

Collateral and Capital Structure*

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Abstract

This paper develops a dynamic model of the capital structure based on the need to collateralize loans with tangible assets. The model provides a unified theory of optimal firm financing in terms of the optimal capital structure, investment, leasing, and risk management policy. Tangible assets are a key determinant of the cross section and dynamic behavior of the capital structure. Firms with low tangible capital are constrained longer, lease more of their physical capital, and borrow less. Leasing of tangible assets enables faster firm growth. The model helps explain the “zero debt puzzle” as well as other stylized facts about the capital structure. For risk management the model implies that incomplete hedging of net worth is optimal.

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1 Introduction

Capital structure has proved elusive. We argue that collateral determines the capital structure. We develop a dynamic agency based model of the capital structure based on the need to collateralize loans with tangible assets. Our model provides a unified theory of optimal firm financing in terms of the optimal capital structure, investment, leasing, and risk management policy.

In the data, we show that tangible assets are a key determinant of firm leverage. Leverage varies by a factor 3 from the lowest to the highest tangibility quartile for Compustat firms. Moreover, tangible assets are an important explanation for the “zero debt puzzle” in the sense that firms with low leverage are largely firms with few tangible assets. We also take firms’ ability to deploy tangible assets by renting or leasing such assets into account. We show that accounting for leased assets reduces the fraction of low leverage firms drastically and that “true” tangibility, that is tangibility adjusted for leased assets, further strengthens our results that firms with low “true” leverage, that is, leverage adjusted for leased assets, are firms with few tangible assets. Finally, we show that accounting for leased capital changes the relation between leverage and size in the cross section of Compustat firms. This relation is essentially flat when leased capital is taken into account. In contrast, when leased capital is ignored, as is done in the literature, leverage increases in size, that is, small firms seem less levered than large firms. Thus, basic stylized facts about the capital structure need to be revisited.

Financing is an inherently dynamic problem. Moreover, we think incentive problems, specifically, the enforcement of repayment, is a critical determinant of the capital structure and develop a dynamic model in which firm financing is subject to collateral constraints due to limited enforcement as in Rampini and Viswanathan (2008). Unlike previous work on dynamic agency models of the capital structure, we explicitly consider firms’ ability to lease capital. We build on the model of Eisfeldt and Rampini (2009), who argue that leasing amounts to a particularly strong form of collateralization due the relative ease with which leased capital can be repossessed, and extend their work by considering a dynamic model. A frictionless rental market for capital would of course obviate financial constraints. Leasing in our model is however costly since the lessor incurs monitoring costs to avoid agency problems due to the separation of ownership and control.

We provide a definition of the user cost of capital in our model of investment with financial constraints. For the frictionless neoclassical model of investment, Jorgenson (1963) defines the user cost of capital. Lucas and Prescott (1971), Abel (1983), and Abel and Eberly (1996) extend Jorgenson’s definition of the user cost of capital to models with adjustment costs. Our definition is closely related to Jorgenson’s. Indeed, the user cost

of capital is effectively the sum of Jorgenson’s user cost and a term which captures the additional cost due to the scarcity of internal funds. We also provide a “weighted average cost of capital” type representation of the user cost of capital. We show how to define the user cost of capital for tangible, intangible, and leased capital. The leasing decision reduces to a comparison between the user costs of (owned) tangible capital and the user cost of leased capital.

Our model predicts that firms only pay out dividends when net worth exceeds a (state-contingent) cut off. In the model, firms require both tangible and intangible capital. The enforcement constraints imply that only tangible capital can be used as collateral. We show that, in the absence of leasing and uncertainty, higher tangibility is equivalent to a better ability to collateralize, that is, only the product of the fraction of assets that is tangible and the fraction of assets that can be collateralized matters. However, firms with less tangible assets are more constrained or constrained for longer. When leasing is taken into account, financially constrained firms, that is, firms with low net worth, lease capital. And over time, as firms accumulate net worth, they grow in size and start to buy capital. Thus, the model predicts that small firms and young firms lease capital. We show that the ability to lease capital enables firms to grow faster. Our model also has implications for risk management. We provide conditions for which incomplete hedging is optimal. That is, we show that it cannot be optimal to hedge net worth to the point where the marginal value of net worth is equated across all states.

