds and projective varieties. Now suppose that $X$ is any symplectic manifold acted on by a compact group $K$ such that a moment map $\mu : X \to \mathfrak{k}^\star$. Then one can obtain almost the same results as for compact manifolds subject only to the condition that

**9.1**. For some metric on $X$ every path of steepest descent under the function $f = ||\mu||^2$ is contained in some compact subset of $X$.

One simply checks that all the arugments used in §§3,4,5 and the appendix are still valid with trivial modifications. The only result which *fails* is theorem 5.8. This says that the rational equivariant cohomology of the total space $X$ is the tensor product of its ordinary rational cohomology with that of the classifying space of $K$; i.e.

$$P_t^K(X) = P_t(X)P_t(BK)$$

Thus in the formulae obtained for the equivariant rational cohomology of $\mu^{-1}(0)$ (see 5.10 and 5.16) one must now always use the equivariant Poincaré series $P_t^K(X)$ rather than the product $P_t^K(X)P_t(BK)$. Otherwise the formulae are correct and in good cases give the Betti numbers of the symplectic quotient $\mu^{-1}(0)/K$.

**Example 9.2. Cotangent bundles**. The examples which motivated the definition of symplectic manifolds and moment maps were phase spaces and conserved quantities such as angular momentum.

The cotangent bundle $T^\star M$ of any manifold $M$ has a natural symplectic structure given by

$$\omega = \sum_i dp_i \wedge dq_i$$

where $(q_1, \ldots, q_n)$ are local coordinates on $M$ and $(p_0, \ldots, p_n)$ are the induced coordinates on the cotangent space at $(q_1, \ldots, q_n)$. Any action of a compact group $K$ on $M$ induces an action of $K$ on $T^\star M$ which preserves this symplectic structure. Moreover it is not hard to check that there is a moment map $\mu : T^\star M \to \mathfrak{k}^\star$ for this action defined as follows. If $m \in M$ and $\xi \in T^\star_m M$ then

9.3

$$\mu(\xi).a = \xi.a_m$$

for all $a \in \mathfrak{k}$ where . on the left hand side denotes the natural pairing between $\mathfrak{k}^\star$ and $\mathfrak{k}$ and on the right denotes the natural pairing between $T^\star_m M$ and $T_m M$. So a general moment map is of the form $\mu + c$ where $c$ lies in the center of $\mathfrak{k}^\star$ (see §2).

Condition 9.1 holds for each of the moment maps on $T^\star M$ provided that $M$ is compact. To see this one fixes a metric on $M$ and uses it to induce a Riemannian metric on $T^\star M$. It can then be shown that the path of steepest descent for $f = ||\mu + c||^2$ from any point $\xi \in T^\star M$ consists of cotangent vectors whose norm is bounded by some number depending only on $\xi$.

The function $||\mu||^2 = f$ where $\mu$ is given by 9.3 is not an interesting Morse function because the only critical points are the points in $\mu^{-1}(0)$. The reason for this is that by lemma 3.1 if $\xi \in T^\star_m M$ is critical for $f$ then the vector field induced by $\mu(\xi)$ on $T^\star M$ vanishes at $\xi$. Thus in particular $\mu(\xi)_m = 0$ so if we put $a = \mu(\xi)$ in 9.3 we obtain $||\mu(\xi)||^2 = 0$. However if $K$ is not semisimple then it is often possible to choose $c$ in the center of $\mathfrak{k}$ such that the norm-square of the moment map $\mu + c$ has non-minimal critical points.

For example, consider the action of the circle $S^1$ on $T^\star S^2$ induced by the rotation of the sphere $S^2$ about some axis. Let $c$ be an element of norm 1 in the Lie algebra of $S^1$ and let $f = ||\mu + c||^2$. Then from 9.3 we have

$$f(\xi) = (c_m.\xi + 1)^2$$

for any $m \in S^2$ and $\xi \in T^\star_m S^2$. So $f(\xi) = 0$ iff $\xi.c_m = -1$ which means that the minimum set for $f$ is homeomorphic to a line bundle over the sphere less two points and hence is homotopically equivalent to $S^1$. Since the circle action on this is free the equivariant cohomology of the minimum set is trivial.

By lemma 3.1 the other critical points $\xi$ for $f$ are those fixed by $S^1$. There are the two points of $S^2$ fixed by the rotation. The index of the Hessian at each of these is 2. Thus we obtain

$$P_t^{S^1}(S^2) = P_t^{S^1}(T^\star S^2) = 1 + 2t^2(1-t^2)^{-1} = (1+t^2)(1-t^2)^{-1} = P_t(S^2)P_t(BS^1)$$

as one expects from proposition 5.8 since $S^2$ has a symplectic structure preserved by the action of $S^1$.

As a second example consider the linear action of the torus

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} : \theta, \varphi \in \mathbb{R} \right\}$$

on the unit sphere $S^3 \subset \mathbb{C}^2$. By 9.3 if $m \in S^3$ and $\xi \in T^\star_m S^3$ then

$$\mu(\xi) = (a_m.\xi)a + (b_m.\xi)b$$

where $a = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$. Consider the function $f = ||\mu + a + b||^2$ on $T^\star S^3$. Any $\xi \in T^\star S^3$ satisfies $f(\xi) = 0$ if $\mu(\xi) = -a - b$, i.e. if $a_m.\xi = -1 = b_m.\xi$. If $a_m = 0$ or $b_m = 0$ these equations for $\xi$ have no solution and otherwise they define an affine line in $T^\star_m S^3$. So the minimum set $f^{-1}(0)$ is acted on freely by $T$ and its equivariant cohomology is isomorphic to the cohomology of the quotient of $T$ by $S^3$ with two circles removed. This quotient is an open interval so its cohomology is trivial.

From lemma 3.1 we see that if $\xi \in T^\star_m S^3$ is a non-minimal critical point for $f$ then either $\xi$ is fixed by $a$ and $\mu(\xi)+a+b$ is a scalar multiple of $a$ or $\xi$ is fixed by $b$ and $\mu(\xi)+a+b$ is a scalar multiple of $b$. In the first case, $\xi.b=-1$ and $\xi \in T^\star S^1$ where $S^1$ is the circle fixed by $a$ and the second case is similar. So the non-minimal critical points form two circles in $T^\star S^3$ each of which is fixed by one copy of $S^1$ in the torus $T$ and is acted on freely by the other. The index of the function $f$ along each of these circles is 2. Thus we obtain

$$P_t^T(S^3) = 1 + 2t^2(1-t^2)^{-1}$$

Note that this is not equal to $P_t(S^3)P_t(BT)$; this does not contradict proposition 5.8 since $S^3$ is not a symplectic manifold!

**Example 9.4. Quasi-projective varieties**. Other obvious examples of non-compact symplectic manifolds are nonsingular quasi-projective complex varieties.

Suppose that $G$ is a complex reductive group with maximal compact subgroup $K$ acting linearly on a nonsingular locally closed subvariety $X$ of some complex projective space $\mathbb{P}_n$. Suppose also that the stabiliser of every semistable point is finite. If condition 9.1 is satisfied then we obtain formulae for the Betti numbers of the symplectic quotient $\mu^{-1}(0)/K$ which is homeomorphic to the quotient variety produced by invariant theory. There is also a more algebraic condition for these formulae to exist which is an alternative to 9.1. It is described as follows.

When $X$ is a closed subvariety of $\mathbb{P}_n$ acted on linearly by $G$ then the stratification of $X$ induced by the action is just the intersection with $X$ of the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ induced on $\mathbb{P}_n$. If $X$ is quasi-projective we can still define a stratification of $X$ with strata $\{X \cap S_\beta\}$. Moreover by 6.18 and 8.10 we have $S_\beta \cong G_\beta \times_{P_\beta} Y_\beta^{ss}$ for each $\beta$ where $Y_\beta^{ss}$ is a nonsingular locally-closed subvariety of $\mathbb{P}_n$ and $P_\beta$ is a parabolic subgroup of $G$. Since $X$ is invariant under $G$ this implies

$$X \cap S_\beta \cong G \times_{P_\beta} (X \cap Y_\beta^{ss})$$

There is also a retraction $p_\beta : Y_\beta^{ss} \to Z_\beta^{ss}$ onto the semistable points of a linear subvariety $Z_\beta$ of $\mathbb{P}_n$ under the action of a subgroup of $G$. Provided that

**9.5**. $p_\beta(x) \in X$ whenever $x \in X \cap Y_\beta^{ss}$ for each $\beta \in \mathcal{B}$, one can check that each $p_\beta$ induces a retraction of $X \cap Y_\beta^{ss}$ onto $X \cap Z_\beta^{ss}$ and that all the results of §8 hold for $X$.

One can use quasi-projective varieties satisfying this condition to rederive Atiyah and Bott's formulae for the cohomology of moduli spaces of vector bundles over Riemann surfaces. For this one considers spaces of holomorphic maps from Riemann surfaces to Grassmannians. These can be embedded as quasi-projective subvarieties of products of Grassmanians.

The results of these notes also apply to reductive group actions on singular varieties satisfying appropriate conditions (see the work of Carrell and Goresky on $\mathbb{C}^\star$-actions [C & G]).

# 10 Appendix. Morse theory extended to minimally degenerate functions

Given any nondegenerate Morse function with isolated critical points on a compact manifold, one has the well-known Morse inequalities which relate the Betti numbers of the manifold to the numbers of critical points of each index. Bott has shown that this classical Morse theory extends to a more general class of Morse functions [Bo]. The functions which are nondegenerate in the sense of Bott are those whose critical sets are disjoint unions of submanifolds along each of which are the Hessian is nondegenerate in normal directions. The associated Morse inequalities relate the Betti numbers of the manifold to the Betti numbers and indices of the critical submanifolds. The purpose of this sectionn is to show that Morse theory can be extended to cover an even larger class of functions.

**Definition 10.1**. A smooth function $f : X \to \mathbb{R}$ on a compact manifold $X$ is called minimally degenerate if the following conditions hold.

1. The set of critical points for $f$ on $X$ is a finite union of disjoint closed subsets $\{C \in \mathbb{C}\}$ on each of which $f$ takes a constant value $f(C)$. The subsets are called critical subsets of $f$. If the critical set of $f$ is reasonably well behaved we can take the subsets $\{C\}$ to be its connected components.
2. For every $C \in \mathbb{C}$ there is a locally closed submanifold $\Sigma_C$ containing $C$ and with orientable normal bundle in $X$ such that
   (a) $C$ is the subset of $\Sigma_C$ on which $f$ takes its minimum value.
   (b) at every point $x \in C$ the tangent space $T_x\Sigma_C$ is maximal among all subspaces $T_xX$ on which the Hessian $H_x(f)$ is positive-definite.

A submanifold satisfying these properties is called a minimising manifold for $f$ along $C$.

Thus minimal degeneracy means that critical sets can be as degenerate as a minimum but no worse.

The purpose behind this definition is to find a condition on $f$ more general than nondegeneracy which ensures that for some choice of metric $f$ induces a Morse stratification whose strata are all smooth. This appendix shows that minimal degeneracy is such a condition. Conversely if $f$ is any function which induces a smooth Morse stratification then the strata themselves are minimising manifolds provided that the Hessian at every critical point is definite in directions normal to the stratum which contains it.

We do not demand that the minimising manifolds be connected. However, this extra condition is always satisfied if we replace each critical subset $C$ by it intersections with the connected components of $\Sigma_C$. Hence we can assume that the index of the Hessian of $f$ takes a constant value $\lambda(C)$ along any $C \in \mathbf{C}$, since by 10.1(b) it coincides with the codimension of the submanifold $\Sigma_C$. We shall call $\lambda(C)$ the index of $f$ along $C$.

Any function which is nondegenerate in the sense of Bott is minimally degenerate. For by definition the set of critical points of $f$ is the disjoint union of connected submanifolds of $X$ and these can be taken as the critical subsets of $f$. If we fix a metric on $X$ then the Hessian of $f$ induces a self-adjoint endomorphism of the normal bundle $N_C$ along each critical submanifold $C$. Because $f$ is nondegenerate $N_C$ splits as a sum $N_C^+ \oplus N_C^-$ where the Hessian is positive definite on $N_C^+$ and negative definite on $N_C^-$. It is easy to check that locally the image of $N_C^+$ under the exponential map induced by the metric is a minimising manifold for $f$ along $C$.

We wish to show that any minimally degenerate Morse function on $X$ induces Morse inequalities in cohomology and also in equivariant cohomology if $X$ is acted on by a compact group $K$ which preserves the function. These inequalities are most easily expressed using the Poincaré polynomials

$$P_t(X) = \sum_{j \geq 0} t^j \dim H^j(X)$$

and equivariant Poincaré polynomials

$$P_t^K(X) = \sum_{j \geq 0} t^j \dim H_K^j(X)$$

Our aim is to prove the following

**Theorem 10.2**. Let $f : X \to \mathbb{R}$ be a minimally degenerate Morse function with critical subsets $\{C \in \mathbf{C}\}$ on a compact manifold $X$. Then the Betti numbers of $X$ satisfy Morse inequalities which can be expressed in the form

$$\sum_{C \in \mathbf{C}} t^{\lambda(C)} P_t(C) - P_t(X) = (1+t)R(t)$$

where $\lambda(C)$ is the index of $f$ along $C$ and $R(t) \geq 0$ in the sense that all its coefficients are nonnegative. If a compact group $K$ acts on $X$ preserving $f$ and the minimising manifolds, then $X$ also satisfies equivariant Morse inequalities of the same form.

When $f$ is nondegeneate one method of obtaining the Morse inequalities is to use a metric to define a smooth startification $\{S_C : C \in \mathbb{C}\}$ of $X$. This is perhaps not the easiest approach but we shall follow it here since the stratification of the particular function relevant to us is interesting in its own right. A point of $X$ lies in a stratum $S_C$ if its trajectory under the gradient field $-\mathrm{grad}\, f$ converges to a point of the corresponding critical subset $C$. For a general function $f$ such a trajectory may not converge to a single point. However the limit set of the trajectory is always a connected nonempty set of critical points for $f$ (see 2.10). Therefore if $f$ is minimally degenerate then any such limit set is contained in a unique critical subset. So we make the following

**Definition 10.3**. Suppose $f : X \to \mathbb{R}$ is a minimally degenerate Morse function with critial subsets $\{C \in \mathbb{C}\}$ and suppose that $X$ is given a fixed Riemannian metric. Then for each $C$ let $S_C$ be the subset of $X$ consisting of all points $x \in X$ such that the limit set $\omega(x)$ of the trajectory of $-\operatorname{grad} f$ from $x$ is contained in $C$.

$X$ is the disjoint union of the subsets $\{S_C : C \in \mathbf{C}\}$. We shall see that if the metric is chosen appropriately they form a smooth stratification of $X$ such that each stratum $S_C$ coincides near $C$ with the minimising manifold $\Sigma_C$. The condition which the metric must satisfy is that the gradient field $\operatorname{grad} f$ should be tangential to each minimising manifold $\Sigma_C$. We shall show that such a metric exists and then prove the following

**Theorem 10.4**. Let $f$ be a minimally degenerate Morse function with critical subsets $\{C \in \mathbb{C}\}$ on a compact Riemannian manifold. Suppose that the gradient flow of $f$ is tangential to each of the minimising manifolds $\{\Sigma_C : C \in \mathbb{C}\}$. Then the subsets $\{S_C : C \in \mathbb{C}\}$ defined at 10.3 form a smooth stratification of $X$ called the Morse stratification of the function $f$ on $X$. For each $C$ the stratum $S_C$ coincides with the minimising submanifold $\Sigma_C$ in some neighbourhood of $C$. Moreover each inclusion $C \to S_C$ is an equivalence of Cech cohomology. If there is a compact group $K$ acting on $X$ such that $f$, the minimising manifolds and the metric are all invariant under $K$ then these inclusions are also equivalences of equivariant cohomology.

In order to be able to apply this result to any minimally degenerate function we need the following

**Lemma 10.5**. Let $f$ be a minimally degenerate function on $X$. Then there is a metric on $X$ such that near each $C$ the gradient flow of $f$ is tangential to the minimising manifold $\Sigma_C$. If $f$ and the minimising manifolds are invariant under the action of a compact group $K$ then the metric may be taken to be $K$-invariant.

*Proof.* A standard argument using partitions of unity shows that it is enough to find such metrics locally. The only point to note is that one should work with dual metrics because $\operatorname{grad}_\rho f$ is linear in $\rho^\star$ but not in $\rho$.

Suppose $x$ is any point of a critical subset $X$. Condition (2) of 10.1 implies that there is a complement to $T_x\Sigma_C$ in $T_xX$ on which the Hessian $H_x(f)$ is negative definite. It follows from the Morse lemma (lemma 2.2 of [Mi]) that there exist local coordinates $(x_1, \ldots, x_n)$ around $x$ such that the minimising manifold $\Sigma_C$ is given locally by

$$0 = x_{d+1} = x_{d+2} = \ldots = x_n,$$

and such that

$$f(x_1,\ldots,x_n) = f(x_1,\ldots,x_d) - (x_{d+1})^2 - \ldots - (x_n)^2$$

(To prove this, regard $x_1,\ldots,x_d$ as parameters and apply the Morse lemma to $x_{d+1},\ldots,x_n$). Then the gradient flow of $f$ with respect to the standard metric on $\mathbb{R}^n$ is tangential to $\Sigma_C$.

Finally, a $K$-invariant metric is obtained by averaging the dual metric over $K$.

Because of this lemma theorem 10.2 can be deduced from theorem 10.4 by the standard argument using Thom-Gysin sequences. The rest of this appendix is devoted to the proof of theorem 10.4.

