# COLLAPSING THREE-MANIFOLDS UNDER A LOWER CURVATURE BOUND 

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#### Abstract

The purpose of this paper is to completely characterize the topology of threedimensional Riemannian manifolds with a uniform lower bound of sectional curvature which converges to a metric space of lower dimension.


## 0. Introduction

We study the topology of three-dimensional Riemannian manifolds with a uniform lower bound of sectional curvature converging to a metric space of lower dimension.

Given a positive integer $n$ and $D>0$, let us consider the set $\mathcal{M}(n, D)$ of isometry classes of $n$-dimensional closed Riemannian manifolds $M$ with sectional curvature $K \geq-1$ and diameter $\operatorname{diam}(M) \leq D$. By the Gromov Precompactness Theorem [16], the closure of $\mathcal{M}(n, D)$ is compact with respect to the Gromov-Hausdorff distance. Thus any sequence $M_{i}, i=1,2, \ldots$, in $\mathcal{M}(n, D)$ has a convergent subsequence whose limit is a compact Alexandrov space $X$ of dimension $\leq n$ and curvature $\geq-1$. We now assume that $M_{i}$ itself converges to $X, i$ is sufficiently large, and consider the following:

Problem 0.1. Describe the topological structure of $M_{i}$ by using the geometry and topology of $X$.

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Some answers are known in the extremal cases: If $\operatorname{dim} X=0$, the fundamental group of $M_{i}$ is almost nilpotent (Fukaya and Yamaguchi [13]) and if $\operatorname{dim} X=n, M_{i}$ is homeomorphic to $X$ (Perelman [26, 27], cf. Grove, Petersen and Wu [19]). In particular, for the above problem it suffices to consider only the case of $\operatorname{dim} X \leq n-1$, the so called collapsing case.

If $X$ has no singular points, then $X$ is a $C^{0}$-Riemannian manifold (Otsu and Shioya [25]), and the Fibration Theorem (Yamaguchi [37]) implies that $M_{i}$ is a fibre bundle over $X$ with almost nonnegatively curved fibre. Actually the Fibration Theorem still holds if $X$ has only 'weak' singularities ([38]) in some sense. According to Perelman ([28]), it is also known that if $X$ has no 'bad' singularities (precisely called extremal subsets), there is an isomorphism $\pi_{k}\left(M_{i}, F_{i}\right) \simeq \pi_{k}(X)$ for homotopy groups, where $F_{i}$ is a 'general fibre' and $i$ is large enough compared with $k$.

When $\operatorname{dim} X \leq n-1$ and $X$ may have 'bad' singularities, no solution to Problem 0.1 is known as of now. In this paper we completely solve it in the case when $n=3$ and $\operatorname{dim} X=1$ or 2 . Note that if $n=3$ and $\operatorname{dim} X=0$ (i.e., $X$ is a single point), it has already been obtained in [13] that some finite cover of $M_{i}$ is homeomorphic to either a homotopy sphere, $S^{1} \times S^{2}, T^{3}$, or a nilmanifold.

From now on, we assume that each element $M_{i} \in \mathcal{M}(3, D)$ in the sequence is orientable and $i$ is sufficiently large. We first state our main results in the case of $\operatorname{dim} X=2$. Recall that $X$ is a topological manifold possibly with boundary in this case.

Theorem 0.2. If $\operatorname{dim} X=2$ and $X$ has no boundary, then $M_{i}$ is homeomorphic to a Seifert fibred space over $X$, for which the orbit type $(\mu, \nu)$ of the singular fibre over a point $p \in X$ satisfies $\mu \leq 2 \pi / L\left(\Sigma_{p}\right)$.

Here, $L\left(\Sigma_{p}\right)$ is the length of the space of directions, $\Sigma_{p}$, at $p$. Observe that every fibre in $M_{i}$ shrinks to a point.

Theorem 0.3. If $\operatorname{dim} X=2$ and $X$ has nonempty boundary, then there is a Seifert fibred space $\operatorname{Seif}_{i}(X)$ over $X$ such that:
(1) $M_{i}$ is homeomorphic to the union $\operatorname{Seif}_{i}(X) \cup\left(\partial X \times D^{2}\right)$ glued along their boundaries, where the fibres of $\operatorname{Seif}_{i}(X)$ over boundary points $x \in \partial X$ are identified with $\{x\} \times \partial D^{2}$.
(2) the orbit type $(\mu, \nu)$ of the singular fibre of $\operatorname{Seif}_{i}(X)$ over a point $x \in \operatorname{int} X$ satisfies $\mu \leq 2 \pi / L\left(\Sigma_{x}\right)$.

It should be noted that the Euler characteristic of $X$ and the number of singular fibres of $M_{i}$ in Theorems 0.2 and 0.3 are estimated by a constant depending only on the upper diameter bound $D$ (see Remark 4.6). Observe that $\partial X \times D^{2} \subset M_{i}$ collapses to $\partial X$.

We have the following corollary of Theorem 0.3.
Corollary 0.4. Let $M_{i}, X$, and $\operatorname{Seif}_{i}(X)$ be as in Theorem 0.3, and let $g$ and $k$ denote the genus of $X$ and the number of components of $\partial X$ respectively. Then we have the following factorization:

$$
M_{i} \simeq S^{3} \# \underbrace{S^{2} \times S^{1} \# \cdots \# S^{2} \times S^{1}}_{f(g, k)} \# L\left(\mu_{1}, \nu_{1}\right) \# \cdots \# L\left(\mu_{\ell}, \nu_{\ell}\right),
$$

where

$$
f(g, k)= \begin{cases}2 g+k-1 & \text { if } X \text { is orientable }, \\ g+k-1 & \text { if } X \text { is non-orientable },\end{cases}
$$

and $\left(\mu_{j}, \nu_{j}\right), 1 \leq j \leq \ell$, denote all the orbit types of singular fibres of $\operatorname{Seif}_{i}(X)$, and $L\left(\mu_{j}, \nu_{j}\right)$ the lens space of type $\left(\mu_{j}, \nu_{j}\right)$.

Notice here that $f(g, k)$ and $\ell$ are both bounded by some constant depending only on $D$. It also follows from Theorem 0.3 and Corollary 0.4 that the set of homeomorphism types of $\left\{M_{i}\right\}$ collapsing to a two-dimensional Alexandrov space with boundary is finite.

Let us next consider the case when $\operatorname{dim} X=1$, i.e., $X$ is isometric to either a circle or a closed interval. When $X$ is isometric to a circle, we readily observe from the Fibration Theorem [37] that $M_{i}$ is a fibre bundle over $S^{1}$ whose fibre is either $S^{2}$ or $T^{2}$. The rest is to investigate the case when $X$ is isometric to a closed interval. We denote by Mö $\tilde{\times} S^{1}$ a twisted $S^{1}$-bundle over a Möbius band (see Section 5 for the definition of twisted bundles).

Theorem 0.5. Assume that $X$ is isometric to a closed interval. Then $M_{i}$ is homeomorphic to a gluing of $B$ and $C$ along their boundaries, where $B$ and $C$ are respectively either $D^{3}$, $P^{3}-\operatorname{int} D^{3}, S^{1} \times D^{2}$ or $M \ddot{o} \tilde{\times} S^{1}$.

Note that there are six combinations for the choice of $B$ and $C$ with a number of gluing $B \cup C$. If we express $M_{i}$ in Theorem 0.5 as $M_{i}=B \cup A \cup C$, where $A=\partial B \times[0, \ell]$, so that as $i \rightarrow \infty, A$, $B$ and $C$ collapse to $[0, \ell],\{0\}$ and $\{\ell\}$ respectively, we call $(B, C)$ the collapsing data of the collapsing $M_{i} \rightarrow[0, \ell]$. For more concrete topological information of $M_{i}$, see Table 1 in Section 8.

Every prism manifolds can be written as a gluing $S^{1} \times D^{2} \cup \mathrm{Mö} \tilde{\times} S^{1}$, and there is an infinite sequence of pairwise non-homeomorphic prism manifolds with constant curvature $K=1$ which collapses to a closed interval with the collapsing data ( $S^{1} \times D^{2}$, Mö $\tilde{x} S^{1}$ ) (Example 7.1). It is unclear if a fixed prism manifold admits a sequence of metrics collapsing to a closed interval under $K \geq-1$.

Theorem 0.6. Let $(M, X)$ be one of the following:
(1) $X$ is a compact surface without boundary, and $M$ a Seifert fibred space over $X$.
(2) $X$ is a compact surface with boundary, and $M$ the union of $a$ Seifert fibred space over $X$ and $\partial X \times D^{2}$ glued as in Theorem 0.3.
(3) $X$ is a closed interval, and $M$ any gluing of $B$ and $C$, where $(B, C)$ is any of the six possible choices as in Theorem 0.5. Suppose that $M$ is not a prism manifold in this case.
Then there exist a sequence of Riemannian metrics $g_{i}$ on $M$ and $a$ smooth orbifold metric $g$ on $X$ such that $\left(M, g_{i}\right)$ collapses to $(X, g)$ under $K \geq-1$.

Combining Theorems $0.2,0.3$ and 0.5 improves the result of [13] in the case $\operatorname{dim} X=0$ previously stated in the following way.

Corollary 0.7. Suppose that $X$ is a point. Then a finite cover of $M_{i}$ is homeomorphic to $S^{1} \times S^{2}, T^{3}$, a nilmanifold or a simply connected Alexandrov space with nonnegative curvature.

Conjecture 0.8. Any three-dimensional compact, simply connected, nonnegatively curved Alexandrov space without boundary which is a topological manifold is homeomorphic to a sphere.

If Conjecture 0.8 is solved, everything will be clear about Problem 0.1 for $n=3$. The above conjecture is certainly true in the Riemannian case ([20]).

It is known by Thurston (cf. [35]) that there are eight geometric structures of three-manifolds modelled on $S^{3}, \mathbb{R}^{3}, H^{3}, S^{2} \times \mathbb{R}^{1}, H^{2} \times \mathbb{R}$, $\widetilde{S L_{2}}(\mathbb{R}), N i l$ and Sol.

Corollary 0.9. For any $D>0$, there exists a constant $\epsilon=\epsilon(D)>$ 0 such that if a closed, prime three-manifold with infinite fundamental group admits a Riemannian metric contained in $\mathcal{M}(3, D)$ with volume $<\epsilon$, then it admits a geometric structure modelled on one of the seven geometries except $H^{3}$.

Observe that a hyperbolic manifold $M$ does not collapse under $K \geq$ -1 because of the non-vanishing property $\|M\| \neq 0$ for simplicial volume (Gromov [17], Thurston [34]).

Combining our results above, we obtain the following corollary on the existence of geometric structures for the elements of $\mathcal{M}(3, D)$.

Corollary 0.10. All elements but finitely many homeomorphism classes in $\mathcal{M}(3, D)$ admit geometric structures.

In the proofs of our results, we essentially use a critical pointrescaling argument to understand the topology of a small neighborhood of $M_{i}$ converging to a small neighborhood of a singular point of $X$. When $\operatorname{dim} X=2$, as the limit space of the rescaled $M_{i}$, we have a three-dimensional complete open nonnegatively curved Alexandrov space $Y$ which is a topological manifold. Here an Alexandrov space is called open if it is noncompact and without boundary. It is significant to determine the topology of such a space $Y$ by using its soul $S$.

Theorem 0.11. Let $Y$ be a three-dimensional complete open Alexandrov space of nonnegative curvature. Suppose that $Y$ is a topological manifold. Then $Y$ is homeomorphic to the normal bundle $N(S)$ of the soul $S$ of $Y$.

This extends the Cheeger-Gromoll Soul Theorem [9] in dimension three. Actually we classify all the three-dimensional complete open Alexandrov spaces with nonnegative curvature which are not necessarily topological manifolds (Theorem 9.6). This seems to be of independent interest.

The organization of this paper is as follows: In Section 1, we sketch the essential idea of the proofs of Theorems $0.2,0.3$ and 0.5 .

In Section 2, we present some basic notions and results on Alexandrov spaces needed in the subsequent sections.

The main body of this paper consists of two parts. In Part 1, we discuss the collapsing of three-dimensional Riemannian manifolds by assuming the Generalized Soul Theorem, which is proved in Part 2.

In Section 3, we prove a key lemma, which is the important first step to understand the topology of a small neighborhood of a point of $M_{i}$ converging to a singular point of $X$.

The proofs of Theorems $0.2,0.3$ and 0.5 are given in Sections 4, 5 and 6 respectively. We also prove Corollaries $0.7,0.9$ and 0.10 in Section 6.

In Section 7, we construct collapsing metrics with a lower curvature bound on three-manifolds to prove Theorem 0.6.

In Section 8, we discuss the bounded curvature collapsing of threemanifolds with some construction of collapsing metrics, and compare our main results with them.

In Part 2, we classify all the three-dimensional complete open Alexandrov spaces $Y$ with nonnegaive curvature (the Generalized Soul Theorem). First in Section 9, we state the main results in Part 2 with some examples, and prove the rigidity part of the Generalized Soul Theorem.

We essentially use the topological Morse theory for distance functions in the proof of non-rigidity part. We divide the proof into the two cases, depending on the dimension of the minimum set $C$ of a Busemann function on $Y$. After some preliminary arguments in Sections 10 and 11, we prove the non-rigidity part of the Generalized Soul Theorem in the case of $\operatorname{dim} C=2$ in Section 12. The case of $\operatorname{dim} C=1$ is proved in Section 13.

In Appendix, we discuss the total curvature, the Gauss-Bonnet Theorem and the Cohn-Vossen Theorem for Alexandrov surfaces, and give a classification of nonnegaitvely curved Alexandrov surfaces. Those are needed in the proof of Theorem 0.5.

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## 1. Idea of proofs

Let a sequence of three-dimensional closed orientable Riemannian manifolds $M_{i}$ with $K \geq-1$ and $\operatorname{diam}\left(M_{i}\right) \leq D$ collapse to a compact Alexandrov space $X$ with $\operatorname{dim} X=1$ or 2 . Assume for simplicity that $i$ is always large enough.

Let us first consider the most basic case when $\operatorname{dim} X=2$ and $X$ has no boundary. For a sufficiently small fixed $\epsilon>0$, take the singular points $x_{j}$ of $X$ with $L\left(\Sigma_{x_{j}}\right) \leq 2 \pi-\epsilon$. There are only finitely many such points $x_{j}$, say $1 \leq j \leq k$. For a sufficiently small $r>0$, we consider $X^{\prime}=X-\left(B\left(x_{1}, r\right) \cup \cdots \cup B\left(x_{k}, r\right)\right)$, where $B(x, r)$ denotes the closed $r$-ball around $x$. Applying the Fibration Theorem (Theorem 2.2) to $X^{\prime}$, we have a domain $M_{i}^{\prime} \subset M_{i}$ converging to $X^{\prime}$ which is a circle bundle
over $X^{\prime}$. We here need some new idea to determine the topology of the components $B_{i j}$ of $M_{i}-M_{i}^{\prime}$ converging to $B\left(x_{j}, r\right)$.

For simplicity we fix a $j$ and set $B_{i}=B_{i j} \subset M_{i}, x=x_{j}$ and $B=B(x, r) \subset X$. To investigate the topology of $B_{i}$, we want to measure the diameter, say $\delta_{i}$, of the 'fibres' of the convergence $B_{i} \rightarrow B$ and consider the convergence of the rescaled pointed manifolds $\left(\frac{1}{\delta_{i}} M_{i}, p_{i}\right)$ as $i \rightarrow \infty$, where $p_{i}$ is the center of $B_{i}$. If this convergence does not collapse, then $\frac{1}{\delta_{i}} B_{i}$ should be homeomorphic to its limit which is a nonnegatively curved complete open Alexandrov space, which could be characterized by generalizing the Soul Theorem.

However, the main problem here is the difficulty to measure the diameter $\delta_{i}$ of the 'fibres'. To find $\delta_{i}$, we shift the point $p_{i}$ slightly to a 'peak' $\hat{p}_{i}$ of $B_{i}$, precisely a point where the average of the distance to all points on $\partial B_{i}$ takes a local maximum. By shifting $p_{i}$, the ball $B_{i}$ isotopically moves, so that the topology does not change. Because of the boundary condition $\partial B_{i} \simeq T^{2}$, there are critical points in $B_{i}$ of the distance function from $\hat{p}_{i}$. We define $\delta_{i}$ to be the furthest distance from $\hat{p}_{i}$ to the critical points in $B_{i}$. It then follows that $\delta_{i} \rightarrow 0$.

By passing to a subsequence, the sequence of the rescaled pointed manifolds ( $\frac{1}{\delta_{i}} M_{i}, \hat{p}_{i}$ ) converges to a pointed noncompact nonnegatively curved Alexandrov space ( $Y, y_{0}$ ). As a crucial lemma (Key Lemma 3.6), we prove that $\operatorname{dim} Y=\operatorname{dim} X+1=3$. Since the convergence $\left(\frac{1}{\delta_{i}} M_{i}, \hat{p}_{i}\right) \rightarrow$ ( $Y, y_{0}$ ) does not collapse, a discussion using Perelman's Stability Theorem (Theorem 2.4) shows that int $B_{i} \simeq Y$.

The next step is to establish the Generalized Soul Theorem for threedimensional Alexandrov spaces to determine the topology of $Y$. For general Alexandrov space, the Soul Theorem as in the Riemannian case does not hold. This happens essentially because of the appearance of topological singular points. We prove the Soul Theorem as in the Riemannian case for three-dimensional complete open nonnegaitvely curved Alexandrov space which is a topological manifold, by generalizing the notion of gradient flows of distance functions with the use of the topological Morse theory.

Applying the Soul Theorem to $Y$ together with the boundary condition $\partial B_{i} \simeq T^{2}$, we conclude that the soul of $Y$ is isometric to a circle, and hence $B_{i} \simeq S^{1} \times D^{2}$. Finally, we put a structure of fibred solid torus on $B_{i}$ which is compatible to the circle bundle structure on $M_{i}^{\prime}$. Thus we obtain the Seifert bundle structure on $M_{i}$ over $X$, and conclude Theorem 0.2.

We summarize the above discussion as follows:
(1) For the almost regular part of $X$, we use Fibration Theorem 2.2 to obtain the circle bundle structure on $M_{i}^{\prime} \subset M_{i}$.
(2) For singular points of $X$, we use the flow chart:
the rescaling argument (Key Lemma 3.6)
$\downarrow$
Stability
$\downarrow$
$\downarrow$
the Generem 2.4
Generalized Theorem
to obtain the topology of a small neighborhood $B_{i} \subset M_{i}$ near a singular point of $X$.
(3) We put a fibred solid torus structure on $B_{i}$ and finally check the compatibility.

Next we consider the case when $\operatorname{dim} X=2$ and $X$ has non-empty boundary. For a sufficiently small fixed $\epsilon>0$, take the boundary points $x_{j} \in \partial X$ with $L\left(\Sigma_{x_{j}}\right) \leq \pi-\epsilon$. There are only finitely many such points $x_{j}$, say $1 \leq j \leq \ell$. For a sufficiently small $r>0$ and $\delta \ll r$, we decompose $X$ into three kinds of parts: $B\left(x_{j}, r\right), 1 \leq j \leq \ell, H=$ $B(\partial X, \delta)-\operatorname{int}\left(B\left(x_{1}, r\right) \cup \cdots \cup B\left(x_{\ell}, r\right)\right)$, and $X^{\prime}$, the complement of int $H$ and int $B\left(x_{j}, r\right)$. Corresponding to these parts, we decompose $M_{i}$ into three kinds of parts $B_{i j}, H_{i}$, and $M_{i}^{\prime}$ which are respectively GromovHausdorff close to $B\left(x_{j}, r\right), H$, and $X^{\prime}$. Applying Theorem 0.2 to $X^{\prime}$, we obtain a Seifert bundle structure on $M_{i}^{\prime}$ over $X^{\prime}$. The generalized Margulis lemma ([13]) implies that $H_{i}$ is homeomorphic to $D^{2} \times(\partial X \cap$ $H)$. Finally by using the critical point-rescaling argument (2) above with the boundary condition $\partial B_{i j} \simeq S^{2}$, we conclude that $B_{i j} \simeq D^{3}$. Combining those topological information, we obtain Theorem 0.3.

When $X$ is isometric to a closed interval $I$, we decompose $M_{i}$ into three parts $A_{i}, B_{i}$, and $C_{i}$, where $B_{i}$ and $C_{i}$ are two metric balls close to the two endpoints of $I$ respectively. The Fibration Theorem implies that $A_{i} \simeq F_{i} \times I$, where $F_{i}$ is either $S^{2}$ or $T^{2}$. To investigate the topologies of $B_{i}$ and $C_{i}$, we use the same discussion as (2) above. However, we only have $3 \geq \operatorname{dim} Y \geq \operatorname{dim} X+1=2$ in this case. If $\operatorname{dim} Y=2$, we apply the critical point-rescaling argument (2) to the convergence $\frac{1}{\delta_{i}} B\left(p_{i}, R \delta_{i}\right) \rightarrow B\left(y_{0}, R\right)$ with large $R>0$ instead of $B_{i} \rightarrow B$. This determines the topology of $B_{i}$ and $C_{i}$ and proves Theorem 0.5.

## 2. Preliminaries

In this section, we present some results on Alexandrov spaces and the Gromov-Hausdorff convergence related with Alexandrov spaces, which will be needed in the subsequent sections. We refer to [4] for the basic materials and the details of the results on Alexandrov spaces mentioned below.

First we give some basic definitions and notations. Let $X$ be a geodesic space in the sense that every two points can be joined by a minimal geodesic. We assume that all geodesic have unit speed unless otherwise stated. For a fixed real number $\kappa$ and a geodesic triangle $\Delta x y z$ in $X$ with vertices $x, y$ and $z$, we denote by $\tilde{\Delta} x y z$ a comparison triangle in the simply connected complete surface $M_{\kappa}$ with constant curvature $\kappa$. This means that each side length of $\tilde{\Delta} x y z$ is equal to the corresponding one of $\Delta x y z$. Here we suppose that the perimeter of $\Delta x y z$ is less than $2 \pi / \sqrt{\kappa}$ if $\kappa>0$. We say that an open set $U$ in $X$ satisfies the Alexandrov convexity if for any geodesic triangle in $U$ with vertices $x, y$ and $z$ and for any point $w$ on the geodesic segment $y z$ joining $y$ to $z$, we have $d(x, w) \geq d(\tilde{x}, \tilde{w})$, where $\Delta \tilde{x} \tilde{y} \tilde{z}=\tilde{\Delta} x y z, \tilde{w}$ is the point on $\tilde{y} \tilde{z}$ corresponding to $w$. The space $X$ is called an Alexandrov space with curvature $\geq \kappa$ if each point of $X$ has a neighborhood satisfying the Alexandrov convexity. Actually it is known that the whole space $X$ satisfies the Alexandrov convexity. From now on we assume that $X$ is of finite dimension.

The angle between the geodesics $x y$ and $y z$ in $X$ is denoted by $\angle x y z$, and the corresponding angle of $\tilde{\Delta} x y z$ by $\tilde{\angle x y z}$. It holds that $\angle x y z \geq \tilde{\angle} x y z$. We denote by $\Sigma_{p}=\Sigma_{p}(X)$ the space of directions at $p \in X$, and by $K_{p}=K_{p}(X)$ the tangent cone at $p$ with vertex $o_{p}$, the Euclidean cone $K\left(\Sigma_{p}\right)$ over $\Sigma_{p}$. It is known that $\Sigma_{p}$ is an Alexandrov space with curvature $\geq 1$.

For a compact set $A \subset X$ and $p \in X-A$, we denote by $A^{\prime}=A_{p}^{\prime}$ the closed set of $\Sigma_{p}$ consisting of all the directions of minimal geodesics from $p$ to $A$. Let us now consider the distance function $d_{x}(\cdot)=d(x, \cdot)$ from $x$. A point $p$ is called a critical point of $d_{x}$ if $\tilde{\angle} x p y \leq \pi / 2$, or equivalently $\angle\left(x_{p}^{\prime}, y_{p}^{\prime}\right) \leq \pi / 2$, for every $y \in X-\{p\}$. Otherwise $p$ is a regular point of $d_{x}$.

A (not necessarily continuous) map $\varphi: Y \rightarrow Z$ between metric spaces is called an $\epsilon$-approximation if:
(1) $|d(x, y)-d(\varphi(x), \varphi(y))|<\epsilon \quad$ for all $x, y \in Y$.
(2) $\varphi(Y)$ is $\epsilon$-dense in $Z$.

The Gromov-Hausdorff distance $d_{G H}(Y, Z)$ between $Y$ and $Z$ is defined to be the infimum of such $\epsilon$ that there exist $\epsilon$-approximations $Y \rightarrow Z$ and $Z \rightarrow Y$. We say that pointed spaces $\left(X_{i}, x_{i}\right)$ converge to $(X, x)$ with respect to the pointed Gromov-Hausdorff topology if the metric balls $B\left(x_{i}, R_{i} ; X_{i}\right)$ converge to $B(x, R ; X)$ with respect to the Gromov-Hausdorff distance for any $R>0$ and some monotone nonincreasing sequence $R_{i} \rightarrow R$. We recall that $K_{p}$ is isometric to the pointed Gromov-Hausdorff limit of $\left(\frac{1}{\epsilon} X, p\right)$ as $\epsilon \rightarrow 0$.

Let $X$ have dimension $n$, and $\delta>0$. A system of $n$ pairs of points, $\left(a_{i}, b_{i}\right)_{i=1}^{n}$ is called an $(n, \delta)$-strainer at $p \in X$ if it satisfies

$$
\begin{array}{r}
\tilde{\angle} a_{i} p b_{i}>\pi-\delta, \quad \tilde{\angle} a_{i} p a_{j}>\pi / 2-\delta \\
\tilde{\angle} b_{i} p b_{j}>\pi / 2-\delta, \quad \tilde{\angle} a_{i} p b_{j}>\pi / 2-\delta
\end{array}
$$

for every $i \neq j$. The number $\min \left\{d\left(a_{i}, p\right), d\left(b_{i}, p\right) \mid 1 \leq i \leq n\right\}$ is called the length of the strainer.