Our paper is part of a recent and growing literature which considers dynamic incentives problems as the main determinant of the capital structure. The incentive problem in our model is limited enforcement of claims. Most closely related to our work is Albuquerque and Hopenhayn (2004) and Lorenzoni and Walentin (2008). Albuquerque and Hopenhayn (2004) study dynamic firm financing with limited enforcement. The specific limits on enforcement differ from our setting and they do not consider the standard neoclassical investment problem.¹ Lorenzoni and Walentin (2008) consider limits on enforcement very similar to ours in a model with constant returns to scale. However, they assume that all enforcement constraints always bind, which is not the case in our model, and focus on the relation between investment and Tobin’s q rather than the capital structure. The aggregate implications of firm financing with limited enforcement are studied by Cooley, Marimon, and Quadrini (2004) and Jermann and Quadrini (2008). Schmid (2008) considers the quantitative implications for the dynamics of firm financing. None of these models consider intangible capital or the option to lease capital.

¹Hopenhayn and Werning (2007) consider a version of this model in which limits on enforcement are stochastic and private information, which results in default occurring in equilibrium.

Capital structure and investment dynamics determined by incentive problems due to private information about cash flows or moral hazard are studied by Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007a), and DeMarzo, Fishman, He, and Wang (2008). Capital structure dynamics subject to similar incentive problems but abstracting from investment decisions are analyzed by DeMarzo and Fishman (2007b), DeMarzo and Sannikov (2006), and Biais, Mariotti, Plantin, and Rochet (2007).²

In Section 2 we report some stylized empirical facts about collateralized financing, tangibility, and leverage. We also show how to take leased capital into account and document the striking effect of doing so. Section 3 describes the model, optimal payout policy, and defines the user cost of tangible, intangible, and leased capital. Section 4 characterizes the optimal leasing and capital structure policy, Section 5 analyzes optimal risk management, and Section 6 concludes. All proofs are in the appendix.

2 Stylized facts

This section provides some aggregate and cross-sectional evidence that highlights the first order importance of tangible assets as a determinant of the capital structure in the data. We first take an aggregate perspective and then document the relation between tangible assets and leverage across firms. We take leased capital into account explicitly and show that it has quantitatively and qualitatively large effects on basic stylized facts about the capital structure, such as the relation between leverage and size.

2.1 Collateralized financing: the aggregate perspective

From the aggregate point of view, the importance of tangible assets is striking. Consider the balance sheet data from the Flow of Funds Accounts of the United States for households, (nonfinancial) corporate businesses, and noncorporate businesses reported in Table 1 (for the second quarter of 2008). Panel A summarizes the balance sheet of households (and nonprofit organizations). In the aggregate, households in the U.S. own tangible assets worth \$26.1 trillion, mainly real estate but also consumer durables. Households' aggregate liabilities are \$14.5 trillion, so considerably less than their tangible assets. Moreover, the bulk of households' liabilities are mortgages, namely \$10.9 trillion or about three quarters of all liabilities. The rest is primarily consumer credit (\$2.6 trillion), of which a large part is explicitly collateralized by consumer durables. Thus, households

²Relatedly, Gromb (1995) analyzes a multi-period version of Bolton and Scharfstein (1990)'s two period dynamic firm financing problem with privately observed cash flows. Atkeson and Cole (2008) consider a two period firm financing problem with costly monitoring of cash flows.

liabilities are largely explicitly collateralized and are substantially less than households' tangible assets.

Similarly, the balance sheets of (nonfinancial) corporate businesses (Panel B) and noncorporate businesses (Panel C) reveal that for both, tangible assets exceed total liabilities. Corporate businesses have tangible assets, including real estate, equipment and software, and inventories, of \$14.9 trillion and total liabilities of \$12.9 trillion, while noncorporate businesses have tangible assets worth \$7.8 trillion and total liabilities of \$5.2 trillion. Note that we are not concerned here with whether these liabilities are explicitly collateralized or only implicitly in the sense that the borrowers concerned have tangible assets exceeding their liabilities. Our reasoning is that even if liabilities are not explicitly collateralized, they are implicitly collateralized since restrictions on further investment, asset sales, and additional borrowing through covenants and the ability not to refinance debt allow lenders to effectively limit borrowing to the value of collateral in the form of tangible assets.

Finally, ignoring the rest of the world and aggregating across all balance sheets implies that U.S. households own tangible assets, either directly or indirectly, worth more than \$48 trillion, which is over 85% of their net worth (\$56 trillion). To be clear, this is at best a coarse picture of aggregate collateral, but we think it highlights the quantitative importance of tangible assets as well as the relation between tangible assets and liabilities in the aggregate.