The most difficult part of the proof of this theorem will be to show that for each $C$ the stratum $S_C$ coincides with the given submanifold $\Sigma_C$ in some neighbourhood of $C$. Once we know that $S_C$ is smooth near $C$ it will follow easily that $S_C$ is smooth everywhere and the cohomology equivalences are not hard to prove.

First we shall show that the subsets $\{S_C\}$ form a stratification of $X$ in the sense of 2.11. It suffices to prove the following lemma which is depends on the assumption 10.1(1) but not on the existence of minimising manifolds.

**Lemma 10.7**. For each $C \in \mathbf{C}$

$$\bar{S}_C \subseteq S_C \cup \bigcup_{f(C')>f(C)} S_{C'}$$

*Proof.* If a point $x$ lies in $S_C$ for some $C \in \mathbf{C}$ then by definition its path of steepest descent for $f$ has a limit point in $C$, and hence $f(x) \geq f(C)$ since $f$ decreases along this path. Moreover, $f(x) > f(C)$ unless $x \in C$.

If $x$ lies in the closure $\bar{S}_C$ of $S_C$ then so does every point of its path of steepest descent. Hence the closure of this path is contained in $\bar{S}_C$. It follows that $x \in S_{C'}$ for some $C'$ with $f(C') \geq f(C)$. So if $x \in \bar{S}_C$ and $x$ is not critical for $f$ then $f(x) > f(C)$.

Since the subsets $\{C \in \mathbf{C}\}$ are compact, there are open sets $\{U_C : C \in \mathbf{C}\}$ whose closures are disjoint such that $U_C \supseteq C$ for each $C$. If $x$ lies on the boundary $\partial U_C$ of some $U_C$ then $x$ is not critical for $f$. Hence, if $x \in \partial U_C \cap \bar{S}_C$ then $f(x) > f(C)$. Since each $\partial U_C \cap \bar{S}_C$ is compact it follows that there is some $\delta > 0$ such that if $C \in \mathbf{C}$ and $x \in \partial U_C \cap \bar{S}_C$ then $f(x) \geq f(C) + \delta$.

Now suppose that $C, C'$ are distinct and that there is some $x \in S_{C'} \cap \bar{S}_C$. Let $\{x_t : t \geq 0\}$ be the path of steepest descent for $f$ with $x_0 = x$; then the limit points of $\{x_t : t \geq 0\}$ as $t \to \infty$ are contained in $C'$. So there exists $T \geq 0$ such that $x_T \in U_{C'}$ and $f(x_T) < f(C') + \delta$. But this implies that there is a neighbourhood $V$ of $x$ such that $y_T \in U_{C'}$ and $f(y_T) < f(C') + \delta$ whenever $y \in V$.

Since $x \in \bar{S}_C$ there is some $y \in V \cap S_C$; then $y_T \in U_{C'}$, but the limit points as $t \to \infty$ of $\{y_t : t \geq 0\}$ are contained in $C$. Since by assumption $\bar{U}_C \cap \bar{U}_{C'} = \emptyset$ there must exist some $t > T$

such that $y_t \in \partial U_C \cap S_C$. This implies that $f(y_t) \geq f(C) + \delta$ by the choice of $\delta$. But $f$ decreases along the path $\{y_t : t \geq 0\}$ and $f(y_T) < f(C') + \delta$ since $y \in V$. Therefore,

$$f(C') + \delta > f(y_T) \geq f(y_t) \geq f(C) + \delta$$

so that $f(C') > f(C)$.

This shows that if $S_{C'} \cap \bar{S}_C$ is nonempty then $f(C) < f(C')$. Since $X$ is the disjoint union of the subsets $\{S_C : C \in \mathbf{C}\}$ the result follows. □

Now we shall begin the proof that each stratum $S_C$ coincides near $C$ with the corresponding minimising manifold $\Sigma_C$.

**Lemma 10.8**. For each $C \in \mathbb{C}$ the intersection of the minimising manifold $\Sigma_C$ with a sufficiently small neighbourhood of $X$ is contained in the Morse stratum $S_C$.

*Proof.* As in the proof of 10.7 choose open subsets $U_C$ of $X$ whose closures are disjoint and $U_C \supseteq C$ for each $C \in \mathbf{C}$. Since each $\Sigma_C$ is a submanifold of some neighbourhood of $C$, if $U_C$ is taken small enough then $\Sigma_C \cap \bar{U}_C$ is closed for each $C$.

If $C \in \mathbf{C}$ then by definition of minimising manifold, $C$ is the subset of $\Sigma_C$ on which $f$ takes on its minimum value. Hence, if $x \in \Sigma_C \cap \partial U_C$ then $f(x) > f(C)$, and so $\Sigma_C \cap \partial U_C$ is compact there exists $\gamma > 0$ such that $f(x) \geq f(C) + \gamma$ whenever $C \in \mathbf{C}$ and $x \in \Sigma_C \cap \partial U_C$. Then, for every $C$, the subset

$$V_C = U_C \cap \{x \in X : f(x) < f(C) + \gamma\}$$

is an open neighbourhood of $C$ in $X$.

Suppose $x$ lies in the intersection of this neighbourhood $V_C$ with $\Sigma_C$. Then as $\operatorname{grad} f$ is tangential to $\Sigma_C$ and $\Sigma_C$ is closed in $\bar{U}_C$ the path $\{x_t : t \geq 0\}$ of steepest descent for $f$ from $x$ stays in $\Sigma_C$ as long as it remains in $U_C$. Hence if the path leaves $U_C$, there exist $t > 0$ such that $x_t \in \partial U_C \cap \Sigma_C$. This implies that

$$f(x) \geq f(x_t) \geq f(C) + \gamma$$

which contradicts the assumption that $x \in V_C$. So the path remains in $U_C$ for all time. Since the only critical points for $f$ in $\bar{U}_C$ are contained in $C$, it follows that the limit points of the path lie in $C$ and so $x \in S_C$.

**Remark 10.9**. Note that the same argument shows that given any neighbourhood $U_C$ of $C$ in $X$ there exists a smaller neighbourhood $V_C$ such that if $x \in V_C \cap S_C$ then the entire path of steepest descent for $f$ from $x$ is contained in $U_C$.

In order to prove the converse to the last lemma we need to investigate the differential equation which defines the gradient flow of $f$ in local coordinates near any critical point $x$. We shall rely on the standard local results to be found in [H].

Recall that if $x \in X$ is a critical point for $f$ then the Hessian $H_x(f)$ of $f$ at $x$ is a symmetric bilinear form on the tangent space $T_xX$ given in local coordinates by the matrix of second partials of $f$. The Riemannian metric provides an inner product on $T_xX$ so that $H_x$ can be identified with a self-adjoint linear endomorphism of $T_xX$. Then all the eigenvalues of $H_x(f)$ are real and $T_xX$ splits as the direct sum of the eigenspaces of $H_x(f)$.

The assumption that the gradient field of $f$ is tangential to $\Sigma_C$ implies that for each $x \in C$ the subspace $T_x\Sigma_C$ of $T_xX$ is invariant under $H_x(f)$ regarded as a self-adjoint endomorphism of $T_xX$. Hence so is its orthogonal complement $T_x\Sigma_C^\perp$. By the definition of a minimising manifold the eigenvalues of $H_x(f)$ restricted to $T_x\Sigma_C$ are all nonnegative while those of $H_x(f)$ restricted to $T_x\Sigma_C^\perp$ are all strictly negative.

Now fix $C \in \mathbb{C}$ and a point $x \in X$. Let $d$ be the dimension of $\Sigma_C$. Then we can find local coordinates $(x_1, \dots, x_n)$ in a neighbourhood $W_x$ of $x$ such that

10.10.

1. $x$ is the origin in these coordinates and the submanifold $\Sigma_C$ is given by $x_{d+1} = x_{d+2} = \ldots = x_n = 0$.

2. The Riemannian metric at $x$ is the standard inner product on $\mathbb{R}^n$.

3. The Hessian $H_x(f)$ is represented by a diagonal matrix

$$H_x(f) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

where $\lambda_1, \ldots, \lambda_d \geq 0$ and $\lambda_{d+1}, \ldots, \lambda_n < 0$.

Let $P$ be the diagonal matrix

$$\operatorname{diag}(-\lambda_1, \ldots, -\lambda_d)$$

and let $Q$ be the diagonal matrix

$$\operatorname{diag}(-\lambda_{d+1}, \ldots, -\lambda_n);$$

then

$$-H_x(f) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

in these coordinates. For $(x_1, \ldots, x_n) \in \mathbb{R}^n$ write $y = (x_1, \ldots, x_d)$ and $z = (x_{d+1}, \ldots, x_n)$. Then the trajectories of $-\operatorname{grad} f$ in these coordinates are the solution curves to the differential equation

10.11

$$y = Py + F_1(y, z); \quad z = Qz + F_2(y, z)$$

where $F_i$ are smooth and their Jacobian matrices $\partial F_i$ vanish at the origin [H, IX §4]. By reducing the neighbourhood $W_x$ of $x$ if necessary we may assume that $F_i$ extend smoothly over $\mathbb{R}^n$ in such a way that there exist complete solution curves to 10.11 through every point $(y_0, z_0)$ given by $t \to (y_t, z_t)$ say, for $t \in \mathbb{R}$ [H, IX §3,4]. Then we have

10.12

$$y_t = e^{Pt} y_0 + Y(t, y_0, z_0); \quad z_t = e^{Qt} z_0 + Z(t, y_0, z_0)$$

for all $t$ where $Y, Z$ and their partial Jacobians vanish at the origin.

We want to show that if a point $x$ does not lie in $\Sigma_C$ then its path of steepest descent stays well away from $C$. If $x$ is sufficiently close to $\Sigma_C$ then it has a well-defined distance $d(x, \Sigma_C)$. It is sufficient to show that near $C$ this distance function is bounded away from zero along all paths of steepest descent not contained in $\Sigma_C$. We can do this by working in local coordinates near each $x \in C$.

The submanifold $\Sigma_C$ is defined in the local coordinates $(y, z)$ in $W_x$ by $z = 0$. Therefore in the standard metric on $\mathbb{R}^n$ the distance from $\Sigma_C$ is given by $||z||$. Moreover the coordinates were chosen so that the given Riemannian metric at $x$ coincides with the standard inner product on $\mathbb{R}^n$. It follows that given any $\epsilon > 0$ we may reduce $W_x$ so that

10.13
$$(1+\epsilon)^{-1}||z|| \leq d((y,z), \Sigma_C) \leq (1+\epsilon)||z||$$

everywhere in $W_x$.

We now need the following technical result.

**Lemma 10.14**. There is a number $b > 1$ which depends only on the critical set $C \in \mathbb{C}$ such that the following property holds for all $x \in C$. If the neighbourhood $W_x$ of $x$ is taken sufficiently small and the extensions of $F_1, F_2$ over $\mathbb{R}^n$ are chosen appropriately, then for every $(y_0, z_0) \in \mathbb{R}^n$ we have $||z_1|| \geq b||z_0||$ where $z_1 = e^{Q z_0} + Z(1, y_0, z_0)$ as in 10.12.

*Proof*. The gradient field of $f$ is tangential to the submanifold $\Sigma_C$ so $F_2(y, 0) = 0$ whenever $(y, 0)$ lies in $W_x$ (see 10.11). Therefore, the extension of $F_2$ to $\mathbb{R}^n$ can be chosen so that $F_2(y, 0) = 0$ for all $y \in \mathbb{R}^d$. This implies that

$$Z(t, y_0, 0) = 0 \qquad \forall y_0 \in \mathbb{R}^d,\ t \in \mathbb{R}$$

(see 10.12).

Now for each $x \in C$ let $c_x$ be the minimum eigenvalue of $e^Q$. Recall that

$$Q = \mathrm{diag}(-\lambda_{d+1}, \ldots, -\lambda_n)$$

where $\lambda_{d+1}, \ldots, \lambda_n$ are the eigenvalues of the Hessian $H_x(f)$ restricted to $T_x\Sigma_C$ and that each of these eigenvalues is strictly negative. Hence $c_x > 1$. Let $c = \inf\{c_x : x \in C\}$; since $C$ is compact and $c_x$ depends continuously on $x$ it follows that $c > 1$. So we can choose $\theta > 0$ such that $c - \theta > 1$. Set $b = c - \theta$; then $b > 1$ and $b$ depends only on $C$.

By 10.12 the partial Jacobian $\partial_{y_0, z_0} Z$ vanishes at the origin for all $t \in \mathbb{R}$. Hence, by reducing the neighbourhood $W_x$ and choosing the extensions of $F_1$ and $F_2$ appropriately we may assume that

$$||\partial_{z_0} Z(1, y_0, z_0)|| \leq \theta$$

for all $(y_0, z_0) \in \mathbb{R}^n$ (cf. [H], IX §4). It follows that

$$||Z(1, y_0, z_0)|| \leq \theta ||z_0||$$

for all $(y_0, z_0) \in \mathbb{R}^n$. Since every eigenvalue of $e^Q$ is at least $c$, for any $(y_0, z_0)$ we have

$$||z_1|| = ||e^{Qz_0} + Z(1, y_0, z_0)|| \geq c||z_0|| - \theta||z_0|| = b||z_0||$$

The result follows. □

**Corollary 10.15**. The intersection of the Morse stratum $S_C$ with a sufficiently small neighbourhood of $C$ in $X$ is contained in the minimising manifold $\Sigma_C$.

*Proof.* It follows from 10.13 and 10.14 that, given $\epsilon > 0$, there is a neighbourhood $W_C$ of $C$ such that if $\{x_t : t \geq 0\}$ is any path of steepest descent with $x_t \in W_C$ when $0 \leq t \leq 1$ then

$$d(x_1, \Sigma_C) \geq b(1+\epsilon)^{-2} d(x_0, \Sigma_C)$$

where $b > 1$ is independent of $\epsilon$. If $\epsilon$ is choen sufficienty small we have

$$b(1+\epsilon)^{-2} > 1$$

By remark 10.9 there is a neighbourhood $V_C$ of $C$ in $X$ such that if $x_0 \in V_C \cap S_C$ its entire path of steepest descent $\{x_t : t \geq 0\}$ is contained in $W_C$. Then for each $n \geq 1$

$$d(x_n, \Sigma_C) \geq (b(1+\epsilon)^{-2})^n d(x_0, \Sigma_C)$$

But we may assume without any loss of generality that $d(x, \Sigma_C)$ is bounded on $W_C$. Hence we must have $d(x_0, \Sigma_C = 0$, i.e. $x_0 \in \Sigma_C$. This shows that $V_C \cap S_C \subseteq \Sigma_C$.

From 10.8 and 10.15 we deduce that each stratum $S_C$ coincides with $\Sigma_C$ in a neighbourhood $U_C$ of $C$ and hence that $S_C \cap U_C$ is smooth. But any point of $S_C$ is mapped into $S_C \cap U_C$ by the diffeomorphism $x \mapsto x_t$ of $S_C$ induced by flowing for some large time $t$ along the gradient field of $f$. So we have the following

**Lemma 10.16**. For each $C \in \mathbb{C}$ the stratum $S_C$ is smooth. It coincides with the minimising manifold $\Sigma_C$ in some neighbourhood of $C$.

We have seen that the subsets $\{S_C : C \in \mathbb{C}\}$ form a smooth stratification of $X$ and it remains only to prove one more result.

**Lemma 10.17**. For each $C \in \mathbb{C}$ the inclusion $C \to S_C$ is an equivalence for Cech cohomology. More generally if a compact connected group $K$ acts on $X$ in such a way that the function $f$ and the Riemannian metric on $X$ are preserved by $K$ then each stratum $S_C$ is $K$-invariant and the inclusion $C \hookrightarrow S_C$ are equivalences of equivariant cohomology.

*Proof.* We need only consider the second statement. It is clear from the definition that the Morse strata $\{S_C\}$ are $K$-invariant.

For each sufficiently small $\delta \geq 0$

$$N_\delta = \{x \in S_C : f(x) \leq f(C) + \delta\}$$

in a compact neighbourhood of $C$ in $S_C$ (see the proof of 10.8). The paths of steepest descent induce retractions of $S_C$ onto each $N_\delta$ with respect to the action of $K$. So each inclusion $N_\delta \times_K EK \to S_C \times_K EK$ is a cohomology equivalence. Also

$$\bigcap_{\delta > 0} N_\delta = C$$

So the continuity of Čech cohomology implies that the inclusion $C \hookrightarrow S_C$ is an equivalence of equivariant Čech cohomology [D, VIII 6.18]. The only problem is that $X \times_K EK$ is not compact. This can be overcome by regarding $EK$ as the union of compact manifolds which are cohomologically equivalent to $EK$ up to arbitrarily large dimensions.

**Remark 10.18**. When $f$ is nondegenerate in the sense of Bott, each path of steepest descent under under $f$ converges to a unique critical point in $X$. Thus the strata retract onto the critical sets along the paths of steepest descent. This fails in general for minimally degenerate functions: there exist minimally degenerate functions with trajectories which spiral in towards a critical subset without ever converging to a unique limit. This is why Čech cohomology is used above. However, it is unlikely that the square of the moment map has such bad behaviour.

# Part II. The Algebraic Approach

## 11 The basic idea

In Part I a formula was obtained in good cases for the Betti numbers of the projective quotient variety associated in geometric invariant theory to a linear action of a complex reductive group $G$ on a nonsingular complex projective variety $X$. The good cases occur when the stabiliser in $G$ of every semistable point of $X$ is finite. The quotient variety is then topologically the quotient $X^{ss}/G$ of the set of semistable points by the group. The formula was obtained by employing the ideas of Morse theory and of symplectic geometry. We shall now approach the same problem using algebraic methods.