Let $X_{\delta}$ denote the set of $(n, \delta)$-strained points of $X$. This has the structure of a Lipschitz $n$-manifold. Note that every point in $X_{\delta}$ has a small neighborhood almost isometric to an open subset of $\mathbb{R}^{n}$ for small $\delta$.

The boundary $\partial X$ of $X$ is inductively defined as the set of points $p$ such that $\Sigma_{p}$ has non-empty boundary.

In dimension two, we have the following result ([4], [1]).
Theorem 2.1. Any two-dimensional Alexandrov space $X$ with curvature bounded below is a topological manifold possibly with boundary. Furthermore for every positive number $\delta$, the set of interior points (resp. boundary points) of $X$ at which the length of the space of directions is smaller than $2 \pi-\delta($ resp. $\pi-\delta)$ is discrete.

Next we recall the Fibration Theorem from [37], [38]. Let $X$ be an Alexandrov space. The $\delta$-strain radius at a point $p \in X_{\delta}$ is defined as the supremum of those $r>0$ that there exists an $(n, \delta)$-strainer at $p$ of length $r$. The $\delta$-strain radius of a closed domain $Y \subset X_{\delta}$ is, by definition,

$$
\delta \text {-str. } \operatorname{rad}(Y)=\inf _{p \in Y} \delta \text {-strain radius at } p
$$

The $\delta$-strain radius plays a role similar to the injectivity radius of a Riemannian manifold. We denote by $\tau\left(\epsilon_{1}, \ldots, \epsilon_{k}\right)$ a function depending on a priori constants and $\epsilon_{i}$ satisfying $\lim _{\epsilon_{i} \rightarrow 0} \tau\left(\epsilon_{1}, \ldots, \epsilon_{k}\right)=0$.

Theorem 2.2 (Fibration Theorem [38]). Given $n$ and $\mu>0$ there exist positive numbers $\delta=\delta_{n}$ and $\epsilon=\epsilon_{n}(\mu)$ satisfying the following : Let $Y \subset X_{\delta} \subset X$ be as above such that $\delta$-str. $\operatorname{rad}(Y)>\mu$. Let $M$ be a complete Riemannian manifold with $K \geq-1$ and suppose that the Gromov-Hausdorff distance $d_{G H}(M, X)<\epsilon$. Then there exists a closed domain $N \subset M$ and a locally trivial fibre bundle $f: N \rightarrow Y$ such that:
(1) It is a $\tau(\delta, \epsilon)$-Lipschitz submersion.
(2) It is a $\tau(\epsilon)$-approximation.

For the definition of $\tau$-Lipschitz submersion, see [38].
Remark 2.3. It is essentially proved in [37] that the first Betti number of the fibre of $f: N \rightarrow Y$ is less than or equal to its dimension, and in [13] that the fundamental group of the fibre is almost nilpotent.

When $X$ is compact, two dimensional and without boundary, one can take as $Y$ the complement of a small neighborhood of the finite set $X-X_{\delta}$. This explains a reason that our methods work in dimension three.

In [38], the general convergence when $M$ is also an Alexandrov space was discussed. Although the theorem above is not stated explicitly in [38], it follows directly from the proof there. However for reader's convenience, we give a sketch of the proof of the above theorem. For the details, see [38].

Let $f_{X}: X \rightarrow L^{2}(X)$ be the embedding of $X$ into the Hilbert space $L^{2}(X)$ defined by using the distance functions from the points of $X$. Since a small neighborhood of each point of $f_{X}\left(X_{\delta}\right)$ can be approximated by an $n$-plane in $L^{2}(X), f_{X}\left(X_{\delta}\right)$ has a normal bundle $\nu$ in a generalized sense. Namely, $\nu$ is a map of $f_{X}\left(X_{\delta}\right)$ into the Grassmann manifold consisting of all subspaces of $L^{2}(X)$ of codimension $n$. This map $\nu$, called a normal bundle of $f_{X}\left(X_{\delta}\right)$, is Lipschitz and $f_{X}\left(X_{\delta}\right)$ has a tubular neighborhood $U$ with respect to $\nu$. The $C^{1}$-map $f_{M}: M \rightarrow L^{2}(X)$ is constructed in a similar way (Remark 4.20 in [38]). Note that $f_{M}(N) \subset U$ for a closed domain $N$ of $M$ with small $d_{G H}(N, Y)$. Thus the map $f=f_{X}^{-1} \circ \pi \circ f_{M}: N \rightarrow Y$ is well defined, where $\pi: U \rightarrow f_{X}\left(X_{\delta}\right)$ is the projection along $\nu$. For $p \in N$, let $T=T_{p}$ be an $n$-plane in $L^{2}(X)$ approximating a small neighborhood of $\pi\left(f_{M}(p)\right)$ in $f_{X}\left(X_{\delta}\right)$. It follows from the proof of Lemma 4.6 in [38] that for every unit vector $\bar{\xi} \in T$ there exists $\xi \in T_{p}(M)$ such that

$$
\left|d f_{M}(\xi)-\bar{\xi}\right|
$$

is small when the given constant $\delta$ and $\epsilon$ are small. This implies that $\pi_{T} \circ f_{M}$ gives a locally trivial fibre bundle on a neighborhood of $p$ over a neighborhood of $\pi_{T} \circ f_{M}(p)$ in $T$, where $\pi_{T}$ denotes the nearest point projection to $T$. Since $\pi: T_{p} \rightarrow f_{X}\left(X_{\delta}\right)$ is homeomorphic on a small neighborhood of $\pi \circ f_{M}(p)\left(\left[38\right.\right.$, Lemma 3.7]), it follows that $\pi \circ f_{M}$ and hence $f$ provides a fibre bundle structure on $N$ over $Y$.

A point $p$ of an Alexandrov space $X$ is called an essential singular point if $\operatorname{rad}\left(\Sigma_{p}\right) \leq \pi / 2$, where

$$
\operatorname{rad}\left(\Sigma_{p}\right)=\min _{\xi \in \Sigma_{p}} \max _{\eta \in \Sigma_{p}} \angle(\xi, \eta)
$$

is the radius of $\Sigma_{p}$. Notice that if a point $p \in X$ is not an essential singular point, then $\Sigma_{p}$ is homeomorphic to a sphere ([18]) and a small metric ball around $p$ is homeomorphic to $\mathbb{R}^{n}([26,27])$, where $n$ is the dimension of $X$. We also say that $p$ is a topological singular point if $\Sigma_{p}$ is not homeomorphic to a sphere.

When no collapsing occurs, we have the following stability result.
Theorem 2.4 (Stability Theorem [26]). Let a sequence of compact n-dimensional Alexandrov spaces $X_{i}$ with curvature $\geq-1$ converge to a compact Alexandrov space $X$ of dimension $n$. Then $X_{i}$ is homeomorphic to $X$ for sufficiently large $i$.

## Part 1. Analyzing collapsed three-manifolds

## 3. Key lemma

Let a sequence of pointed complete $n$-dimensional Riemannian manifolds ( $M_{i}, p_{i}$ ) with $K \geq-1$ converge to a pointed $k$-dimensional Alexandrov space $(X, p)$, where $k \leq n$. In this section, we investigate the topology of the metric ball $B\left(p_{i}, r\right)$ for $i$ large enough compared to a fixed small $r>0$ under some assumption for $p$.

Let $A$ be a metric space and $\epsilon>0$ a number. A discrete subset $N$ of $A$ is called an $\epsilon$-discrete net of $A$ if $d(x, y) \geq \epsilon$ for any $x \neq y \in N$. Set

$$
\beta_{A}(\epsilon)=\max \{\# N \mid N \text { is an } \epsilon \text {-discrete net of } A\} .
$$

An $\epsilon$-discrete net $N$ of $A$ is said to be maximal if $\# N=\beta_{A}(\epsilon)$. If $A$ is a relatively compact open subset of an $n$-dimensional Alexandrov
space, there are two constants $c_{1}$ and $c_{2}$ depending on $A$ such that $0<c_{1} \leq \epsilon^{n} \beta_{A}(\epsilon) \leq c_{2}<\infty$ for any $\epsilon>0$ (see [4]).

Let $\phi_{i}: X \rightarrow M_{i}$ be a $\mu_{i}$-approximation, where $\mu_{i} \rightarrow 0$ as $i \rightarrow \infty$. For any $\epsilon>0$, we take a maximal $\epsilon$-discrete net $\left\{\xi^{j}\right\}_{j=1, \ldots, \beta_{\Sigma_{p}}(\epsilon)}$ of $\Sigma_{p}$. For a small enough $r>0$ compared to $p$ and $\epsilon$, there are $x^{j} \in \partial B(p, r)$, $j=1, \ldots, \beta_{\Sigma_{p}}(\epsilon)$, such that the direction $\eta^{j}$ at $p$ of a minimal segment from $p$ to $x^{j}$ satisfies $\angle\left(\xi^{j}, \eta^{j}\right)<\epsilon^{2}$. Set $x_{i}^{j}=\phi_{i}\left(x^{j}\right)$ and

$$
f_{i}=\frac{1}{\beta_{\Sigma_{p}}(\epsilon)} \sum_{j} d\left(x_{i}^{j}, \cdot\right): M_{i} \rightarrow \mathbb{R} .
$$

By letting $\delta_{x}$ be Dirac's $\delta$-measure, there exists a sequence $\epsilon_{\ell} \rightarrow 0$ such that the measure $\frac{1}{\beta_{\Sigma_{p}}\left(\epsilon_{\ell}\right)} \sum_{j} \delta_{\xi^{j}}$ for $\epsilon=\epsilon_{\ell}$ converges to some Borel measure $m_{p}$ on $\Sigma_{p}$ as $\ell \rightarrow 0$ in the weak* topology. Remark that the measure $m_{p}$ is (possibly) not unique because of the variety of the choices of $\xi^{j}$ and $\epsilon_{\ell}$. However, we observe that $m_{p}$ coincides with the normalized Hausdorff measure over $\Sigma_{p}$ if $k \leq 2$. Let $\psi_{r}: K_{p} \rightarrow \frac{1}{r} B(p, r)$ be an $\nu_{r}$-approximation, $\lim _{r \rightarrow 0} \nu_{r}=0$, and let

$$
\bar{f}=\int_{\xi \in \Sigma_{p}} d(\iota(\xi), \cdot) d m_{p}: K_{p} \rightarrow \mathbb{R},
$$

where $\iota: \Sigma_{p} \rightarrow K_{p}$ is the natural embedding.
Lemma 3.1. We have

$$
\lim _{\ell \rightarrow \infty} \lim _{r \rightarrow 0} \lim _{i \rightarrow \infty} f_{i} \circ \phi_{i} \circ \psi_{r}=\bar{f}
$$

where the convergence is uniform on any compact set.
Proof. Since, as $i \rightarrow \infty, d\left(x_{i}^{j}, \phi_{i}(\cdot)\right)$ converges to $d\left(x^{j}, \cdot\right)$, the function $f_{i} \circ \phi_{i}: X \rightarrow \mathbb{R}$ converges to $\frac{1}{\beta_{\Sigma_{p}}(\epsilon)} \sum_{j} d\left(x^{j}, \cdot\right)$ uniformly on any compact set. Therefore, since $\left(\frac{1}{r} X, p\right) \rightarrow\left(K_{p}, o_{p}\right)$ as $r \rightarrow 0$, we have

$$
\begin{aligned}
\lim _{r \rightarrow 0} \lim _{i \rightarrow \infty} f_{i} \circ \phi_{i} \circ \psi_{r} & =\lim _{r \rightarrow 0} \frac{1}{\beta_{\Sigma_{p}}(\epsilon)} \sum_{j} d\left(x^{j}, \psi_{r}(\cdot)\right) \\
& =\frac{1}{\beta_{\Sigma_{p}}(\epsilon)} \sum_{j} d\left(\iota\left(\xi^{j}\right), \cdot\right) .
\end{aligned}
$$

This completes the proof. q.e.d.

Lemma 3.2. If $k=1$ or 2 , and if $\operatorname{diam}\left(\Sigma_{p}\right)<\pi$, the function $\bar{f}$ takes a strictly local maximum at the vertex $o_{p}$ of $K_{p}$.

Proof. If $k=1$, the lemma is trivial. Assume $k=2$. The directionally derivative of the function $\bar{f}$ with the direction $v \in \Sigma_{o_{p}}\left(K_{p}\right)$ is

$$
v(\bar{f})=-\frac{\sin \operatorname{diam}\left(\Sigma_{p}\right)}{\operatorname{diam}\left(\Sigma_{p}\right)}<0,
$$

which proves the lemma. q.e.d.
From now on we assume the following:
Assumption 3.3. The function $\bar{f}$ takes a strictly local maximum at the vertex $o_{p}$ of $K_{p}$.

Assume that $\epsilon>0$ is small enough compared to the point $p \in X$ and that $0<r \ll \epsilon$. The precise conditions for $\epsilon$ will be exposed in the proof of Lemma 3.5 below. The assumption together with Lemma 3.1 directly implies the following:

Lemma 3.4. For every large $i$ there is a point $\hat{p}_{i} \in M_{i}$ where $f_{i}$ takes a local maximum such that $d\left(p_{i}, \hat{p}_{i}\right) \rightarrow 0$ as $i \rightarrow \infty$.

We define the metric annulus

$$
A\left(x ; r_{1}, r_{2}\right)=B\left(x, r_{2}\right)-\operatorname{int} B\left(x, r_{1}\right)
$$

for $r_{1}<r_{2}$ and a point $x$ in a metric space. Letting $r$ be small and $i$ large, we may assume that $\hat{p}_{i}$ as in Lemma 3.4 exists and satisfies $d\left(p_{i}, \hat{p}_{i}\right) \ll r$, and that the annulus $A\left(p_{i} ; r / 1000,2 r\right)$ contains no critical points of $d\left(p_{i}, \cdot\right)$ (resp. $d\left(\hat{p}_{i}, \cdot\right)$ ). Denote by $q_{i}$ one of the critical points of $d\left(\hat{p}_{i}, \cdot\right)$ in $B\left(p_{i}, r\right)$ which are furthest from $\hat{p}_{i}$ if it exists, and set

$$
\delta_{i}=d\left(\hat{p}_{i}, q_{i}\right) .
$$

Notice that $B\left(p_{i}, r\right) \simeq D^{n}$ if such $q_{i}$ does not exist. Clearly, $\lim _{i \rightarrow \infty} \delta_{i}=$ 0 . Therefore, if $i$ is large enough compared to a given $\lambda>1$, the balls $B\left(p_{i}, r\right), B\left(\hat{p}_{i}, r\right)$, and $B\left(\hat{p}_{i}, \lambda \delta_{i}\right)$ are all homeomorphic each other. By replacing with a subsequence of $\left(M_{i}, p_{i}\right)$, it may be assumed that the rescaled pointed manifold $\left(\frac{1}{\delta_{i}} M_{i}, \hat{p}_{i}\right)$ converges to a noncompact pointed Alexandrov space ( $Y, y_{0}$ ) of nonnegative curvature. The following lemma is important.

Lemma 3.5. We have $\operatorname{dim} Y \geq k+1$.

Proof. Taking a subsequence if necessarily, we assume that $q_{i} \in \frac{1}{\delta_{i}} M_{i}$ tends to some point $z \in Y$ under the convergence $\left(\frac{1}{\delta_{i}} M_{i}, \hat{p}_{i}\right) \rightarrow\left(Y, y_{0}\right)$. Since $q_{i}$ is a critical point of $d\left(\hat{p}_{i}, \cdot\right)$, the point $z$ is a critical point of $d\left(y_{0}, \cdot\right)$. For any fixed number $a>1$, set $R_{i}^{j}=d\left(\hat{p}_{i}, x_{i}^{j}\right)-a \delta_{i}$ and $B_{i}^{j}=B\left(x_{i}^{j}, R_{i}^{j}\right)$. Taking a subsequence, we assume that for each $j, B_{i}^{j}$ converges to some closed subset $B^{j}$ of $Y$ as $i \rightarrow \infty$. Since the function $\left(d\left(x_{i}^{j}, \cdot\right)-R_{i}^{j}\right) / \delta_{i}=d\left(B_{i}^{j}, \cdot\right) / \delta_{i}$ on $M_{i}-B_{i}^{j}$ tends to the function $d\left(B^{j}, \cdot\right)$ on $Y-B^{j}$ and since $\hat{p}_{i}$ takes a local maximum of $f_{i}$, the point $y_{0}$ takes a local maximum of the function

$$
f=\frac{1}{\beta_{\Sigma_{p}}(\epsilon)} \sum_{j} d\left(B^{j}, \cdot\right) .
$$

For each $j$, let $y_{0} b^{j}, b^{j} \in \partial B^{j}$, be a minimal segment from $y_{0}$ to $B^{j}$ which is a limit of $\hat{p}_{i} x_{i}^{j}-\operatorname{int} B_{i}^{j}$. The direction $v^{j}$ of $y_{0} b^{j}$ at $\Sigma_{y_{0}}$ satisfies

$$
\angle\left(v^{j}, v^{j^{\prime}}\right) \geq \tilde{\angle} b^{j} y_{0} b^{j^{\prime}} \geq \lim _{i \rightarrow \infty} \tilde{\angle} x_{i}^{j} \hat{p}_{i} x_{i}^{j^{\prime}} \geq \epsilon / 2 \quad \text { for all } j \neq j^{\prime}
$$

Since $z$ is a critical point of $d\left(y_{0}, \cdot\right)$, we have $\tilde{\angle} y_{0} z b^{j} \leq \pi / 2$ and hence $\angle z y_{0} b^{j} \geq \tilde{\angle} z y_{0} b^{j} \geq \pi / 2-\arcsin (1 / a)$. Therefore, fixing a direction $u$ to $z$ at $\Sigma_{y_{0}}$ we have

$$
\angle\left(u, v^{j}\right) \geq \pi / 2-\arcsin (1 / a) \quad \text { for all } j .
$$

Since $f$ takes a local maximum at $y_{0}$, it follows that

$$
0 \geq u(f)=-\frac{1}{\beta_{\Sigma_{p}}(\epsilon)} \sum_{j} \cos \angle\left(u, v^{j}\right),
$$

which implies that the number of $j$ 's with $v^{j} \in A_{a}$ is not less than $\beta_{\Sigma_{p}}(\epsilon) / 2$, where we set

$$
A_{a}=A(u ; \pi / 2-\arcsin (1 / a), \pi / 2+\arcsin (1 / a)) .
$$

Therefore, $\beta_{A_{a}}(\epsilon / 2) \geq \beta_{\Sigma_{p}}(\epsilon) / 2$. Taking $a \rightarrow \infty$ yields

$$
\beta_{\partial B(u, \pi / 2)}(\epsilon / 2) \geq \beta_{\Sigma_{p}}(\epsilon) / 2
$$

Consider the map which assigns to each $x \in \partial B(u, \pi / 2)$ the direction at $u$ of a minimal geodesic joining $u$ and $x$. It follows from the Alexandrov convexity that this map is expanding, i.e., distance nondecreasing. Since the curvature of $\Sigma_{u}\left(\Sigma_{y_{0}}\right)$ is $\geq 1$, there is an expanding map from $\Sigma_{u}\left(\Sigma_{y_{0}}\right)$
to the $(\ell-2)$-dimensional unit sphere $S^{\ell-2}(1), \ell:=\operatorname{dim} Y$. Combining these two expanding maps, we have $\beta_{\Sigma_{p}}(\epsilon) / 2 \leq \beta_{\partial B(u, \pi / 2)}(\epsilon / 2) \leq$ $\beta_{S^{\ell-2}(1)}(\epsilon / 2)$. Since the order of $\beta_{\Sigma_{p}}(\epsilon)$ (resp. $\left.\beta_{S^{k-2}(1)}(\epsilon)\right)$ as $\epsilon \rightarrow 0$ is exactly $\epsilon^{1-k}$ (resp. $\epsilon^{2-k}$ ), there is an $\epsilon_{p}>0$ depending only on $p$ such that $\beta_{\Sigma_{p}}\left(\epsilon_{p}\right) / 2>\beta_{S^{k-2}(1)}\left(\epsilon_{p} / 2\right)$. We may take $\epsilon=\epsilon_{p}$. Thus we obtain $\beta_{S^{k-2}(1)}(\epsilon / 2)<\beta_{S^{\ell-2}(1)}(\epsilon / 2)$, which implies $k<\ell$. q.e.d.

We put the above results together into the following:
Lemma 3.6 (Key Lemma). Assume that the dimensions satisfy $n=\operatorname{dim} M_{i}=3, k=\operatorname{dim} X=1$ or 2 , and that $\operatorname{diam}\left(\Sigma_{p}\right)<\pi$ for $a$ point $p \in X$. Then, there exists a small number $r_{p}>0$ such that if $B\left(p_{i}, r\right)$ for a number $0<r \leq r_{p}$ is not homeomorphic to $D^{3}$, there are sequences $\hat{p}_{i} \in B\left(p_{i}, r\right)$ and $\delta_{i} \rightarrow 0$ satisfying the following (1)-(3).
(1) $d\left(p_{i}, \hat{p}_{i}\right) \rightarrow 0$ as $i \rightarrow \infty$; in particular, $\hat{p}_{i}$ converges to $p$ in the convergence $\left(M_{i}, p_{i}\right) \rightarrow(X, p)$.
(2) For the limit $\left(Y, y_{0}\right)$ of any convergent subsequence of $\left(\frac{1}{\delta_{i}} M_{i}, \hat{p}_{i}\right)$ we have

$$
k+1 \leq \operatorname{dim} Y \leq 3
$$

(3) If $S_{i} \subset M_{i}, S \subset Y$ are compact subsets such that $S_{i}$ converges to $S$ under the convergence $\left(\frac{1}{\delta_{i}} M_{i}, \hat{p}_{i}\right) \rightarrow\left(Y, y_{0}\right)$, then for every sufficiently large $R>0$ we have $B\left(p_{i}, r\right) \simeq B\left(S_{i}, R \delta_{i}\right)$ for all $i$ large enough compared to $R$.

Proof. (1) and (2) are the direct consequences of the discussion above.

We will prove (3). Since $Y$ is noncompact and of nonnegative curvature, $\left(\frac{1}{R} Y, y_{0}\right)$ converges to the limit cone of $Y$ as $R \rightarrow \infty$. Therefore, for a $\mu>1$ and a sufficiently large $R>0$, the rescaled annulus $\frac{1}{R \delta_{i}} A\left(\hat{p}_{i} ; \mu^{-1} R \delta_{i}, \mu R \delta_{i}\right)$ in $M_{i}$ is $d_{G H}$-close to the annulus $A\left(o_{\infty} ; \mu^{-1}, \mu\right)$ in the limit cone for $i$ large, where $o_{\infty}$ is the vertex of the limit cone. This together with a standard argument of critical point theory proves $B\left(\hat{p}_{i}, R \delta_{i}\right) \simeq B\left(S_{i}, R \delta_{i}\right)$. This completes the proof. q.e.d.

Next we shortly discuss the ideal boundary of $Y$, where $Y$ is as in Key Lemma 3.6. The ideal boundary $Y(\infty)$ of $Y$ and the Tits metric $L_{\infty}$ on $Y(\infty)$ were defined in [2] (cf. [22], [32]). Let $K$ be the asymptotic cone of $Y$ defined as the pointed Gromov-Hausdorff limit of $\left(\epsilon Y, y_{0}\right)$ as $\epsilon \rightarrow 0$ for a point $y_{0} \in Y$. Then $K$ is the Euclidean cone over $\left(Y(\infty), L_{\infty}\right)$.

Lemma 3.7. There is an expanding map $\Sigma_{p} \rightarrow Y(\infty)$.
Proof. For a fixed $\epsilon>0$ and for each $x \in \partial B(p, \epsilon)$, take a minimal geodesic $\gamma_{i}$ from $\hat{p}_{i}$ to $x_{i}$, where $x_{i}$ is a point in $\partial B\left(p_{i}, \epsilon\right)$ converging to $x$. Passing to a subsequence, we may assume that $\frac{1}{\delta_{i}} \gamma_{i}$ converges to a geodesic ray $\gamma_{x}$ from $y_{0}$ under the convergence $\left(\frac{1}{\delta_{i}} B\left(\hat{p}_{i}, r\right), \hat{p}_{i}\right) \rightarrow\left(Y, y_{0}\right)$. Thus we have a map $\varphi_{\epsilon}: \partial B(p, \epsilon) \rightarrow Y(\infty)$ defined by $\varphi_{\epsilon}(x):=\gamma_{x}(\infty)$. Since the lower bound of the sectional curvature of $\frac{1}{\delta_{i}} M_{i}$ goes to zero, we get

$$
\begin{aligned}
厶_{\infty}\left(\gamma_{x}(\infty), \gamma_{y}(\infty)\right) & =\lim _{t \rightarrow \infty} 2 \sin ^{-1}\left(\frac{d\left(\gamma_{x}(t), \gamma_{y}(t)\right)}{2 t}\right) \\
& \geq 2 \sin ^{-1}\left(\frac{d(x, y)}{2 \epsilon}\right)>0
\end{aligned}
$$

Letting $\epsilon \rightarrow 0$, we obtain an expanding $\operatorname{map} \varphi: \Sigma_{p} \rightarrow Y(\infty)$. q.e.d.

## 4. Seifert bundle structure

Let a sequence of pointed complete orientable three-manifolds ( $M_{i}, p_{i}$ ) with $K \geq-1$ converge to a pointed complete Alexandrov space $(X, p)$ of dimension two with respect to the pointed Gromov-Hausdorff convergence. In this section, we study the topology of a neighborhood of $p_{i}$ in the case when $p$ is an interior singular point of $X$ and define a compatible Seifert bundle structure on the neighborhood.