2.2 Tangibility and leverage

To document the relation between tangibility and leverage, we analyze data for a cross section of Compustat firms. We sort firms into quartiles by tangibility measured as the value of property, plant, and equipment divided by the market value of assets. The results are reported in Table 2, which also provides a more detailed description of the construction of the variables. We measure leverage as long term debt to the market value of assets.

The first observation that we wanted to stress is that across tangibility quartiles, (median) leverage varies from 7.4% for low tangibility firms (that is, firms in the lowest quartile by tangibility) to 22.6% for high tangibility firms (that is, firms in the highest quartile by tangibility). This is a factor 3!³ Tangibility also varies substantially across quartiles; the cut-off value of the first quartile is 6.3% and the cut-off value of the fourth quartile is 32.2%.

³Mean leverage varies somewhat less, by a factor 2.2 from 10.8% to 24.2%.

To assess the role of tangibility as an explanation for the observation that some firms have very low leverage (the so-called “zero debt puzzle”), we compute the fraction of firms in each tangibility quartile which have low leverage, specifically leverage less than 10%.⁴ The fraction of firms with low leverage decreases from 58.3% in the low tangibility quartile to 14.9% in the high tangibility quartile. Thus, low leverage firms are largely firms with relatively few tangible assets.

2.3 Leased capital and leverage

Thus far, we have ignored leased capital which is the conventional approach in the literature. To account for leased (or rented) capital, we simply capitalize the rental expense (Compustat item #47).⁵ This allows us to impute capital deployed via operating leases, which are the bulk of leasing in practice.⁶ To capitalize the rental expense, recall that Jorgenson (1963)’s user cost of capital is $u \equiv r + \delta$, that is, the user cost is the sum of the interest cost and the depreciation rate. Thus, the frictionless rental expense for an amount of capital k is

$$\text{Rent} = (r + \delta)k.$$

Given data on rental payments, we can hence infer the amount of capital rented by capitalizing the rental expense using the factor $1/(r + \delta)$. For simplicity, we capitalize the rental expense by a factor 10. We adjust firms’ assets, tangible assets, and liabilities by adding 10 times rental expense to obtain measures of “true” assets, “true” tangible assets, and “true” leverage.

We proceed as before and sort firms into quartiles by true tangibility. The results are reported in Table 3. True debt leverage is somewhat lower as we divide by true assets here. There is a strong relation between true tangibility and true leverage (as before), with the median true debt leverage varying again by a factor of about 3. Rental leverage also increases with true tangibility by about a factor 2 for the median and more than 3 for the mean. Similarly, true leverage, which we define as the sum of debt leverage and rental leverage, also increases with tangibility by a factor 3.

Taking rental leverage into account reduces the fraction of firms with low leverage drastically, in particular for firms outside the low tangibility quartile. True tangibility is an even more important explanation for the “zero debt puzzle.” Indeed, less than 4% of firms with high tangibility have low true leverage.

⁴We do not think that our results change if lower cutoff values are considered.

⁵A similar procedure is known in accounting as constructive capitalization.

⁶Note that capital leases are already accounted for as they are capitalized on the balance sheet for accounting purposes.

It is also worth noting that the median rental leverage is on the order of half of debt leverage or more, and is hence quantitatively important. Overall, we conclude that tangibility, when adjusted for leased capital, emerges as a key determinant of leverage and the fraction of firms with low leverage.

2.4 Leverage and size revisited

Considering leased capital changes basic cross-sectional properties of the capital structure. Here we document the relationship between firm size and leverage (see Table 4 and Figure 1). We sort Compustat firms into deciles by size. We measure size by true assets here, although using unadjusted assets makes our results even more stark. Debt leverage is increasing in size, in particular for small firms, when leased capital is ignored. Rental leverage, by contrast, decreases in size, in particular for small firms. Indeed, rental leverage is substantially larger than debt leverage for small firms! True leverage, that is, the sum of debt and rental leverage, is roughly constant across Compustat size deciles. In our view, this evidence provides a strong case that leased capital cannot be ignored if one wants to understand the capital structure.