The basic idea common to both approaches is to associate to the group action a canonical stratification of the variety $X$. The unique open stratum of this stratification coincides with the set $X^{ss}$ (provided this is nonempty) and the other strata are all $G$-invariant locally-closed nonsingular subvarieties of $X$. There then exist equivariant Morse-type inequalities relating the $G$-equivariant Betti numbers of $X$ to those of the strata. It turns out that these inequalities are in fact equalities, i.e. that the stratification is equivariantly perfect over $\mathbb{Q}$. From this an inductive formula can be derived fro the equivariant Betti numbers of the semistable stratum $X^{\,ss}$ which in good cases coincide with the ordinary Betti numbers of the quotient variety.

The difference between the two approaches lies in the way the stratification of $X$ is defined. In Part I symplectic geometry was used to define a function $f$ (the norm-square of the moment map) which induced a Morse stratification of $X$. In Part II the stratification will be defined purely algebraically. The main advantage of this method is that it applies to varieties defined over any algebaically closed field. On the other hand the approach in Part I generalizes to Kähler and symplectic manifolds.

The algebraic definition of the stratification is based on the work of Kempf. It also has close links with the paper [Ne] by Ness. Suppose that we are given a linear action of a reductive group $G$ on any projective variety $X$, singular or nonsingular, defined over any algebraically closed field. Kempf shows that for each unstable point $x \in X$ there is a conjugacy class of virtual one-parameter groups of a certain parabolic subgroup of $G$ which are most responsible for the instability of $x$. (The term canonical destabilizing flags is also used). The stratum to whcih $x$ belongs is determined by the conjugacy class of these virtual one-parameter subgroups in $G$. Over the complex field, the stratification is the same as the one already defined in Part I.

Just as in Part I the indexing set $\mathcal{B}$ of the stratification may be described in terms of the weights of the representation of $G$ which defines the action. An element $\beta \in \mathcal{B}$ may be thought of as the closest point to the origin of the convex hull of some nonempty set of weights when the weights are regarded as elements of an appropriate normed space (see §12.8).

In §13 it is shown that if $X$ is nonsingular then the strata $S_\beta$ are also nonsingular and have the same structure as in the complex case. That is, to each $\beta$ in $\mathbf{B}$ there is a smooth locally closed subvariety $Y_\beta^{ss}$ of $X$ acted on by a parabolic subgroup $P_\beta$ of $G$ such that

11.1

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$$

There is also a nonsingular closed subvariety $Z_\beta$ of $X$ and a locally trivial fibration

11.2

$$P_\beta : Y_\beta^{ss} \to Z_\beta^{ss}$$

whose fibres are all affine spaces. Here $Z_\beta^{ss}$ is the set of semistable points of $Z_\beta$ under the action of a reductive subgroup of $P_\beta$.

These results were precisely what was needed in Part I to show that the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is equivariantly perfect and hence to derive an inductive formula for the equivariant Betti numbers of $X^{ss}$. Thus the reader who is interested in solely in complex algebraic varieties can avoid the detailed analytic arguments needed for symplectic and Kähler manifolds by using the definitions and results from these two sections. It will be found that at times the algebraic method is neater while at others it is more elegant to argue analytically.

In §14 we shall see how the formulae for the Betti numbers of the quotient variety $M$ can be refined to given the Hodge numbers as well. We use Deligne's extension of Hodge theory to complex varieties which are not necessarily compact and nonsingular.

In §15 an alternative method for obtaining the formulae is described though without detailed proofs. This method was suggested by work of Harder and Narasimhan. It uses the Weil conjectures which were establised by Deligne. These enable one to calculate the Betti numbers of a nonsingular complex projective variety by counting the points of associated varieties defined over finite fields. In our case it is possible to count points by decomposing these varieties into strata using 11.1-2. However the Weil conjectures apply only when the quotient variety is nonsingular.

Finally in §16 some examples of stratifications and of calculating the Betti numbers of quotients are considered in detail. The first example is given by the action of SL(2) on the space $\mathbb{P}_n$ of binary forms of degree $n$ which can be identified with the space of unordered sets of poitns on $\mathbb{P}_1$. We also consider the space $(\mathbb{P}_1)^n$ of ordered sets of points on $\mathbb{P}_1$. These have been used as examples throughout part I. The good cases occur when $n$ is odd and then the Hodge numbers of the quotient varieties $M$ are given by

$$h^{p,p} = \left[1 + \frac{1}{2}\min(p, n-3-p)\right]$$

for the case of unordered points, and

$$h^{p,p} = 1 + (n-1) + \binom{n-1}{2} + \dots \binom{n-1}{\min(p, n-3-p)}$$

for ordered points. The Hodge numbers $h^{p,q}$ with $p \neq q$ all vanish. Then we generalize $(\mathbb{P}_1)^n$ to an arbitrary product of Grassmannians. That is, we consider for any $m$ the diagonal action of $\mathrm{SL}(m)$ on a product of $G(\ell_i, m)$ ($\ell_i$-dimensional subspaces of $\mathbb{C}^m$). The good cases occur when $m$ is coprime to $\sum \ell_i$. The associated stratification is described in Proposition 16.9 and it is shown how in good cases this provides an inductive formula for the equivariant Betti numbers of the semistable stratum in terms of the equivariant Betti numbers of the semistable strata of products of the same form but with smaller values of $m$. Explicit calculations are made for some products of $\mathbb{P}_2$.

One reason for studying products of Grassmanians in depth is that it is possible to rederive the formulae obtained in [H& N] and [A & B] for the Betti numbers of moduli spaces of vector bundles over Riemann surfaces by applying the results of these notes to subvarieties of products of Grassmannians [Ki3].

# 12 Stratifications over arbitrary algebraically closed fields

Let $k$ be an algebraically closed field. Suppose that $X$ is a $k$-variety acted on linearly by a reductive $k$-group $G$. In this section we shall define a stratification of $X$ which generalizes the definition given in Part I for the case when $X$ is nonsingular and $k$ is the field of complex numbers.

The set $X^{ss}$ of semistable points of $X$ under the action will form one stratum of the stratification. To define the others we shall use work of Kempf as expounded in a paper by Hesselink (see [K] and [Hes]). Kempf associates to each unstable point $x$ of $X$ a conjugacy class of virtual one-parameter subgroups in a parabolic subgroup of $G$. These are the one's most responsible for the instability of the point $x$. The stratum to which $x$ belongs will be determined by the conjugacy class of these virtual one-parameter subgroups. We shall find that each stratum $S_\beta$ can be described in the form

$$S_\beta = GY_\beta^{ss}$$

where $Y_\beta^{ss}$ is a locally-closed subvariety of $X$, itself defined in terms of the semistable points of a small variety under the action of a subgroup of $G$. From this it will be obvious that the stratification coincides with the one defined in Part I in the complex nonsingular case.

First we shall review briefly Hesselink's definition and results and relate them to what we have already done in the complex case: this is completed in lemma 12.13. Note that in [Hes] arbitrary ground fields are considered. We shall restrict ourselves to algebraically closed fields for the sake of simplicity.

**Remark**. The definition of the stratification given at 12.14 makes sense when $k$ is any field. This is also the stratification of the variety $X \times_k K$ defined over the algebraic closure $K$ of $k$. When $k$ is perfect it follows from [Hes] that this last stratification is defined over $k$ and coincides with the first stratification on $X$. However this fails in general. In §15 where finite fields occur it will be necessary to avoid certain characteristics when things go wrong.

Hesselink studies reductive group actions on affine pointed varieties. We shall apply his results to the action of $G$ on the affine cone $X^\star \subset k^{n+1}$ on $X$. For each nonzero $x^\star \in X^\star$ and one-parameter subgroup $\lambda : k^\star \to G$ of $G$. Hesselink defines a measure of instability $m(x^\star, \lambda)$. This really only depends on the point $x$ determined by $x^\star$ and hence can also be written as $m(x, \lambda)$. The following two facts determine $m$ for every $x, \lambda$.

**12.1**. If $\lambda : k^\star \to \mathrm{GL}(n+1)$ is given by

$$z \mapsto \mathrm{diag}(z^{r_0}, \ldots, z^{r_n})$$

with $r_i \in \mathbb{Z}$ then

$$m(x, \lambda) = \min\{r_j : x_j \neq 0\}$$

if this is non-negative and

$$m(x; \lambda) = 0$$

otherwise. Also for any $g \in G$

$$m(x; g\lambda g^{-1}) = m(gx; \lambda)$$

**Definition 12.2**. $x \in X$ is unstable for the action of $G$ if $m(x, \lambda) > 0$ for some one-parameter subgroup $\lambda$ of $G$.

Mumford proves that

**12.3**. $x \in X$ is semistable iff $m(x, \lambda) \leq 0$ for every one-parameter subgroup $\lambda$ of $G$; that is, iff $x$ is not unstable!

**Definition 12.4**. Let $Y(G)$ denote the set of one-parameter subgroups $\lambda : k^\star \to G$ of $G$ and let $M(G)$ be the quotient of the product of $Y(G)$ with the natural numbers by the equivalence relation $\sim$ such that $(\lambda, \ell) \sim (\mu, m)$ if $\lambda, \mu$ satisfy

$$\lambda(t^m) = \mu(t^\ell)$$

If $T$ is a torus then $Y(T)$ is a free $\mathbb{Z}$-module of finite rank and $M(T)$ is a $\mathbb{Q}$-vector space. Moreoever, there is a natural corespondence between one-parameter sungroups of a torus $T$ over the complex field and lattice points in the Lie algebra $\mathfrak{t}$ of its maximal compact subgroup. Hence in the case $M(T)$ may be identified with the rational points of $\mathfrak{t}$.

The adjoint action of $G$ on $Y(G)$ extends to an action on $M(G)$. Let $q$ be a norm on $M(G)$ as defined in [Hes] §1; that is, $q$ is a $G$-invariant map from $M(G)$ to $\mathbb{Q}$ which restricts to a quadratic form on $M(T)$ for any torus $T \subset G$. If $T$ is a maximal torus of $G$ a norm on $M(G)$ is the square of an inner product on $M(T)$ invariant under the Weyl group and any such inner product determines a unique norm on $M(G)$. When $k = \mathbb{C}$ any invariant rational inner product on the Lie algebra of a maximal compact subgroup of $G$ induces a norm on $M(G)$ in Hesslink's sense.

**Definition 12.5** ([Hes] 4.1). For any $x \in X$ let

$$q^{-1}(x)_G = \inf \{q(\lambda) : \lambda \in M(G),\, m(x, \lambda) \geq 1\}$$

and

$$\Lambda_G(x) = \left\{\lambda \in M(G) : m(x, \lambda) \geq 1,\, q(\lambda) = q_G^{-1}\right\}$$

Thus $x$ is unstable iff $q_G^{-1} < \infty$ or equivalently $\Lambda_G(x) \neq \emptyset$. The definition of $m$ can be extended uniquely over all $\lambda$ in $M(G)$ to satisfy $m(x, r\lambda) = m(x, \lambda)$ for every $r \in \mathbb{Q}$.

The set $\Lambda_G(x)$ will be used to determine the stratum to which the point $x \in X$ belongs.

Let $T$ be a maximal torus of $G$. As when $k = \mathbb{C}$ the representation of $T$ on $k^{n+1}$ splits as the sum of scalar representations given by characters $\alpha_0, \dots, \alpha_n$, say. These $\alpha_i$ are elements of the dual of $M(T)$ and may be identified with elements of $M(T)$ by using the inner product on $M(T)$ whose square is the norm $q$.

Fix $x = (x_0, \dots, x_n) \in X$ and let $\beta$ be the closest point to the origin for the norm $q$ of the convex hull $C(x)$ of the set $\{\alpha_i : x_i \neq 0\}$ in the $\mathbb{Q}$-vector space $M(T)$. Then

$$(\xi - \beta).\beta \geq 0$$

I.e. $\xi.\beta \geq q(\beta)$ for all $\xi \in C(x)$ where . denotes the inner product on $M(T)$ whose square is $q$. In Part I this point $\beta$ indexed the stratum containing $x$. The next two lemmas show how $\beta$ is related to the set $\Lambda_T(x)$.

**Lemma 12.6**. If $\beta \neq 0$ then $\Lambda_T(x) = \{\beta/q(\beta)\}$.

*Proof.* By 12.1 if $\lambda \in M(T)$ then $m(x;\lambda) = \min\{\alpha_i.\lambda : x_i \neq 0\}$ if this is nonnegative and $m(x;\lambda) = 0$ otherwise. Therefore, $m(x;\lambda) \geq 1$ iff $\lambda.\alpha_i \geq 1$ for every $i$ such that $x_i \neq 0$. But if $x_i \neq 0$ thene $\alpha_i.\beta \geq q(\beta)$ by the choice of $\beta$. Therefore $\beta/q(\beta).\alpha_i \geq 1$ for such $i$. Moreover, if $\lambda$ satisfies $\lambda.\alpha_i \geq 1$ whenever $x_i \neq 0$ then $\lambda.\beta \geq 1$ since $\beta$ lies in the convex hull of the set $\{\alpha_i : x_i \neq 0\}$. This means that

$$q(\lambda)q(\beta) \geq (\lambda.\beta)^2 \geq 1$$

with equality iff $\lambda = \beta/q(\beta)$. Thus it follows straight from definition 12.5 that $q_G^{-1}(x) = q(\beta)^{-1}$ and that the set $\Lambda_T(x)$ consists of the single point $\beta/q(\beta)$. □

**Lemma 12.7**. If $\beta = 0$ then $\Lambda_T(x) \neq \emptyset$.

*Proof.* If $\Lambda_T(x) \neq \emptyset$ then there is some $\lambda \in M(T)$ such that $m(x;\lambda) \geq 1$ and hence such that $\lambda.\alpha_i \geq 1$ whenever $x_i \neq 0$. This implies that $0 \notin \mathrm{Conv}\{\alpha_i : x_i \neq 0\}$ and hence that $\beta \neq 0$. □

Thus the set $\Lambda_T(x)$ determines and is determined by the point $\beta$.

**Definition 12.8**. Call the closest point to 0 of the convex hull in $M(T)$ of any nonempty subset of $\{\alpha_0, \dots, \alpha_n\}$ a minimal combination of weights. Let **B** be the set of all minimial combinations of weights lying in some positive Weyl chamber (i.e. some convex fundamental do-main for the action of the Weyl group on $M(T)$). **B** will be the indexing set for the stratification.

**Definition 12.9**. A subgroup $H$ of $G$ is optimal for $x$ if

$$q_H^{-1}(x) = q_G^{-1}(x)$$

It is clear from definition 12.5 that if $H$ is optimal then

$$\Lambda_H(x) = M(H) \cap \Lambda_G(x)$$

and that $\Lambda_H(x)$ is nonempty precisely when $\Lambda_G(x)$ is nonempty. By [Hes] there is always some maximal torus $T'$ of $G$ which is optimal for $x$ and $T' = g^{-1}Tg$ for some $g \in G$ where $T$ is the fixed maximal torus of $G$. This implies that

**12.10**. For every $x \in X$ there exists some $g \in G$ such that $T$ is optimal for $gx$.

Next note that $G$ acts on itself by conjugation and hence $G$ becomes an affine pointed $G$-variety. So we can make the following definition.

**Definition 12.11**. If $\lambda \in M(G)$ let

$$P_\lambda = \{g \in G : m(g, \lambda) \geq 0\}$$

Clearly if $r > 0$ is rational then $P_\lambda = P_{r\lambda}$. Moreover if $\lambda : k^\star \to G$ is actually a one-parameter subgroup of $G$ then $P_\lambda$ consists of those $g \in G$ such that

$$\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$$

exists in $G$. Then Lemma 5.1(a) of [Hes] shows that

**12.12**. $P_\lambda$ is a parabolic subgrop of $G$ for each $\lambda \in M(G)$.

The main result needed from Kempf's work can now be stated.

**Lemma 12.13**.

1. For each unstable $x$ there is a unique parabolic subgroup $P(x)$ of $G$ such that $P(x) = P_\lambda$ for all $\lambda$ in $\Lambda_G(x)$.
2. $\Lambda_G(x)$ is a single $P(x)$ orbit under the adjoint action of $G$ on $M(G)$.
3. If $\lambda \in \Lambda_G(x)$ then $g^{-1}\lambda g$ also lies in $\Lambda_G(x)$ iff $g \in P(x)$. In particular $P(x)$ contains the stabiliser of $x$ in $G$.
4. $\Lambda_G(x) \subset M(P(x))$.

5. If $T$ is optimal for $x$ and $\Lambda_T(x) = \{\beta/q(\beta)\}$ then $P(x) = P_\beta$.

*Proof.* (1) and (2) are [Hes], Theorem 5.2 applied to any nonzero $x^\star \in X^\star$ lying over $x$. If $\lambda : k^\star \to G$ is a one-parameter subgroup of $G$ and if $g \in G$ is such that $g^{-1}\lambda g = \lambda$, then $g$ commutes with every element of $\lambda(k^\star)$, so $m(g;\lambda) = 0$ and hence $g \in P_\lambda$. The first part of (3) follows from this together with (1) and (2). The second part is an immediate consequence since $m(x; g^{-1}\lambda g) = m(gx;\lambda)$ by 12.1. Also, (4) follows because if $\lambda \in \Lambda_G(x)$ then $r\lambda Y(G)$ for some positive integer $r$. Since $r\lambda$ commutes with $\lambda$, it represents a one-parameter subgroup of $P_\lambda = P(x)$ and so $\lambda \in M(P(x))$. Finally, if $T$ is optimal for $x$ and $\Lambda_T(x) = \beta/q(\beta)$, then $\beta/q(\beta) \in \Lambda_G(x)$, hence $P(x) = P(\beta/q(\beta)) = P_\beta$ by (1) and 12.1. □

This lemma completes the review of the results needed from [Hes].