Let $p \in \operatorname{int} X$ be an interior singular point of $X$, and $r=r_{p}$ a fixed small positive number given in Key Lemma 3.6. We may assume that $B(p, 10 r)-\{p\} \subset X_{\delta}$, where $\delta=\delta_{2}$ is a constant given in Fibration Theorem 2.2. Applying Fibration Theorem 2.2 to the convergence $B\left(p_{i}, r\right) \rightarrow B(p, r)$, we see that

$$
\begin{equation*}
\partial B\left(p_{i}, r\right) \simeq S^{1} \times S^{1} \tag{4.1}
\end{equation*}
$$

Lemma 4.1. $B\left(p_{i}, r\right)$ is homeomorphic to $S^{1} \times D^{2}$ for large $i$.
Proof. We prove the lemma by contradiction. Suppose that it does not hold. Passing to a subsequence, we may assume that all $B\left(p_{i}, r\right)$ are not homeomorphic to $S^{1} \times D^{2}$. Note that the assumption in Key Lemma 3.6 is satisfied because of (4.1). Let $\hat{p}_{i} \in M_{i}$ and $\delta_{i} \rightarrow 0$ be as in Key Lemma 3.6. Then the limit $\left(Y, y_{0}\right)$ of a subsequence of $\left(\frac{1}{\delta_{i}} B\left(\hat{p}_{i}, r\right), \hat{p}_{i}\right)$ has dimension three. The space $Y$ contains an essential information on the topology of $B\left(p_{i}, r\right)$. We note that by Theorem 9.6 the topology of $Y$ is determined by its soul $S$.

Assertion 4.2. $\quad S$ is a circle.
Suppose this assertion for a moment. Then we can obtain the topological type of $B\left(p_{i}, r\right)$ as follows. By Theorem $9.6, Y$ is isometric to the form $Y=\left(\mathbb{R} \times N^{2}\right) / \mathbb{Z}$, where $N^{2}$ is a nonnegatively curved Alexandrov surface homeomorphic to $\mathbb{R}^{2}$. Let a compact set $S_{i} \subset M_{i}$ converges to $S \subset Y$ under the convergence $\left(\frac{1}{\delta_{i}} B\left(\hat{p}_{i}, r\right), \hat{p}_{i}\right) \rightarrow\left(Y, y_{0}\right)$. For a large $R$, we then have

$$
\begin{equation*}
B\left(p_{i}, r\right) \simeq B\left(S_{i}, R ; \frac{1}{\delta_{i}} M_{i}\right) \simeq B(S, R ; Y) \simeq S^{1} \times D^{2} \tag{4.2}
\end{equation*}
$$

where the second $\simeq$ follows from Stability Theorem 2.4 and the third $\simeq$ follows from Corollary 9.7. q.e.d.

Proof of Assertion 4.2. This is done by an argument similar to the above. If $\operatorname{dim} S=0$, then $B\left(p_{i}, r\right) \simeq B\left(y_{0}, R \delta_{i}\right) \simeq D^{3}$. However this is impossible because of (4.1). Next suppose that $\operatorname{dim} S=2$. Then by Theorem $9.6, Y$ would be isometric to the normal bundle $N(S)$ of $S$. It turns out that the ideal boundary $Y(\infty)$ of $Y$ consists of at most two points. However this is impossible because of Lemma 3.7. Therefore we have $\operatorname{dim} S=1$ and hence $S$ is a circle. q.e.d.

Our next purpose is to study the limit of the universal covering spaces $\pi_{i}: \widetilde{B}\left(p_{i}, r\right) \rightarrow B\left(p_{i}, r\right)$ to define the Seifert bundle structure on $B\left(p_{i}, r\right)$. This will immediately provide the proof of Theorem 0.2. To do this, we need a rescaling argument.

Let $\tilde{p}_{i} \in \widetilde{B}\left(p_{i}, r\right)$ be a point over $p_{i}$, and $\Gamma_{i} \simeq \mathbb{Z}$ the deck transformation group. Since our argument is by contradiction, we can take a subsequence if necessary. For $\epsilon_{i}=d_{G H}\left(B\left(p_{i}, r\right), B(p, r)\right)$, take a sequence $r_{i} \rightarrow 0$ such that $\epsilon_{i} / r_{i} \rightarrow 0$. Passing to a subsequence, we may assume that $\left(\frac{1}{r_{i}} \widetilde{B}\left(p_{i}, r\right), \tilde{p}_{i}, \Gamma_{i}\right)$ converges to a triple $(Z, z, G)$ with respect to the pointed equivariant Gromov-Hausdorff convergence ([13]), where $Z$ is a simply connected, complete Alexandrov space with nonnegative curvature.

Proposition 4.3. Under the situation above, we have:
(1) There exists a locally trivial fibration

$$
f_{i}: A\left(p_{i} ; r_{i} / 2, r\right) \rightarrow A\left(p ; r_{i} / 2, r\right)
$$

satisfying the conclusion of Fibration Theorem 2.2.
(2) $\left(\frac{1}{r_{i}} B\left(p_{i}, r\right), p_{i}\right)$ converges to $\left(K_{p}, o_{p}\right)$.
(3) $Z$ is isometric to a product $Z_{0} \times \mathbb{R}$ and $G$ is isomorphic to $\mathbb{Z}_{\mu} \times \mathbb{R}$ for some integer $\mu \leq\left[2 \pi / L\left(\Sigma_{p}\right)\right]$, where $Z / G \equiv Z_{0} / \mathbb{Z}_{\mu} \equiv K_{p}$ (isometric).
(4) The space $Z_{0}$ is isometric to a flat cone, say $Z_{0}=K\left(S_{\ell}^{1}\right)$ with cone angle $\ell \leq 2 \pi$. Thus the generator $\gamma$ of $\mathbb{Z}_{\mu}$ is given by

$$
\gamma\left(r e^{\ell \theta i}\right)=r e^{\ell(\theta+\nu / \mu) i}
$$

where $(\mu, \nu)=1$ and we make an obvious identification

$$
K\left(S_{\ell}^{1}\right)=\left\{r e^{\ell \theta i} \mid 0 \leq \theta \leq 1, r \geq 0\right\} .
$$

(5) $\left(\frac{1}{r_{i}} \widetilde{B}\left(p_{i}, r_{i}\right), \tilde{p}_{i}\right)$ converges to $\left(B\left(z_{0}, 1\right) \times \mathbb{R}, z\right)$ under the convergence $\left(\frac{1}{r_{i}} \widetilde{B}\left(p_{i}, r\right), \tilde{p}_{i}\right) \rightarrow(Z, z)$, where $\widetilde{B}\left(p_{i}, r_{i}\right)=\pi_{i}^{-1}\left(B\left(p_{i}, r_{i}\right)\right)$ and $z_{0}$ is the vertex of the cone $Z_{0}$.

Proof. Since

$$
d_{G H}\left(\frac{1}{r_{i}} A\left(p_{i} ; r_{i} / 2, r\right), \frac{1}{r_{i}} A\left(p ; r_{i} / 2, r\right)\right) \leq \epsilon_{i} / r_{i}
$$

and the $\delta_{2}$-strain radius of $\frac{1}{r_{i}} A\left(p ; r_{i} / 2, r\right)$ is greater than a constant independent of $i$, (1) follows from Fibration Theorem 2.2. (2) is clear from the choice of $r_{i}$.

Using the limit $G$-action, one can construct a line in $Z$. It follows from the splitting theorem that $Z$ is isometric to a product $Z_{0} \times \mathbb{R}$. Since $G$ is a Lie group ([14]), by using Lemma 3.10 of [13] it is possible to take a subgroup $\Gamma_{i}^{\prime}$ of $\Gamma_{i}$ such that:
(1) $\left(\frac{1}{r_{i}} \widetilde{B}\left(p_{i}, r\right), \tilde{p}_{i}, \Gamma_{i}^{\prime}\right)$ converges to $\left(Z, z, G_{0}\right)$, where $G_{0}$ is the identity component of $G$.
(2) $\Gamma_{i} / \Gamma_{i}^{\prime} \simeq G / G_{0}$ for large $i$.

Since $Z / G \equiv K_{p}$ is of dimension two, $\operatorname{dim} Z-\operatorname{dim} G=2$. If $G_{0}$ were trivial, $G \simeq \mathbb{Z}$ and $\operatorname{dim} Z=2$. It turns out that $Z$ is isometric to one of $\mathbb{R}^{2},[0, \ell] \times \mathbb{R}$ and $[0, \infty) \times \mathbb{R}$. It is now an easy exercise to show that none of those cases implies that $Z / G$ is a flat cone $K_{p}$, a contradiction. Thus $\operatorname{dim} G_{0}=1$ and $\operatorname{dim} Z=3$. It follows from Stability Theorem 2.4 that $Z_{0}$ is a complete open Alexandrov surface homeomorphic to $\mathbb{R}^{2}$. If $G_{0} \simeq$ $S^{1}$, then $G \simeq \mathbb{Z} \times S^{1}$ and $Z / G$ cannot be a flat cone, a contradiction. Hence $G_{0} \simeq \mathbb{R}$. It is now easy to show that $G \simeq \mathbb{Z}_{\mu} \times \mathbb{R}$ and $Z / G \equiv$ $Z_{0} / \mathbb{Z}_{\mu} \equiv K_{p}$. Thus $Z_{0}$ is a flat cone with cone angle $\mu \times L\left(\Sigma_{p}\right)$, and all the conclusions of the proposition follow. q.e.d.

Lemma 4.4. There exists a topological Seifert bundle structure on $B\left(p_{i}, r\right)$ of orbit type $(\mu, \nu)$ over $B(p, r)$ which is compatible to the circle bundle structure on $A\left(p_{i} ; r_{i} / 2, r\right)$ defined by $f_{i}$.

Proof. For any $x \in \partial B\left(p_{i}, r / 2\right)$ consider the fibre $f_{i}^{-1}(x)$.
Sublemma 4.5. $f_{i}^{-1}(x)$ represents a generator of $\Gamma_{i}^{\prime}$, where $\Gamma_{i}^{\prime}$ is as in the proof of Proposition 4.3.

Proof. We put $U=B(x, \ell r / 10)$. Any non-trivial geodesic loop at $x_{i} \in f_{i}^{-1}(x)$ of length $\leq \ell r / 100$ is contained in $f_{i}^{-1}(U)$. Since $f_{i}^{-1}(U) \simeq U \times S^{1}$, it follows that $\Gamma_{i}^{\prime}$ is contained in the image $H_{i}$ of the inclusion homomorphism $\pi_{1}\left(f_{i}^{-1}(U)\right) \rightarrow \Gamma_{i}$. Conversely Proposition $4.3(3)$ implies that $H_{i} \subset \Gamma_{i}^{\prime}$. Therefore $H_{i}=\Gamma_{i}^{\prime}$. q.e.d.

Consider $B_{i}^{\prime}=\widetilde{B}\left(p_{i}, r_{i}\right) / \Gamma_{i}^{\prime} \simeq D^{2} \times S^{1}$. Let $\gamma_{i}$ be a generator of $\Gamma_{i}$, and $\gamma_{i}^{\prime}$ the generator of $\Gamma_{i} / \Gamma_{i}^{\prime} \simeq \mathbb{Z}_{\mu}$ represented by $\gamma_{i}$. Let $C_{i}$ be a path on $\partial B_{i}^{\prime}$ joining a point $x \in \partial B_{i}^{\prime}$ to $\gamma_{i}^{\prime} x$ in a suitable direction. Then Proposition 4.3(4) implies that the union of $\left(\gamma_{i}^{\prime}\right)^{k}\left(C_{i}\right), k=0, \ldots, \mu-1$, is a loop rotating $\nu$-times in the meridian direction. Hence the $\mathbb{Z}_{\mu}$-action on $B_{i}^{\prime}$ defines a Seifert bundle structure on $B\left(p_{i}, r_{i}\right)$ which is isomorphic to the Seifert bundle structure defined by the standard $\mathbb{Z}_{\mu}$-action on $D^{2} \times S^{1}$ :

$$
\tau_{\mu \nu}\left(r e^{i \theta}, e^{i \phi}\right)=\left(r e^{i\left(\theta+\frac{2 \nu}{\mu} \pi\right)}, e^{i\left(\phi+\frac{2 \pi}{\mu}\right)}\right)
$$

Thus we can put the topological Seifert bundle structure on $B\left(p_{i}, r_{i}\right)$ of orbit type $(\mu, \nu)$ which is compatible to the circle bundle structure $f_{i}$ on $A\left(p_{i} ; r_{i} / 2, r\right)$. This completes the proof of Lemma 4.4. q.e.d.

Proof of Theorem 0.2. Take finitely many points $\left\{p_{1}, \ldots, p_{m}\right\}$ such that $X-\left\{p_{1}, \ldots, p_{m}\right\} \subset X_{\delta}$. Let $r$ be so small that the balls $B\left(p_{j}, 10 r\right)$, $1 \leq j \leq m$, are disjoint. Applying Lemma 4.4 to each $B\left(p_{j}, r\right)$, we complete the proof of Theorem 0.2 . q.e.d.

Remark 4.6. Let $X$ be as in Theorems 0.2 or 0.3 . Applying the Gauss-Bonnet Theorem (Proposition 14.1) and the volume comparison to $X$ yields that the Euler characteristic of $X$ satisfies

$$
2 \geq \chi(X) \geq-v_{-1}^{2}(D) / 2 \pi
$$

where $v_{-1}^{2}(D)$ denotes the volume of a $D$-ball in the hyperbolic plane. Moreover, as a consequence of Theorem 0.2 and 0.3 , the number of singular fibres of $M_{i}$ (resp. of $\operatorname{Seif}_{i}(X)$ in Theorem 0.3) is at most $4+$ $v_{-1}^{2}(D) / \pi$ (see Corollary 14.3).

## 5. Topology near boundary

In this section, we consider the case when the limit space $X$ is a two-dimensional Alexandrov space with boundary. The argument in the previous section shows that the part $M_{i}^{\prime}$ of $M_{i}$ converging to a part $X_{0}$ of $X$ away from the boundary $\partial X$ is a Seifert fibred space over $X_{0}$. Hence the essential point of the proof is to describe the topology of $M_{i}-M_{i}^{\prime}$. Actually we prove that it is homeomorphic to $\partial X \times D^{2}$.

For reader's convenience, we give the definition of twisted bundles over surfaces. For the details, see [21]. Let $S^{1}=\{z \in \mathbb{C}| | z \mid=1\}$ and $I=[0,1]$. A twisted $S^{1}$-bundle Mö $\tilde{\times} S^{1}$ over a Möbius band Mö is defined as the quotient space $\left(S^{1} \times I \times S^{1}\right) / \tau$, where $\tau$ is the involution of $S^{1} \times I \times S^{1}$ defined by $\tau\left(e^{i \theta}, t, e^{i \eta}\right)=\left(e^{i(\theta+\pi)}, 1-t, e^{-i \eta}\right)$. Let $N$ be a non-orientable surface and $\hat{N}$ the orientable double cover of $N$ with the nontrivial deck transformation $\sigma$ on $\hat{N}$. Then a twisted $I$-bundle $N \tilde{\times} I$ over $N$ is defined as the quotient space $(\hat{N} \times I) / \tau$, where $\tau$ is the involution of $\hat{N} \times I$ defined by $\tau(x, t)=(\sigma(x), 1-t)$. Note that $\partial \mathrm{Mö} \tilde{\times} S^{1} \simeq T^{2}, \partial N \tilde{\times} I \simeq \hat{N}$ and Mö $\tilde{\times} S^{1} \simeq K^{2} \tilde{\times} I$, where $K^{2}$ denotes a Klein bottle.

Proof of Theorem 0.3. For a small $\epsilon>0$ we take $\left\{x_{1}, \ldots, x_{N}\right\}$ contained in a fixed connected component $C$ of $\partial X$ such that:
(1) $x_{j}$ is adjacent to $x_{j-1}$.
(2) $L\left(\Sigma_{x}\right)>\pi-\epsilon$ for any $x \in C-\left\{x_{1}, \ldots, x_{N}\right\}$.
(3) There exist positive numbers $r$ and $\delta \ll r$ such that:
(a) $\tilde{\angle} x_{j} y z>\pi-\epsilon$ for every $y \in B\left(x_{j}, r\right)$ and for some $z \in X$.
(b) $\tilde{\angle} x_{j} x x_{j+1}>\pi-\epsilon$ for every point $x \in B\left(\widehat{x_{j} x_{j+1}}, \delta\right)-B\left(x_{j}, r\right)-$ $B\left(x_{j+1}, r\right)$, where $\widehat{x_{j} x_{j+1}}$ is the arc joining $x_{j}$ and $x_{j+1}$ in $C$.
(c) If $\operatorname{diam}\left(\sum_{x_{j}}\right)=\pi$, then $\tilde{\angle} x_{j-1} x x_{j+1}>\pi-\epsilon$ for every point $x \in B\left(\widehat{x_{j-1} x_{j+1}}, \delta\right)-B\left(x_{j-1}, r\right)-B\left(x_{j+1}, r\right)$.

Now suppose $d_{G H}\left(M_{i}, X\right)<\epsilon_{i}$ and $\epsilon_{i} \ll \delta$. For a fixed $j$ we take $p_{i}, p_{i}^{\prime} \in M_{i}$ such that $p_{i}$ and $p_{i}^{\prime}$ converge to $x_{j}$ and $x_{j+1}$ respectively under the convergence $M_{i} \rightarrow X$. Let $B_{i}$ and $B_{i}^{\prime}$ be $C^{\infty}$-approximations of $B\left(p_{i}, r\right)$ and $B\left(p_{i}^{\prime}, r\right)$ respectively. Let $C_{i}^{\prime}$ be a compact domain which converges to $B\left(\widehat{x_{j} x_{j+1}}, \delta\right), C_{i}$ the closure of $C_{i}^{\prime}-B_{i}-B_{i}^{\prime}$, and $N_{i}$ the closure of $\partial C_{i}-B_{i}-B_{i}^{\prime}$. Applying Fibration Theorem 2.2 to a neighborhood
of $\partial B\left(\widehat{x_{j} x_{j+1}}, \delta\right)$, we can take such $C_{i}^{\prime}$ that for every $x \in\left(B_{i} \cup B_{i}^{\prime}\right) \cap N_{i}$

$$
\left|\angle\left(\xi_{1}(x), \xi_{2}(x)\right)-\pi / 2\right|<\tau(r)+\tau(r \mid \delta)+\tau(r, \delta \mid \epsilon),
$$

where $\xi_{1}$ and $\xi_{2}$ denote the unit normal vector fields to $\partial\left(B_{i} \cup B_{i}^{\prime}\right)$ and $N_{i}$ respectively, and $\tau\left(r_{1}, \ldots, r_{k} \mid \epsilon\right)$ a function depending on $r_{i}, \epsilon$ satisfying $\lim _{\epsilon \rightarrow 0} \tau\left(r, \ldots, r_{k} \mid \epsilon\right)=0$ for fixed $r_{i}$. Thus both $\partial B_{i}$ and $\partial B_{i}^{\prime}$ meet $N_{i}$ transversally, and $B_{i} \cap N_{i} \simeq S^{1}, B_{i}^{\prime} \cap N_{i} \simeq S^{1}$. It follows from Fibration Theorem 2.2 that

$$
N_{i} \simeq S^{1} \times I
$$

We next show that $C_{i} \simeq D^{3}$. Let $\gamma_{i}$ be a geodesic in $B_{i} \cup C_{i} \cup B_{i}^{\prime}$ converging to a geodesic joining $x_{j}$ and $x_{j+1}$. Now we consider the functions

$$
f_{i}=d\left(\gamma_{i}, \cdot\right), \quad g_{i}=d\left(p_{i}, \cdot\right)-d\left(p_{i}^{\prime}, \cdot\right) .
$$

Note that $f_{i}$ is regular on $f_{i}^{-1}([\delta / 100, \delta])$ and the gradient of $f_{i}$ is almost perpendicular to $N_{i}$. Note also that $g_{i}$ is regular on $C_{i}$. Set $F_{i}=f_{i}^{-1}([0, \delta / 2]) \cap g^{-1}(0)$ and denote by $H_{i}$ the set consisting of all flow curves of the gradient of $g_{i}$ contained in $C_{i}$ through $F_{i}$. Clearly,

$$
H_{i} \simeq F_{i} \times I .
$$

Note that the gradient of $f_{i}$ is almost perpendicular to that of $g_{i}$ on $f_{i}^{-1}([\delta / 100, \infty)) \cap C_{i}$. It follows that $\partial F_{i} \simeq S^{1}$. By the generalized Margulis lemma ([13]), $\pi_{1}\left(F_{i}\right) \simeq \pi_{1}\left(H_{i}\right)$ is almost nilpotent, and therefore by the orientability, $F_{i} \simeq D^{2}$.

It is easy to construct a smooth vector field $V_{i}$ on a neighborhood of $C_{i}-H_{i}$ such that:
(1) $V_{i}=\operatorname{grad} f_{i}$ outside a small neighborhood of $\partial B_{i} \cup \partial B_{i}^{\prime}$.
(2) $V_{i}$ is tangent to $\partial B_{i} \cup \partial B_{i}^{\prime}$.
(3) $f_{i}$ is strictly decreasing along the flow curves of $V_{i}$.

Thus we have

$$
C_{i} \simeq H_{i} \simeq D^{3} .
$$

Next we show that $B_{i} \simeq D^{3}$. Suppose that $\operatorname{diam}\left(\Sigma_{x_{j}}\right)=\pi$. Let $\hat{C}_{i}^{\prime}$ be a compact domain which converges to $B\left(\widehat{x_{j-1} x_{j}+1}, \delta\right)$, and $\hat{C}_{i}$ the closure of $\hat{C}_{i}^{\prime}-B_{i}-B_{i}^{\prime}$. Applying the previous argument, we obtain that $\hat{C}_{i} \simeq D^{3}$. It is now easy to see that $B_{i} \simeq \hat{C}_{i} \cap B_{i} \simeq \hat{C}_{i} \simeq D^{3}$. If $\operatorname{diam}\left(\Sigma_{x_{j}}\right)<\pi$, take a point $p_{i}^{\prime \prime} \in M_{i}$ converging to $x_{j-1}$. For the points
$p_{i}$ and $p_{i}^{\prime \prime}$ we construct a compact domain $C_{i}^{\prime}$ in the same way as the construction of $C_{i}$. From Fibration Theorem 2.2, $\partial B_{i}-C_{i}-C_{i}^{\prime} \simeq S^{1} \times I$, which implies that

$$
\begin{equation*}
\partial B_{i} \simeq S^{2} \tag{5.1}
\end{equation*}
$$

Then we show
Assertion 5.1. $B_{i} \simeq D^{3}$.
Proof. This is done by an argument similar to the proof of Lemma 4.1 as follows. Suppose that it does not hold. Passing to a subsequence, we may assume that all $B_{i}$ are not homeomorphic to $D^{3}$. We may assume that the assumption in Key Lemma 3.6 is satisfied. Let $\hat{p}_{i} \in B_{i}$ and $\delta_{i} \rightarrow 0$ be as in Key Lemma 3.6. Then the limit ( $Y, y_{0}$ ) of a subsequence of $\left(\frac{1}{\delta_{i}} B_{i}, \hat{p}_{i}\right)$ has dimension three. We show that the soul $S$ of $Y$ is a point. If $\operatorname{dim} S=1$, then $S$ is a circle and $B_{i} \simeq B(S, R) \simeq S^{1} \times D^{2}$. However this is impossible because of (5.1). Next suppose that $\operatorname{dim} S=2$. Then $Y$ would be isometric to the normal bundle $N(S)$ of $S$. It turns out that the ideal boundary $Y(\infty)$ of $Y$ consists of at most two points. However this is impossible because we have an expanding map $\Sigma_{x_{j}} \rightarrow Y(\infty)$ as before. Thus $S$ is a point and we see that $B_{i} \simeq B(S, R) \simeq D^{3}$. q.e.d.

Now we change the notation. Let $p_{j}^{i} \in M_{i}$ be a point converging to $x_{j}$. Let $B_{j}^{i}$ and $C_{j}^{i}$ denote $B_{i}$ and $C_{i}$ respectively. Then the previous argument shows that

$$
A_{i}=\bigcup_{j=1}^{N}\left(B_{j}^{i} \cup C_{j}^{i}\right) \simeq \partial X \times D^{2}
$$

By Theorem 0.2, $M_{i}-A_{i}$ is homeomorphic to a Seifert fibred space, say $\operatorname{Seif}_{i}(X)$ over $X-A \simeq X$, where $A=\cup_{i=1}^{N} B\left(x_{j}, r\right) \cup B\left(\widehat{x_{j-1} x_{j}}, \delta\right)$. Thus

$$
M_{i} \simeq \operatorname{Seif}_{i}(X) \cup \partial X \times D^{2}
$$

where the identification is made by $\{$ the fibre over $x \in \partial X\}=\{x\} \times$ $\partial D^{2}$. This completes the proof of Theorem 0.3. q.e.d.

For the proof of Corollary 0.4, it suffices to prove the following
Proposition 5.2. Let $M$ be a closed orientable three-manifold and $X$ a compact surface with boundary such that

$$
M \simeq \operatorname{Seif}(X) \cup \partial X \times D^{2}
$$

for a Seifert fibred space $\operatorname{Seif}(X)$ over $X$, where the fibre over $x \in \partial X$ is identified with $\{x\} \times \partial D^{2}$. Let $g$ and $k$ denote the genus and the number of components of $\partial X$. Then we have

$$
M \simeq S^{3} \# \underbrace{S^{2} \times S^{1} \# \cdots \# S^{2} \times S^{1}}_{f(g, k)} \# L\left(\mu_{1}, \nu_{1}\right) \# \cdots \# L\left(\mu_{\ell}, \nu_{\ell}\right)
$$

where $f(g, k)$ and $\left(\mu_{j}, \nu_{j}\right)$ are as in Corollary 0.4.
We need a lemma.
Lemma 5.3. Let $M$ be an orientable three-manifold containing a surface $S$ homeomorphic to $S^{2}$ such that $M-S$ is connected. Then:
(1) $M$ has a decomposition, $M \simeq S^{2} \times S^{1} \# N$.
(2) Let $P$ be the result of cutting of $M$ along $S$. Then $N=\operatorname{Cap}(P)$, where $\operatorname{Cap}(P)$ denotes the closed three-manifold obtained from $P$ by attaching $D^{3}$ along their boundary spheres.