3 Model

This section describes our infinite horizon dynamic agency based model of the capital structure. Dynamic financing is subject to collateral constraints due to limited enforcement. We extend previous work by considering both tangible and intangible capital as well as borrower's ability to lease capital. We characterize the dividend policy and define the user cost of tangible, intangible, and leased capital. We provide a weighted average cost of capital type representation of the user cost of capital. The user cost of capital definitions allow a very simple description of the leasing decision, which can be reduced to a comparison of the user cost of tangible capital and the user cost of leased capital.

3.1 Borrower's problem

A risk neutral borrower, who is subject to limited liability and discounts the future at rate $\beta \in (0, 1)$, requires financing for investment. The investment problem has an infinite horizon and we write the problem recursively. The borrower starts the period with net worth w . The borrower has access to a standard neoclassical production function with decreasing returns to scale. An investment of capital k' yields stochastic cash flow $A(s')f(k')$ next period, where $A(s')$ is the realized idiosyncratic total factor productivity

of the technology in state s' , which we assume follows a Markov process described by the transition function $\Pi(s, s')$ on $s' \in S$. Capital k' is the total amount of capital of the firm, which will have three components, intangible capital, purchased physical capital, and leased physical capital, described in more detail below. Capital depreciates at rate $\delta \in (0, 1)$ and there are no adjustment costs.

There are two types of capital, physical capital and intangible capital (k'_i). Either type of capital can be purchased at a price normalized to 1 and both are fully reversible. Physical and intangible capital are assumed to depreciate at the same rate δ . Moreover, physical capital can be either purchased (k'_p) or leased (k'_l), while intangible capital can only be purchased. Physical capital which the borrower owns can be used as collateral for state-contingent one period debt up to a fraction θ of its resale value. These collateral constraints are motivated by limited enforcement. We assume that enforcement is limited in that borrowers can abscond with all cash flows, all intangible capital, and $1 - \theta$ of purchased physical capital k'_p . We further assume that borrowers cannot abscond with leased capital k'_l , that is, leased capital enjoys a repossession advantage. Moreover, and importantly, we assume that borrowers who abscond cannot be excluded from the market for intangible capital, physical capital, or loans, nor can they be prevented from renting capital. Extending the results in Rampini and Viswanathan (2008) one can show that these dynamic enforcement constraints imply the above collateral constraints, which are described in more detail below.⁷

Our model of leased capital extends the work of Eisfeldt and Rampini (2009) to a dynamic environment. The assumption that borrowers cannot abscond with leased capital mirrors their assumption that leased capital is more easily repossessed. Leased capital involves monitoring costs m per unit of capital incurred by the lessor at the beginning of the period, which are reflected in the user cost of leased capital u_l . A competitive lessor with a cost of capital $R \equiv 1 + r$ charges a user cost of

$$u_l \equiv R^{-1}(r + \delta) + m$$

per unit of capital at the beginning of the period.⁸ Without loss of generality, the user

⁷These collateral constraints are very similar to the ones in Kiyotaki and Moore (1997), albeit state contingent. However, they are derived from a explicitly dynamic model of limited enforcement similar to the one considered by Kehoe and Levine (1993). The main difference to their limits on enforcement is that we assume that borrowers who abscond cannot be excluded from future borrowing whereas they assume that borrowers are in fact excluded from intertemporal trade after default. Similar constraints have been considered by Lustig (2007) in an endowment economy and by Lorenzoni and Walentin (2007) in a production economy with constant returns to scale.

⁸Equivalently, we could assume that leased capital depreciates faster; specifically, $\delta_l \equiv \delta + Rm$ implies $u_l = R^{-1}(r + \delta_l)$.

cost of leased capital is charged up front due to the constraints on enforcement. Recall that in the frictionless neoclassical model, the rental cost of capital is Jorgenson (1963)'s user cost $u = r + \delta$. There are two differences to the rental cost in our model. First, there is a positive monitoring cost. Second, due to limited enforcement, the rental charge is paid in advance and hence discounted to time 0.⁹

The total amount of capital is $k' \equiv k'_i + k'_p + k'_l$. We assume that physical and intangible capital are required in fixed proportions and denote the fraction of physical capital required by φ , implying that $k'_i = (1 - \varphi)k'$ and $k'_p + k'_l = \varphi k'$. Using these two equations, the borrower's investment problem simplifies to the choice of k' and k'_l only.