**Definition 12.14**. For each nonzero $\beta \in M(T)$ let

$$S_\beta = G\{x \in X : \beta/q(\beta) \in \Lambda_G(x)\}$$

and let

$$S_0 = G\{x \in X : \Lambda_G(x) = \emptyset\}$$

Then by 12.3 $S_0 = X^{ss}$. Also $\beta/q(\beta) \in \Lambda_G(x)$ iff $T$ is optimal for $x$ and $\Lambda_T(x) = \{\beta/q(\beta)\}$ by 12.6-7.

**Lemma 12.15**. $X$ is the disjoint union of the subsets $\{S_\beta : \beta \in \mathbf{B}\}$.

*Proof.* Suppose that $x \in X$ is unstable, i.e. that $\Lambda_G(x) \neq \emptyset$. By 12.10 there is some $g \in G$ such that $T$ is optimal for $gx$. By 12.6 and 12.7, $\Lambda_T(gx) = \{\beta/q(\beta)\}$ where $\beta \neq 0$ is the closest point to 0 of $\mathrm{Conv}\{\alpha_i : (gx)_i \neq 0\}$. Therefore

$$\bigcup_\beta S_\beta = X,$$

where $\beta$ runs over all minimal combinations of weights.

Since $x$ is unstable if $\Lambda_T(gx) \neq \emptyset$ for any $g \in G$,

$$S_0 \cap \bigcap_{\beta\neq 0} S_\beta \neq \emptyset$$

If $\beta, \beta'$ are nonzero and the intersection $S_\beta \cap S_{\beta'} \neq \emptyset$, then there exist $x \in X$ and $g \in G$ such that both $\beta/q(\beta)$ and $\mathrm{Ad}\, g(\beta'/q(\beta'))$ lie in $\Lambda_G(x)$. Therefore, by 12.13 (2), $\beta/q(\beta)$ and $\beta'/q(\beta')$

are equivalent under the adjoint representation of $G$ on $M(G)$. This implies that $q(\beta) = q(\beta')$, so $\beta, \beta'$ are also equivalent. As $\beta, \beta' \in M(T)$, if follows that they lie in the same orbit of the Weyl group in $M(T)$.

Conversely, suppose that $\beta, \beta'$ are equivalent under the action of the Weyl group, so that there is some $g \in G$ normalising $T$ such that $\beta' = \mathrm{Ad}\, g(\beta)$. Then, for any $x$ we have $\beta/q(\beta) \in \Lambda_G(x)$ iff $\beta'/q(\beta') \in \Lambda_G(gx)$, so $S_\beta = S_{\beta'}$. The result follows. □

Write $\beta' > \beta$ if $q(\beta') > q(\beta)$. In order to show that we have a stratification of $X$ (in the Zariski topology) it now suffices to show that

**Lemma 12.16**.

$$\bar{S}_\beta \subset \bigcup_{\beta' \geq \beta} S_{\beta'}$$

for each $\beta \in \mathbf{B}$.

*Proof.* For each $\beta \in \mathcal{B}$ let

$$W_\beta = \{x \in X : x_i = 0 \,\mathrm{if}\, \alpha_i.\beta < q(\beta)\}$$

By 12.6 and 12.7 and the preceding remark, the stratum $S_\beta$ consists of all points of the form $gx$ such that $T$ is optimal for $x$ and $\beta$ is the closest point to 0 of $\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$. This implies $S_\beta \subseteq GW_\beta$ for each $\beta \in \mathcal{B}$. It is easy to check that $W_\beta$ is invariant under $P_\beta$ (see 12.23 belownoting that $W_\beta = \bar{Y}_\beta$ when $X$ is projective space).By a standard argument using the completeness of $G/P_\beta$ (see e.g. [B] 11.9(i), [Hes] 6.3 or theorem 13.7 below) it follows $GW_\beta$ is closed in $X$ so that $\bar{S}_\beta \subseteq GW_\beta$.

Suppose $x \in W_\beta$ and let $\beta'$ be the closest point to 0 of $\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$; then either $\beta' = \beta$ or $q(\beta') > q(\beta)$. It $T$ is optimal for $x$ then $x \in S_{\beta''}$ for some $\beta''$ with $q(\beta'') > q(\beta') \geq q(\beta)$. Therefore

**12.17**. If $x \in W_\beta$ then either $T$ is optimal for $x$ and $\beta$ is the closest point to 0 of $\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$ or there is some $\beta' > \beta$ such that $x$ belongs to $S_{\beta'}$.

Hence

$$\bar{S}_\beta \subseteq GW_\beta \subseteq \bigcup_{\beta' \geq \beta} S_{\beta'}$$

so the proof is complete. □

This lemma shows that the subsets $\{S_\beta : \beta \in \mathcal{B}\}$ form a stratification of $X$ in the sense of definition 2.11, and in particular

$$S_\beta = \bar{S}_\beta - \bigcup_{\beta' > \beta} \bar{S}_{\beta'}$$

is open in its closure $\bar{S}_\beta$ for each $\beta \in \mathcal{B}$.

We next want to describe the stratum $S_\beta$ in such a way that it is clear that when $k = \mathbb{C}$ this stratification coincides with the one defined in Part I.

**Definition 12.18**. Let

$$Z_\beta = \{(x_0, \ldots, x_n) \in X : x_j = 0 \text{ if } \alpha_j.\beta \neq q(\beta)\}$$

and let

$$Y_\beta = \{(x_0, \ldots, x_n) \in X : x_j = 0 \,\text{if}\; \alpha_j.\beta < q(\beta),\; x_j \neq 0, \text{ some } j \text{ with } \alpha_j.\beta = q(\beta)\}$$

$Z_\beta$ is a closed subvariety of $X$ and $Y_\beta$ is a locally-closed subvariety. Define $p_\beta : Y_\beta \to Z_\beta$ by

$$p_\beta(x_0, \ldots, x_n) = (x'_0 : \ldots : x'_n)$$

where $x'_j = x_j$ if $\alpha_j.\beta = q(\beta)$ and $x'_j = 0$ otherwise. This is well defined as a map since if $y \in Y_\beta$ then $p_\beta(y) \in \overline{Gy}$ and in particular, lies in $X$. Let $\operatorname{Stab}\beta$ be the stabilizer of $\beta$ under the adjoint action of $G$ on $M(G)$. $\operatorname{Stab}\beta$ is a reductive subgroup of $G$ which acts on $Z_\beta$.

The definitions of $Z_\beta, Y_\beta, p_\beta$ depend only on $\beta$. They are independent of the choice of coordinates and indeed of the maximal torus $T$ chosen except that $\beta$ must lie in $M(T)$. Moreover by 6.5, when $k = \mathbb{C}$ and $X$ nonsingular, they coincide with the definitions made in Part I.

**Lemma 12.19**. If $x \in Z_\beta$ then $\operatorname{Stab}\beta$ is optimal for $x$.

*Proof.* If $x \in Z_\beta$ then $\beta$ fixes $x$ so $\beta \in M(P(x))$ by 12.13 (3). Also, $\Lambda_G(x) \subseteq M(P(x))$ by 12.13 (4) so that if $\lambda \in \Lambda_G(x)$ there is some $p \in P(x)$ such that $p\lambda p^{-1}$ and $\beta$ commute. But this implies that $p\lambda p^{-1} \in M(\operatorname{Stab}\beta) \cap \Lambda_G(x)$ by 12.13 (2), so $\operatorname{Stab}\beta$ is optimal for $x$ as required.

Note that if $x \in Z_\beta$ then by definition

$$m(x;\beta) = \min\{\alpha_i.\beta : x_i \neq 0\} = q(\beta)$$

Thus in particular when $\beta \neq 0$ no point in $Z_\beta$ is semistable. However, there is an open subset of $Z_\beta$ whose elements are unstable *only insofar as $\beta$ makes them unstable.* The neatest definition of this subset is the following.

**Definition 12.20**. Let $Z_\beta^{ss}$ be the subset of $Z_\beta$ consisting of those $x \in Z_\beta$ such that $\beta/q(\beta) \in \Lambda_G(x)$.

Since $\mathrm{Stab}\,\beta$ is optimal for $x$ the condition that $\beta/q(\beta) \in \Lambda_G(x)$ is equivalent to the condition

$$m(x;\lambda) \leq \lambda.\beta, \qquad \forall\lambda \in M(\mathrm{Stab}\,\beta)$$

Note that $\lambda.\beta$ makes sense for $\lambda \in M(\mathrm{Stab}\,\beta)$ since there is some maximal torus $T'$ such that $\lambda$ and $\beta$ both lie in $M(T')$ and there is a unique inner product on $M(T')$ whose square is the norm $q$.

Let $Y_\beta^{ss}$ be the inverse image of $Z_\beta^{ss}$ under the map $p_\beta : Y_\beta \to Z_\beta$ defined at 12.18.

**Remark 12.21**. It is not hard to give alternative definitions of $Z_\beta^{ss}$ and $Y_\beta^{ss}$ directly in terms of semistability (cf. 8.11). One can show that there is a unique connected reductive subgroup $G_\beta$ of $\mathrm{Stab}\,\beta$ such that

$$M(G_\beta) = \{\lambda \in M(\mathrm{Stab}\,\beta) : \lambda.\beta = 0\}$$

Then $Z_\beta^{ss}$ consists precisely of those $x \in Z_\beta$ which are semistable under the action of $G_\beta$ on $Z_\beta$ via the restriction of the homomorphism $G \to \mathrm{GL}(n+1)$ to $G_\beta$. This is easily seen by using lemmas 12.6-7 together with 12.3.

Alternatively there exists a positive integer $r$ such that when $M(T)$ is identified with its dual $r\beta$ corresponds to a character of $T$ which extends to a character $\chi$ of $\mathrm{Stab}\,\beta$. Then the action of $\mathrm{Stab}\,\beta$ on $Z_\beta$ is linearized with respect to the $r$th tensor power of the hyperplane bundle by the $r$th tensor power of the homomorphism $G \to \mathrm{GL}(n+1)$ multiplied by the character $\chi^{-1}$. It is not hard to check that a point $x$ lies in $Z_\beta^{ss}$ iff $x$ is semistable for this linear action of $\mathrm{Stab}\,\beta$ on $Z_\beta$.

It is clear from the definition that

**12.22**. $Z_\beta^{ss}$ is invariant under $\mathrm{Stab}\,\beta$

and it follows that

**12.23**. $Y_\beta$ and $Y_\beta^{ss}$ are invariant under $P_\beta$.

The proof is essentially that of 6.10. It depends on two facts: firstly that if $\lambda : k^\star \to T$ is any 1-PS which is a positive scalar multiple of $\beta$ in $M(T)$ then

$$\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$$

exists in $G$, and secondly that if $y \in Y_\beta$ then

$$p_\beta(y) = \lim_{t \to 0} \lambda(t) y$$

for any such $\lambda$.

Our aim is to show that $S_\beta = GY_\beta^{\mathrm{ss}}$. For this the following lemma is needed.

**Lemma 12.24**. Suppose that $\beta \neq 0$. If $y \in Y_\beta$ and $x = p_\beta(y)$ then the following are equivalent:

1. $T$ is optimal for $y$ and $\Lambda_T(y) = \{\beta/q(\beta)\}$
2. $y \in S_\beta$
3. $y \in Y_\beta^{\mathrm{ss}}$
4. $x \in Z_\beta^{\mathrm{ss}}$
5. $x \in S_\beta$

*Proof.* (3) and (4) are equivalent by definition, while (1) implies (2) by definition 12.14 and the converse follows from 12.17 since $Y_\beta \subseteq W_\beta$. By 12.17 again, if $y \notin S_\beta$ then $y \in S_{\beta'}$ for some $\beta'$ satisfying $q(\beta') > q(\beta)$; then $x \in \bar{S}_{\beta'}$ since $x \in \overline{Gy}$, and by Lemma 12.16 this implies that $x \notin S_\beta$. Therefore, (5) implies (2). It follows straight from the definitions that (4) implies (5).

Finally, suppose that $x \notin Z_\beta^{\mathrm{ss}}$. Since $T$ is a maximal torus of $\operatorname{Stab}\beta$ there is some $s \in \operatorname{Stab}\beta$ such that $T$ is optimal for $sx$. By 12.6, $\beta$ is not the closest point to 0 of $\operatorname{Conv}\{\alpha_i : (sx)_i \neq 0\}$. Moreover $(sx)_i \neq 0$ iff both $(sy)_i \neq 0$ and $\alpha_i.\beta = q(\beta)$ because $p_\beta(sy) = sx$ (see definition 12.18). So it follows from the geometry of convex sets that $\beta$ is not the closest point to 0 of $\operatorname{Conv}\{\alpha_i : (sy)_i \neq 0\}$. (This is best seen by drawing a picture). Thus by 12.6 and 12.7, $\Lambda_T(sy) \neq \{\beta/q(\beta)\}$ and hence by 12.7, $sy \in S_{\beta'}$ for some $\beta' > \beta$. So $y \notin S_\beta$. Thus (2) implies (4). $\square$

**Corollary 12.25**. If $\beta \neq 0$ then $y \in Y_\beta^{\mathrm{ss}}$ iff $T$ is optimal for $y$ and $\Lambda_T(y) = \{\beta/q(\beta)\}$ or equivalently iff $\beta/q(\beta) \in \Lambda_G(x)$. Thus $S_\beta = GY_\beta^{\mathrm{ss}}$ for any $\beta \in \mathbf{B}$.

*Proof.* It is obvious that $GY_0^{\mathrm{ss}} = X^{\mathrm{ss}} = S_0$. If $\beta \neq 0$ and $\Lambda_T(y) = \{\beta/q(\beta)\}$ then by 12.6 and 12.7, $\beta$ is the closest point to 0 of $\operatorname{Conv}\{\alpha_i : y_i \neq 0\}$. Thus $y \in Y_\beta$ so the result follows straight from Lemma 12.24.

We have now proved the following

**Theorem 12.26**. *Let $X \subset \mathbb{P}_n$ be a projective vareity over $k$ and let $G$ be a reductive $k$-group. Fix a norm $q$ on the space $M(G)$ of virtual one-parameter subgroups of $G$. Then to any linear action of $G$ on $X$ there is associated a stratification $\{S_\beta : \beta \in \mathbf{B}\}$ of $X$ by $G$-invariant locally closed subvarieties described as follows. If $T$ is a maximal torus of $G$ the indices $\beta$ are minimal combinations of weights in a fixed Weyl chamber of $M(T)$ and $S_0 = X^{\mathrm{ss}}$ while if $\beta \neq 0$, $S_\beta = GY_\beta^{\mathrm{ss}}$ where*

$$Y_\beta^{\mathrm{ss}} = \{x \in X : \beta/q(\beta) \in \Lambda_G(x)\}$$

*when $k = \mathbb{C}$ and $X$ is nonsingular the strata $S_\beta$ and the subvarieties $Y_\beta^{ss}$ coincide with those defined in Part I.*

# 13 The strata of a nonsingular variety

Now suppose that $X$ is a nonsingular projective variety over $k$. In this section we shall see that the strata $\{S_\beta\}$ of the stratification associated in §12 to the action of a reductive group $G$ are all nonsingular subvarieties of $X$. To prove this we shall show firstly that the subvarieties $Z_\beta$ and $Y_\beta$ at 12.20 are all nonsingular and secondly that each $S_\beta$ is isomorphic to $G \times_{P_\beta} Y_\beta^{ss}$. In addition we shall see that each morphism $p_\beta : Y_\beta^{ss} \to Z_\beta^{ss}$ is an algebraic locally trivial fibration such that every fibre is an affine space.

The following facts about linear actions of the multiplicative group $k^\star$ on nonsingular projective varieties such as $X$ will be needed. These are due to Bialynicki-Birula [B-B]. We shall apply them to certain one-parameter subgroups of $G$.

**13.1**. Suppose that $k^\star$ acts linearly on $X$. Then the set of fixed points is a finite disjoint union of closed connected nonsingular subvarieties of $X$; let $Z$ be one of these. For every $x \in X$ the morphism $k^\star \to X$ given by $t \mapsto tx$ extends uniquely to a morphism $k \to X$; the image of 0 will be denoted by $\lim_{t\to 0} tx$. Let $Y$ consist of all $x \in X$ such that $\lim tx \in Z$. Then $Y$ is a connected locally-closed nonsingular subvariety of $X$ and the map $p : Y \to Z$ defined by

$$p(x) = \lim_{t\to 0} tx$$

is an algebraic locally trivial fibration with fibre some affine space over $k$.

**Corollary 13.2**. For each $\beta \in \mathbf{B}$ the subvarieties $Y_\beta, Z_\beta$ defined at 12.18 are nonsingular. The morphism $p_\beta : Y_\beta \to Z_\beta$ is an algebraic locally trivial fibration whose fibre at any point is an affine space. The same is therefore true of its restriction

$$p_\beta : Y_\beta^{ss} \longrightarrow Z_\beta^{ss}$$

to the open subset $Y_\beta^{ss} \subset Y_\beta$.