Proof. The first part follows from for instance Lemma 3.8 in [21]. The second part is an easy exercise. q.e.d.

Proof of Proposition 5.2. The proof is done by induction on $m=$ $g+k+\ell$. First we suppose that $X$ is orientable. If $m=1$, then $k=1$ and $g=\ell=0$, and we have that $M \simeq D^{2} \times S^{1} \cup S^{1} \times D^{2}$, where $(x, y) \in \partial D^{2} \times S^{1}$ is identified with $(x, y) \in S^{1} \times \partial D^{2}$. Hence $M \simeq S^{3}$.

Next we consider the case $m=2$.
Case I. $g=k=1, \ell=0$.
Let $T^{2}=S^{1} \times S^{1}, B=\left\{\left(e^{i \theta}, e^{i \varphi}\right) \mid-\epsilon<\theta, \varphi<\epsilon\right\}$. We identify $X \simeq T^{2}-B$. Consider the two curves, $\gamma_{1}(\theta)=\left(e^{i \theta}, 1\right), \gamma_{2}(\varphi)=\left(1, e^{i \varphi}\right)$, $\epsilon \leq \theta, \varphi \leq 2 \pi-\epsilon$. Let $S$ be the part of $M$ "over $\gamma_{1}$ ";

$$
S \simeq \gamma_{1} \times S^{1} \cup \partial \gamma_{1} \times D^{2} \simeq S^{2} .
$$

Note that $M-S$ is connected. Then we have the decomposition $M=$ $S^{2} \times S^{1} \# M^{\prime}$, where $M^{\prime}=\operatorname{Cap}\left(P_{i}\right)$, and $P_{i}$ is the result of cutting $M$ along $S$. Let $S^{\prime}$ be the part of $M$ over $\gamma_{2}$, which is homeomorphic to $S^{2}$ as before. We may assume that $S^{\prime} \subset M^{\prime}$. Note that $M^{\prime}-S^{\prime}$ is connected. Then we have a decomposition $M^{\prime}=S^{2} \times S^{1} \# M^{\prime \prime}$, where $M^{\prime \prime}=\operatorname{Cap}\left(Q_{i}\right)$, and $Q_{i}$ is the result of cutting $M^{\prime}$ along $S^{\prime}$. Now one can verify that $M^{\prime \prime}=\operatorname{Cap}\left(Q_{i}\right) \simeq S^{3}$. Thus $M \simeq S^{2} \times S^{1} \# S^{2} \times S^{1}$.

Case II. $g=0, k=\ell=1$.
Let $\left(\mu_{1}, \nu_{1}\right)$ be the orbit type of the unique singular orbit in $\operatorname{Seif}(X)$. Then from the definition of lens spaces, we have $M \simeq L\left(\mu_{1}, \nu_{1}\right)$.

Now consider the general case. Let $\gamma$ be a path joining two points of $\partial X$ which divides $X$ into two compact domains $X_{1}, X_{2}$ in such a way that each $X_{j}$ contains at least one of handles, boundary components and singular loci of $X$. The part of $M$ over $\gamma$ is homeomorphic to $S^{2}$. If we denote by $M^{\prime}$ and $M^{\prime \prime}$ the part of $M$ over $X_{1}$ and $X_{2}$ respectively, then we have $M=M^{\prime} \# M^{\prime \prime}$. By applying the induction to $M^{\prime}$ and $M^{\prime \prime}$ we obtain the required form for $M$.

We next consider the case when $X$ is non-orientable. Suppose first $g=k=1, \ell=0$. Then $M=$ Mö $\tilde{\times} S^{1} \cup S^{1} \times D^{2}$, where $(x, y) \in \partial$ Mö $\tilde{\times} S^{1}$ is identified with $(x, y) \in S^{1} \times \partial D^{2}$. Let $\gamma$ be a path in Mö cutting Mö open to a disk. Let $S$ be the part of $M$ over $\gamma$, which is homeomorphic to $S^{2}$. Note that $M-S$ is connected. Then by a similar argument, we can conclude that $M \simeq S^{2} \times S^{1}$. The rest of the inductive argument follows in the same way. q.e.d.

Corollary 5.4. Let $M_{i}$ and $X$ be as in Theorem 0.3. Then the set of homeomorphism classes of $M_{i}$ is finite.

Proof. This follows from Theorem 0.3 and Proposition 5.2. q.e.d.

## 6. Collapsing to a closed interval

For a Riemannian manifold $M$ with boundary, we denote by $\mathrm{dbl}(M)$ the double of $M$, i.e., the gluing of two copies of $M$ along their boundaries by the original identification.

Proof of Theorem 0.5. Let $M_{i}$ be a sequence of closed orientable three-manifolds with $K \geq-1$ collapsing to a closed interval [ $0, \ell$ ]. By Fibration Theorem 2.2, we have

$$
M_{i}=B_{i} \cup A_{i} \cup C_{i},
$$

where $A_{i} \simeq F_{i} \times[0,1], B_{i}$ and $C_{i}$ are metric balls Gromov-Hausdorff close to the endpoints of $[0, \ell]$ and $F_{i}$ is homeomorphic to $S^{2}$ or $T^{2}$. If $B_{i}$ is not homeomorphic to $D^{3}$, then by Key Lemma 3.6, we have a sequence $\hat{p}_{i}$ converging to the end point 0 of $[0, \ell]$, and $\delta_{i} \rightarrow 0$ such that for a subsequence, $\left(\frac{1}{\delta_{i}} B_{i}, \hat{p}_{i}\right)$ converges to a pointed noncompact Alexandrov space ( $Z, z_{0}$ ) with nonnegative curvature, where $\operatorname{dim} Z \geq 2$. In what follows, assuming that $B_{i}$ is not homeomorphic to $D^{3}$, we analyze the
topology of $B_{i}$ from the information on the fibre data $F_{i}$, the dimension and the boundary data of $Z$.

Case I. $F_{i} \simeq S^{2}$.
We have to show that $B_{i}$ is homeomorphic to $D^{3}$ or $P^{3}-\operatorname{int} D^{3}$. Assume that $B_{i}$ is not homeomorphic to $D^{3}$. If $\operatorname{dim} Z=3, Z$ has no boundary. Let $S$ be a soul of $Z$. If $\operatorname{dim} S=1$, then $B_{i} \simeq S^{1} \times D^{2}$, a contradiction to $\partial B_{i} \simeq S^{2}$. If $\operatorname{dim} S=2$, we see that $S \simeq P^{2}$ and $Z$ is isometric to a flat line bundle $P^{2} \tilde{\times} \mathbb{R}$. Therefore $B_{i} \simeq P^{2} \tilde{\times} I \simeq$ $P^{3}-\operatorname{int} D^{3}$.

Next we consider the case $\operatorname{dim} Z=2$.
We claim
Assertion 6.1. $Z$ is isometric to the double $\operatorname{dbl}([0, \infty) \times[0, \infty)) \cap$ $\{(x, y) \mid y \leq h\}$.

Proof. First we show that $Z$ has non-empty boundary. If $Z$ has empty boundary, then take a large metric ball $D$ around $z_{0}$. It turns out from Fibration Theorem 2.2 that $\partial B_{i}$ is homeomorphic to $S^{1} \times S^{1}$, a contradiction. Since $\partial B_{i}$ is connected, $Z$ has one end. It follows from Corollary 14.4 that $Z$ is homeomorphic to $[0, \infty) \times \mathbb{R}$. Note that $B_{i}$ is a Seifert fibred space over $D$. If $B_{i}$ has no singular fibre, then Theorem 0.3 implies that $B_{i} \simeq D^{3}$, a contradiction. Thus $B_{i}$ has a singular fibre, say one over $z \in Z$. Theorem 0.3 shows that $z$ is an essential singular point. Hence Corollary 14.4 yields the conclusion. q.e.d.

We put

$$
\begin{aligned}
D & =\operatorname{dbl}([0, \infty) \times[0, \infty)) \cap\{(x, y) \mid x \leq 1, y \leq h\} \subset Z, \\
D_{1} & =\operatorname{dbl}([0, \infty) \times[0, \infty)) \cap\left\{(x, y) \mid x \leq 1, y \leq h_{1}\right\}
\end{aligned}
$$

for some $h_{1}<h$. Then we obtain a Seifert fibration $f_{i}: B_{i}^{\prime} \rightarrow D$ for a closed domain $B_{i}^{\prime} \simeq B_{i}$. Let $N_{i}=\operatorname{Cap}\left(B_{i}^{\prime}\right)=B_{i}^{\prime} \cup_{S^{2}} D^{3}$. Note that the closure of $N_{i}-f_{i}^{-1}\left(D_{1}\right)$ is homeomorphic to $S^{1} \times D^{2}$. It is then easy to see that

$$
N_{i} \simeq S^{1} \times D^{2} \cup S^{1} \times D^{2} \simeq L(2,1) \simeq P^{3}
$$

Thus $B_{i} \simeq P^{3}-\operatorname{int} D^{3}$ as required.
Case II. $F_{i} \simeq T^{2}$.
We have to show that $B_{i}$ is homeomorphic to $S^{1} \times D^{2}$ or Mö $\tilde{\times} S^{1}$. Let us first assume $\operatorname{dim} Z=3$. If the soul $S$ of $Z$ has dimension zero, the Soul Theorem would imply that $B_{i} \simeq D^{3}$, a contradiction. If $\operatorname{dim} S=1$,
we have $B_{i} \simeq S^{1} \times D^{2}$. If $\operatorname{dim} S=2$, then $Z$ must be isometric to a flat twisted line-bundle $K^{2} \tilde{\times} \mathbb{R}$ since $\partial B_{i} \simeq T^{2}$. Therefore $B_{i} \simeq K^{2} \tilde{\times} I \simeq$ Mö $\tilde{\times} S^{1}$.

Next suppose that $\operatorname{dim} Z=2$. We first consider the case when $Z$ has empty boundary. If the soul of $Z$ is not a point, in view of the connectedness of $\partial B_{i}, Z$ must be isometric to a Möbius strip, and therefore $B_{i} \simeq \mathrm{Mö} \tilde{\times} S^{1}$. If the soul of $Z$ is a point, then $Z$ is homeomorphic to a plane. Note that $B_{i}$ is a Seifert fibred space over a large metric ball $D \subset Z$. The Cohn-Vossen formula then implies that the number $r$ of singular orbits in $B_{i}$ is at most two. If $r=1$, then $B_{i} \simeq S^{1} \times D^{2}$. If $r=2$, then a straightforward argument shows that $B_{i}$ is homeomorphic to $K^{2} \tilde{\times} I \simeq \mathrm{Mö} \tilde{\times} S^{1}$.

Suppose next that $Z$ has non-empty boundary. If $\partial Z$ is compact, then Corollary 14.4 implies that $Z$ is isometric to $S^{1} \times[0, \infty)$. It follows from Theorem 0.3 that $B_{i} \simeq S^{1} \times D^{2}$. If $\partial Z$ is noncompact, it follows from a way similar to the previous argument that $Z \simeq[0, \infty) \times \mathbb{R}$ and that the number $r$ of singular orbits in $B_{i}$ is at most one. If $r=1$, then $B_{i} \simeq P^{3}-D^{3}$. If $r=0$, then $B_{i} \simeq D^{3}$. In any case, we have a contradiction to $\partial B_{i} \simeq T^{2}$. This completes the proof of Theorem 0.5.
q.e.d.

Proof of Corollary 0.7. Rescale the metric of $M_{i}$ so that the diameter of the new metric is equal to one. Passing to a subsequence, we may assume that $M_{i}$ with the new metric converges to a nonnegatively curved Alexandrov space $Y$ with positive dimension. If $\operatorname{dim} Y \leq 2$ or $\pi_{1}\left(M_{i}\right)$ is infinite, then Theorems $0.2,0.3$ and 0.5 together with the result of [13] mentioned in Introduction imply that a finite cover of $M_{i}$ is homeomorphic to either $S^{3}, S^{1} \times S^{2}, T^{3}$ or a nilmanifold. If $\operatorname{dim} Y=3$, then Stability Theorem 2.4 yields that $M_{i}$ is homeomorphic to $Y$. q.e.d.

Proof of Corollary 0.9. This is done by contradiction. Suppose that the corollary does not hold. Then we have a sequence $M_{i} \in \mathcal{M}(3, D)$ of closed, prime three-dimensional Riemannian manifolds with infinite fundamental groups such that the volume of $M_{i}$ goes to zero as $i \rightarrow \infty$ and that $M_{i}$ does not admit a geometric structure. We may assume that each $M_{i}$ is orientable. Passing to a subsequence, we may assume that $M_{i}$ collapses to a compact Alexandrov space $X$. If $X$ is a point, it follows from [13] together with the infiniteness assumption on the fundamental groups that $M_{i}$ admits a geometric structure modelled on either $S^{1} \times \mathbb{R}$, $\mathbb{R}^{3}$ or Nil, a contradiction. If $\operatorname{dim} X=1$ or 2 , then Theorems 0.2 ,
0.3 and 0.5 together imply that $M_{i}$ is homeomorphic to a Seifert fibred space or a infrasolvmanifold, and hence admits a geometric structure (see [29]). q.e.d.

In view of the above proof, the infiniteness assumption on the fundamental groups in Corollary 0.9 can be replaced by a lower diameter bound. Namely we have the following corollary by the same argument.

Corollary 6.2. For any positive numbers $\delta \leq D$, there exists a constant $\epsilon=\epsilon(\delta, D)>0$ such that if a closed, prime three-manifold admits a Riemannian metric contained in $\mathcal{M}(3, D)$ with diameter $\geq \delta$ and volume $<\epsilon$, then it admits a geometric structure modelled on one of the seven geometries except $H^{3}$.

Proof of Corollary 0.10. This is done by contradiction. Suppose that the corollary does not hold. Then we have a sequence $M_{i} \in$ $\mathcal{M}(3, D)$ of pairwise non-homeomorphic closed three-dimensional Riemannian manifolds admitting no geometric structures. We may assume that each $M_{i}$ is orientable. Passing to a subsequence, we may assume that $M_{i}$ converges to a compact Alexandrov space $X$. By Stability Theorem 2.4 we only have to consider the collapsing case $\operatorname{dim} X \leq 2$. If $\operatorname{dim} X=0$, we can rescale the metric of $M_{i}$ so that the new metric has diameter $=1$. Thus we may consider that $\operatorname{dim} X=1$ or 2 . Theorems $0.2,0.3$ and 0.5 then imply that $M_{i}$ is homeomorphic to a Seifert fibred space or a infrasolvmanifold, and hence admits a geometric structure. q.e.d.

## 7. Construction of collapsing metrics

In this section, we prove Theorem 0.6 by constructing collapsing metrics together with some examples. First we show that an infinite sequence of pairwise non-homeomorphic prism manifolds collapses to a closed interval under $K=1$.

Example 7.1. Let $M$ be a prism manifold $S^{3} / \Gamma$, where $\Gamma \subset S O(4)$ is one of the following two types (see [36]):

Type 1) $\Gamma$ is generated by

$$
\gamma_{1}=\left(\begin{array}{cc}
R(1 / m) & 0 \\
0 & R(r / m)
\end{array}\right), \quad \gamma_{2}=\left(\begin{array}{cc}
0 & I \\
R(2 \ell / n) & 0
\end{array}\right)
$$

where $n$ is even, $(n(r-1), m)=1, r \not \equiv r^{2} \equiv 1 \bmod m,(\ell, n / 2)=1$ and

$$
R(\theta)=\left(\begin{array}{cc}
\cos 2 \pi \theta & -\sin 2 \pi \theta \\
\sin 2 \pi \theta & \cos 2 \pi \theta
\end{array}\right)
$$

Type 2) $\Gamma$ is generated by

$$
\sigma_{1}=\left(\begin{array}{cc}
R\left(\frac{u+v}{u v}\right) & 0 \\
0 & R\left(\frac{v-u}{u v}\right)
\end{array}\right), \quad \sigma_{2}=\left(\begin{array}{cc}
0 & I \\
-I & 0
\end{array}\right),
$$

where $v$ is even, $(u, v)=1$.
For Type 1), take distinct prime numbers $p$ and $q$ and consider the group $\Gamma_{p q}$ of Type 1) defined by $m=p, r=p-1, n=2 q$ and $\ell=1$. The group generated by $\gamma_{1}$ and the group generated by $\gamma_{2}^{2}$ converge to distinct circle groups. This implies that the subgroup $\Lambda_{p q}$ generated by $\gamma_{1}$ and $\gamma_{2}^{2}$ has index two in $\Gamma_{p q}$ and converges to $T^{2}$. Since the limit of the $\mathbb{Z}_{2}$-action on $S^{3}(1) / \Lambda_{p q}$ induced by $\gamma_{2}$ changes the orientation of $[0, \pi / 2]$, we conclude that $S^{3}(1) / \Gamma_{p q}$ converge to $[0, \pi / 4]$ under $K \equiv 1$.

For Type 2), consider a sequence $\Gamma_{p}$ of groups of Type 2) defined by $u=p$, an odd prime number and $v=p+1$. Then similarly one can verify that as $p \rightarrow \infty, S^{3}(1) / \Gamma_{p}$ converges to $[0, \pi / 4]$ under $K \equiv 1$.

The case $\operatorname{dim} X=2$ is covered by the following two examples. We denote by $D^{n}(\epsilon)$ and $S^{n-1}(\epsilon)=\partial D^{n}(\epsilon)$ the $n$-disk and the ( $n-1$ )-sphere of radius $\epsilon$.

Proof of Theorem 0.6. (I) Let $M^{n}$ be a Seifert fibred space over an ( $n-1$ )-dimensional smooth compact orbifold $X$ without boundary. Then it admits a local $S^{1}$-action, which defines a pure-polarized Fstructure on $M^{n}$. Therefore $M^{n}$ admits a sequence of metrics which collapses to $X$ with bounded curvature $|K| \leq \Lambda$ for some constant $\Lambda>0$. See [10] for details.
(II) Let $N^{3}$ be a Seifert fibred space over two-dimensional smooth compact orbifold $X$ with boundary. We suppose that $X$ has a product metric $\partial X \times[0, \delta)$ near the boundary $\partial X$. As in (I), one can construct a Riemannian metric $h_{\epsilon}$ on $N$ such that:
(a) $\left(N, h_{\epsilon}\right)$ collapses to $X$ with $|K| \leq \Lambda$.
(b) Near the boundary $\left(N, h_{\epsilon}\right)$ is the product of a collar neighborhood of $\partial X$ and $S^{1}(\epsilon)$.

Let $D_{\epsilon}^{2}$ denote a disk with a metric such that:
(c) The diameter of $D_{\epsilon}^{2}$ is less than $10 \epsilon$.
(d) The curvature of $D_{\epsilon}^{2}$ is nonnegative and its metric is a product metric $S^{1}(\epsilon) \times[0, \delta)$ near the boundary.
If $\left(M, g_{\epsilon}\right)$ denotes the union $\left(N, h_{\epsilon}\right) \cup\left(\partial X \times D_{\epsilon}^{2}\right)$ glued along their boundaries, then it converges to $X$ under a lower curvature bound $K \geq-\Lambda$. Note that $\lim _{\epsilon \rightarrow 0} \sup K_{g_{\epsilon}}=+\infty$.
(III) Let $M, X=[0, \ell], F=S^{2}$ or $=T^{2}, A, B, C$ be as in Theorem 0.6. Namely, $F=\partial B=\partial C, A=F \times[0, \ell]$ and $M \simeq B \cup_{\varphi} C$, where $\varphi: \partial B \rightarrow \partial C$ is the gluing map.

Let $T_{\epsilon}^{2}$ be the flat torus of square of length $\epsilon$. Let $F_{\epsilon}$ denote $S^{2}(\epsilon)$ or $T_{\epsilon}^{2}$, and $A_{\epsilon}=F_{\epsilon} \times[0, \ell]$.

Case (1) $F=S^{2}$.
Then $M$ is homeomorphic to $S^{3}, P^{3}$ or $P^{3} \# P^{3}$. Let $D^{3}(\epsilon)$ be a three-disk with a metric satisfying conditions similar to (c), (d) in (II). Let $B_{\epsilon}$ and $C_{\epsilon}$ be either $D^{3}(\epsilon)$ or a projective space $P^{3}-\operatorname{int} D^{3}$ with a disk removed equipped with a metric satisfying conditions similar to (c), (d) in (II). In either case, $\left(M, g_{\epsilon}\right)=B_{\epsilon} \cup A_{\epsilon} \cup C_{\epsilon}$ is a smooth Riemannian manifold and converges to $[0, \ell]$ under $K \geq 0$.

Case (2) $F=T^{2}$.
First consider the following
Case (2-i) $(B, C)=\left(S^{1} \times D^{2}, S^{1} \times D^{2}\right)$.
Then $M$ is homeomorphic to either $S^{3}, S^{2} \times S^{1}$ or a lens space. If $M \simeq S^{2} \times S^{1}$, then the union $\left(M, g_{\epsilon}\right)=S^{1}(\epsilon) \times D^{2}(\epsilon) \cup A_{\epsilon} \cup S^{1}(\epsilon) \times D^{2}(\epsilon)$ is a smooth Riemannian manifold with nonnegative curvature which converges to $[0, \ell]$ as $\epsilon \rightarrow 0$.

Lemma 7.2. Let $T^{m}$ act effectively on a compact smooth manifold $M^{n}$, and $g$ a $T^{m}$-invariant metric on $M^{n}$. Then there exists a sequence of $T^{m}$-invariant metrics $g_{i}$ on $M^{n}$ such that $\left(M^{n}, g_{i}\right)$ collapses to $\left(M^{n} / T^{m}, \bar{g}\right)$ under a lower curvature bound $K \geq-\Lambda$, where $\bar{g}$ is the quotient metric.

Proof. Let $g_{i}$ be the metrics constructed in Example 1.2(c) in [37] so that ( $M^{n}, g_{i}$ ) collapses to ( $M^{n} / T^{m}, \bar{g}$ ) under a lower curvature bound $K \geq-\Lambda$. Then the commutativity of $T^{m}$ implies the $T^{m}$-invariance of $g_{i}$. q.e.d.

Let $S^{3} / \Gamma$ be any lens space and $T^{2} \subset S O(4)$ the maximal torus. By Lemma 7.2, we have a sequence $g_{i}$ of $T^{2}$-invariant metrics on $S^{3}$ collapsing to $S^{3}(1) / T^{2}=[0, \pi / 2]$ under $K \geq-\Lambda$. Note that $g_{i}$ descends to a metric $\bar{g}_{i}$ on $S^{3} / \Gamma$. Thus, $\left(S^{3} / \Gamma, \bar{g}_{i}\right)$ collapses to $[0, \pi / 2]$ under $K \geq-\Lambda$.

Case (2-ii) $(B, C)=\left(S^{1} \times D^{2}\right.$, Mö $\left.\tilde{\times} S^{1}\right)$.
In this case we show that $M$ is homeomorphic to either $S^{1} \times S^{2}$, $P^{3} \# P^{3}$ or a prism manifold.

Case (2-ii-a) $\varphi\left(\partial D^{2}\right.$-factor) $=S^{1}$-factor.
In this case we come to the situation of Propositon 5.2. Namely it is the case when the base surface $X$ is a Möbius band with no singular points $(g=k=1, \ell=0)$. It follows that $M \simeq S^{1} \times S^{2}$. Let $B_{\epsilon}=$ $S^{1}(\epsilon) \times D^{2}(\epsilon)$ and $C_{\epsilon}=\left(\mathrm{Mö} \tilde{\times} S^{1}, h_{\epsilon}\right)$ such that $\partial C_{\epsilon}$ is isometric to $T_{\epsilon}^{2}, \operatorname{diam}\left(C_{\epsilon}\right)<10 \epsilon$ and $K_{h_{\epsilon}}=0$. Consider now the union $\left(M, g_{\epsilon}\right)=$ $B_{\epsilon} \cup A_{\epsilon} \cup C_{\epsilon}$. Since $\varphi: T_{\epsilon}^{2} \rightarrow T_{\epsilon}^{2}$ is an isometry in this case, $\left(M, g_{\epsilon}\right)$ is a smooth Riemannian manifold and converges to the closed interval $[0, \ell]$ under $K \geq 0$.

Case (2-ii-b) The case other than Case (2-ii-a).
In this case $M$ is a Seifert fibred space over $P^{2}$, where the number $r$ of singular fibres satisfies $r \leq 1$. If $r=0$, namely, $\varphi\left(S^{1}\right.$-factor $)=$ $S^{1}$-factor, $\varphi\left(\partial D^{2}\right.$-factor) $=\partial$ Mö-factor, then $M \simeq P^{2} \tilde{\times} S^{1}=P^{3} \# P^{3}$. Consider the union $M_{\epsilon}=B_{\epsilon} \cup A_{\epsilon} \cup C_{\epsilon}$, where $B_{\epsilon}$ and $C_{\epsilon}$ are as in Case (2-ii-a). Note that $\varphi: T_{\epsilon}^{2} \rightarrow T_{\epsilon}^{2}$ is an isometry in this case. Hence $M_{\epsilon}$ is a smooth Riemannian manifold and converges to the closed interval [ $0, \ell]$ under $K \geq 0$.

If $r=1$, it follows (see [24]) that $M$ is a prism manifold.

## Case (3) $(B, C)=\left(\right.$ Mö $\tilde{\times} S^{1}$, Мö $\left.\tilde{\times} S^{1}\right)$.

In this case, $M$ is doubly covered by a $T^{2}$-bundle over $S^{1}$. In particular $M$ admits a geometric structure modelled on $\mathbb{R}^{3}$, Nil or Sol (see [29]). We show that every such $M \simeq B \cup_{\varphi} C$ actually admits a sequence of metrics collapsing to a closed interval with bounded curvature $|K| \leq \Lambda$.

First consider the monodoromy matrix $J \in S L(2, \mathbb{Z})$ induced by the homomorphism $\varphi_{*}$ on the first homology group of the torus. If $J$ can be represented by

$$
\left(\begin{array}{cc}
1 & n \\
0 & 1
\end{array}\right)
$$

then $M$ is finitely covered by $T^{3}(n=0)$ or a nilmanifold $(n \neq 0)$. Suppose now that $n \neq 0$. Then $M$ is a circle bundle over the Klein bottle

$$
K^{2}=\mathrm{Mö} \cup_{S^{1}} \mathrm{Mö} .
$$

It follows from a straightforward argument that a suitable double cover $\hat{M}$ admits a sequence of metrics $g_{i}$ such that:
(a) $\lim _{i \rightarrow \infty}\left|K_{g_{i}}\right|=0$.
(b) $\left(\hat{M}, g_{i}\right)$ collapses to $S^{1}$.
(c) $g_{i}$ is invariant under the non-trivial deck transformation of $\hat{M} \rightarrow$ $M$.

Thus $g_{i}$ descends to a metric on $M$ collapsing to a closed interval.
For a specific example, see Example 7.3.
Finally consider the other case that $J$ is of hyperbolic type:

$$
J \sim\left(\begin{array}{cc}
e^{t} & 0 \\
0 & e^{-t}
\end{array}\right)
$$

where $t \neq 0$. In this case, $M$ is doubly covered by a solvmanifold $\hat{M}$, which is a $T^{2}$-bundle over a circle. It is standard to construct a sequence of metrics on $\hat{M}$ such that:
(a) $\left|K_{g_{i}}\right| \leq \Lambda$.
(b) $\left(\hat{M}, g_{i}\right)$ converges to $S^{1}$.
(c) $g_{i}$ is invariant under the non-trivial deck transformation of $\hat{M} \rightarrow$ $M$.

Thus $g_{i}$ descends to a metric on $M$ converging to a closed interval. This completes the proof of Theorem 0.6. q.e.d.

Example 7.3. Let us consider the Heisenberg group $N$ consisting of all $3 \times 3$ real upper triangular matrices of the form

$$
\left(\begin{array}{lll}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{array}\right)
$$

with the left invariant metric

$$
g_{\epsilon}=d x^{2}+\epsilon^{2} d y^{2}+\epsilon^{4}(d z-x d y)^{2} .
$$

Let $\Lambda$ be the integer lattice of $N$ and $\gamma$ the isometry of $\left(N, g_{\epsilon}\right)$ defined as

$$
\gamma(x, y, z)=(-x, y+1,-z) .
$$

Then the group $\Gamma$ generated by $\Lambda$ and $\gamma$ is discrete and contains $\Lambda$ as a normal subgroup of index two. It is easily verified that as $\epsilon \rightarrow 0$, $\left(N / \Gamma, g_{\epsilon}\right)$ collapses to a closed interval under $\lim _{\epsilon \rightarrow 0}\left|K_{g_{\epsilon}}\right|=0$.

Corollary 7.4. Let $M_{i}$ be a convergent sequence in $\mathcal{M}(3, D)$ of closed orientable three-dimensional Riemannian manifolds with finite fundamental groups. Suppose that the limit $X$ of $M_{i}$ has boundary as an Alexandrov space. Then $M_{i}$ is homeomorphic to either $S^{3}$, a lens space or a prism manifold.

Proof. This follows from the discussion above and Corollary 0.4.
q.e.d.

## 8. Comparison with bounded curvature collapsing

In the bounded curvature case, the collapsing phenomena are well understood (see [12], [11], [8], etc.). Since we know no reference for the following result however, we give a proof.

Proposition 8.1. Let $M_{i}, i=1,2, \ldots$, be a sequence of closed $n$-dimensional Riemannian manifolds with $|K| \leq 1, \operatorname{diam}\left(M_{i}\right) \leq D$ converging to a space $X$ of dimension $(n-1)$. Then:
(1) $M_{i}$ is a Seifert fibred space over $X$ for large $i$.
(2) If each $M_{i}$ is orientable, then the Alexandrov space $X$ has no boundary.

Proposition 8.1 (2) explains a difference between the bounded curvature collapsing and the lower curvature collapsing.

Proof. We follow an argument in [12]. Let $B(r)$ denote the metric ball $B\left(0, r ; \mathbb{R}^{n}\right)$. In particular, we use the notation $B=B(1)$ and $B^{\prime}=B(2)$. For $p \in X$ and $p_{i} \in M_{i}$ with $p_{i} \rightarrow p$, let $g_{i}$ denote the pullback metric on $B^{\prime}$ of the metric of $M_{i}$ via the exponential map $f_{i}=\exp _{p_{i}}: B^{\prime} \rightarrow M_{i}$. Let $\Gamma_{i}$ denote the pseudogroup of isometric
imbeddings $\gamma:\left(B, g_{i}\right) \rightarrow\left(B^{\prime}, g_{i}\right)$ such that $f_{i} \circ \gamma=f_{i}$. Then $\left(B, g_{i}\right) / \Gamma_{i}$ is isometric to $B\left(p_{i}, 1 ; M_{i}\right)$. Passing to a subsequence, we may assume that $\left(\left(B, g_{i}\right), \Gamma_{i}\right)$ converges to $\left(\left(B, g_{\infty}\right), G\right)$, where $g_{\infty}$ is a $C^{1, \alpha}$-metric, $G$ is a pseudogroup of isometric imbeddings $\gamma:\left(B, g_{\infty}\right) \rightarrow\left(B^{\prime}, g_{\infty}\right)$ and $\left(B, g_{\infty}\right) / G$ is isometric to $B(p, 1 ; X)$. Note that $G$ is locally isomorphic to a Lie group ([12]). We consider the isotropy group $I_{p}=\{g \in G \mid g \bar{p}=$ $\bar{p}\}$, where $\bar{p}$ denote the origin of $B$. Since $\operatorname{dim} G=1, I_{p}$ should be finite. Otherwise $I_{p}$ would contain $G_{0}$, the identity component of $G$, and the orbit $G \bar{p}$ would be a finite set, a contradiction. Then it is straightforward ([12]) to show that there exists a $\delta>0$ such that $B(p, \delta ; X)$ is isometric to $\left(B(\delta), g_{\infty}\right) / I_{p} G_{0}$. Let $V$ be the $(n-1)$-plane in $T_{\bar{p}} B$ perpendicular to $G_{0} \bar{p}$. Then $I_{p}$ acts on $V$ as linear isotoropy representation. If $U$ is a small ball in $V$ around the origin, then $U / I_{p}$ is almost isometric to $\left(B(\delta), g_{\infty}\right) / I_{p} G_{0} \equiv B(p, \delta ; X)$. If each $M_{i}$ is orientable, $\Gamma_{i}$ preserves the orientation, and so does $G$. This implies that $I_{p}$ preserves the orientation of $U$ and hence $X$ has empty boundary.

By [12], we have a map $f: M_{i} \rightarrow X$ such that $f^{-1}(p) \simeq S^{1} / I_{p} \simeq S^{1}$. Thus $M_{i}$ is foliated by circles and $I_{p}$ is a cyclic group, say $\mathbb{Z}_{m}$. Note that $I_{p}$ is lower semicontinuous with respect to $p$, namely $I_{q} \subset I_{p}$ for any $q$ sufficiently close to $p$. This implies that there exists a small neighborhood $D$ of $p$ in $X$ such that an $m$-fold cyclic covering of $f^{-1}(D)$ is isomorphic to the product $D \times S^{1}$ as $S^{1}$-foliated manifolds. q.e.d.

Remark 8.2. By using the argument above, we can also prove Theorem 0.5 in the case when $|K| \leq 1$ and the limit is a closed interval. We leave it as reader's exercise.

By the construction of metrics in Section 7 and the following examples, we obtain that for any given fibre and collapsing data a collapsing satisfying the data actually occurs in the bounded curvature. To see this, it suffices to consider the situation that a sequence of closed orientable three-manifolds $M_{i}$ collapses to a closed interval with $|K| \leq 1$. In this case, we have the fibre and collapsing data: $F=T^{2},(B, C)$ is one of $\left(S^{1} \times D^{2}, S^{1} \times D^{2}\right),\left(S^{1} \times D^{2}\right.$, Мö $\left.\tilde{\times} S^{1}\right)$ and (Мö $\tilde{\times} S^{1}$, Мö $\left.\tilde{\times} S^{1}\right)$.

Lemma 8.3. Let $\left(T^{2}, g_{0}\right)$ be the flat torus of rectangle with side lengths $a$ and $b$, and let $\left(T^{2}, g_{1}\right)$ be the flat torus of parallelogram with the same sidelengths $a, b$ and angle $\theta$. Then there exists a metric $g$ on
$T^{2} \times[0,1]$ such that $\left|K_{g}\right| \leq \tau(\cos \theta)$ and

$$
\left.g\right|_{T^{2} \times t}= \begin{cases}g_{0} & \text { near } t=0 \\ g_{1} & \text { near } t=1\end{cases}
$$

Proof. Putting $\epsilon=\cos \theta$, choose a monotone non-decreasing function $\psi:[0,1] \rightarrow \mathbb{R}$ with

$$
\psi(t)= \begin{cases}0 & \text { near } t=0 \\ \epsilon & \text { near } t=1\end{cases}
$$

and $\sup \left\{\left|\psi^{\prime}(t)\right|, \mid \psi^{\prime \prime}(t)\right\}<\tau(\epsilon)$. Let $T^{2}=\mathbb{R}^{2} / \Gamma$, where $\Gamma=a \mathbb{Z} \times b \mathbb{Z}$. Using the canonical coordinates $(x, y, t) \in T^{2} \times[0,1]$, we define the metric $g$ by

$$
g=d t^{2}+d x^{2}+\psi(t) d x d y+d y^{2} .
$$

Then a standard calculation shows that $\left|K_{g}\right| \leq \tau(\epsilon)$. q.e.d.
Example 8.4. First consider the collapsing of $S^{2} \times S^{1}$ with possible data on $(B, C)$. Let $\Gamma_{i}, i \geq 1$, be the subgroup of $\operatorname{Isom}\left(S^{2}(1) \times \mathbb{R}\right)$ generated by

$$
\gamma(x, t)=\left(R(1 / i) x, t+1 / i^{2}\right),
$$

where

$$
R(\theta)=\left(\begin{array}{ccc}
\cos 2 \pi \theta & -\sin 2 \pi \theta & 0 \\
\sin 2 \pi \theta & \cos 2 \pi \theta & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

Then $\left(S^{2}(1) \times \mathbb{R}\right) / \Gamma_{i}$ is homeomorphic to $S^{2} \times S^{1}$ and converges to $[0, \pi]$ under $0 \leq K \leq 1$. In this case, $(B, C)=\left(S^{1} \times D^{2}, S^{1} \times D^{2}\right)$.

In the above construction, we slightly change the metric of $S^{2}(1)$ so that the new metric, say $g$, has the product metric $S^{1}(1) \times(-\epsilon, \epsilon)$ near the equator $S^{1}(1) \subset S^{2}(1)$. Note that $\left(\left(S^{2}, g\right) \times \mathbb{R}\right) / \Gamma_{i}$ still converges to a closed interval under $0 \leq K \leq 1$. Next we consider $P_{i}=\left(\left(S_{+}^{2}, g\right) \times\right.$ $\mathbb{R}) / \Gamma_{i}$, where $S_{+}^{2}$ denotes the closed upper hemisphere. The boundary of $P_{i}$ is isometric to the flat torus of parallelogram with side lengths $a_{i}=\sqrt{4 \pi^{2} / i^{2}+1 / i^{4}}, b_{i}=1 / i$ and angle $\theta_{i}$, where $\lim _{i \rightarrow \infty} \theta_{i}=\pi / 2$. Note that $a_{i}$ (resp. $b_{i}$ ) corresponds to the $\partial D^{2}$-factor (resp. the $S^{1}$ factor) of $\partial P_{i} \simeq \partial D^{2} \times S^{1}$. Choose a flat metric $h_{i}$ on Mö $\tilde{x} S^{1}$ such that:
(1) The boundary of ( $\mathrm{M} \ddot{\times} \tilde{x} S^{1}, h_{i}$ ) is isometric to the flat torus of rectangle with side lengths $a_{i}$ and $b_{i}$.
(2) The length of $S^{1}$-factor is $a_{i}$.
(3) The length of $\partial \mathrm{Mö}$-factor is $b_{i}$.

Let $\left(T^{2} \times[0,1], g_{i}\right)$ be the metric constructed in Lemma 8.3 for $\left(a_{i}, b_{i}, \theta_{i}\right)$. Then the union $\left(\mathrm{Mö} \tilde{\times} S^{1}, h_{i}\right) \cup\left(T^{2} \times[0,1], g_{i}\right) \cup P_{i}$ defined in an obvious way is a smooth Riemannian manifolds homeomorphic to $S^{2} \times S^{1}$ (see Proposition 5.2) and collapses to a closed interval under $-\tau\left(\cos \theta_{i}\right) \leq$ $K \leq 1$. In this case, $(B, C)=\left(S^{1} \times D^{2}\right.$, Mö $\left.\tilde{\times} S^{1}\right)$.

Next take a flat metric $k_{i}$ on Mö $\tilde{\times} S^{1}$ such that:
(1) The boundary of ( $\mathrm{Mö} \tilde{x} S^{1}, k_{i}$ ) is isometric to the flat torus of rectangle with side lengths $a_{i}$ and $b_{i}$.
(2) The length of $\partial$ Mö-factor is $a_{i}$.
(3) The length of $S^{1}$-factor is $b_{i}$.

Then the union $\left(\mathrm{Mö} \tilde{\times} S^{1}, k_{i}\right) \cup\left(T^{2} \times[0,1], g_{i}\right) \cup P_{i}$ defined in an obvious way is a smooth Riemannian manifolds homeomorphic to $P^{3} \# P^{3}$ and collapses to a closed interval under $-\tau\left(\cos \theta_{i}\right) \leq K \leq 1$. In this case, $(B, C)=\left(S^{1} \times D^{2}, \mathrm{Möx} S^{1}\right)$.

By our argument, we can summarize the results on both lower curvature collapsingand bounded curvature collapsing of oriented threemanifolds in Table 1.

## Part 2. Classification of complete open Alexandrov three-spaces of nonnegative curvature

The argument in Section 3 shows that the geometry of complete open Alexandrov spaces of dimension three is important to obtain an essential topological information on a small neighborhood of the manifold near the singular point of the limit space. In Part 2, we give a classification of three-dimensional complete open Alexandrov spaces with nonnegative curvature extending the Cheeger-Gromoll classification ([9]) of three-dimensional complete open Riemannian manifolds with nonnegative curvature.

| $X$ | $\|K\| \leq 1$ | $K \geq-1$ | fibre data |
| :---: | :---: | :---: | :---: |
| $\partial X^{2}=\emptyset$ | Seifert bundle | Seifert bundle | $F=S^{1}$ |
| $\partial X^{2} \neq \emptyset$ |  | $\begin{gathered} S^{3} \# S^{2} \times S^{1} \# \cdots \\ \# L\left(\mu_{1}, \nu_{1}\right) \# \cdots \end{gathered}$ | $F=S^{1}$ |
| $S^{1}$ |  | $S^{2} \times S^{1}$ | $F=S^{2}$ |
|  | Flat, Nil, Sol | Flat, Nil, Sol | $F=T^{2}$ |
| [0, थ] |  | $S^{3}, P^{3}, P^{3} \# P^{3}$ | $F=S^{2}$ |
|  | $S^{3} / \mathbb{Z}_{p}, S^{2} \times S^{1}$ | $S^{3} / \mathbb{Z}_{p}, S^{2} \times S^{1}$ | $\begin{gathered} B, C=S^{1} \times D^{2}, \\ F=T^{2} \end{gathered}$ |
|  | $\begin{aligned} & S^{2} \times S^{1}, P^{3} \# P^{3}, \\ & \text { prism manifolds } \end{aligned}$ | $\begin{gathered} S^{2} \times S^{1}, P^{3} \# P^{3}, \\ \text { prism manifolds } \end{gathered}$ | $\begin{gathered} B=S^{1} \times D^{2}, \\ C=\mathrm{Mö} \tilde{\times} S^{1}, \\ F=T^{2} \end{gathered}$ |
|  | Flat, Nil, Sol | Flat, Nil, Sol | $\begin{gathered} B, C=\mathrm{Mö} \tilde{\times} S^{1}, \\ F=T^{2} \end{gathered}$ |
| point | Flat, Nil | $\begin{gathered} S^{2} \times S^{1}, P^{3} \# P^{3}, \\ \text { Flat, Nil, } \\ \sim S^{3} / \Gamma \\ \hline \end{gathered}$ |  |

Table 1: Collapsing in dimension three

## 9. Examples, results and rigidity

First we recall the basic construction in [9] for complete open Riemannian manifold of nonnegative curvature. This can be done with the same procedure for Alexandrov spaces. Let $X$ be an $n$-dimensional complete noncompact Alexandrov space with nonnegative curvature. For a geodesic ray $\gamma:[0, \infty) \rightarrow X$, consider the Busemann function $b_{\gamma}: X \rightarrow \mathbb{R}$ defined by

$$
b_{\gamma}(x)=\lim _{t \rightarrow \infty} t-d(x, \gamma(t)) .
$$

It is straightforward to see
Lemma 9.1. $b_{\gamma}$ is convex.
Now we consider the Busemann function associated with a point $p \in X$ defined by $b(x)=\sup _{\gamma} b_{\gamma}(x)$, where $\gamma$ runs over all the geodesic rays emanating from $p$. The function $b$ is convex and the sublevel sets $b^{-1}(-\infty, a]$ are compact.

Let $C \subset X$ be a closed totally convex subset, and $\partial C$ denote the boundary of $C$ as an Alexandrov space. We consider the distance function $\rho_{C}=\operatorname{dist}(\partial C, \cdot)$ on $C$.

Lemma 9.2 ([26]). $\quad \rho_{C}$ is concave on $C$.
Let $\mu$ be the minimum of $b$ and put $C^{0}=b^{-1}(\mu)$. Note that the totally convex set $b^{-1}(\mu)$ has empty interior. If $C^{0}$ has boundary, consider the distance function $\rho_{C^{0}}$ on $C^{0}$, and put $C^{1}$ to be the maximum set. By iteration, we have a sequence of finitely many non-empty compact totally convex sets:

$$
C^{0} \supset C^{1} \supset C^{2} \cdots \supset C^{k}
$$

where $n>\operatorname{dim} C^{0}, \operatorname{dim} C^{i}>\operatorname{dim} C^{i+1}$ and $C^{k}$ has no boundary. Then a soul $S$ of $X$ is defined as $S=C^{k}$. It was proved in [26] that $X$ is homotopy equivalent to $S$.

By the following example, the Soul Theorem ([9]) does not hold for three-dimensional Alexandrov spaces. See [26] for such an example in 5-dimension.

Example 9.3. Let $\varphi_{1}(x, y)=(-x,-y)$ and $\varphi_{2}(x, y)=(x,-y)$ be the isometric involutions of $\mathbb{R}^{2}$ and consider the product $X=[0,1] \times \mathbb{R}^{2}$. The pairs $\left(\varphi_{i}, \varphi_{j}\right)$ acts on $\partial X$ in such a way that $\varphi_{i}$ acts on $\{0\} \times \mathbb{R}^{2}$ and $\varphi_{j}$ acts on $\{1\} \times \mathbb{R}^{2}$. Now we consider the quotient spaces

$$
X_{1}=X /\left(\varphi_{1}, \varphi_{1}\right), \quad X_{2}=X /\left(\varphi_{1}, \varphi_{2}\right), \quad X_{3}=X /\left(\varphi_{2}, \varphi_{2}\right)
$$

It is easy to verify that each $X_{i}, 1 \leq i \leq 3$, is a complete open Alexandrov space with nonnegative curvature and without boundary. Note that each $X_{i}$ has one point soul $S=\{(1 / 2,0,0)\}$, and that the normal bundle $N(S)$ is homeomorphic to $\mathbb{R}^{3}$. Clearly we have

$$
X_{1} \simeq K_{1}\left(P^{2}\right) \cup_{D^{2}} K_{1}\left(P^{2}\right), \quad X_{2} \simeq K\left(P^{2}\right), \quad X_{3} \simeq \mathbb{R}^{3}
$$

where $K_{1}\left(P^{2}\right)$ is the open unit cone over the projective plane $P^{2}$ and $D^{2}$ is a disk in $\partial K_{1}\left(P^{2}\right)$. Note also that $X_{1}$ is isometric to $\mathbb{R}^{3} / \Gamma$, where $\Gamma$ is the discrete subgroup of isometries of $\mathbb{R}^{3}$ generated by $\gamma(x, y, z)=$ $-(x, y, z)$ and $\sigma(x, y, z)=(x+2, y, z)$.

We recall that in the case when $X$ is a Riemannian manifold, $X$ is isometric to the normal bundle $N(S)$ if the soul $S$ is of codimension one (see [9]). As the discussion below shows, this does not hold for Alexandrov spaces even in the three-dimensional case.

Let $S$ be a compact nonnegatively curved Alexandrov surface without boundary, and $p_{1}, \ldots, p_{k} \in S$ essential singular points of $S$. Let us consider a three-dimensional Alexandrov space of nonnegative curvature (if it exits), denoted by $L\left(S ; p_{1}, \ldots, p_{k}\right)$ or simply $L(S ; k)$, satisfying the following :
(1) $S$ is isometrically imbedded as a totally convex set of $L\left(S ; p_{1}, \ldots\right.$, $\left.p_{k}\right)$.
(2) $\left\{p_{1}, \ldots, p_{k}\right\}$ are the set of topological singular points of $L\left(S ; p_{1}\right.$, $\left.\ldots, p_{k}\right)$, and hence the space of directions at $p_{i}$ is homeomorphic to $P^{2}$.
(3) The space of directions at each $x \in S-\left\{p_{1}, \ldots, p_{k}\right\}$ is isometric to the spherical suspension over $\Sigma_{x}(S)$.
(4) The normal bundle $\mathcal{N}$ of $S-\left\{p_{1}, \ldots, p_{k}\right\}$ in $L\left(S ; p_{1}, \ldots, p_{k}\right)$ has a locally product metric.
(5) The normal exponential map $\exp : \mathcal{N} \rightarrow L\left(S ; p_{1}, \ldots, p_{k}\right)$ carries each ray in the fibre from the zero section to a geodesic ray in $L\left(S ; p_{1}, \ldots, p_{k}\right)$.
(6) There is a unique geodesic ray $\gamma_{i}$ from $p_{i}$ perpendicular to every direction in $\Sigma_{p_{i}}(S)$, and $L\left(S ; p_{1}, \ldots, p_{k}\right)-\exp (\mathcal{N})$ consists of $\gamma_{i}$, $1 \leq i \leq k$.

We note that $S$ is a soul of $L\left(S ; p_{1}, \ldots, p_{k}\right)$.
Example 9.4. We take $S_{1} \simeq P^{2}, S_{2} \simeq S^{2}, S_{4} \simeq S^{2}$ with nonnegative curvature. We assume that each $S_{i}$ has $i$ essential singular points. Note that $S_{4}$ is isometric to $\operatorname{dbl}([0, a] \times[0, b])$ (see Proposition 14.4). Then one can define the spaces $L\left(S_{i} ; i\right)$ in an obvious way.

Proposition 9.5. If $k \geq 1, L\left(S ; p_{1}, \ldots, p_{k}\right)$ is isometric to one of $L\left(S_{1} ; 1\right), L\left(S_{2} ; 2\right)$ or $L\left(S_{4} ; 4\right)$ in Example 9.4.

Proof. Since $p_{i}$ is an essential singular point of $S$, it follows from Corollary 14.3 that $k \leq 4$. Note that the union of fibers in $\mathcal{N}$ over a small circle around $p_{i}$ is isometric to a flat Möbius strip with respect to the induced metric. It follows from a cutting and gluing argument that if $S \simeq S^{2}$ (resp. $S \simeq P^{2}$ ), then $k=2$ or 4 (resp. $k=1$ ). q.e.d.

The main purpose of Part 2 is to prove the following

Theorem 9.6 (Generalized Soul Theorem). Let $X$ be a threedimensional complete noncompact Alexandrov space with nonnegative curvature and without boundary, and $S$ a soul of $X$. Then:
(1) If $\operatorname{dim} S=0$, then $X$ is homeomorphic to $\mathbb{R}^{3}$, or the cone $K\left(P^{2}\right)$ over the projective plane $P^{2}$, or $X_{1}=\mathbb{R}^{3} / \Gamma$ in Example 9.3 .
(2) If $\operatorname{dim} S=1$, then $X$ is isometric to a quotient $(\mathbb{R} \times N) / \Lambda$, where $N$ is an Alexandrov space with nonnegative curvature homeomorphic to $\mathbb{R}^{2}$ and $\Lambda$ is an infinite cyclic group. The $\Lambda$-action is diagonal; $\Lambda$ acts on $\mathbb{R}$ by translation and on $N$ by isometries fixing a point of $N$. In particular, $X$ is homeomorphic to an $N$-bundle over a circle.
(3) If $\operatorname{dim} S=2$, then $X$ is isometric to either the normal bundle $N(S)$ of $S$ in $X$ or one of types $L\left(S_{i}, i\right), i=1,2,4$, in Proposition 9.5.

The metric of $N(S)$ in Theorem 9.6 is defined in an obvious way.
The proof of Theorem 9.6 (1) is given in Sections 10-13.
The following corollary is the direct consequence of Theorem 9.6 together with the Morse theory given in Section 10.

Corollary 9.7. Under the assumption of Theorem 9.6, suppose further that $X$ is a topological manifold. Then:
(1) $X$ is homeomorphic to the normal bundle $N(S)$.
(2) The topology of the closed ball $B=B(S, r)$ around $S$ is determined as follows:
(a) If $\operatorname{dim} S=0$, then $B$ is homeomorphic to $D^{3}$.
(b) If $\operatorname{dim} S=1$, then $B$ is homeomorphic to $S^{1} \times D^{2}$ or a solid Klein bottle.
(c) If $\operatorname{dim} S=2$, then $B$ is homeomorphic to the normal I-bundle $N_{1}(S)$ of $S$ in $X$.