*Proof.* Fix $\beta \in \mathcal{B}$ and let $r > 0$ be an integer such that $r\beta \in M(T)$ corresponds to a 1-PS of $T$. This 1-PS act on $X$ as

$$t \mapsto \mathrm{diag}(t^{r\alpha_0+\beta}, \dots, t^{r\alpha_n+\beta})$$

where $\alpha_0, \dots, \alpha_n$ are the weights of the representation of $T$ on $k^{n+1}$. The definition of $Z_\beta$ and $Y_\beta$ shows that $Z_\beta$ is a union of components of the fixed point set of this action and that $x \in Y_\beta$ iff

$$\lim_{t\to 0} tx \in Z_\beta,$$

in which case this limit coincides with $p_\beta(x)$. So the result is an immediate consequence of 13.1. □

Now we want to show that each stratum $S_\beta$ is isomorphic to $G \times_{P_\beta} Y_\beta^{ss}$ where $P_\beta$ is the parabolic subgroup of $G$ defined at 12.11. For simplicity we shall assume that the homomorphism $\phi : G \to GL(n+1)$ which defines the action of $G$ on $X$ is faithful. The general result follows immediately from this except that $P_\beta$ must be replaced by $\phi^{-1}(\phi(P_\beta))$ which is also a parabolic subgroup of $G$.

**Definition 13.3**. ([B], 3.3) Let $\mathfrak{g}$ be the Lie algebra of the $k$-group $G$ and for each $\beta \in \mathbf{B}$ let $\mathfrak{p}_\beta$ be the Lie algebra of the parabolic subgroup $P_\beta$.

As a $k$-vector space $\mathfrak{g}$ is just the tangent space to the group $G$ at the origin. The action of $G$ on $X$ induces a $k$-linear map $\xi \to \xi_x$ from $\mathfrak{g}$ to the Zariski tangent space $T_xX$ for each $x \in X$.

**Lemma 13.4** Suppose $G$ is a subgroup of $GL(n+1)$. If $x \in Y_\beta^{ss}$ then

$$\{g \in G : gx \in Y_\beta^{ss}\} = P_\beta; \quad \{\xi \in \mathfrak{g} : \xi_x \in T_x(Y_\beta^{ss})\} = \mathfrak{p}_\beta$$

*Proof* (compare with Lemma 6.15). By 12.3, $Y_\beta^{ss}$ is invariant under $P_\beta$ so

$$P_\beta \subseteq \{g \in G : gx \in Y_\beta^{ss}\} \qquad \text{and} \qquad \mathfrak{p}_\beta \subseteq \{\xi \in \mathfrak{g} : \xi_x \in T_xY_\beta^{ss}\}$$

By 12.24, $x \in Y_\beta^{ss}$ iff $T$ is optimal for $x$ and $\Lambda_T(x) = \{\beta/q(\beta)\}$. Suppose that $x$ and $gx$ both lie in $Y_\beta^{ss}$ for some $g \in G$; then $\beta/q(\beta) \in \Lambda_G(gx)$ so that $\beta/q(\beta)$ and $\mathrm{Ad}(g^{-1})\beta/q(\beta)$ both lie in $\Lambda_G(x)$. Therefore, $g \in P_\beta$ by 12.23 (3).

It remains to show that $\{\xi \in \mathfrak{g} : \xi_x \in T_xY_\beta^{ss} \subseteq P_\beta$. As in the proof of 13.2, if $r$ is any positive integer such that $r\beta$ is a 1-PS of $T$ then $r\beta$ acts on $X$ as

$$t \mapsto \mathrm{diag}(t^{r\alpha_0+\beta}, \ldots, t^{r\alpha_n+\beta})$$

By 12.11 the subgroup $P_\beta$ consists of all $g \in G$ such that $(r\beta(t))g(r\beta(t))^{-1}$ tends to some limit in $G$ as $t \in k^\star$ tends to 0. Hence, an element $g \in G$ lies in $P_\beta$ iff it is of the form $g = (g_{ij})$ with $g_{ij} = 0$ when $\alpha_i.\beta < \alpha_j.\beta$.

Let

$$\mathfrak{g} = \mathfrak{t} + \sum_\alpha \mathfrak{g}^\alpha$$

be the root space decomposition of $\mathfrak{g}$ with respect to the Lie algebra $\mathfrak{t}$ of the maximal torus $T$ (see [B] Theorem 13.18). If $\xi \in \mathfrak{g}^\alpha$ has a nonzero $ij$-component then, as $[\eta, \xi] = \alpha(\eta)\xi$ for all $\eta \in \mathfrak{t}$, it follows that $\alpha = \alpha_i - \alpha_j$. So $\mathfrak{g}^\alpha \subseteq \mathfrak{p}_\beta$ whenever $\alpha.\beta \geq 0$. Hence it suffices to show that if

$$\xi \in \sum_{\alpha.\beta<0} \mathfrak{g}^\alpha$$

and $\xi_x \in T_xY_\beta^{ss}$ then $\xi \in P_\beta$.

Let $V_+$ (resp. $V_0, V_-$) be the sum of all the subspaces of $k^{n+1}$ on which $T$ acts as multiplication by some character $\alpha_i$ with $\alpha_i.\beta > q(\beta)$ (resp. $\alpha_i.\beta = q(\beta), \quad \alpha_i.\beta < q(\beta)$). Then any element of

$$\sum_{\alpha.\beta<0} g^\alpha$$

is of block form

$$\begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{bmatrix}$$

with respect to the decomposition of $k^{n+1}$ as $V_+ \oplus V_0 \oplus V_-$.

If $x \in Z_\beta^{\mathrm{ss}}$ then $x$ is represented by a vector of the form $(0, v, 0)$ in $k^{n+1}$. We have

$$\begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{bmatrix} \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dv \end{bmatrix}$$

and so by definition of $Y_\beta^{\mathrm{ss}}$, if $\xi_x \in T_x Y_\beta^{\mathrm{ss}}$ then $dv = 0$ and hence $\xi_x = 0$. But this means that $\xi$ is contained in the Lie algebra of the stabiliser of $x$ in $G$, and by the first part of the lemma, the stabiliser of $x$ is contained in $P_\beta$. Therefore, $\xi \in P_\beta$, as required.

Thus it has been shown that $\mathfrak{p}_\beta \subseteq \{\xi \in \mathfrak{g} : \xi_x \in T_x Y_\beta^{\mathrm{ss}}\}$ and that equality holds when $x \in Z_\beta^{\mathrm{ss}}$. But the subset of $Y_\beta^{\mathrm{ss}}$ where equality holds is open and is invariant under the action of $P_\beta$. So it suffices to show that the only $P_\beta$-invariant neighbourhood of $Z_\beta^{\mathrm{ss}}$ in $Y_\beta^{\mathrm{ss}}$ is $Y_\beta^{\mathrm{ss}}$ itself. This follows easily from the fact that if $y \in Y_\beta^{\mathrm{ss}}$ then the point $p_\beta(y) \in Z_\beta^{\mathrm{ss}}$ lies in the closure of the orbit of $x$ under any 1-PS of $T$ which is an integer multiple of $\beta \in M(T)$. This completes the proof. □

Now we can state the result we're aiming for.

**Theorem 13.5**. Suppose $X \subset \mathbb{P}_n$ is a nonsingular projective variety over $k$ and $G$ is a reductive subgroup of $\mathrm{GL}(n+1)$ defined over $k$ which acts on $X$. Then the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$ defined in §12 is smooth. For each $\beta$ the stratum $S_\beta$ is isomorphic to $G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$ where $Y_\beta^{\mathrm{ss}}$ is a nonsingular locally-closed subvariety of $X$ and $P_\beta$ is a parabolic subgroup of $G$. Moreover there is an algebraic locally trivial fibration $p_\beta : Y_\beta^{\mathrm{ss}} \to Z_\beta^{\mathrm{ss}}$ with affine fibres where $Z_\beta^{\mathrm{ss}}$ consists of the semistable points of a closed nonsingular subvariety of $X$ under the action of a maximal reductive subgroup of $P_\beta$.

*Proof.* By 12.26 for each $\beta \in \mathcal{B}$ the stratum $S_\beta$ coincides with $GY_\beta^{\mathrm{ss}}$ where $Y_\beta^{\mathrm{ss}}$ is defined as in 12.10. Moreoever, by 12.23, $Y_\beta^{\mathrm{ss}}$ is invariant under the action of the parabolic subgroup $P_\beta$ of $G$ defined in 12.11. So there is a morphism $\sigma : G \times_{P_\beta} Y_\beta^{\mathrm{ss}} \to X$ whose image is $S_\beta$. We shall show using Lemma 13.4 that $\sigma$ is an isomorphism onto its image. The proof is a standard one (cf. e.g. [B] 11.9).

Recall that

$$W_\beta = \{x \in X : x_i = 0 \text{ for } \alpha_i.\beta < q(\beta)\};$$

it is invariant under $P_\beta$. Consider the morphisms

$$G \times W_\beta \xrightarrow{\gamma} G \times X \xrightarrow{\delta} (G/P_\beta) \times X$$

given by $\gamma(gx, x) = (g, gx)$ and $\delta(g, x) = (gP_\beta, x)$. Let

$$M = \delta\gamma(G \times W_\beta) \qquad \text{and} \qquad M' = \delta\gamma(G \times Y_\beta^{\text{ss}})$$

Since $W_\beta$ is invariant under $P_\beta$ we have

$$\delta^{-1}(M) = \{(g, y) : g^{-1}y \in W_\beta\}$$

which is closed in $G \times X$ and is isomorphic to $G \times W_\beta$ via $\gamma$. As $\delta$ is a quotient morphism, $M$ is therefore closed in $(G/P_\beta) \times X$

Now $GW_\beta$ is the image of $M$ under the projection

$$p_X : (G/P_\beta) \times X \to X$$

Since $G/P_\beta$ is complete, this shows that $GW_\beta$ is closed (we have already used this). Furthremore

$$G(W_\beta - Y_\beta^{\text{ss}}) \subseteq \bigcup_{\beta' > \beta} S_{\beta'}$$

by 12.17 and it follows that

$$M' = M \cap p_X^{-1}(S_\beta)$$

and hence is an open subset of $M$. We have

$$M' = \{(gP_\beta, y) : g^{-1}y \in Y_\beta^{\text{ss}}\}$$

which is isomorphic to $G \times_{P_\beta} Y_\beta^{\text{ss}}$ and hence is nonsingular. Morever, by Lemma 13.4, the restriction $p_X|M'$ is a bijection onto $S_\beta$. Indeed, since $G/P_\beta$ is complete, $p_X$ is a closed map, so that $p_X|M' : M' \to S_\beta$ is a homeomorphism because $M'$ is locally closed in $G/P_\beta \times X$. To show that $p_X|M'$ is an isomorphism it therefore suffices (by [Ha] Ex. I.3.3 and Lemma II.7.4) to check that the induced maps of Zariski tangent spaces $(p_X)_\star : T_m M' \to T_{p_X(m)} S_\beta$ are all injective.

It is only necessary to consider the case when $m = (P_\beta, y)$ for some $y \in Y_\beta^{\text{ss}}$; then an element of $T_m M'$ is of the form $(a + \mathfrak{p}_\beta, \xi)$ where $a + \mathfrak{p}_\beta \in \mathfrak{g}/\mathfrak{p}_\beta$, $\xi \in T_y X$ and $-a_y + \xi \in T_y Y_\beta^{\text{ss}}$. So if

$$0 = (p_X)_\star(a + \mathfrak{p}_\beta, \xi) = \xi$$

then $a_y \in T_y Y_\beta^{\mathrm{ss}}$, and hence by Lemma 13.4, $a \in \mathfrak{p}_\beta$ so that $(a+\mathfrak{p}_\beta, \xi)$ is the zero element of $T_m M'$. It follows that $(p_X)_\star$ is injective everywhere on $M'$ and hence that $p_X|M'$ is an isomorphism. We conclude that for each $\beta \in \mathbf{B}$ the stratum $S_\beta$ is nonsingular and isomorphic to $G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$.

Thanks to Corollary 13.2 the proof is now complete. □

# 14 Hodge numbers

Suppose now that $X \subseteq \mathbb{P}_n$ is a nonsingular complex projective variety acted on linearly by a connected complex reductive group $G$. Suppose that the stabilizer in $G$ of every semistable point of $X$ is finite. We have obtained a formula for the Betti numbers of the quotient variety $M$ associated in invariant theory to the action of $G$ on $X$. In this section we shall see that this formula can be refined to give a formula for the Hodge numbers of $M$.

We shall use Deligne's extension of Hodge theory which applies to algebraic varieties which are not necessarily compact and nonsingular [D1, D2]. If $Y$ is a variety which is not nonsingular and projective it may not be possible to decompose $H^n(Y, \mathbb{C})$ as the direct sum of subspaces $H^{p,q}(Y)$ in a way which generalizes the classical Hodge decomposition. However Deligne shows that there are to canonical filtrations $H^n(Y, \mathbb{C})$ the weight filtration

$$\dots W_{k-1} \subset W_k \subset W_{k+1} \subset \dots$$

which is defined over $\mathbb{Q}$ and the Hodge filtration

$$\dots \supset F_{p-1} \supset F_p \supset F_{p+1} \supset \dots$$

giving what Deligne calls a mixed Hodge structure on $H^n(Y)$. One can then define the Hodge numbers $h^{p,q}(H^n(Y))$ on $H^n(Y)$ to be the dimension of appropriate quotients associated to these filtrations. The Hodge numbers satisfy

$$\dim H^n(Y, \mathbb{C}) = \sum_{p,q} h^{p,q}(H^n(Y))$$

if $h^{p,q}(H^n(Y)) \neq 0$ then $p, q$ lie between $\max(0, n - \dim(Y))$ and $\min(n, \dim(Y))$ and $p + q \neq n$ if $Y$ is projective while $p + q \geq n$ if $Y$ is nonsingular. When $Y$ is nonsingular and projective the $h^{p,q}(H^n(Y))$ with $p + q = n$ are the same as the classical Hodge numbers $h^{p,q}$. If $f : Y_1 \to Y_2$ is a morphism of nonsingular quasi-projective varieties then the induced homomorphism $f^\star$ : $H^(Y_2) \to H^\star(Y_1)$ is strictly compatible with both the Hodge filtration and the weight filtration.

Suppose now that $Y$ is acted on by a group $G$. Recall that the equivariant cohomology is defined to be

$$H^\star_G(Y, \mathbb{Z}) = H^\star(Y \times_G EG, \mathbb{Z})$$

where $EG \to BG$ is the universal classifying bundle for $G$. Although $BG$ is not a finite dimensional manifold there is a natural Hodge structure on its cohomology. Indeed, $BG$ may be regarded as the union of finite dimensional varieties $M_n$ such that for any $n$ the inclusion of $M_n$ in $BG$ induces isomorphisms of cohomology in dimensions less than $n$ which preserve the Hodge structure. In the same way $Y \times_G EG$ is the union of finite dimensional varieties whose Hodge structures induce a natural Hodge structure on the cohomology of $Y \times_G EG$. Thus we can define the equivariant Hodge numbers

$$h_G^{p,q;n}(Y) = h^{p,q}(H_G^n(Y))$$

for $Y$.

In particular, there are equivariant Hodge numbers for each stratum $S_\beta$ of the stratification associated in §12 to the action of $G$ on the projective variety $X$. These strata may be disconnected so its convenient to refine the stratification as follows. For each integer $m > 0$ let $S_{\beta,m}$ be the union of those components of $S_\beta$ whose complex codimension in $X$ is $\frac{1}{2}d(\beta,m)$ where

$$d(\beta,m) = m - \dim(G) + \dim(\mathrm{Stab}(\beta))$$

In §8 we saw that

14.1

$$\dim\ H_G^n(X,\mathbb{Q}) = \sum_{\beta,m} \dim\ H_G^{n-d(\beta,m)}(S_{\beta,m},\mathbb{Q})$$

for each $n \geq 0$ where the sum is over all $\beta \in \mathbf{B}$ and integers $0 \leq m \leq \dim X$. The argument for this goes as follows. First, because $\{S_{\beta,m} : \beta \in \mathcal{B}, 0 \leq\leq \dim X\}$ is a stratification of $X$ the elements of the indexing set $\mathbf{B} \times \{\mathbf{0},\ldots,\dim \mathbf{X}\}$ can be ordered as $1,\ldots,M$ for some $M$ in such a way that $S_1 \cup \ldots S_i$ is open in $X$ for $1 \leq i \leq M$ (see definition 2.11). Let $T_i$ denote this open subset (for $1 \leq i \leq M$). Then as each stratum $S_i$ is smooth the Thom isomorphism theorem tells us that

$$H_G^n(T_i, T_{i-1};\mathbb{Q}) \cong H_G^{n-2\lambda_i}(S_i;\mathbb{Q})$$

where $\lambda_i$ is the complex codimension of $S_i$ in $X$. Thus for each $i$ there is a long exact sequence (the Gysin sequence)

$$\ldots \to H_G^{n-2\lambda_i}(S_i,\mathbb{Q}) \to H_G^n(T_i,\mathbb{Q}) \to H^n(T_{i-1},\mathbb{Q}) \to H^{n+1-2\lambda_i}(S_i,\mathbb{Q}) \to \ldots$$

In §5 we showed that the stratification is equivariantly perfect over $\mathbb{Q}$ which means exactly that each of these long exact sequences splits into short exact sequences

14.2

$$0 \to H_G^{n-2\lambda_i}(S_i,\mathbb{Q}) \to H^n(T_i,\mathbb{Q}) \to H_G^n(T_{i-1},\mathbb{Q}) \to 0$$

Then

14.3

$$\dim\ H_G^n(T_i,\mathbb{Q}) = \dim\ H_G^n(T_{i-1},\mathbb{Q}) + \dim\ H_G^{n-2\lambda_i}(S_i,\mathbb{Q})$$

for each $n$ and we obtain the formula 14.1 by using induction on $i$.