Theorem 9.6 (2) and a part of (3) are the special cases of the following:

Theorem 9.8. Let $X$ be an $n$-dimensional complete noncompact Alexandrov space with nonnegative curvature and empty boundary, and $S$ a soul of $X$.
(1) If $\operatorname{dim} S=1$, then $X$ is isometric to a quotient $(\mathbb{R} \times N) / \Lambda$, where $N$ is an Alexandrov space with nonnegative curvature whose soul is a point and $\Lambda \simeq \mathbb{Z}$. The $\Lambda$-action is diagonal; $\Lambda$ acts on $\mathbb{R}$ by translation and on $N$ by isometries fixing a point of $N$. Thus $X$ is homeomorphic to an $N$-bundle over a circle.
(2) If $\operatorname{dim} S=n-1$ and $X$ is a topological manifold, then $\Sigma_{p}$ is the spherical suspension over $\Sigma_{p}(S)$ for every $p \in S$ and $X$ is isometric to the normal bundle $N(S)$ of $S$ in $X$.

Proof of Theorem 9.8 (1). Suppose that $\operatorname{dim} S=1$ and let $\Lambda$ be the deck transformation group of the universal covering $\pi: \widetilde{X} \rightarrow X$. Then $S$ is isometric to a circle and $\Lambda \simeq \mathbb{Z}$. It is easy to see that $\pi^{-1}(S)$ is totally convex set isometric to $\mathbb{R}$. The splitting theorem then implies that $\widetilde{X}$ is isometric to a product $\mathbb{R} \times P$. Let $p_{1}: \Lambda \rightarrow \operatorname{Isom}(\mathbb{R}), p_{2}: \Lambda \rightarrow \operatorname{Isom}(P)$ be the projections. Then $p_{1}(\Lambda)$ acts on $\mathbb{R}$ by translation. If $\pi^{-1}(S)$ corresponds to $\mathbb{R} \times\{p\}$, then the point $p$ is a fixed point of $p_{2}(\Gamma)$. This completes the proof of Theorem 9.8(1). q.e.d.

In the Riemannian case, the Berger Comparison Theorem (cf. [7]) was used for the proof of Theorem 9.8(2). It is unknown if the Berger Comparison Theorem holds for Alexandrov spaces. Hence we need another argument for the proof.

For the proof of Theorem 9.8(2), we consider the situation that $C^{0}$ is of dimension $n-1$. In what follows, we put $C=C^{0}$ for simplicity.

We say that a direction $\xi \in \Sigma_{p}=\Sigma_{p}(X)$ at $p \in C$ is normal to $C$ if $\angle(\xi, v)=\pi / 2$ for all $v \in \Sigma_{p}(C)$.

Lemma 9.9. Suppose that $C=C^{0}$ has dimension $n-1$.
(1) Every point of $C$ has at most two normal directions to $C$.
(2) For a point $p \in \operatorname{int} C$ and a normal direction $\xi \in \Sigma_{p}(X)$, there exists a locally isometric covering map

$$
\Sigma_{\xi}\left(\Sigma_{p}(X)\right) \rightarrow \Sigma_{p}(C)
$$

of order $r \leq 2$, where:
(a) $r=1$ if and only if $p$ has two normal directions to $C$ and $\Sigma_{p}(X)$ is isometirc to the spherical suspension over $\Sigma_{p}(C)$.
(b) $r=2$ if and only if $p$ has exactly one normal direction to $C$. In this case, $p$ is an essential singular point of $X$ and $\Sigma_{p}(X)-\Sigma_{p}(C)$ is connected.

Proof. For $p \in C$, let $\xi \in \Sigma_{p}(X)$ be normal to $C$. The Alexandrov convexity implies that for every $v_{1}, v_{2} \in \Sigma_{p}(C), \xi, v_{1}$ and $v_{2}$ are the vertices of a geodesic triangle isometric to a comparison triangle in the unit sphere. Let $v \in \Sigma_{p}(C)$ be a regular point of $\Sigma_{p}(C)$. It follows that $\xi_{v}^{\prime} \in \Sigma_{v}\left(\Sigma_{p}(X)\right)$ is perpendicular to $\Sigma_{v}\left(\Sigma_{p}(C)\right)$. Thus if there were three normal directions to $C$ at $p$, we would have three directions in $\Sigma_{v}\left(\Sigma_{p}(X)\right)$ perpendicular to $\Sigma_{v}\left(\Sigma_{p}(C)\right)$, a contradiction.

By the argument above, we have a locally isometric covering map $\pi: \Sigma_{\xi}\left(\Sigma_{p}(X)\right) \rightarrow \Sigma_{p}(C)$ of order $r \leq 2$. The rest of the proof is now clear. q.e.d.

Setting $X^{t}=b^{-1}(-\infty, \mu+t]$, we have the filtration $\left\{X^{t}\right\}_{t \geq 0}$ by compact totally convex sets such that:
(1) $X^{s}=\left\{x \in X^{t} \mid d\left(x, \partial X^{t}\right) \geq t-s\right\} \quad$ for $s \leq t$.
(2) $X^{0}=C^{0}$.

A point $p \in C$ is called a one-normal point (resp. a two-normal point) if $\Sigma_{p}$ contains exactly one (resp. two) normal direction to $C$. Recall that if $p \in C$ is a one-normal point, then it is an essential singular point of $X$.

Proposition 9.10. Under the assumption of Theorem 9.8, suppose further that $\operatorname{dim} C=n-1$. Let $p, q, r, s \in X$ be such that:
(1) $p, r \in \partial X^{t}$, and $q, s \in C$.
(2) $d(p, q)=d(r, s)=t$.
(3) A minimal geodesic pr does not meet $C$.

Then $p, q, r, s$ are the vertices of a totally geodesic flat rectangle.
Proof. First we show that for every $p \in \partial X^{t}$ and $q \in C$ with $d(p, q)=t, \Sigma_{p}(X)$ is the spherical suspension over $\partial \Sigma_{p}\left(X^{t}\right)$. From the basic construction, there is a geodesic ray $\gamma$ starting from $q$, through $p$ and perpendicular to $C$. Let $p_{1}$ be the intersection point of $\gamma$ with $\partial X^{t^{\prime}}$ for a $t^{\prime}>t$. Since $p$ is the foot of both $q$ and $p_{1}$ to $\partial X^{t}$, we see that $\angle\left(q_{p}^{\prime}, \partial \Sigma_{p}\left(X^{t}\right)\right) \geq \pi / 2, \angle\left(\left(p_{1}\right)_{p}^{\prime}, \partial \Sigma_{p}\left(X^{t}\right)\right) \geq \pi / 2$. Since $\angle\left(q_{p}^{\prime},\left(p_{1}\right)_{p}^{\prime}\right)=$
$\pi$, it follows from the splitting theorem applied to $K_{p}$ that $\Sigma_{p}$ is the spherical suspension over $\partial \Sigma_{p}\left(X^{t}\right)$.

For given $p, q, r, s$, let $\gamma:[0, a] \rightarrow X$ be a minimal geodesic joining $p$ to $s$. We consider the concave function $f(u)=d\left(\gamma(u), \partial X^{t}\right)$. Put $\alpha=$ $\angle\left(\dot{\gamma}(0), \partial \Sigma_{p}\left(X^{t}\right)\right)$, where $\dot{\gamma}(0)$ denotes the direction at $\gamma(0)$ represented by $\gamma$.

Assertion 9.11. $f^{\prime}(0)=\sin \alpha$.
Proof. For arbitrary $\epsilon>0$, take a geodesic $\sigma(u)$ emanating from $p$ such that

$$
\angle\left(\dot{\sigma}(0), \partial \Sigma_{p}\right)<\epsilon, \quad|\angle(\dot{\gamma}(0), \dot{\sigma}(0))-\alpha|<\epsilon .
$$

Consider now the function $g(u)=d(\gamma(u), \sigma(u \cos \alpha))$. We note that

$$
g^{\prime}(0)=\sin \angle(\dot{\gamma}(0), \dot{\sigma}(0)) .
$$

It suffices to show that $\left|f^{\prime}(0)-g^{\prime}(0)\right|<2 \epsilon$. Clearly, $f(u) \leq g(u)+\epsilon u$ and hence $f^{\prime}(0) \leq g^{\prime}(0)+\epsilon$. Suppose that $f\left(u_{n}\right) \leq u_{n}\left(g^{\prime}(0)-2 \epsilon\right)$ for some sequence $u_{n} \rightarrow 0$. Let $p_{n}$ be a point of $\partial X^{t}$ which is closest from $\gamma\left(u_{n}\right)$. Then we would have $\tilde{\angle} \gamma\left(u_{n}\right) p p_{n} \leq \angle(\dot{\gamma}(0), \dot{\sigma}(0))-2 \epsilon$, and that $\angle \gamma\left(u_{n}\right) p p_{n} \leq \angle(\dot{\gamma}(0), \dot{\sigma}(0))-\epsilon$ for large $n$. Taking a subsequence if necessary, we may assume that the direction $v_{n}=\left(p_{n}\right)_{p}^{\prime}$ converges to a direction $v \in \Sigma_{p}$. It turns out from the lower semi-continuity of angle that $\angle(\dot{\gamma}(0), v) \leq \angle(\dot{\gamma}(0), \dot{\sigma}(0))-\epsilon$, and hence $\angle(\dot{\gamma}(0), \dot{\sigma}(0)) \geq \alpha+\epsilon$, a contradiction. q.e.d.

We put $b=d(q, s)$ and consider the triangle $\triangle p^{\prime} q^{\prime} s^{\prime}$ on $\mathbb{R}^{2}$ such that $d\left(p^{\prime}, q^{\prime}\right)=t, d\left(q^{\prime}, s^{\prime}\right)=b$ and $\angle p^{\prime} q^{\prime} s^{\prime}=\pi / 2$. Set $a^{\prime}=d\left(p^{\prime}, s^{\prime}\right)$, $\alpha^{\prime}=\angle p^{\prime} s^{\prime} q^{\prime}, \theta^{\prime}=\angle q^{\prime} p^{\prime} s^{\prime}$, and $\theta=\angle q p s=\pi / 2-\alpha$. Since $\angle p q s=\pi / 2$, we have $a^{\prime} \geq a=d(p, s)$. It follows from the concavity of $f$ that

$$
f^{\prime}(0) \geq \frac{t}{a} \geq \frac{t}{a^{\prime}} .
$$

Thus from the previous assertion, we obtain that

$$
\begin{equation*}
\alpha \geq \alpha^{\prime} \quad \text { and } \quad \theta \leq \theta^{\prime} \tag{9.1}
\end{equation*}
$$

Consider now a comparison triangle $\tilde{\triangle} p q s$ in $\mathbb{R}^{2}$ and put $\tilde{\theta}=\tilde{\angle} q p s$, $\tilde{\alpha}=\tilde{\angle} p s q$. Since we may assume for our purpose that $t>b$, it follows from an obvious consideration with $a^{\prime} \geq a>t$ that $\alpha^{\prime} \leq \tilde{\alpha} \leq \pi / 2$, $\theta^{\prime} \leq \tilde{\theta}$ and hence

$$
\begin{equation*}
\theta^{\prime}=\theta=\tilde{\theta}, \quad \alpha^{\prime}=\tilde{\alpha}=\alpha, \quad a=a^{\prime} \quad \text { and } \quad \tilde{\angle} p q s=\pi / 2 . \tag{9.2}
\end{equation*}
$$

It follows from the rigidity argument(cf.[30]) that $\triangle p q s$ spans a totally geodesic flat triangle isometric to $\tilde{\triangle} p q s$. Furthermore, $f^{\prime}(0)=t / a$. It follows from the concavity of $f$ that $f(u)=t u / a$ for all $u$. Let $x_{u}$ and $y_{u}$ be the points on $\partial X^{t}$ and $q s$ respectively such that $f(u)=d\left(\gamma(u), x_{u}\right)$ and $d\left(q, y_{u}\right)=u b / a$. Then it follows together with the comparison argument that $d\left(x_{u}, y_{u}\right) \leq d\left(x_{u}, \gamma(u)\right)+d\left(\gamma(u), y_{u}\right) \leq t$. Thus $\gamma$ lies on the minimal connections from the points of $q s$ to $\partial X^{t}$.

By repeating the argument above for $x_{u}, y_{u}, r, s$ in place of $p, q, r, s$, we conclude that the set of minimal connections $x_{u} y_{u}, 0 \leq u \leq a$, provides a totally geodesic flat rectangle. q.e.d.

Proof of Theorem $9.8(2)$. Since $\Sigma_{p} \simeq S^{n-1}$ for every $p \in S$, it follows from Lemma 9.9 that $p$ is a two-normal point, and hence the conclusion follows from Lemma 9.9(2) and Proposition 9.10. q.e.d.

Proof of Theorem 9.6 (3). If $S$ contains no one-normal point, then $X$ is isometric to $N(S)$ by Proposition 9.10. Now let $p_{1}, \ldots, p_{k}$ be the one-normal points of $S$. Proposition 9.10 implies that $X$ can be written as $X=L\left(S ; p_{1}, \ldots, p_{k}\right)$. Then the conclusion follows from Proposition 9.5. q.e.d.

## 10. Preliminaries on Morse theory

In the rest of Part 2, we shall prove Theorem 9.6(1).
We need the Morse theory for distance functions on Alexandrov spaces.

A map $\pi: E \rightarrow B$ is a topological submersion if for each $p \in E$ there are a neighborhood $U$ of $p$ in the fibre $\pi^{-1}(\pi(p))$, a neighborhood $N$ of $\pi(p)$ in $B$ and a topological imbedding $\varphi: U \times N \rightarrow E$ onto a neighborhood of $p$ such that $\pi \circ \varphi$ is the projection $U \times N \rightarrow N$. The map $\varphi$ is a product chart about $U$ for $\pi$, and the image $\varphi(U \times N)$ is a product neighborhood around $p$.

A finite dimensional topological space $Y$ is said to be a $W C S$-space if it satisfies the following (1) and (2):
(1) $Y$ is a stratified space, i.e., it has a stratification

$$
Y \supset \cdots \supset Y^{(n)} \supset \cdots \supset Y^{-1}=\phi
$$

such that $Y^{(n)}-Y^{(n-1)}$ is a topological $n$-manifold without boundary.
(2) For each $x \in Y^{(n)}-Y^{(n-1)}$ there is a cone $C$ with vertex $v$ and a homeomorphism $\rho: \mathbb{R}^{n} \times C \rightarrow Y$ onto an open neighborhood of $x$ in $Y$ such that $\rho^{-1}\left(Y^{(n)}\right)=\mathbb{R}^{n} \times\{v\}$.

Theorem 10.1 ([33]). Let $\pi: E \rightarrow B$ be a topological submersion, and $F=\pi^{-1}\left(x_{0}\right)$ the fibre over a point $x_{0}$. We assume that $F$ is a $W C S$-space.
(1) For given compact sets $A_{1}, A_{2}$ of $F$ and for open neighborhoods $U_{i}$ of $A_{i}$ in $F$, let $\varphi_{i}: U_{i} \times N_{i} \rightarrow E$ be product charts about $U_{i}$ for $\pi$. Then there exists a product chart $\varphi: U \times N \rightarrow E$ about an open set $U \supset A_{1} \cup A_{2}$ in $F$ such that

$$
\varphi= \begin{cases}\varphi_{1} & \text { near } A_{1} \times\left\{x_{0}\right\} \\ \varphi_{2} & \text { near }\left(A_{2}-U_{1}\right) \times\left\{x_{0}\right\}\end{cases}
$$

(2) If $\pi$ is proper in addition, then $F \hookrightarrow E \xrightarrow{\pi} B$ is a locally trivial fibre bundle.

Let $f=\left(f_{1}, \ldots, f_{m}\right): U \rightarrow \mathbb{R}^{m}$ be a map on an open set $U$ of $X$ defined by $f_{i}(x)=d\left(A_{i}, x\right)$ for compact subsets $A_{i} \subset X$. The map $f$ is said to be $(c, \epsilon)$-regular at $p \in U$ if there is a point $w \in X$ such that:
(1) $\angle\left(\left(A_{i}\right)_{p}^{\prime},\left(A_{j}\right)_{p}^{\prime}\right)>\pi / 2-\epsilon$.
(2) $\angle\left(w_{p}^{\prime},\left(A_{i}\right)_{p}^{\prime}\right)>\pi / 2+c$.

Theorem 10.2 ([26]). Let $X$ be an Alexandrov space with curvature bounded below, $U \subset X$ an open subset, and $f: U \rightarrow \mathbb{R}^{m}(c, \epsilon)$ regular at each point of $U$. If $\epsilon$ is small compared with $c$, then we have:
(1) $f$ is a topological submersion.
(2) If $f$ is proper in addition, then the fibres of $f$ are WCS-spaces. Hence $f$ is a locally trivial fibre bundle over its image.

We simply say that $f$ is regular on $U$ if it satisfies the assumption in Theorem 10.2.

In what follows, we use the notation in Section 9. In particular, $X$ denotes a complete open Alexandrov space with nonnegative curvature without boundary.

We first observe the following simple:

Lemma 10.3. For any compact set $K$ in $C^{0}$, let $f=d(K, \cdot)$. Then $f$ is regular on $X-C^{0}$.

Proof. For any $t>b(x)$ there is a point $y \in \partial X^{t}$ such that $d(x, y)=$ $t-b(x)$. Then it follows from the total convexity that $\angle y x K>\pi / 2$. q.e.d.

Lemma 10.4. If $\operatorname{dim} C^{0}=0$, then $X$ is homeomorphic to the tangent cone $K_{p}$, where $\{p\}=C^{0}$.

Proof. This follows from Lemma 10.3 and Theorems 10.2 and 2.4. q.e.d.

The purpose of the rest of this section is to obtain the topological type of $X^{t}$ for any small $t>0$ in the case when $\operatorname{dim} X=3$ and the soul $S$ is a point. From now on we assume that $X$ is of dimension three and the dimension of $C^{0}$ is one or two. We put $C=C^{0}$ for simplicity.

In the Riemannian case, the distance function from the soul $S$ has no critical points outside $S$, which implies that any complete open Riemannian manifold with nonnegative curvature is diffeomorphic to the normal bundle over its soul ([9]). For Alexandrov spaces however, we have the following counterexamples:

Example 10.5. Let $X=\operatorname{dbl}([0, a] \times[0, \infty))$. Then the distance function from the soul point has two critical points $(0,0)$ and $(a, 0)$.

Example 10.6. We construct a complete open Alexandrov space $X$ with nonnegative curvature consisting of some building blocks each of which is a convex polyhedron in the $(x, y, z)$-space. Let $C$ be a convex polygon on the $(x, y)$-plane. We denote by $\left\{e_{i}\right\}, 1 \leq i \leq k$, the edges of $\partial C$. The largest building block is $C \times[0, \infty)$. Let $\ell_{i}$ be the ray starting from $\left(p_{i}, 1\right)$ and parallel to the positive direction of $z$-axis, where $p_{i}$ is a point on $\mathbb{R}^{2}-\operatorname{int} C$ sufficiently close to the midpoint of $e_{i}$. Let $B_{i}$ be the convex hull of the union $\ell_{i} \cup e_{i} \times[0, \infty)$, and $B_{i}^{\prime}$ the identification space $B_{i} \cup_{E_{i}} B_{i}$, where $E_{i}=\partial B_{i}-\operatorname{int} e_{i} \times(0, \infty)$. Note that $B_{i}^{\prime}$ has nonnegative curvature and its boundary consists of two copies, say $F_{i}$, $F_{i}^{\prime}$, of $e_{i} \times[0, \infty)$. Let $K$ denote the quotient space of the disjoint union of $C \times[0, \infty)$ and $B_{i}^{\prime}, 1 \leq i \leq k$, where $e_{i} \times[0, \infty)$ is identified with $F_{i}$. Finally we take the double $X=\operatorname{dbl}(K)$. Since $K$ has nonnegative curvature, so does $X$ (see [26]). Note that $C^{0}=C$ and the soul of $X$ is an interior point of $C$.

To treat such cases as in the previous example, it is convenient to consider the following notion of pseudo-gradient flows.

For an open set $U \subset X$, a continuous local $\mathbb{R}$-action $\psi$ on $U$ is called a local flow on $U$, and denoted by $\psi(x, s)$ for $(x, s) \in U \times \mathbb{R}$ as long as it can be defined. For a continuous function $f$ on $X$, a local flow $\psi$ on $U$ is a gradient for $f$ on an open subset $V \subset U$ if $f(\psi(x, s))$ is strictly decreasing in $s$ as long as $\psi(x, s) \in V$. A local gradient flow $\psi$ on $U$ for $f$ is a pseudo-gradient if it is a gradient outside a compact subset of $U$.

Later we shall consider local pseudo-gradient flows for the following functions on $X$ :

$$
f(x)=d(C, x), \quad f_{\epsilon}(x)=d\left(C_{\epsilon}, x\right),
$$

where

$$
C_{\epsilon}=\{x \in C \mid d(\partial C, x) \geq \epsilon\} .
$$

By Theorem 10.2, we have the following lemma in a similar way to Lemma 10.3.

Lemma 10.7. For every positive numbers $\epsilon$ and $\delta$, there exists a local gradient flow $\psi(x, s)$ for $f_{\epsilon}$ on a neighborhood of $f_{\epsilon}^{-1}([\epsilon+\delta, \infty))$ which provides a homeomorphism $f_{\epsilon}^{-1}([\epsilon+\delta, \infty)) \simeq f_{\epsilon}^{-1}(\epsilon+\delta) \times[\epsilon+$ $\delta, \infty)$.

The idea of the proof of Theorem 9.6 (1) is to push a small neighborhood of $\partial C$ into $\mathcal{N} \subset X$ by using a pseudo-gradient flow of $f_{\epsilon}$ for small $\epsilon$, where $\mathcal{N}$ is the normal bundle over int $C$ with essential singular points removed and assumed to be imbedded in $X$ as in Section 9. Examples 10.5 and 10.6 suggest that the difficulty in the construction of such a pseudo-gradient flow occurs near $\partial C$. In the next section, we shall study the local topological structure near $\partial C$.

## 11. Local structure at $\partial C$

Let $X$ be as in Theorem 9.6 and $S$ the soul of $X$ which is a point. Since $f_{\epsilon}$ may have critical points on $\partial C$, we need to understand the topology of a small neighborhood of a point of $\partial C$. In this section, we assume that $C$ has dimension two and non-empty boundary. First note that $C$ is homeomorphic to $D^{2}$.

The following lemma is an easy consequence of Corollary 14.4.
Lemma 11.1. The number of one-normal points in $\operatorname{int} C$ is less than or equal to two.

Example 11.2. Let $C_{k}$ be a nonnegatively curved Alexandrov surface homeomorphic to $D^{2}$ and having distinct essential singular points
$p_{1}, \ldots, p_{k}$ in $\operatorname{int} C_{k}(k \leq 2)$. Then we construct a three-dimensional complete noncompact Alexandrov space $L^{\prime}\left(C_{k} ; p_{1}, \ldots, p_{k}\right)$ in a similar way to the construction of $L\left(S_{k} ; k\right)$ in Section 9 . Note that the boundary of $L^{\prime}\left(C_{1} ; p_{1}\right)$ (resp. of $L^{\prime}\left(C_{2} ; p_{1}, p_{2}\right)$ ) is a non-trivial (resp. trivial) line bundle over $\partial C_{1}$ (resp. over $\partial C_{2}$ ). In the fibre over every point $p \in \partial C_{k}$, we naturally identify the two rays emanating from $p$, and obtain a complete open nonnegatively curved Alexandrov space $L\left(C_{k} ; p_{1} \ldots, p_{k}\right)$ without boundary such that $C=C_{k}$ and that the topological singular points are $p_{1}, \ldots, p_{k}$.

Proposition 11.3. Every point p of $X$ except the one-normal points in int $C$ is a manifold-point of $X$, in other words, $\Sigma_{p}$ is homeomorphic to a sphere.

From Proposition 9.10 and the proof of Lemma 10.3, we know that $\operatorname{diam}\left(\Sigma_{p}\right)>\pi / 2$ for every point $p \in X-\partial C$ and hence $\Sigma_{p}$ is homeomorphic to a sphere ([26]). Note that from the basic construction, $\operatorname{diam}\left(\Sigma_{p}\right) \geq \pi / 2$ for every $p \in C$ (if $\operatorname{dim} C \geq 1$ ). Thus for the proof of Proposition 11.3 , we only have to care a point $p \in \partial C$ with $\operatorname{diam}\left(\Sigma_{p}\right)=$ $\pi / 2$.

Let $\Sigma$ be a two-dimensional compact Alexandrov surface with curvature $\geq 1$ and without boundary. Suppose that $\operatorname{diam}(\Sigma)=\pi / 2$. For a subset $B \subset \Sigma$ such that $\hat{B}=\{x \in \Sigma \mid d(B, x)=\pi / 2\}$ is non-empty, we consider $A_{1}=\hat{B}$ and $A_{2}=\hat{A}_{1}$. Then we have $\hat{A}_{2}=A_{1}$ (see [15] for details).

Proposition 11.4. Let $\Sigma, A_{1}$ and $A_{2}$ be as above. Then we have:
(1) If both $A_{1}$ and $A_{2}$ are contractible, then $\Sigma$ is homeomorphic to a sphere.
(2) If one of $A_{1}$ and $A_{2}$ is not contractibe, then $\Sigma$ is isometric to the projective plane,
(the spherical suspension over $\left.S_{\ell}^{1}\right) / \mathbb{Z}_{2}$,
where the length $\ell$ of the circle $S_{\ell}^{1}$ is less than or equal to $2 \pi$ and $\Sigma$ has constant curvature $K=1$ outside the possible singular vertex.

Proof. First we note that from the Alexandrov convexity:
(a) $A_{i}$ are convex sets.
(b) Any distinct three points $x, y \in A_{i}$ and $z \in A_{j}, i \neq j$ span a geodesic triangle isometric to $\widetilde{\triangle} x y z$ in $S^{2}(1)$.
(c) For any $x \in \Sigma-\left(A_{1} \cup A_{2}\right)$, minimal geodesic segments from $x$ to $A_{1}$ and to $A_{2}$ make an angle greater than $\pi / 2+c$ for some $c=c(x)>0$.