In order to extend the formula 14.1 to Hodge numbers all we need is the following

**Lemma 14.4**. The homomorphism $H^n_G(T_i) \to H^n_G(T_{i-1})$ induced by the inclusion of $T_{i-1}$ in $T_i$ is strictly compatible with the Hodge structures. So is the homomorphism $H^{n-2\lambda_i}_G(S_i) \to H^n_G(T_i)$ except that the Hodge structure of $H^{n-2\lambda_i}_G(S_i)$ must be shifted up by $\lambda_i$; that is, the weight filtration $\{W_k\}_{k\in\mathbb{Z}}$ is replaced by $\{W_{k+2\lambda_i}\}_{k\in\mathbb{Z}}$ and the Hodge filtration $\{F_p\}$ by $\{F_{p+\lambda_i}\}_{p\in\mathbb{Z}}$.

*Proof*. The first statement follows from [D1] II 2.3.5 and 3.2.11.1.

When $Y$ is nonsingular of complex dimension $N$, Poincaré duality gives an isomorphism

$$H^n(Y;\mathbb{Q}) \cong \mathrm{Hom}(H_c^{2N-n}(Y;\mathbb{Q}),\, H^{2N}(Y;\mathbb{Q})) = (H_c^{2N-n}(Y;\mathbb{Q}))^\star$$

where $H_c$ is cohomology with compact supports and $\star$ indicates duality. There is a natural Hodge structure on $(H_c^{2N-n}(Y))^\star$ (see [D1] II) and Poincaré duality carries the Hodge structure on $H^n(Y)$ to the natural Hodge structure on $(H_c^{2N-n}(Y))^\star$ shifted up by $N$ (see [D2] 8.2). If $i : Y' \to Y$ is the inclusion in $Y$ of a smooth closed subvariety $Y'$ of codimension $\lambda$ then the composition

$$H^{n-2\lambda}(Y';\mathbb{Q}) \overset{\text{Thom}}{\cong} H^n(Y,Y';\mathbb{Q}) \to H^n(Y;\mathbb{Q})$$

is the composition of two Poincaré duality maps with the dual f the map induced by $i$ on cohomology with compact supports:

$$H^{n-2\lambda}(Y';\mathbb{Q}) \cong (H_c^{2N-n}(Y';\mathbb{Q}))^\star \xrightarrow{(i^\star)^\star} H_c^{2N-n}(Y;\mathbb{Q}) \cong H^n(Y;\mathbb{Q})$$

Since $(i^\star)^\star$ is strictly compatible with the Hodge structure, we deduce that this composition carries the usual Hodge structure on $H^n(Y)$ to the Hodge structure on $H^{n-2\lambda}(Y')$ shifted up by $\lambda$.

The result follows by applying this to finite dimensional approximations to the inclusion of $S_i \times_G EG$ in $T_i \times_G EG$. □

It follows from this lemma and the exact sequence 14.2 that

$$h_G^{p,q;n}(T_i) = h_G^{p,q;n}(T_{i-1}) + h_G^{(p,q;n)-\lambda_i}(S_i)$$

where $(p,q;n)-\lambda$ is shorthand for $p-\lambda, q-\lambda; n-2\lambda$. Thus by induction we obtain

14.5

$$h_G^{p,q;n}(X) = h_G^{p,q;n}(X^{\rm ss}) + \sum_{\beta,m} h_G^{(p,q;n)-\frac{1}{2}d(\beta,m)}(S_{\beta,m})$$

where the sum is over all nonzero $\beta \in \mathbf{B}$ and integers $0 \leq m \leq \dim X$.

By theorem 13.7 for each $\beta$ we have

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{\rm ss}$$

and there is a locally trivial fibration

$$p_\beta : Y_\beta^{\rm ss} \to Z_\beta^{\rm ss}$$

with contractible fibre which respects the action of Stab$\beta$. Since Stav$\beta$ is homotopically equivalent to $P_\beta$ it follows that

$$H_G^\star(S_\beta, \mathbb{Q}) \cong H_{P_\beta}^\star(Y_\beta^{ss}, \mathbb{Q}) \cong H_{\mathrm{Stab}\,\beta}^\star(Z_\beta^{\rm ss}, \mathbb{Q})$$

and it is easily checked that these are isomorphism of the Hodge structures. By looking at components we also get

$$H_G^\star(S_{\beta,m}(S_{\beta,m}, \mathbb{Q}) \cong H_{\mathrm{Stab}\,\beta}^\star(Z_{\beta,m}^{\rm ss}, \mathbb{Q})$$

for each $\beta$ and $m$, where $Z_{\beta,m}^{\rm ss}$ is the set of semistable points of a nonsingular subvariety $Z_{\beta,m}$ of $X$ under a suitable linearization of the action of Stab$\beta$. Hence

14.6

$$h_G^{p,q;n}(S_{\beta,m}) = h_{\mathrm{Stab}\,\beta}^{p,q;n}(Z_{\beta,m}^{\rm ss})$$

for each $p, q, n$. Therefore

14.7

$$h_G^{p,q;n}(X^{\rm ss}) = h_G^{p,q;n}(X) - \sum_{\beta,m} h_{\mathrm{Stab}\,\beta}^{(p,q;n)-\frac{1}{2}d(\beta,m)}(Z_{\beta,m}^{\rm ss})$$

This gives an inductive formula for the equivariant Hodge numbers of $X^{\rm ss}$ in terms of those $X$ itself and of the semistable strata of smaller varieties acted on by reductive groups.

We also know that the fibration $X \times_G EG \to BG$ with fibre $X$ is cohomologically trivial over $\mathbb{Q}$ (see theorem 5.4) so that

14.8

$$H_G^\star(X, \mathbb{Q}) \cong H^\star(X, \mathbb{Q}) \bigotimes H^\star(BG, \mathbb{Q})$$

This isomorphism is an isomorphism of Hodge structures.

Using 14.7 and 14.8 an explicit formula can be derived for the equivariant Hodge numbers of the semistable stratum $X^{ss}$. This formula involves the Hodge numbers of $X$ and certain nonsingular subvarieties of $X$, and also the Hodge numbers of the classifying space of $G$ and various reductive subgroups of $G$ (cf. §5).

By assumption the stabiliser in $G$ of every $x \in X^{ss}$ is finite. This implises that the quotient variety $M$ coincides with the topological quotient $X^{ss}/G$. Moreover, the obvious map $X^{ss} \times_G EG \to X^{ss}/G$ induces an isomorphism

$$H^\star(X^{ss}/G;\mathbb{Q}) \to H^\star_G(X^{ss},\mathbb{Q})$$

which is strictly compatible with the Hodge structures and hence is an isomorphism of Hodge structures. Thus we obtain a formula for calculating the Hodge numbers $h^{p,q}(M)$ of the quotient $M = X^{ss}/G$ which are the classical Hodge numbers $h^{p,q}(M)$ when $M$ is smooth.

Note that since $h^{p,q;n}_G(X)$ is nonzero only when $p+q=n$ the same is true by induction on $h^{p,q;n}(X^{ss})$ and each $h^{p,q;n}_G(S_{\beta,m})$ and hence also of $h^{p,q;n}(M)$ when the stabilizer of each $x \in X^{ss}$ is finite. This last fact could be of course also be deduced directly from [D1] and the fact that $X^{ss}/G$ is a compact rational homology manifold.

Finally, note that 14.2 shows that the map $H^n_G(X,\mathbb{Q}) \to H^n_G(X^{ss},\mathbb{Q})$ induced by the inclusion of $X^{ss}$ in $X$ is surjective since it is the composition of the surjective maps $H^n_G(T_i,\mathbb{Q}) \to H^n_G(T_{i-1},\mathbb{Q})$ for $1 \le i \le M$. Thus we have a surjective homomorphism

14.9

$$H^\star(X,\mathbb{Q}) \bigotimes H^\star(BG,\mathbb{Q}) \longrightarrow H^\star(M,\mathbb{Q})$$

which is strictly compatible with the Hodge structures. In particular if $h^{p,q}(X) = 0$ for $p \neq q$ then the same is true for $M$, because by [D1] III 9.1.1, only the even Betti numbers of $BG$ are nonzero and $H^{2n}(BG,\mathbb{C})$ is purely of type $(n,n)$ for every $n$.

# 15 Calculating cohomology by counting points

Again let $M$ be the projective quotient variety associated to the linear action of a complex reductive group $G$ on a nonsingular complex projective variety $X$. When the action of $G$ on a semistable stratum $X^{\text{ss}}$ is free there is an alternative method for deriving the formulae already obtained for the Betti numbers of $M$ which uses the Weil conjectures. These conjectures which were verified by Deligne enable one to calculate the Betti numbers of a nonsingular projective variety by counting the number of points in associated varieties defined over finite fields. In our case we can count points by using the stratifications defined in §12 of varieties over the algebraic closures of finite fields, $\bar{\mathbb{F}}_q$. The idea was suggested by work of Harder and Narasimhan who used the Weil conjectures to calculate Betti numbers of moduli spaces of bundles over Riemann surfaces. Their formulae were subsequently rederived in the paper of Atiyah and Bott which motivated Part I.

The idea of the alternative method is explained in this section but the arguments are not given in detail because nothing new is being proved. Unless the Weil conjectures can be extended in an appropriate way to projective varieties which are locally the quotients of nonsingular varieties by finite groups, the same method will not work in all cases where the stabilizer of each semistable point is finite. It is necessary that the action of $G$ or at least some quotient of $G$ on $X^{\text{ss}}$ be free.

First let us summarize what we shall need of the Weil conjectures

Let $Y$ be a nonsingular complex projective variety. Then $Y$ is defined over a finitely generated subring $R$ of $\mathbb{C}$ so that there is a an $R$-scheme $Y_R$ such that $Y = Y_R \times_R \operatorname{Spec} \mathbb{C}$. Let $\pi$ be a maximal ideal of $R$. Then $R/\pi$ is a finite field with $q$ elements for some prime power $q$. Let

15.1

$$Y_\pi = Y_R \times_R \operatorname{Spec} R/\pi$$

be the reduction of $Y$ mod $\pi$. For most choices of $\pi$ if $\ell$ is any prime number different from the characteristic of $R/\pi$ then the $\ell$-adic numbers of $Y_\pi$ and $Y$ are equal. But the $\ell$-adic Betti numbers of $Y$ are the same as the ordinary Betti numbers of $Y$ regarded as a complex manfiold, by the comparison theorem of $\ell$-adic cohomology.

Provided that the characteristic of $R/\pi$ is not one of finitely many bad primes, $Y_\pi$ is a nonsingular projective variety over the finite field with $q$ elements. Then the Weil conjectures enable us to calculate its $\ell$-adic Betti numbers. In fact, there exist complex numbers $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_s$ such that for any integer $n \geq 1$ the number of points of $Y_\pi$ defined over the finite field with $q^n$ elements is

15.2

$$\sum_i (\alpha_i)^n - \sum_j (\beta_j)^n$$

We may assume that $\alpha_i \neq \beta_j$ for every $i, j$. Then the absolute value of each $\alpha_i$ is of the form $q^{n(i)}$ and the absolute value of each $\beta_j$ is of the form $q^{n(j)+\frac{1}{2}}$ where $n(i), n(j)$ are non-negative integers. Moreover the $(2k)$th $\ell$-adic Betti number of $Y_\pi$ is equal to the number of $\alpha_i$ with absolute value $q^k$ and its $(2k+1)$st $\ell$-adic Betti number is the number of $\beta_j$ with absolute value $q^{k+\frac{1}{2}}$

We shall use the Weil conjectures in a slightly different but equivalent form.

**Definition 15.3**. For $r \geq 1$ let $N_r(Y)$ be the number of points of $Y_\pi$ which are defined over the field of $q^r$ elements. If $n$ is the dimension of $Y$ let $N_r = q^{-rn} N_r(Y)$.

**15.4**. If follows easily from Poincaré duality and the Weil conjectures as stated above that we can write the series

$$\exp\left(\sum_{r\geq 1} \tilde{N}_r(Y) t^r / r\right) \in \mathbb{Q}[[t]]$$

in the form

$$\frac{Q_1(t) \cdots Q_{2n-1}(t)}{Q_0(t) \cdots Q_{2n}(t)}$$

where

$$Q_i(t) = \prod_j (1 - \gamma_{ij} t)$$

for complex numbers $\gamma_{ij}$ satisfying

$$|\gamma_{ij}| = q^{-i/2}$$

and where $\deg Q_i$ is the $i$th Betti number of $Y$.

We shall use 15.4 to calculate the rational Poincaré polynomial of the quotient variety $M$ associated to the action of $G$ on $X$. It seems to be natual to use this dual form of the Weil conjectures here. This is what AB do when comparing their methods with those of [H&N]. Using the ordinary form corresponds to using cohomology with compact supports and it is difficulty to make sense of this for the infinite dimensional manifolds in [A&B].

For simplicity suppose that $G$ is a subgroup of $GL(n+1)$. We assume that $G$ acts freely on $X^{\mathrm{ss}}$. The argument we shall use runs as follows.

We may assume throughout that the action of $G$ on $X$ is defined over $R$ and that all the finitely many quasi-projective nonsingular varieties of $X$ and subgroups of $G$ which we shall need to consider are also defined over $R$ and have nonsingular reduction mod $\pi$. We may also assume that their dimensions are unaltered by reduction mod $\pi$. Moreover the Weil conjectures still hold if $q$ is replaced by some power $q^s$. Hence we may assume that all subvarieties of $X_\pi$ and subgroups of $G_\pi$ under consideration are defined over $\mathbb{F}_q$.

We shall find that the stratification of $X_\pi$ induced by the action of $G_\pi$ is the reduction mod $\pi$ of the stratification of $X$ induced by the action of $G$, and hence using the results of §13 that

15.5

$$\tilde{N}_r(X^{\mathrm{ss}}) = \tilde{N}_r(X) - \sum_{\beta,m} q^{-\frac{1}{2}rd(\beta,m)} \tilde{N}_r(Z^{\mathrm{ss}}_{\beta,m}) \tilde{N}_r(G/P_\beta)$$

where the sum is over all nonzero $\beta$ and integers $0 \leq m \leq \dim X$. This gives us an inductive formula for $\tilde{N}_r(X^{\mathrm{ss}})/\tilde{N}_r(G)$ which is analogous to the formula for the Poincaré series $P_t^G(X^{\mathrm{ss}})$ obtained in Part I. From it an explicit formula can be derived for $\tilde{N}_r(M)$ by the arguments used in §5. This formula is such that if $q$ is replaced by $t^{-2}$ and $\tilde{N}_r(Y)$ by $P_t(Y)$ for each projective variety $Y$ which appears in it, then the result is the formula for $P_t(M)$ already derived! It then remains only to justify this substitution.

Let us now examine the details of this argument more closely.

Let $T$ be a maximal torus of $G$ defined over $R$ and let $T \subseteq B$ be a Borel subgroup also defined over $R$. By extending $R$ if necessary we may assume that $T$ acts diagonally on $R^{n+1}$. It follows from our assumptions that the group $G_\pi$ is reductive and has $T_\pi$ as a maximal torus and $B_\pi$ as a Borel subgroup.

Theorem 12.6 can be applied to the action of $G_\pi$ on $X_\pi$ and to that of $G$ on $X$ to obtain stratifications of $X_\pi$ and $X$. It is necessary to investigate the relationship between these stratifications. First we must check that they can be indexed by the same set $\mathcal{B}$. Recall that the indexing set for the stratification of $X$ is a finite subset of the $\mathbb{Q}$-vector space $M(T) = Y(T) \otimes \mathbb{Q}$ where $Y(T)$ is the free abelian group consisting of all 1-PS of the maximal torus $T$. Since $T_\pi$ has the same ramk as $T$, there is a natural identification of $M(T)$ with $M(T_\pi)$. The Weyl group actions coincide under these identifications, and so do the weights $\alpha_0, \ldots, \alpha_n$ of the representations of $T$ and $T_\pi$ which define their actions on $X$ and $X_\pi$. Hence the stratifications of $X$ and $X_\pi$ may be indexed by the same set $\mathcal{B}$ (see 12.8).

Let $\{S_\beta : \beta \in \mathcal{B}\}$ be the stratification of $X$ and let $\{S_{\beta,\pi} : \beta \in \mathcal{B}\}$ be the stratification of $X_\pi$. Under the assumptions already made, the following lemma follows without difficulty from the definitions of §12.

**Lemma 15.6**. The stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is defined over $R$ and

$$(S_\beta)_\pi = S_{\beta,\pi} \qquad \forall \beta \in \mathcal{B}$$

Moreover, $(Y_\beta^{\mathrm{ss}})_\pi$, $(Z_\beta^{\mathrm{ss}})_\pi$ and $(P_\beta^{\mathrm{ss}})_\pi$ coincide with the subvarieties of $X_\pi$ and parabolic subgroup of $G_\pi$ defined in the corresponding way for the action of $G_\pi$ on $X_\pi$. Finally, the quotient variety $M = X^{\mathrm{ss}}/G$ satisfies

$$\tilde{N}_r(M) = \tilde{N}_r(X^{\mathrm{ss}})\tilde{N}_r(G)^{-1}$$

for each $r \geq 1$.