We show that
Assertion 11.5. $\operatorname{dim} A_{1}+\operatorname{dim} A_{2} \leq 1$.
Proof. Suppose that the assertion does not hold. Then $\operatorname{dim} A_{1}=$ $\operatorname{dim} A_{2}=1$. Let $x_{i}$ be any interior point of $A_{i}$. Using (b) above, we see that a neighborhood of a minimal geodesic segment $x_{1} x_{2}$ has constant curvature $=1$. Then applying a standard parallel translation technique along $x_{1} x_{2}$ together with curvature $K=1$, we would have a contradiction to $d\left(A_{1}, A_{2}\right)=\pi / 2$. q.e.d.

Making use of Theorem 10.2 together with (c) above, we obtain that $\Sigma-B\left(A_{1}, \epsilon\right)-B\left(A_{2}, \epsilon\right)$ is homeomorphic to $\partial B\left(A_{1}, \epsilon\right) \times[0,1]$ for any small $\epsilon>0$. This implies the conclusion (1). If $A_{1}$ is a point and $A_{2}$ is a circle, in view of the above (b), it is easy to see that $B\left(A_{2}, \epsilon\right)$ is homeomorphic to a Möbius band and that $\Sigma$ is isometric to the required one. q.e.d.

Lemma 11.6. If $p \in \partial C$ is a two-normal point, then $\operatorname{diam}\left(\Sigma_{p}\right)>$ $\pi / 2$.

Proof. Let $\xi_{0}, \xi_{1}$ be the directions at $p$ normal to $C, v$ any point of $\Sigma_{p}(C)$, and $\xi_{t}$ a minimal geodesic segment in $\Sigma_{p}$ joining $\xi_{0}$ and $\xi_{1}$. Suppose that $\operatorname{diam}\left(\Sigma_{p}\right)=\pi / 2$. Applying the Alexandrov convexity to the triangle $\triangle v \xi_{0} \xi_{1}$, we have $\angle\left(v, \xi_{t}\right) \geq \pi / 2$ and hence $\angle\left(v, \xi_{t}\right)=\pi / 2$ by the assumption. It turns out that $\xi_{t}$ are normal at $p$, a contradiction to Lemma 9.9. q.e.d.

Proof of Proposition 11.3. We only have to consider a point $p \in \partial C$ with a unique normal $\xi$ to $C$. Let us consider subsets of $\Sigma=\Sigma_{p}$, $B=\Sigma_{p}(C), A_{1}=\{\xi\}=\hat{B}$ and $A_{2}=\hat{A}_{1}$. By Proposition 11.4, it suffices to show that $A_{2}$ is a segment. Suppose that $A_{2}$ is a circle. Then Proposition 11.4 implies that the length $\ell$ of $A_{2}$ is less than or equal to $\pi$. On the other hand, from construction, $K\left(\Sigma_{p}(C)\right)$ is totally convex in $K_{p}$ and hence in $K\left(A_{2}\right)$. This is however impossible since $\ell \leq \pi$.
q.e.d.

## 12. Deformation of local flows

Let $X$ and $C$ be as in the previous section, and $p \in \partial C$. From the filtration $\left\{X^{t}\right\}_{t \geq 0}$, we obtain the filtration $\left\{K_{p}^{t}\right\}_{t \geq 0}$ of $K_{p}$ by totally convex sets, where $K_{p}^{t}$ is the limit of $X^{t / n}$ under the convergence $(n X, p) \rightarrow\left(K_{p}, o_{p}\right)$. We put $C_{\infty}=K_{p}^{0}$, and

$$
C_{\infty \epsilon}=\left\{x \in C_{\infty} \mid d\left(\partial C_{\infty}, x\right) \geq \epsilon\right\},
$$

which is the limit of $n C_{\epsilon / n}$ under the convergence $(n X, p) \rightarrow\left(K_{p}, o_{p}\right)$. Note that $C_{\infty}$ is isometric to the flat cone over the segment $\Sigma_{p}(C)$ and that Proposition 9.10 holds for $K_{p}^{t}$ and $C_{\infty}$ in place of $X^{t}$ and $C$.

We shall consider the function

$$
f_{\infty \epsilon}(x)=d\left(C_{\infty \epsilon}, x\right), \quad x \in K_{p} .
$$

Lemma 12.1. Given $p \in \partial C$, there exist positive numbers $\epsilon_{p}$ and $\delta_{p}$ such that for any $\epsilon$ and $\delta \leq \delta_{p}$ with $\epsilon / \delta \leq \epsilon_{p}$ :
(1) $f_{\epsilon}$ is regular on $U(p, \epsilon, \delta)-V(p, \epsilon, \delta)$.
(2) $\left(f_{\epsilon}, d_{p}\right)$ is $(c, \tau(\epsilon / \delta))$-regular on $(U(p, \epsilon, \delta)-V(p, \epsilon, \delta)) \cap B(p, \delta / 10)^{c}$.
where $c>0$ is a uniform constant and
$U(p, \epsilon, \delta)=B(\partial C, \epsilon / 10) \cap B(p, \delta), \quad V(p, \epsilon, \delta)=B(\partial C, \epsilon / 100) \cap B(p, \delta)$.
Proof. (1) is clear. We set

$$
U_{\infty \epsilon}=B\left(\partial C_{\infty}, \epsilon / 10\right) \cap B\left(o_{p}, 1\right), \quad V_{\infty \epsilon}=B\left(\partial C_{\infty}, \epsilon / 100\right) \cap B\left(o_{p}, 1\right) .
$$

By a simple convergence argument, for the proof of (2) it suffices to prove that $\left(f_{\infty \epsilon}, d_{o_{p}}\right)$ is $(c, \tau(\epsilon))$-regular on $\left(U_{\infty \epsilon}-V_{\infty \epsilon}\right) \cap B\left(o_{p}, 1 / 10\right)^{c}$ for a uniform constant $c>0$ and a small $\epsilon>0$. For every

$$
x \in\left(U_{\infty \epsilon}-V_{\infty \epsilon}\right) \cap B\left(o_{p}, 1 / 10\right)^{c},
$$

let $y \in \partial C_{\infty}, z \in \partial C_{\infty \epsilon}$ and $u \in C_{\infty}$ be the nearest points of $\partial C_{\infty}$, of $\partial C_{\infty \epsilon}$ and of $C_{\infty}$ respectively from $x$. First note that $\tilde{\angle} o_{p} x z>$ $\pi / 2-\tau(\epsilon), \tilde{\angle} z x a>\pi / 2-\tau(\epsilon)$, where $a$ is a point on the ray from $o_{p}$ through $x$ with $d\left(o_{p}, a\right)>d\left(o_{p}, x\right)$. We consider the following three cases.

Case 1. $d(y, u) \geq \epsilon / 1000$.

Let $b \in \partial C_{\infty}$ and $v \in C_{\infty}$ be such that $d\left(o_{p}, b\right)=d\left(o_{p}, y\right)+d(y, b)$ and $y b v u$ forms a square in $C_{\infty}$. Now observe that the normal bundle $\mathcal{N}$ over int $C_{\infty}$ is naturally imbedded in $K_{p}$. Let $y_{1}, b_{1}$ and $v_{1}$ be points in $\mathcal{N}$ such that uybvxy $y_{1} b_{1} v_{1}$ forms a parallelepiped in $\mathcal{N}$. Then we have

$$
\tilde{\angle} z x b_{1}>\pi / 2+c_{1}, \quad \tilde{\angle} o_{p} x b_{1}>\pi / 2+c_{1}
$$

for some uniform constant $c_{1}>0$. This implies that $\left(f_{\epsilon}, d_{o_{p}}\right)$ is $\left(c_{1}, \tau(\epsilon)\right)$ regular at $x$.

Case 2. $d(y, u) \leq \epsilon / 1000$ and $\tilde{\angle} x z y \geq 1 / 100$.
Let $x_{1}$ be the point on $x z \cap \overline{\mathcal{N}}$ such that $\tilde{\angle} z u x_{1}=\pi / 2\left(x_{1}=x\right.$ if $y \neq u)$. Let $x_{2}$ be the point on the ray from $o_{p}$ through $x_{1}$ such that $\left(x_{2}\right)_{u}^{\prime}$ is a direction at $u$ normal to $C_{\infty}$. We show that $\angle x_{1} u x_{2}<\tau(\epsilon)$. Let $q$ be the point on the ray from $o_{p}$ through $u$ with $d\left(o_{p}, q\right)=2 d\left(o_{p}, u\right)$. Obviously,

$$
\begin{array}{cl}
\tilde{\angle} o_{p} x q>\pi-\tau(\epsilon), & \tilde{\angle} o_{p} z q>\pi-\tau(\epsilon), \\
\left|\tilde{\angle} o_{p} x z-\pi / 2\right|<\tau(\epsilon), & \left|\tilde{\angle} o_{p} z x-\pi / 2\right|<\tau(\epsilon) .
\end{array}
$$

It follows from Corollary 5.7 in [4] that

$$
\left|\tilde{\angle} o_{p} x_{1} z-\pi / 2\right|<\tau(\epsilon), \quad\left|\tilde{\angle} q x_{1} z-\pi / 2\right|<\tau(\epsilon),
$$

which implies that $\angle x_{1} u x_{2}<\tau(\epsilon)$.
Take $v \in \mathcal{N}$ such that $d(u, v)=d\left(u, x_{1}\right)+d\left(x_{1}, v\right)$ and $d\left(x, x_{1}\right) / d(v, x)$ is sufficiently small. Then $\tilde{\angle} z x v>\pi / 2+c_{2}$. Let $w$ be a point such that $w_{x}^{\prime}$ is a midpoint between $v_{x}^{\prime}$ and $a_{x}^{\prime}$. Then we have

$$
\tilde{\angle} z x w>\pi / 2+c_{2}^{\prime}, \quad \tilde{\angle} o_{p} x w>\pi / 2+c_{2}^{\prime},
$$

for some uniform constant $c_{2}^{\prime}>0$. This implies that $\left(f_{\epsilon}, d_{o_{p}}\right)$ is $(c, \tau(\epsilon))$ regular at $x$.

Case 3. $d(y, u) \leq \epsilon / 1000$ and $\tilde{\angle} x z y \leq 1 / 100$.
Note that $y=u$ in this case. Let $\delta_{1}$ be a small positive number. Take a small $\epsilon>0$ so that for every $x \in U_{\infty \epsilon} \cap A\left(o_{p} ; 1 / 10,1\right)$ and $y \in$ $\partial C_{\infty} \cap A\left(o_{p} ; \underset{\sim}{1 / 10,1)}\right.$ with $d(x, y)=d\left(x, \partial C_{\infty}\right)$, there exists a point $v$ such that $\tilde{\angle} y x v>\pi-\delta_{1}$. Then in the above situation we have $\tilde{\angle} z x v>\pi / 2+c_{3}$. Taking $w$ as in Case 2, we obtain the required regularity of $\left(f_{\epsilon}, d_{o_{p}}\right)$. q.e.d.


Figure 1: Canonical neighborhoods pair
Definition 12.2. We say that $(U, V)$ is a canonical neighborhoods pair of a point $p \in \partial C$ for $\delta \gg \epsilon>0$ (see Figure 1) if:
(1) $U \supset V$ are neighborhoods of $p$ with $U \cap \partial C=V \cap \partial C=B(p, \delta) \cap$ $\partial C$.
(2) $\left(f_{\epsilon}, d_{p}\right)$ is $(c, \tau(\epsilon / \delta))$-regular on $(U-V) \cap B(p, \delta / 10)^{c}$.
(3) There is a pseudo-gradient flow $\psi$ on $U$ for $f_{\epsilon}$.
(4) There is a homeomorphism $h: U \cap \partial C \times\left(I \times J-I^{\prime} \times J^{\prime}\right) \rightarrow U-V$, where $I=\left[s_{0}, s_{1}\right], I^{\prime}=\left[s_{0}^{\prime}, s_{1}^{\prime}\right], s_{0}<s_{0}^{\prime}, s_{1}>s_{1}^{\prime}$ and $J \supset \supset J^{\prime}$ are closed intervals, such that for each $x \in U \cap \partial C$ :
(a) $h(\{x\} \times I \times\{t\})$ provides the flow curve of $\psi$ for each $t \in J-J^{\prime}$, that is,

$$
\psi\left(h\left(x, s_{1}, t\right), s\right)=h\left(x, s_{1}-s, t\right) .
$$

(b) $h\left(\{x\} \times\left\{s_{1}\right\} \times J\right) \subset f_{\epsilon}^{-1}(21 \epsilon / 20), \quad h\left(\{x\} \times\left\{s_{0}\right\} \times J\right) \subset$ $f_{\epsilon}^{-1}(19 \epsilon / 20)$.
(c) $h\left(\{x\} \times\left\{s_{1}^{\prime}\right\} \times J^{\prime}\right) \subset f_{\epsilon}^{-1}(51 \epsilon / 50), \quad h\left(\{x\} \times\left\{s_{0}^{\prime}\right\} \times J^{\prime}\right) \subset$ $f_{\epsilon}^{-1}(49 \epsilon / 50)$.
(d) Each $\psi$-flow curve in $(U-V) \cap B(p, \delta / 10)^{c}$ is contained in a $d_{p}$-level set.
(e)

$$
\begin{aligned}
& \partial U-h((\operatorname{int} B(p, \delta) \cap \partial C) \times \partial(I \times J)) \subset \partial B(p, \delta), \\
& \partial V-h\left((\operatorname{int} B(p, \delta) \cap \partial C) \times \partial\left(I^{\prime} \times J^{\prime}\right)\right) \subset \partial B(p, \delta)
\end{aligned}
$$

Lemma 12.3. Each point $p \in \partial C$ has a canonical neighborhoods pair for arbitrarily small $\delta \gg \epsilon>0$.

For the proof, we need the following
Lemma 12.4. Let $D^{2}$ be the standard unit disk in $\mathbb{R}^{2}$ with $\partial D^{2}=$ $S^{1}$ the union $I_{0} \cup I_{1}$ of closed hemi-circles. Let $\psi_{0}$ and $\psi_{1}$ be local flows on $D^{2}$ such that:
(1) The flow curves start from $I_{0}$ and reach $I_{1}$.
(2) $\psi_{i}$ leaves $I_{0} \cap I_{1}$ fixed.

Then there exists a continuous family of flows $\psi_{t}$ on $D^{2}$ satisfying the conditions above and joining $\psi_{0}$ and $\psi_{1}$.

Proof. Let $r$ be the reflection about the line through $I_{0} \cap I_{1}$. Let $\phi_{0}$ be the canonical straight flow on $D^{2}$ starting from $x \in I_{0}$ and reaching $r(x) \in I_{1}$. Now consider a flow $\psi$ on $D^{2}$ satisfying (1), (2). Making use of the flow curves of $\phi_{0}$ and $\psi$, we can think of $\psi$ as such a homeomorphism $f=f_{\psi}$ of $D^{2}$ that the restriction to $I_{0}$ is the identity. Now let $\psi^{\prime}$ be the straightening of $\psi$. Namely, it can be defined as the natural flow formed by straight line segments from $x \in I_{0}$ to $\psi(x, s) \in I_{1}$. Put $h=f_{\psi^{\prime}}^{-1} \circ f_{\psi}$. Since $h$ is the identity on $\partial D^{2}$, it is isotopic to the identity of $D^{2}$ while keeping $\partial D^{2}$ fixed. Let $h_{t}, 0 \leq t \leq 1$, be the isotopy. The isotopy $f_{\psi^{\prime}} \circ h_{t}$ induces a deformation $\psi_{t}$ of flows from $\psi$ to $\psi^{\prime}$. For the given flows $\psi_{0}$ and $\psi_{1}$, it is now obvious that $\psi_{0}^{\prime}$ is deformable to $\psi_{1}^{\prime}$. This completes the proof. q.e.d.

Proof of Lemma 12.3. Let $\delta_{p}$ and $\epsilon_{p}$ be as in Lemma 12.1. Then for $\epsilon<\delta$ with $\delta \leq \delta_{p}$ and $\epsilon / \delta \leq \epsilon_{p}$, consecutive use of Theorem 10.1 together with Lemma 12.1 provides neighborhoods $U \supset V$ of $p$ and a gradient flow $\psi$ on $U-V$ for $f_{\epsilon}$ satisfying the conditions (1), (2) and (4) in Definition 12.2. For the proof, it suffices to extend $\psi$ to a pseudo-gradient flow on $U$ for $f_{\epsilon}$. By Proposition 11.3, we may assume that $U \supset V$ satisfy (e). Let $E_{0}$ and $E_{1}$ be the two component of $\partial V-h\left((\operatorname{int} B(p, \delta) \cap \partial C) \times \partial\left(I^{\prime} \times J^{\prime}\right)\right) \subset \partial B(p, \delta)$. Since $\partial V \simeq S^{2}$, it follows from Proposition 11.3 and the generalized Schoenflies Theorem
([3]) that $V \simeq I^{\prime} \times J^{\prime} \times[0,1]$, where $E_{i}$ corresponds to $I^{\prime} \times J^{\prime} \times\{i\}$. Let $\phi_{i}$ be a flow on $E_{i}$ starting from $\left\{s_{1}^{\prime}\right\} \times J^{\prime} \times\{i\}$, reaching $\left\{s_{0}^{\prime}\right\} \times J^{\prime} \times\{i\}$ and extending $\psi, i=0,1$. Then by Lemma 12.4, we have a flow $\phi$ on $V$ extending $\phi_{i}$ and $\psi$ restricted to $\partial V \cap \operatorname{int} B(p, \delta)$. The flow defined by the union of $\psi$ and $\phi$ gives a required flow on $U$. q.e.d.

Proposition 12.5. There exist an $\epsilon_{0}>0$, a neighborhood $U$ of $\partial C$ and a pseudo-gradient flow $\psi$ on $U$ for $f_{\epsilon_{0}}$ together with a homeomorphism $h: \partial C \times I \times J \rightarrow U$ such that for each $x \in \partial C$ :
(1) $h(\{x\} \times I \times \partial J)$ gives flow curves of $\psi$, that is,

$$
\psi\left(h\left(x, s_{1}, t_{j}\right), s\right)=h\left(x, s_{1}-s, t_{j}\right)
$$

(2) $h\left(\{x\} \times\left\{s_{1}\right\} \times J\right) \subset f_{\epsilon_{0}}^{-1}\left(21 \epsilon_{0} / 20\right)$.
(3) $h\left(\{x\} \times\left\{s_{0}\right\} \times J\right) \subset f_{\epsilon_{0}}^{-1}\left(19 \epsilon_{0} / 20\right)$.

Here, $I=\left[s_{0}, s_{1}\right], J=\left[t_{0}, t_{1}\right]$.
Proof. By a straightforward compactness argument using Lemma 12.1, we have finitely many points $p_{1}, \ldots, p_{N}$ of $\partial C$ such that for some $\delta_{i} \leq \delta_{p_{i}}:$
(1) $\left\{B\left(p_{i}, \delta_{i}\right)\right\}$ covers $\partial C$.
(2) $\left\{B\left(p_{i}, \delta_{i} / 10\right)\right\}$ is disjoint.

For each $p_{i}$, take $\epsilon_{i} \ll \delta_{i}$ with $\epsilon_{i} / \delta_{i} \leq \epsilon_{p_{i}}$ as in Lemma 12.1, and choose a small $\epsilon_{0}$ with $\epsilon_{0} / \min \left\{\delta_{i}\right\} \leq \min \left\{\epsilon_{p_{i}}\right\}$. Let $\left(U_{i}, V_{i}\right)$ be a canonical neighborhoods pair of $p_{i}$ for $\epsilon_{0} \ll \delta_{i}$, and $\psi_{i}$ the psuedo-gradient flow on $U_{i}-V_{i}$ for $f_{\epsilon_{0}}$ as in Definition 12.2. We denote by $U^{\prime}$ the union of $U_{i}$. To prove the proposition, we have to deform those local flows $\psi_{i}$ to obtain a local flow on a neighborhood of $\partial C$. Suppose that $p_{i}$ is adjacent to $p_{j}$. We use the deformation theory to obtain a pseudo-gradient flow $\psi_{i j}$ on a neighborhood of $\partial C \cap\left(U_{i} \cup U_{j}\right)$ which differs from $\psi_{i}$ and $\psi_{j}$ only on a neighborhood of a compact set of $\partial C \cap U_{i} \cap U_{j}$.

We put $g_{i}=d\left(p_{i},.\right)$ which is regular on $U_{i}-B\left(p_{i}, \delta_{i} / 10\right)$. Let $T$ be the component of $\partial C \cap U_{i} \cap U_{j}-B\left(p_{i}, \delta_{i} / 10\right)-B\left(p_{j}, \delta_{j} / 10\right)$ on which $\left(f_{\epsilon_{0}}, g_{i}\right)$ is regular. Let $h_{i}: U_{i} \cap \partial C \times\left(I \times J-I^{\prime} \times J^{\prime}\right) \rightarrow U_{i}-V_{i}$ be as in Definition $12.2(4)$. We consider a pair $(U, V)$ with $U \subset U_{i}, V \subset V_{i}$ defined by

$$
U-V=h_{i}\left(T \times\left(I \times J-I^{\prime} \times J^{\prime}\right)\right)
$$

By Definition 12.2 (4)-(d), we have a homeomorphism $\hat{h}: U-V \rightarrow K \times$ $\left(I \times J-I^{\prime} \times J^{\prime}\right)$ such that pro $\hat{h}=\left(g_{i}, f_{\epsilon_{0}}\right)$, where $I=\left[19 \epsilon_{0} / 20,21 \epsilon_{0} / 20\right]$, $I^{\prime}=\left[49 \epsilon_{0} / 50,51 \epsilon_{0} / 50\right], K=[r, R]$, and $p r: K \times I \times J \rightarrow K \times I$ is the projection. Here we make an identification for simplicity

$$
U-V \equiv T \times\left(I \times J-I^{\prime} \times J^{\prime}\right) \equiv K \times\left(I \times J-I^{\prime} \times J^{\prime}\right)
$$

Recall that the flow $\psi_{i}$ restricted to $U-V$ is gradient for $f_{\epsilon_{0}}$ and that the flow curves lie on the $g_{i}$-level sets. Take $r<r_{1}<R_{1}<R$. For simplicity, we set

$$
U\left(r_{1}, R_{1}\right)=U \cap g_{i}^{-1}\left(\left[r_{1}, R_{1}\right]\right)
$$

Assertion 12.6. We can take a pseudo-gradient flow $\psi_{i j}$ on $U$ which is gradient for $f_{\epsilon_{0}}$ outside $V$ satisfying

$$
\psi_{i j}= \begin{cases}\psi_{i} & \text { on } \quad U\left(r, r_{1}\right)  \tag{12.1}\\ \psi_{j} & \text { on } \quad U\left(R_{1}, R\right)\end{cases}
$$

Proof. We have to join the flow $\psi_{i}$ on $U\left(r, r_{1}\right)$ and the flow $\psi_{j}$ on $U\left(R_{1}, R\right)$. Let $r_{1}<r_{2}<R_{1}$. By Theorem 10.1, we have a flow $\psi_{j}^{\prime}$ on $U\left(r_{1}, R\right)$ such that

$$
\psi_{j}^{\prime}= \begin{cases}\psi_{i} & \text { on } \quad U\left(r_{1}, r_{2}\right)-V_{0}  \tag{12.2}\\ \psi_{j} & \text { on } \quad U\left(R_{1}, R\right) \cup\left(U\left(r_{1}, R\right) \cap V\right)\end{cases}
$$

where $V \subset V_{0} \subset U$. Let $K_{1}$ be the union of $\psi_{j}^{\prime}$-flow curves starting from $\left\{R_{1}\right\} \times\left\{21 \epsilon_{0} / 20\right\} \times J$ to $\left\{f_{\epsilon_{0}}=19 \epsilon_{0} / 20\right\}$. Put

$$
K_{2}=\left\{r_{2}\right\} \times I \times J, \quad K_{3}=\left[r_{2}, R_{1}\right] \times\left\{21 \epsilon_{0} / 20\right\} \times J
$$

Let $K_{4}$ be the domain on $\left\{f_{\epsilon_{0}}=19 \epsilon_{0} / 20\right\}$ bounded by the curves $\left\{r_{2}\right\} \times$ $\left\{19 \epsilon_{0} / 20\right\} \times J,\left[r_{2}, R_{1}\right] \times\left\{19 \epsilon_{0} / 20\right\} \times \partial J$ and $F$, where $F=K_{1} \cap\left\{f_{\epsilon_{0}}=\right.$ $\left.19 \epsilon_{0} / 20\right\}$. Retaking $\epsilon_{0}$ smaller and $r_{1}, r_{2}, r_{3}$ suitably if necessary, we may assume that $K_{4} \simeq D^{2}$ and $g_{i}\left(K_{1}\right) \subset\left(r_{2}, R\right)$. Let $D$ be the domain bounded by $K_{1}, \ldots, K_{4}$, and $\left[r_{2}, R_{1}\right] \times I \times \partial J$. Note that $D \simeq D^{3}$ and observe that we have the $\psi_{i}$-flow on $K_{2}$ and the $\psi_{j}^{\prime}$-flow on $K_{1}$. By Lemma 12.4 , we can construct a new flow $\psi_{i j}$ on $D$ such that

$$
\psi_{i j}=\left\{\begin{array}{lll}
\psi_{i} & \text { on } & K_{2}  \tag{12.3}\\
\psi_{j} & \text { on } & K_{1} \cup V
\end{array}\right.
$$

Now the flows $\psi_{i}, \psi_{i j}$, and $\psi_{j}^{\prime}$ provide a required flow on $U$. q.e.d.
Repeating the above procedure finitely many times, we obtain a pseudo-gradient flow for $f_{\epsilon_{0}}$ on a neighborhood of $\partial C$ as required. q.e.d.