In orde to apply 15.4 we need to calculate $\tilde{N}_r(M)$ for each $r \geq 1$. The last lemma suggests that we should investigate $\tilde{N}_r(X^{\mathrm{ss}})$. It also tells us that for each $\beta \in \mathcal{B}$ that $N_r(S_\beta)$ is the number of points in the stratum $S_{\beta,\pi}$ of $X_\pi$ which are defined over the field of $q^r$ elements, and so

15.7

$$N_r(X^{\mathrm{ss}}) = N_r(X) - \sum_{\beta \neq 0} N_r(S_\beta)$$

Moreover

$$S_{\beta,\pi} \cong G_\pi \times_{(P_\beta)_\pi} (Y_\beta^{\mathrm{ss}})_\pi$$

by the lemma together with Theorem 2.26, and so

15.8

$$N_r(S_\beta) = N_r(Y_\beta^{\mathrm{ss}})N_r(G/P_\beta)$$

for each $\beta$. As in §4 we can decompose $Y_\beta^{\mathrm{ss}}$ into a disjoint union of open subsets $\{Y_{\beta,m}^{\mathrm{ss}} : 0 \leq m \leq \dim X\}$ such that each component of $Y_{\beta,m}^{\mathrm{ss}}$ has real codimension $m$ in $X$. Then $S_\beta$ is the disjoint union of open subsets $GY_{\beta,m}^{\mathrm{ss}}$ which have complex codimension

$$\frac{1}{2}d(\beta,m) = \frac{1}{2}m - \dim G/P_\beta$$

There is also a locally trivial fibration

$$p_\beta : (Y_{\beta,m}^{\mathrm{ss}})_\pi \longrightarrow (Z_{\beta,m}^{\mathrm{ss}})_\pi$$

such that each fibre is an affine space (see 13.2), from which it follows that

$$\tilde{N}_r(Y_{\beta,m}^{\mathrm{ss}}) = \tilde{N}_r(Z_{\beta,m}^{\mathrm{ss}})$$

for each $r \geq 1$. So by 15.7 and 15.8 we have

15.9

$$\tilde{N}_r(X^{\mathrm{ss}} = \tilde{N}_r(X) - \sum_{\beta,m} q^{-\frac{1}{2}rd(\beta,m)}\tilde{N}_r(Z_{\beta,m}^{\mathrm{ss}})\tilde{N}_r(G/P_\beta)$$

for each $r \geq 1$ where the sum is over nonzero $\beta \in \mathcal{B}$ and integers $0 \leq m \leq \dim X$.

Next we consider $\tilde{N}_r(G/P_\beta)$. As in 6.9 we have $P_\beta = B\mathrm{Stab}\,\beta$ where $B$ is the Borel subgroup of $G$ and $\mathrm{Stab}\,\beta$ is the stabiliser of $\beta$ under the adjoint action of $G$ Since $\mathrm{Stab}\,\beta$ contains the maximal torus $T$ it follows that $P_\beta = B_u\mathrm{Stab}\,\beta$, where $B_u$ is the unipotent part of $B$ (see [B] 10.6 (4)).

**Lemma 15.10**. If $H$ is a unipotent subgroup of $G$ defined over $R$ such that $\dim H_\pi = \dim H$, then

$$N_r(H) = q^{r\,\dim H}$$

and hence $\tilde{N}_r(H) = 1$ for all $r \geq 1$.

*Proof.* The remark at the end of [B] 14.4 shows that $H_\pi$ is isomorphic as a variety over $\mathbb{F}_q$ to an affine space. The result follows. □

Under our assumptions this lemma applies to the unipotent subgroups $B_u i$ and $B_u \cap \mathrm{Stab}\,\beta$ of $G$. Hence

15.11

$$\tilde{N}_r(P_\beta) = \tilde{N}_r(\mathrm{Stab}\,\beta)$$

From this together with 5.9 it follows that

15.12

$$\tilde{N}_r(X)\tilde{N}_r(G)^{-1} = \tilde{N}_r(X^{\mathrm{ss}})\tilde{N}_r(G)^{-1} + \sum_{\beta,m} q^{-\frac{1}{2}rd(\beta,m)}\tilde{N}_r(Z^{\mathrm{ss}}_{\beta,m}\tilde{N}_r(\mathrm{Stab}\,\beta)^{-1}$$

for all $r \geq 1$ where the sum is over all nonzero $\beta \in \mathcal{B}$ and $0 \leq m \leq \dim X$. This is an inductive formula for $\tilde{N}_r(X^{\mathrm{ss}})\tilde{N}_r(G)^{-1}$ (which coincides with $\tilde{N}_r(M)$ under the assumption that $G$ acts freely on $X^{\mathrm{ss}}$ by Lemma 15.6). By the argument used in §5 we can derive from it the following explicit formula.

15.13

$$\tilde{N}_r(M) = \tilde{N}_r(X)\tilde{N}_r(G)^{-1} + \sum_{\underline{\beta},m}(-1)^{q(\underline{\beta})}q^{-\frac{1}{2}rd(\underline{\beta},m)}\tilde{N}_r(Z_{\underline{\beta},m})\tilde{N}_r(\mathrm{Stab}\,\underline{\beta})^{-1}$$

for each $r \geq 1$ where the sum is over all integers $0 \leq m \leq \dim X$ and $\beta$-sequences $\underline{\beta}$ defined as in §5. If $\underline{\beta} = (\beta_1, \ldots, \beta_q)$ is a $\beta$-sequence then $q(\underline{\beta}) = q$ is the length of $\underline{\beta}$. Each $Z_{\underline{\beta},m}$ is a nonsingular closed subvariety of $X$ and

$$\operatorname{Stab}\underline{\beta} = \bigcap_j \operatorname{Stab}\beta_j$$

is a reductive subgroup of $G$.

From 15.4 we know that the Poincaré polynomial $P_t(Y)$ of any nonsingular projective variety $Y$ can be calculated from the numbers $\tilde{N}_r(Y)$. Our aim is to apply this to the quotient variety $M$ and use formula 15.13 to obtain an expression for $P_t(M)$. However the groups which appear in 15.13 are not projective varieties, so we need to modify the formula a little as follows.

The Borel subgroup is the product of its unipotent part $B_u$ and the maximal torus $T$ so lemma 15.10 implies that

15.14
$$\tilde{N}_r(G) = \tilde{N}_r(G/B)\tilde{N}_r(T) = \tilde{N}_r(G/B)(1-q^{-r})^{\dim T}$$

for $t \geq 1$.

If we apply this to each of the subgroups $\operatorname{Stab}\beta$ of $G$ and substitute in 15.13, we obtain an expression for $\tilde{N}_r(M)$ as a rational function of $q$ and the numbers $\tilde{N}_r(Y)$ for certain nonsingular projective varieties $Y$. The varieties involved here are $X$ and its subvarieties $Z_{\underline{\beta},m}$. This gives us a formula for the Poincaré polynomial $P_t(M)$ of the quotient $M$ because of the following

**Lemma 15.15**. Suppose that $Y_1, \dots, Y_k$ are smooth complex projective varieties defined over $R$ whose reductions modulo $\pi$ are also smooth. Suppose that $f$ is a rational function of $s+1$ variables with integer coefficients such that

$$f(q^{-r}, \tilde{N}_r(Y_1), \dots, \tilde{N}_r(Y_k)) = 0$$

for all $r \geq 1$. Then

$$f(t^2 P_t(Y_1), \dots, P_t(Y_k)) = 0$$

*Proof.* Call a sequence $N = \{n_r : r \geq 1\} \subseteq \mathbb{Z}$ a Weil sequence if there exist finitely many polynomials $Q_i(t)$ of the form

$$Q_i(t) = \prod_j (1-\gamma_{ij}t), \qquad |\gamma_{ij}| = q^{-\frac{i}{2}}$$

for each $i, j$, and such that

$$\exp\left(\sum_{r\geq 1} n_r t^r / r\right) = \frac{Q_1(t)\dots Q_{2n-1}(t)}{Q_0(t)\dots Q_{2n}(t)}$$

for some $n \geq 0$. These conditions determine each nontrivial $Q_i$ uniquely, so we may define a polynomial $P_t(N)$ by

$$P_t(N) = \sum_{i \geq 0} (\deg Q_i) t^i$$

It is easy to check that if $N$ and $M$ are Weil sequences then so are $NM$ and $N + M$ and $q^{-1}N := \{q^{-r}n_r : r \geq 1\}$, and that

$$P_t(NM) = P_t(N)P_t(M), \qquad P_t(N + M) = P_t(N) + P_t(M), \qquad P_t(q^{-1}N) = t^2 P_t(N)$$

For each positive integer $j \leq k$ let $N_j$ be the sequence $\{\tilde{N}_r(Y_j) : r \geq 1\}$. It follows from 5.4 that each of the sequences $N_j$ is a Weil sequence and that the polynomial $P_t(N_j)$ coincides with $P_t(Y_j)$.

To prove the lemma it is enough to consider the case when $f \in \mathbb{Z}[x_0, \ldots, x_n]$. We can write such an $f$ as $f = g - h$ where $g, h$ are sums of monomials with positive integer coefficients. Since $N_1, \ldots, N_k$ are Weil sequences, so are the sequences whose $r$th terms are

$$g(q^{-r}, \tilde{N}_r(Y_1), \ldots, \tilde{N}_r(Y_k)) \qquad \text{and} h(q^{-r}, \tilde{N}_r(Y_1), \ldots, \tilde{N}_r(Y_k))$$

and their corresponding polynomials are

$$g(t^2, P_t(Y_1), \ldots, P_t(Y_k)) \qquad \text{and} \qquad h(t^2, P_t(Y_1), \ldots, P_t(Y_k))$$

But by assumption these sequences are equal, and hence so are the corresponding polynomials. □

This lemma may be applied to the equation obtained from 15.13 by using 15.14 to substitute for $\tilde{N}_r(G)$ and for each $\tilde{N}_r(\operatorname{Stab}\beta)$. This gives us the following formula for the Betti numbers of the quotient $M$:

15.16

$$P_t(M) = (1-t^2)^{-\dim T} \left\{ P_t(X)P_t(G/B)^{-1} + \sum_{\underline{\beta},m} (-1)^{q(\underline{\beta})} t^{d(\beta,m)} P_t(Z_{\underline{\beta},m}) P_t(\mathrm{Stab}\,\underline{\beta}/(B \cap \mathrm{Stab}\,\underline{\beta}))^{-1} \right\}$$

As before, let $BG$ denote the classifying space for the group $G$. There is a fibration $BG \to BT$ whihc has fibe $G/B$ and is cohomologically trivial. Thus

$$P_t(BG) = P_t(BT)P_t(G/B)^{-1} = (1-t^2)^{-\dim T} P_t(G/B)^{-1}$$

By applying this to all the reductive subgroups $\mathrm{Stab}\,\underline{\beta}$ of $G$, we find that the formula for $P_t(M)$ in 15.16 coincides with the formula derived in Part I.

# 16 Examples

In this section the stratifications induced by some particular group actions will be described and the Betti numbers of their quotients will be calculuated.

We shall start by reviewing the diagonal action of SL(2) on a power $(\mathbb{P}_1)^n$. This was used as an example throughout Part I. When SL(2) acts on $\mathbb{P}_n$ identified with the space of binary forms of degree $n$ very similiar results hold. Then we shall consider the action of SL($m$) on a product of the form

$$X = \prod_j G(\ell_i, m)$$

where $G(\ell, m)$ is the Grassmannian of $\ell$-dimensional subspces of $\mathbb{C}^m$. The subvarieties $Z_\beta$ which appear in the inductive formula for $P_t^{\mathrm{SL}(m)}(X^{\mathrm{ss}})$ are all products of varieties of the same form as $X$ but with smaller values of $m$. Thus although the calculation of $P_t^G(X^{\mathrm{ss}})$ for large $m$ would be extremely lengthy by hand, it cold be carried out by a computer. We do some explicit calculations for the special of products $(\mathbb{P}_2)^n$ of the projective plane. These examples are more intricate than $(\mathbb{P}_1)^n$ and are more typical of the general case.

### 16.1. Ordered points on the projective line

For fixed $n \geq 1$ consider the diagonal action of the special linear group SL(2) on $(\mathbb{P}_1)^n$. This is linear with respect to the Segre embedding; the corresponding representation of SL(2) is the $n$th tensor power of its standard representation on $\mathbb{C}^2$. Let $T_c$ be the complex maximal torus consisting of all diagonal elements of SL(2) and let $\alpha$ be the one-parameter subgroup of $T_c$ given by

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

The weights of the representation with respect to this torus are of the form

$$r\alpha - (n - r)\alpha$$

where $r$ is any integer such that $0 \leq r \leq n$. If we choose the positive Weyl chamber to contain $\alpha$ the it follows that the indexing set for the stratification is

$$\mathcal{B} = \left\{(2r - n)\alpha : n \geq r > \frac{n}{2}\right\} \bigcup \{0\}$$

Suppose $\beta = (2r - n)\alpha$ where $r > \frac{n}{2}$. Then it is easy to check from definition 12.8 that a sequence in $(\mathbb{P}_1)^n$ lies in $Z_\beta$ iff it contains $r$ copies of 0 and $n - r$ copies of $\beta$. Also $Y_\beta$ consists of sequences containing precisely $r$ copies of 0.

It follows from definition 12.20 that $Z_\beta^{\mathrm{ss}} = Z_\beta$ and hence $Y_\beta^{\mathrm{ss}} = Y_\beta$. Since the stratum $S_\beta$ indexed by $\beta$ is $GY_\beta^{\mathrm{ss}}$ (see 2.26) it follows that $S_\beta$ consists of all sequences $(x_1, \ldots, x_n)$ such that $r$ but no more of the points $x_i$ coincide. Thus $S_\beta$ has $\binom{n}{r}$ components each of which has complex codimension $r - 1$.

Therefore the semistable elements of $(\mathbb{P}_1)^n$ are those which contain no point of $\mathbb{P}_1$ with multiplicity strictly greater than $n/2$ [N]. If $x \in (\mathbb{P}_1)^n$ is not semistable, the stratum to which $x$ belongs is determined by the multiplicity of the unique point of $\mathbb{P}_1$ which occurs as a component of $x$ strictly more than $n/2$ times.

SL(2) is the complexification of the compact group SU(2) which preserves the standard Kähler structure on $(\mathbb{P}_1)^n$. Since SU(2) is semisimple there is a unique moment map $\mu : (\mathbb{P}_1)^n \to \mathfrak{su}(2)$. The adjoint action of SU(2) of its Lie algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$ is via the double cover $\theta : \mathrm{SU}(2) \to SO(3)$. Use the standard inner product on $\mathbb{R}^3$ to identify $\mathfrak{su}(2)$ with its dual. The complex projective line $\mathbb{P}_1$ may be identified with the unit sphere in $\mathbb{R}^3$ which is an orbit of the adjoint representation of SU(2). By [Ar], the moment map for the action of SU(2) on $\mathbb{P}_1$ is then the inclusion $\mathbb{P}_1 \to \mathbb{R}^3$. It follows easily from this or from 2.7 that the moment map $\mu : (\mathbb{P}_1)^n \to \mathbb{R}^3$ is given by

$$\mu(x_1, \ldots, x_n) = x_1 + \ldots + x_n$$

$$f(x_1, \ldots, x_n) = ||x_1 + \ldots + x_n||^2$$

where $|| \, ||$ is the standard norm on $\mathbb{R}^3$. A point $(x_1, \ldots, x_n)$ is critical for $f$ if either $f(x_1, \ldots, x_n) = 0$ or each $x_i$ is one of a fixed pair of antipodal points of $\mathbb{P}_1$.

It is intuitively reasonable that the Morse stratification of this function should coincide with the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ already described. For by symmetry if two components $x_i, x_j$ of $x = (x_1, \ldots, x_n)$ agree then these components will remain the same on the path of steepest descent for $f$ from $x$. On the other hand, it is possible to move a configuration of $n$ points into a balanced position (i.e. a position with centre of gravity at the origin) without splitting up points which coincide iff no point has multiplicity strictly more than $n/2$.

Note that the stratification for the action of GL(2) is the same as that for SL(2) although labelled differently. This is because GL(2) is the quotient by a finite subgroup of the product of SL(2) with a central one-parameter subgroup which acts trivially on $\mathbb{P}_1$.

The stabilizer in PGL(2) of a point $x \in (\mathbb{P}_1)^n$ is nontrivial precisely when at most two distinct points of $\mathbb{P}_1$ occur as components of $x$. So if $n$ is odd PGL(2) acts freely on the semistable points of $(\mathbb{P}_1)^n$. Then as SL(2) is a finite cover of PGL(2) we can use theorem 8.12 to calculate the Betti numbers and Hodge numbers of the quotient variety $M$ as follows.

Since the rank of SU(2) is 1 each $\beta$-sequence has length 1 (see definition 5.11) and so is just a nonzero element of $\mathbf{B}$. Thus by 8.10 and 5.17

$$\begin{aligned} P_t(M) &= P_t((\mathbb{P}_1)^n)P_t(B\mathrm{SU}(2)) - \sum_{\frac{n}{2} < r \leq n} \binom{n}{r} t^{2(r-1)} P_t(BS^1) \\ &= (1+t^2)^n(1-t^4)^{-1} - \sum \binom{n}{r} t^{2(r-1)}(1-t^2)^{-1} \\ &= 1 + nt^2 + \ldots + \left\{ 1 + (n-1) + \binom{n-1}{2} + \ldots + \binom{n-1}{\min(j, n-3-j)} \right\} t^{2j} + \ldots + t^{2n-6} \end{aligned}$$

This obeys Poincaré duality as expected. Note that the equivariant cohomology of the semistable stratum of $(\mathbb{P}_1)^n$ is given by the series above for any $n$, even or odd. However, this is not a polynomial when $n$ is even!