Proof of Theorem 9.6(1). Let $k \leq 2$ be the number of one-normal points in int $C$. It follows from the previous proposition that

$$
\begin{aligned}
X & \simeq f_{\epsilon_{0}}^{-1}\left(\left[0,21 \epsilon_{0} / 20\right)\right) \\
& \simeq f_{\epsilon_{0}}^{-1}\left(\left[0,19 \epsilon_{0} / 20\right)\right) \\
& \simeq \begin{cases}\mathbb{R}^{3} & \text { if } k=0, \\
K\left(P^{2}\right) & \text { if } k=1, \\
X_{1}=\mathbb{R}^{3} / \Gamma & \text { if } k=2 .\end{cases}
\end{aligned}
$$

This completes the proof of Theorem 9.6 (1) in the case when $\operatorname{dim} C=2$.
q.e.d.

## 13. The case of $\operatorname{dim} C=1$

Let $X$ be as in Theorem 9.6 with $\operatorname{dim} C=1$. As Example 9.3 shows, $X$ is not necessarily a topological manifold near $\partial C$. Hence we cannot apply the deformation theory developed in the previous section to a neighborhood of $\partial C$.

Proof of Theorem 9.6 in the case $\operatorname{dim} C=1$. Let $q, q^{\prime}$ be the boundary points of the geodesic segment $C$. Under the convergence $(n X, q) \rightarrow\left(K_{q}, o_{q}\right)$, let $C$ converge to $C_{\infty}$, and consider the functions

$$
\begin{array}{rll}
f=d\left(C_{\infty}, \cdot\right), & & g=d\left(o_{q}, \cdot\right), \\
f_{n}=d_{n}(C, \cdot), & & g_{n}=d_{n}(q, \cdot) .
\end{array}
$$

where $d_{n}$ is the $n$-times rescaling of the original metric of $X$. Notice that $f$ and $g$ are regular on $K_{q}-C_{\infty}$ and $K_{q}-\left\{o_{q}\right\}$ respectively.

Lemma 13.1. There exist $n_{0}, \epsilon_{0}>0$ such that for every $n \geq n_{0}$ and $\epsilon \leq \epsilon_{0}$, we have a homeomorphism

$$
H:(f, g)^{-1}([0, \epsilon] \times[0,1]) \rightarrow\left(f_{n}, g_{n}\right)^{-1}([0, \epsilon] \times[0,1])
$$

with $f_{n} \circ H=f$.

Proof. Observe that there exist $n_{0}, \epsilon_{0}>0$ and $c_{0} \gg \epsilon_{0}$ such that for every $\epsilon \leq \epsilon_{0}, n \geq n_{0}$ and for every $x \in\left(f_{n}, g_{n}\right)^{-1}((0, \epsilon] \times[1 / 2,1])$ there is a point $w$ with $d_{n}(w, C) \geq c_{0}$ satisfying

$$
\tilde{\angle} q x w>\pi / 2+c, \quad \tilde{\angle} C x w>\pi / 2+c, \quad \tilde{\angle} q x C>\pi / 2-\tau(\epsilon)
$$

for some uniform constant $c>0$. For instance, the point $w$ can be taken as follows: Take $y \in C$ and $y_{1}$ with $d_{n}(x, y)=d_{n}(x, C), \tilde{\angle} y x y_{1}>$ $\pi-\tau(\epsilon)$ and $d_{n}\left(y, y_{1}\right) \gg \epsilon$. Take a point $z$ such that $\tilde{\angle} q x z>\pi-\tau_{n}$ and $d_{n}(x, z)=d_{n}\left(x, y_{1}\right)$, where $\lim \tau_{n}=0$. Then the mid point $w$ of $y_{1} z$ satisfies the above conditions. In particular, $\left(f_{n}, g_{n}\right)$ and $(f, g)$ are $(c, \tau(\epsilon))$-regular on $\left(f_{n}, g_{n}\right)^{-1}((0, \epsilon] \times[1 / 2,1])$ and $(f, g)^{-1}((0, \epsilon] \times$ $[1 / 2,1])$ respectively.

Applying Theorem B and its complement in [26], we then have a homeomorphism

$$
h_{11}:(f, g)^{-1}([\epsilon / 2, \epsilon] \times[1 / 2,1]) \rightarrow\left(f_{n}, g_{n}\right)^{-1}([\epsilon / 2, \epsilon] \times[1 / 2,1])
$$

such that $\left(f_{n}, g_{n}\right) \circ h_{11}=(f, g)$. The $(c, \tau(\epsilon))$-regularities of $\left(f_{n}, g_{n}\right)$ and $(f, g)$ enables us to extend $h_{11}$ to a homeomorphism

$$
\hat{h}_{1}:(f, g)^{-1}((0, \epsilon] \times[1 / 2,1]) \rightarrow\left(f_{n}, g_{n}\right)^{-1}((0, \epsilon] \times[1 / 2,1])
$$

with $\left(f_{n}, g_{n}\right) \circ \hat{h}_{1}=(f, g)$. Now it is easy to extend $h_{1}$ to a homeomorphism

$$
h_{1}:(f, g)^{-1}([0, \epsilon] \times[1 / 2,1]) \rightarrow\left(f_{n}, g_{n}\right)^{-1}([0, \epsilon] \times[1 / 2,1])
$$

with $\left(f_{n}, g_{n}\right) \circ h_{1}=(f, g)$. We put

$$
\begin{aligned}
K_{i} & =(f, g)^{-1}\left(\left[0, \epsilon / 2^{i-1}\right] \times\left[1 / 2^{i}, 1 / 2^{i-1}\right]\right) \\
K_{i}^{n} & =\left(f_{n}, g_{n}\right)^{-1}\left(\left[0, \epsilon / 2^{i-1}\right] \times\left[1 / 2^{i}, 1 / 2^{i-1}\right]\right)
\end{aligned}
$$

Noting $2 K_{i}$ and $2 K_{i}^{n}$ are isometric to $K_{i-1}$ and $K_{i-1}^{2 n}$ respectively, we can inductively construct homeomorphisms $h_{i}: K_{i} \rightarrow K_{i}^{n}$ such that $\left(f_{n}, g_{n}\right) \circ h_{i}=(f, g)$ and $h_{i}=h_{i-1}$ on $K_{i} \cap K_{i-1}$. Thus we obtain a homeomorphism $h: \cup_{i=1}^{\infty} K_{i} \rightarrow \cup_{i=1}^{\infty} K_{i}^{n}$ with $\left(f_{n}, g_{n}\right) \circ h=$ $(f, g)$. Note that $f$ and $f_{n}$ are regular on $(f, g)^{-1}((0, \epsilon] \times[0,1])$ and $\left(f_{n}, g_{n}\right)^{-1}((0, \epsilon] \times[0,1])$ respectively. We put

$$
\begin{aligned}
L_{i} & =(f, g)^{-1}\left(\left[\epsilon / 2^{i}, \epsilon / 2^{i-1}\right] \times\left[0,1 / 2^{i}\right]\right) \\
L_{i}^{n} & =\left(f_{n}, g_{n}\right)^{-1}\left(\left[\epsilon / 2^{i}, \epsilon / 2^{i-1}\right] \times\left[0,1 / 2^{i}\right]\right)
\end{aligned}
$$

By Theorem 10.1 and the complement of Theorem B in [26], there exists $n_{1}$ such that for each $n \geq n_{1}$, we have a homeomorphism $k_{1}: L_{1} \rightarrow L_{1}^{n}$ such that:
(1) $f_{n} \circ k_{1}=f$.
(2) $k_{1}$ is compatible with $h$.

Now with the use of Theorem 10.1, we can inductively construct homeomorphisms $k_{i}: L_{i} \rightarrow L_{i}^{n}$ such that:
(3) $f_{n} \circ k_{i}=f$.
(4) $k_{i}$ is compatible with $h, k_{1}, \ldots, k_{i-1}$.

Thus we have a homeomorphism $k: \cup_{i=1}^{\infty} L_{i} \rightarrow \cup_{i=1}^{\infty} L_{i}^{n}$ with $f_{n} \circ k=f$, and $h$ and $k$ define a homeomorphism

$$
H:(f, g)^{-1}([0, \epsilon] \times[0,1]) \rightarrow\left(f_{n}, g_{n}\right)^{-1}([0, \epsilon] \times[0,1])
$$

with $f_{n} \circ H=f$. q.e.d.
Since obviously $B\left(o_{q}, 1\right) \simeq(f, g)^{-1}([0, \epsilon] \times[0,1])$, it follows that $\left(f_{n}, g_{n}\right)^{-1}([0, \epsilon] \times[0,1]) \simeq K_{q}$. Thus,

$$
\left(f_{1}, g_{1}\right)^{-1}([0, \epsilon / n] \times[0,1 / n]) \simeq K_{q}
$$

for all sufficiently large $n$. Similarly we have

$$
\left(f_{1}, g_{1}^{\prime}\right)^{-1}([0, \epsilon / n] \times[0,1 / n]) \simeq K_{q^{\prime}}
$$

where $g_{1}^{\prime}=d\left(q^{\prime}, \cdot\right)$.
For simplicity, we put $\epsilon_{1}=\epsilon / n, \delta_{1}=1 / n$ and $R=d\left(q, q^{\prime}\right) / 2$. Taking a larger $n$ if necessary, we may also assume that $\left(f_{1}, g_{1}\right)$ (resp. $\left.\left(f_{1}, g_{1}^{\prime}\right)\right)$ is $(c, \tau(\epsilon))$-regular on $\left(f_{1}, g_{1}\right)^{-1}\left(\left\{\epsilon_{1}\right\} \times\left[\delta_{1}, R\right]\right)$ (resp. on $\left(f_{1}, g_{1}^{\prime}\right)^{-1}\left(\left\{\epsilon_{1}\right\} \times\right.$ [ $\left.\left.\delta_{1}, R\right]\right)$ ). It follows that for any $R_{1} \leq R$

$$
\begin{aligned}
\left(f_{1}, g_{1}\right)^{-1}\left(\left[0, \epsilon_{1}\right]\right. & \left.\times\left[\delta_{1}, R_{1}\right]\right) \\
& \simeq\left(f_{1}, g_{1}\right)^{-1}\left(\left[0, \epsilon_{1}\right] \times \delta_{1}\right) \times\left[\delta_{1}, R_{1}\right] \\
& \simeq D^{2} \times\left[\delta_{1}, R_{1}\right] \\
& \simeq D^{3} .
\end{aligned}
$$

Similarly, we have

$$
\left(f_{1}, g_{1}^{\prime}\right)^{-1}\left(\left[0, \epsilon_{1}\right] \times\left[\delta_{1}, R_{1}\right]\right) \simeq D^{3}
$$

We put

$$
E\left(\epsilon_{1}, \delta_{1}\right)=f_{1}^{-1}\left[0, \epsilon_{1}\right]-\operatorname{int} B\left(q, \delta_{1}\right)-\operatorname{int} B\left(q^{\prime}, \delta_{1}\right) .
$$

Since $B\left(q, \delta_{1}\right) \cap E\left(\epsilon_{1}, \delta_{1}\right) \simeq D^{2}$ and $B\left(q^{\prime}, \delta_{1}\right) \cap E\left(\epsilon_{1}, \delta_{1}\right) \simeq D^{2}$, it suffices to show that $E\left(\epsilon_{1}, \delta_{1}\right) \simeq D^{3}$. Let $z$ be the midpoint of $C$. We may assume that

$$
B\left(z, 10 \delta_{1}\right) \simeq D^{3} .
$$

Note that:
(1) $F_{q}\left(\epsilon_{1}, \delta_{1}\right):=\left(f_{1}, g_{1}\right)^{-1}\left(\left[0, \epsilon_{1}\right] \times\left[\delta_{1}, R-\delta_{1}\right]\right) \simeq D^{3}$.
(2) $F_{q^{\prime}}\left(\epsilon_{1}, \delta_{1}\right):=\left(f_{1}, g_{1}^{\prime}\right)^{-1}\left(\left[0, \epsilon_{1}\right] \times\left[\delta_{1}, R-\delta_{1}\right]\right) \simeq D^{3}$.
(3) $G\left(\epsilon_{1}, \delta_{1}\right):=E\left(\epsilon_{1}, \delta_{1}\right)-\operatorname{int} F_{q}\left(\epsilon_{1}, \delta_{1}\right)-\operatorname{int} F_{q^{\prime}}\left(\epsilon_{1}, \delta_{1}\right) \subset B\left(z, 10 \delta_{1}\right)$ has boundary homeomorphic to $D^{2} \cup S^{1} \times I \cup D^{2} \simeq S^{2}$.

It follows from the generalized Schoenflies Theorem ([3]) that $G\left(\epsilon_{1}, \delta_{1}\right) \simeq$ $D^{3}$. Therefore

$$
E\left(\epsilon_{1}, \delta_{1}\right) \simeq F_{q}\left(\epsilon_{1}, \delta_{1}\right) \cup_{D^{2}} G\left(\epsilon_{1}, \delta_{1}\right) \cup_{D^{2}} F_{q^{\prime}}\left(\epsilon_{1}, \delta_{1}\right) \simeq D^{3}
$$

This completes the proof. q.e.d.
Remark 13.2. In the case when $X$ is a Riemannian manifold, one can use the flow curves of a gradient-like vector field of $d_{q}-d_{q^{\prime}}$ to conclude that $E\left(\epsilon_{1}, \delta_{1}\right) \simeq D^{3}$. In our case however, it is not certain if one can apply the above argument to a function of type $d_{q}-d_{q^{\prime}}$ on a general Alexandrov space.

## 14. Appendix: Total curvature on Alexandrov surfaces

In this section, we explain the total curvature and the Gauss-Bonnet Theorem on Alexandrov surfaces, originally studied by Alexandrov [1]. We also formulate the Cohn-Vossen Theorem and investigate Alexandrov surfaces admitting essential singular points, which are needed for the proof of Theorem 0.5.

Throughout this section, let $X$ be an Alexandrov surface of curvature $\geq a \in \mathbb{R}$ and $\mathcal{H}^{2}$ the Hausdorff measure over $X$. Recall that such an $X$ is a two-dimensional topological manifold possibly with boundary. A polygonal region of $X$ is by definition a subset of $X$ whose (topological) boundary is a union of finitely many broken geodesics. The rotation $\kappa(\partial D)$ of the (topological) boundary $\partial D$ of a polygonal region $D \subset X$ is defined by

$$
\kappa(\partial D):=\sum_{x \in \partial D}\left(\pi-\angle_{x} D\right),
$$

where $\angle_{x} D$ denotes the inner angle of $D$ at $x \in \partial D$, which is equal to $\pi$ if $x$ is not a vertex of $D$, so that the sum here is indeed a finite sum. Fixing one of the two sides of a broken geodesic $\sigma=x_{0} x_{1} \ldots x_{k}$ in $X$, we define the rotation $\kappa(\sigma)$ of $\sigma$ with respect to the chosen side in the same manner.

Let $\boldsymbol{\Delta}$ denote the open disk domain bounded by a triangle $\triangle$ in $X$. The total curvature (or excess) $\omega(\mathbf{\Delta})$ of $\mathbf{\Delta}$ in $X$ is defined by

$$
\omega(\mathbf{\Delta}):=\alpha+\beta+\gamma-\pi,
$$

where $\alpha, \beta, \gamma$ are the inner angles of $\boldsymbol{\Delta}$ at its three vertices. Let $D \subset$ int $X$ be a relatively compact polygonal open region and find a triangulation of $D$ with triangles $\{\triangle\}$. Then, the total curvature (or total excess) $\omega(D)$ of $D$ is defined by

$$
\omega(D):=\sum_{\Delta} \omega(\mathbf{\Delta})+\sum_{x \in V \cap \mathrm{int} D}\left(2 \pi-L\left(\Sigma_{x}\right)\right),
$$

where $V$ is the set of vertices of the triangulation of $D$. According to [1] (see also [23]), the total curvature $\omega$ is independent of the triangulation and extends to a signed Radon measure over $X$ with the following properties:
(1) For any $D \subset \operatorname{int} X$ as above, we have the Gauss-Bonnet formula:

$$
\omega(D)+\kappa(\partial D)=2 \pi \chi(D)
$$

where $\chi(D)$ denotes the Euler characteristic of $D$.
(2) Any $\mathcal{H}^{2}$-measurable subset of $X$ is $\omega$-measurable and we have

$$
\omega \geq a \mathcal{H}^{2}
$$

so that $\omega-a \mathcal{H}^{2}$ is a (nonnegative) Radon measure.
(3) The restriction of $\omega$ onto $\partial X$ is

$$
\left.\omega\right|_{\partial X}=0 .
$$

(4) For any minimal segment $x y$ in $X$,

$$
\left.\omega\right|_{x y-\{x, y\}}=0 .
$$

(5) For any $x \in \operatorname{int} X$,

$$
\omega(\{x\})=2 \pi-L\left(\Sigma_{x}\right) .
$$

We now introduce the rotation measure $\kappa$ over $\partial X$. Let $c$ be a subarc of $\partial X$ from $p \in \partial X$ to $q \in \partial X$. Take a division $\left\{p=x_{0}, x_{1}, \ldots, x_{m}=\right.$ $q\}$ of the arc $c$, where $x_{0}, \ldots, x_{m}$ are points lying on $c$ in this order. We obtain a broken geodesic $\sigma:=x_{0} x_{1} \ldots x_{m}$, which approximates $c$. Choose the side of $\sigma$ for which we can measure the inner angle. It then follows that

$$
\kappa(\sigma)=\sum_{i=1}^{m-1}\left(\pi-\angle x_{i-1} x_{i} x_{i+1}\right) .
$$

Clearly, $\kappa(\sigma)$ is nonnegative. If a point $y \in c$ is taken to be between $x_{k-1}$ and $x_{k}$ for a $k$ and if $\boldsymbol{\Delta}$ is the open disk domain surrounded by $\triangle x_{k-1} y x_{k}$, then

$$
\begin{aligned}
\kappa\left(x_{0} \ldots x_{k-1} y x_{k} \ldots x_{m}\right) & =\kappa(\sigma)-\angle_{x_{k-1}} \mathbf{\Delta}-\angle_{x_{k}} \mathbf{\Delta}-\angle_{y} \mathbf{\Delta}+\pi \\
& =\kappa(\sigma)-\omega(\mathbf{\Delta}) .
\end{aligned}
$$

Therefore, for any subdivision $\left\{p=y_{0}, y_{1}, \ldots, y_{n}=q\right\}$ of $\left\{x_{0}, \ldots, x_{m}\right\}$ and for $\tau:=y_{0} y_{1} \ldots y_{n}$, we have

$$
\kappa(\tau)=\kappa(\sigma)-\omega\left(E_{\sigma, \tau}\right),
$$

where $E_{\sigma, \tau}$ denotes the union of open disk domains of $X$ between $\sigma$ and $\tau$. If the subdivision $\left\{y_{0}, \ldots, y_{n}\right\}$ is getting finer and finer, then $\tau$ tends to $c$ and hence $E_{\sigma, \tau}$ to the open domain, say $E_{\sigma}$, bounded by $\sigma$ and $c$, so that

$$
\lim _{\tau \rightarrow c} \kappa(\tau)=\kappa(\sigma)-\omega\left(E_{\sigma}\right) .
$$

We define $\kappa(c)$ to be the above and call this the rotation of $c$. It follows that $\kappa(c)$ is nonnegative and independent of $\sigma$. A standard measure construction argument yields that the rotation $\kappa$ extends to the (nonnegative) Radon measure over $\partial X$.

For a polygonal region $D \subset X$, we set $\hat{\partial} D:=\partial D \cup(\partial X \cap D)$ (where $\partial D$ is the topological boundary of $D$ ). The rotation $\kappa$ is naturally defined over $\hat{\partial} D$ as a signed Radon measure. We now extend the GaussBonnet formula to the case where $D$ may touch $\partial X$.

Proposition 14.1 (The Gauss-Bonnet Theorem). For any relatively compact polygonal open region $D \subset X$ we have

$$
\omega(D)+\kappa(\hat{\partial} D)=2 \pi \chi(D)
$$

In particular, if $X$ is compact,

$$
\omega(X)+\kappa(\partial X)=2 \pi \chi(X)
$$

Proof. An easy calculation using $\kappa(c)=\kappa(\sigma)-\omega\left(E_{\sigma}\right)$. q.e.d.
Remark that for the Gauss-Bonnet formula $\omega(D)+\kappa(\hat{\partial} D)=2 \pi \chi(D)$ to hold, the region $D$ has to be an open subset, because $\omega(\partial D)>0$ may happen.

The total curvature $\omega$ over $X$ is a signed Radon measure which splits into two nonnegative Radon measures $\omega_{+}$and $\omega_{-}$over $X$ such that $\omega=\omega_{+}-\omega_{-}$. When a Borel subset $D \subset X$ is not relatively compact, it may happen that $\omega_{+}(D)=\omega_{-}(D)=\infty$, in which case $\omega(D)$ is not defined. The rotation measure $\kappa$ over the boundary $\hat{\partial} D$ of a polygonal region $D \subset X$ also splits into two nonnegative Radon measures $\kappa_{ \pm}$ such that $\kappa=\kappa_{+}-\kappa_{-}$. When $\hat{\partial} D-\partial X$ is unbounded, $\kappa(\hat{\partial} D)$ is not necessarily defined as well. Notice that $\kappa$ over $\partial X$ is nonnegative and $\kappa(\partial X) \in[0, \infty]$ is always defined.

We say that a topological manifold is finitely connected if it can be contracted to a compact submanifold (with boundary) by strong deformation retraction.

Proposition 14.2 (The Cohn-Vossen Theorem). If $\omega(D)$ and $\kappa(\hat{\partial} D)$ are both defined for a finitely connected polygonal open region $D \subset X$, then we have

$$
2 \pi \chi(D)-\pi \chi(\hat{\partial} D)-\omega(D)-\kappa(\hat{\partial} D) \geq 0
$$

In particular, if $X$ is finitely connected and if $\omega(X)$ is defined, then

$$
2 \pi \chi(X)-\pi \chi(\partial X)-\omega(D)-\kappa(\partial X) \geq 0
$$

Notice here that $\chi(\hat{\partial} D)$ is equal to the number of components of $\hat{\partial} D$ homeomorphic to $\mathbb{R}$.

Proof. The proof is by the same discussion as for Riemannian manifolds (see [31]). q.e.d.

Corollary 14.3. If $X$ is compact, the number of essential singular points in $X-\partial X$ is at most $4-a \mathcal{H}^{2}(X) / \pi$.

Proof. Denoting by $E$ the set of essential singular points in $X-\partial X$, we have

$$
\begin{aligned}
\pi \# E & \leq \omega(E)=\omega(X)-\omega(X-E) \\
& \leq 2 \pi \chi(X)-a \mathcal{H}^{2}(X) \leq 4 \pi-a \mathcal{H}^{2}(X)
\end{aligned}
$$

q.e.d.

Corollary 14.4. Let $X$ be a two-dimensional Alexandrov space of nonnegative curvature. Then, the following hold.
(1) $X$ is homeomorphic to either $\mathbb{R}^{2},[0,+\infty) \times \mathbb{R}, S^{2}, P^{2}, D^{2}$, or isometric to $[0, \ell] \times \mathbb{R},[0, \ell] \times S^{1}(r),[0,+\infty) \times S^{1}(r), \mathbb{R} \times S^{1}(r)$, $\mathbb{R} \times S^{1}(r) / \mathbb{Z}_{2}$, a flat torus, or a flat Klein bottle for some $\ell, r>0$.
(2) int $X$ contains at most four essential singular points, and denoting by $n$ the number of essential singular points in int $X$, we have the following for some $\ell, h>0$.
(a) If $n \geq 1, X$ is either homeomorphic to $\mathbb{R}^{2}, S^{2}, P^{2}, D^{2}$, or isometric to $\mathrm{dbl}([0, \infty) \times[0, \infty)) \cap\{(x, y) \mid y \leq h\}$.
(b) If $n \geq 2, X$ is either homeomorphic to $S^{2}$, or isometric to $\operatorname{dbl}([0, \infty) \times[0, h]), \operatorname{dbl}([0, \infty) \times[0, h]) \cap\{(x, y) \mid x \leq \ell\}$, or $\operatorname{dbl}([0, \ell] \times[0, h]) / \mathbb{Z}_{2}$.
(c) If $n \geq 3, X$ is homeomorphic to $S^{2}$.
(d) If $n=4, X$ is isometric to $\operatorname{dbl}([0, \ell] \times[0, h])$.

Proof. Since $\partial X$ can be approximated by broken geodesics as is seen before, there is a triangulation of $X$ with countably many geodesic triangles $\{\triangle\}$ such that int $X \subset \bigcup_{\triangle} \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is the open disk domain bounded by $\triangle$. Recall that

$$
\omega(X)=\sum_{\Delta} \omega(\mathbf{\Delta})+\sum_{x \in V \cap \operatorname{int} X}\left(2 \pi-L\left(\Sigma_{x}\right)\right),
$$

where $V$ is the set of vertices of the triangulation. The nonnegativity of the curvature of $X$ implies that $L\left(\Sigma_{x}\right) \leq 2 \pi, \omega(\mathbf{\Delta}) \geq 0$, and that $\omega(\mathbf{\Delta})=0$ if and only if $\boldsymbol{\Delta}$ is isometric to a triangular disk domain of $\mathbb{R}^{2}$. Therefore, we have $\omega(X) \geq 0$ and the equality holds if and only if $X$ is flat everywhere. If $X$ is flat and $\kappa(\partial X)=0$, then it is easy to observe that $\partial X$ is totally geodesic. Thus, applying the Gauss-Bonnet or Cohn-Vossen Theorem to $X$ proves (1).

To prove (2), we suppose that int $X$ contains $n$ different essential singular points $x_{1}, \ldots, x_{n}$. Then, it follows that $\omega(X) \geq n \pi$ and that the equality holds if and only if $X-\left\{x_{1}, \ldots, x_{n}\right\}$ is flat and if $L\left(\Sigma_{x_{i}}\right)=\pi$ for all $i$. This together with the Gauss-Bonnet or Cohn-Vossen Theorem completes the proof. q.e.d.

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