When $n$ is odd it is also possible to obtain the Hodge numbers of the quotient $M$. Indeed, 14.9 shows that $h^{p,q}(M) = 0$ for $p \neq q$, and, for each $p$,

$$h^{p,p}(M) = 1 + (n-1) + \ldots + \binom{n-1}{\min(p, n-3-p)}$$

### 16.2. **Binary forms** (cf. [M] 4 §1 and [N] 4 §1, §3)

An example which is familiar to 16.1 is the action of SL(2) on the projective space $\mathbb{P}_n$ identified with the $n$th symmetric product of $\mathbb{P}_1$.

The maximal torus $T_c$ acts on $\mathbb{P}_n$ via the homomorphism

$$\mathrm{diag}(z, z^{-1}) \to \mathrm{diag}(z^n, z^{n-2}, \ldots, z^{-n})$$

so that as in 16.1

$$\mathcal{B} = \{(2r-n)\alpha : n \geq r > \frac{n}{2}\} \cup \{0\}$$

If $\beta = (2r-n)\alpha \in \mathcal{B}$ then $Z_\beta^{ss} = Z_\beta$ consists of the single configuration in which 0 has multiplicity $r$ and $\infty$ has multiplicty $n-r$. The stratum $S_\beta$ consists of those configuations with a point of multiplicity precisely $r$ and has codimension $r-1$ in $\mathbb{P}_n$.

Thus the stratifications of $(\mathbb{P}_1)^n$ and $\mathbb{P}_n$ correspond under the quotient map $h : (\mathbb{P}_1)^n \to \mathbb{P}_n$. However the moment maps do not correspond. This reflects the fact that the symplectic structure is not preserved by $h$. The Kähler form on $\mathbb{P}_n$ pulls back via $h$ to a form on $(\mathbb{P}_1)^n$ which is symplectic except that it degenerates along a subset of positive codimension. In faact, such forms give moment maps in the same way as nondegenerate ones. Thus we have two different moment maps on $(\mathbb{P}_1)^n$ which induce the same stratification.

When $n$ is odd the stabilizers of all semistable points are finite so there is a singular projective quotient $M = \mathbb{P}_n^{ss}/\mathrm{SL}(2)$ such that

$$\begin{aligned}
P_t(M) &= (1+t^2+\ldots+t^{2n})(1-t^4)^{-1} - \sum_{n \geq r > \frac{n}{2}} t^{2(r-1)}(1-t^2)^{-1} \\
&= (1-t^2)^{-1}(1+t^4+\ldots+t^{n-1}-t^{n+1}-\ldots-t^{2(n-1)}) \\
&= 1+t^2+2t^4+2t^6+3t^8+\ldots+\left[1+\frac{1}{2}\min(j, n-3-j)\right]t^{2j}+\ldots+t^{2n-6}
\end{aligned}$$

### 16.3. **Products of Grassmannians**

This example is a generalization of 16.1. If $V$ is a complex vector space let $G(\ell, V)$ be the Grassmannian of $\ell$-dimensional linear subspaces of $V$ or equivalently of $(\ell-1)$-dimensional linear subspaces of the projective space $\mathbb{P}(V)$. We can embed $G(\ell, V)$ in $\mathbb{P}(\Lambda^\ell V)$ by using Plücker coordinates. Thus any product of the form

$$X = G(\ell_1, \mathbb{C}^m) \times \ldots \times G(\ell_r, \mathbb{C}^m)$$

can be embedded as a subvariety of the projective space

$$\mathbb{P}\left(\bigotimes_{1\le j\le r} \Lambda^{\ell_j}\mathbb{C}^m\right)$$

Since the central one-parameter subgroup of $\mathrm{GL}(m)$ acts trivially on

$$\bigotimes_{1\le j\le r} \Lambda^{\ell_j}\mathbb{C}^m$$

the stratification of $X$ arising from this action of $\mathrm{GL}(m)$ coincides wtih the $\mathrm{SL}(m)$ stratification except that a stratum labelled by $\beta$ for $\mathrm{GL}(m)$ is labelled in the $\mathrm{SL}(m)$ stratification by the projection

$$\beta - (n+1)^{-1}\left(\sum_{1\le j\le r} \ell_j\right)(1,\ldots,1)$$

of $\beta$ onto $\mathfrak{su}(m)$.

By [M] or [N] 4.17 a sequence of subspaces $(L_1,\ldots,L_r) \in X$ is semistable for $\mathrm{SL}(M)$ iff

$$\sum_{1\le j\le r} (\dim\ L_j \cap M)/(\dim\ M) \le \sum_{1\le j\le r} \ell_j/m$$

for every proper subspace $M \subset \mathbb{C}^m$. The stabiliser of $(L_1,\ldots,L_r)$ is finite if strict inequality always holds. Therefore if $\sum \ell_j$ is coprime to $m$, every semistable point of $X$ has finite stabilizer and so theorem 8.12 will give us a formula for the Betti numbers of the quotient variety.

Suppose that $(L_1,\ldots,L_r)$ is not semistable. Let $\mathbf{M}$ be the set of proper subspaces $M$ of $\mathbb{C}^m$ such that the ratio

$$\sum_{1\le j\le r} (\dim L_j \cap M)/(\dim M)$$

is maximal. Then by 16.5 for each $M \in \mathbf{M}$ the sequence $(L_1 \cap M,\ldots,L_r \cap M)$ is semistable in

$$\prod_{1\le j\le r} G(\dim L_j \cap M, M)$$

Let $M_1$ be a maximal element of $\mathbf{M}$. If $M \in \mathbf{M}$ it is easy to check that $M + M_1 \in \mathbf{M}$ and hence $M \subset M_1$ by the maximality of $M_1$. In particular, $M_1$ is uniquely determined.

By induction we find that any $(L_1,\ldots,L_r) \in X$ determines a unique sequence

$$0 = M_0 \subset M_1 \subset \ldots \subset M_s = \mathbb{C}^m$$

of subspaces satisfying the following conditions.

**16.7**(a). The sequence $(L_{i1},\ldots,L_{ir})$ is semistable in

$$\prod_{1\le j\le r} G(\ell_{ij}, M_i/M_{i-1})$$

for $1\le i\le s$ where

$$L_{ij} = (L_j\cap M_i + M_{i-1})/M_{i-1}$$

is the image of $L_j$ in $M_i/M_{i-1}$ and $\ell_{ij} = \dim L_{ij}$.

(b) Each $M_i$ is maximal among subspaces with property (a).

(c)

$$\sum_{1\le j\le r} (\dim L_j\cap M_i)/\dim M_i) > \sum_{1\le j\le r} (\dim L_j\cap M_{i-1})/\dim M_{i-1})$$

for $1\le i\le s$ or equivalently

$$k_1/m_1 > k_2/m_2 > \ldots > k_s/m_s$$

where

$$k_i = \sum_{1\le j\le r} \ell_{ij}$$

and $m_i = \dim M_i/M_{i-1}$.

**Remark 16.8**. The equivalence in (c) comes from the fact that if $a,b,c,d>0$ then $a/b<c/d$ if $(a+c)/(b+d)<c/d$.

Let $T_c$ be the complex maximal torus of $\mathrm{GL}(m)$ consisting of the diagonal matrices and let $T=T_c\cap \mathrm{U}(m)$ Denote by $\mathfrak{t}_+$ the standard positive Weyl chamber in the Lie algebra of $T$.

**Proposition 16.9**. Suppose $(L_1,\ldots,L_r)\in X$. Let

$$0=M_0\subset M_1\ldots\subset M_s=\mathbb{C}^m$$

be the unique sequence of subspaces of $\mathbb{C}^m$ satisfying 16.7 and let the integers $k_i, m_i$ and $\ell_{ij}$ be defined as at 16.7. Then the stratum of the $\mathrm{GL}(m)$ stratification of $X$ to which $(L_1,\ldots,L_r)$ belongs is labelled by the vector

$$\beta = (k_1/m_1, \ldots, k_1/m_1, k_2/m_2, \ldots, k_s/m_s) \in \mathfrak{t}_+$$

in which $k_i/m_i$ appears $m_i$ consecutive times for each $i$.

Note that for convenience in 16.9 we worked with $\mathrm{GL}(m)$ not $\mathrm{SL}(m)$. However when considering quotients it is better to work with $\mathrm{SL}(m)$. For the central one-parameter subgroup of $\mathrm{GL}(m)$ acts trivially on $X$ and makes every point of $X$ unstable for $\mathrm{GL}(m)$.

This proposition gives us an inductive formula for the equivariant Betti numbers of the semistable stratum in

$$X = \prod_{1 \le j \le r} G(\ell_j, \mathbb{C}^m)$$

under the action of $\mathrm{SL}(m)$. It is

16.10

$$\begin{aligned} P_t^{\mathrm{SL}(m)}(X^{\mathrm{ss}}) &= P_t(X) P_t(B\mathrm{SL}(m)) \\ &- \sum_{\beta,\underline{\ell}} (1-t^2)^{1-s} t^{d(\beta)} \prod_{1 \le j \le s} P_t^{\mathrm{SL}(m_i)} \left( \left( \prod_{1 \le j \le r} G(\ell_{ij}, m_i) \right)^{\mathrm{ss}} \right) \end{aligned}$$

The sum is over vectors $\beta \in \mathfrak{t}_+$ and sequences

$$\ell = \{\ell_{ij} : 1 \le i \le s, 1 \le j \le r\}$$

such that there are integers $k_i \ge 0$ and $m_i > 0$ satisfying

$$k_1/m_1 > \ldots k_s/m_s,$$
$$\sum_i m_i = m, \;\; \sum_j \ell_{ij} = k_i, \;\; \sum_i \ell_{ij} = \ell_j$$

and

$$\beta = (k_1/m_1, \ldots, k_1/m_1, k_2/m_2, \ldots, k_s/m_s)$$

with each $k_i/m_i$ appearing $m_i$ times. Also

$$d(\beta) = \sum_{1 \le i < j \le s} 2(k_i - m_i) m_j$$

The factor $(1-t^2)^{1-s}$ appears since in 16.9 since we worked with $\mathrm{GL}(m)$ not $\mathrm{SL}(m)$.

**Remark 16.11**. In this example it is possible to show that the stratification is equivariantly perfect for any field of coefficients, not just the rationals. The proof is essentially the same as for

$\mathbb{Q}$. It works for all fields because GL($m$) is torsion-free, and because it is possible to find for each $\beta \in \mathcal{B}$ a subtorus $T_\beta$ which fixes $Z_\beta$ pointwise and whose action on $T_x S_\beta$ is $\mathbb{Z}$-primitive, not just $\mathbb{Q}$-primitive, for each $x \in Z_\beta$ (cf. [A&B] theorem 7.14). We deduce that the GL($m$)-equivariant cohomology of the semistable stratum has no torsion. Since PGL($m$) acts freely on the semistable points, it follows from considering spectral sequences that the quotient variety has $p$-torsion for the same primes $p$ as GL($m$), that is, for $p \leq m$.

### 16.12. Ordered points in a projective plane

As a special case of the last example consider the diagonal action of SL(3) on $(\mathbb{P}_2)^n$. The first value of $n$ for which the quotient is interesting is $n = 5$. Then $3 \not| n$ so by 16.6 the stabilizer of every semistable point is finite.

Suppose $x \in (\mathbb{P}_2)^5$. By 16.5 $x$ is semistable if no point of $\mathbb{P}_2$ occurs in $x$ with multiplicity greater than $n/3$ and no line contains more than $2n/3$ components of $x$. If a point occurs with multiplicity $k > n/3$ and no line in $\mathbb{P}_2$ contains more than $2k$ components then $x$ lies in the stratum labelled for GL(3) by

$$\beta = (k, (5-k)/2, (5-k)/2)$$

If a line contains $k \geq 2n/3$ components then either a point of this line occurs with multiplicity $k_1 > k/2$ so that

$$\beta = (k_1, k - k_1, 5 - k)$$

or else no such point occurs and

$$\beta = (k/2, k/2, 5 - k)$$

So the stratification is given by the following table. The indices $\beta$ here are indices for the GL(3)-stratification; the indices for SL(3) are given by replacing each $\beta$ by $\beta - (5/3, 5/3, 5/3)$.

| $\beta$ | points lying in $S_\beta$ | contribution to $P_t^{\mathrm{SL}(3)}(X)$ |
|---|---|---|
| $(5/3, 5/3, 5/3)$ | semistable for SL(3) | $P_t(X^{\mathrm{ss}})$ |
| $(5, 0, 0)$ | all components coincide | $t^{16}(1-t^2)^{-1}(1-t^4)^{-1}$ |
| $(4, 1, 0)$ | 4 components coincide | $5t^{12}(1-t^2)^{-2}$ |
| $(3, 1, 1)$ | 3 components coincide; | $10t^8(1-t^2)^{-2}$ |
| | others linearly independent | |
| $(5/2, 5/2, 0)$ | all components lie in a line; at most 2 coincide | $t^6(1-t^2)^{-1}(1+5t^2+t^4)$ |
| $(3, 2, 0)$ | all components lie in a line; at most 3 coincide | $10t^{10}(1-t^2)^{-2}$ |
| $(2, 2, 1)$ | 4 components lie in a line; at most 2 coincide | $5t^4(1-t^2)^{-2}(1+3t^2-t^4)$ |
| $(2, 3/2, 3/2)$ | 2 components coincide; no 4 lie in a line | $10t^4(1-t^2)^{-1}$ |

By applying 16.10 we obtain the Betti numbers of the quotient $M = X^{\rm ss}/{\rm SL}(3)$. The Poincaré series of $M$ is

$$\begin{aligned} P_t(M) &= (1+t^2+t^4)^5(1-t^4)^{-1}(1-t^6)^{-1} - t^{16}(1-t^2)^{-1}(1-t^4)^{-1} \\ &- (1-t^2)^{-2}\{5t^{12}+10t^8+10t^{10}+5t^4(1+3t^2-t^4)\} \\ &- (1-t^2)^{-1}\{t^6+5t^8+t^{10}+10t^4\} \end{aligned}$$

which works outs as $1+5t^2+t^4$. Here the inductive formula 16.10 was used rather than an explicit formula involving $\beta$-sequences. The former was quicker because the Poincaré series $P_t^{\rm SL(2)}(((\mathbb{P}_1)^n)^{\rm ss})$ have already been calculated.

When $n=6$ we are no longer in a good case and the series $P_t^{\rm SL(3)}(X^{\rm ss})$ is not polynomial. When $n=7$

$$P_t(M) = 1+7t^2+29t^4+64t^6+29t^8+7t^{10}+t^{12}$$

In general one finds that if $3 \not| n$ the Betti numbers of the quotient $M$ for the action of SL(3) on $(\mathbb{P}_2)^n$ are given by

$$b_{2j} = a_j + 2a_{j-1} + \ldots + (j+1)a_0$$

for $0 \le j \le 2(n-5)$ where $a_d$ is given by

$$\begin{aligned} a_d &= \sum_{0\le b\le d/2} \frac{(n-2)!}{(d-2b)!b!(n-2-d+b)!}\left\{b+1-\frac{n(n-1)}{b+1}\left[\frac{\chi_1(b)-\chi_2(b)}{n-b-1}+\frac{\chi_3(b)}{n-d-b-1}\right]\right\} \\ &- \sum_{n/3\le k\le n} \frac{n!(\chi_4(k)-\chi_5(k))}{k(n-k)!(d-n-k+2)!(d+1)!} \end{aligned}$$

and $\chi_i$ are the characteristic functions of the intervals

1. $[\max(n/3, d-n, d/2]$
2. $[n/3, \min(d/2, (2d-n+1)/3)]$
3. $[\max(d+1-n, (d-1)/3), \min(d+1-2n/3, d/2-1]$
4. $[n-d-2, 2(n-d-2)]$
5. $[2(n-d-2), n]$

# References

[A2] M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. 14, 1-15 (1982)

[B] A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969

[Bo] R. Bott, Nondegenerate critical manifolds, Annals of Math. 60, 248-261 (1954)

[C & G] J. B. Carell and R. M. Goresky, A decomposition theorem for the integral cohomology of a variety, preprint

[D] A. Dold, *Lectures on algebraic topology*, Springer-Verlag Berlin-Heidelberg, 1972

[G & H] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978

[H] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964

[Ha] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977

[M] D. Mumford, *Geometric invariant theory*, Springer-Verlag, New York, 1965

[Ki] Kirwan, Sur la cohomologie des espaces quotients, C. R. Acad. Sci. Paris, **295**, Serie I, 1982

[Ki2] Kirwan, Desingularizations of quotients of nonsingular varieties and their Betti numbers, preprint

[Ki3] Kirwan, On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles, preprint

[C&S] Carrell, Sommese, Some topological aspects of $\mathbb{C}^\star$-actions on compact Kähler manifolds, Comment. Math. Helevicti **54**, 1979

[C&G] Carrell, Goresky, A decomposition theorem for the integral homology of a variety, preprint

[Bi] Birkes, Orbits of linear algebraic groups, Annals of Math. **93**, 1971

[B-B] Bialynicki-Birula, Some theorems on actions of algebraic groups, Annals of Math., **98**, 1973

[Hes] Hesselink, Uniform instability in reductive groups, J. Reine Angew. Math. **304**, 1978

[H&N] Harder and Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, Math. Annalen **212**, 1975

[K] Kempf, Instability in invariant theory, Annals of Math. **108**, 1978

[N] Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute Lectures **51**, Springer-Verlag, Heidelberg, 1978

[M] Mumford and Fogarty, *Geometric invariant theory*, Springer-Verlag, Berlin-New York, 1982

[Ne] Ness, A stratification of the null cone via the moment map, Amer. J. Math.

[L] Lang, Algebraic groups over finite fields, Amer. J. Math. **78**, 1956