We assume that the borrower has access to lenders who have deep pockets in all dates and states and discount the future at rate $R \in (\beta, 1)$. These lenders are thus willing to lend in a state contingent way at an expected return R . The assumption that $R > \beta$ implies that borrowers are less patient than lenders and will imply that borrowers will never be completely unconstrained in our model. This assumption is important to understand the dynamics of firm financing, in particular the fact that firms pay dividends even if they are not completely unconstrained and that firms may stop dividend payments and switch back to leasing capital, as we discuss below.¹⁰

The borrower's problem can hence be written as the problem of maximizing the discounted expected value of future dividends by choosing the current dividend d , capital k' , leased capital k'_l , net worth $w'(s')$ in state s' , and state-contingent debt $b(s')$:

$$V(w, s) \equiv \max_{\{d, k', k'_l, w'(s'), b(s')\} \in \mathbb{R}_+^{3+S} \times \mathbb{R}^S} d + \beta \sum_{s' \in S} \Pi(s, s') V(w'(s'), s') \quad (1)$$

subject to the budget constraints

$$w + \sum_{s' \in S} \Pi(s, s') b(s') \geq d + k' - (1 - u_l) k'_l \quad (2)$$

$$A(s') f(k') + (k' - k'_l)(1 - \delta) \geq w'(s') + R b(s'), \forall s' \in S, \quad (3)$$

⁹To impute the amount of capital rented from rental payments, we should hence capitalize rental payments by $1/(R^{-1}(r + \delta) + m)$. In documenting the stylized facts, we assumed that this factor takes a value of 10. The implicit debt associated with rented capital is $R^{-1}(1 - \delta)$ times the amount of capital rented, so in adjusting liabilities, we should adjust by $R^{-1}(1 - \delta)$ times 10 to be precise. In documenting the stylized facts, we ignored the correction $R^{-1}(1 - \delta)$, implicitly assuming that it is approximately equal to 1.

¹⁰While we do not explicitly consider taxes here, our assumption about discount rates can also be interpreted as a reduced form way of taking into account the tax-deductibility of interest, which effectively lowers the cost of debt finance.

the collateral constraints

$$\theta(\varphi k' - k'_l)(1 - \delta) \geq Rb(s'), \quad \forall s' \in S, \quad (4)$$

and the constraint that only physical capital can be leased

$$\varphi k' \geq k'_l. \quad (5)$$

Note that the program in (1)-(5) requires that dividends d and net worth $w'(s')$ are non-negative which is due to limited liability. Furthermore, capital k' and leased capital k'_l have to be non-negative as well. We write the budget constraints as inequality constraints, despite the fact that they bind at an optimal contract, since this makes the constraint set convex as shown below. There are only two state variables in this recursive formulation, net worth w and the state of productivity s . This is due to our assumption that there are no adjustment costs of any kind and greatly simplifies the analysis.

We make the following assumptions about the stochastic process describing productivity and the production function:

Assumption 1 For all $\hat{s}, s \in \mathcal{S}$ such that $\hat{s} > s$, **(i)** $A(\hat{s}) > A(s)$, **(ii)** $A(s) > 0$, and **(iii)** $\Pi(\hat{s}, s')$ strictly first order stochastically dominates $\Pi(s, s')$.

Assumption 2 f is strictly increasing and strictly concave, $f(0) = 0$ and $\lim_{k \rightarrow 0} f'(k) = +\infty$.

We first show that the firm financing problem is a well-behaved convex dynamic programming problem and that there exists a unique value function V which solves the problem. To simplify notation, we introduce the shorthand for the choice variables x , where $x \equiv [d, k', k'_l, w'(s'), b(s')]'$, and the shorthand for the constraint set $\Gamma(w, s)$ given the state variables w and s , defined as the set of $x \in \mathbb{R}_+^{3+S} \times \mathbb{R}^S$ such that (2)-(5) are satisfied. Define operator T as

$$(Tf)(w, s) = \max_{x \in \Gamma(w, s)} d + \beta \sum_{s' \in S} \Pi(s, s') f(w'(s'), s').$$

We prove the following result about the firm financing problem in (1)-(5):

Proposition 1 **(i)** $\Gamma(w, s)$ is convex, given (w, s) , and convex in w and monotone in the sense that $w \leq \hat{w}$ implies $\Gamma(w, s) \subseteq \Gamma(\hat{w}, s)$. **(ii)** The operator T satisfies Blackwell's sufficient conditions for a contraction and has a unique fixed point V . **(iii)** V is strictly increasing and concave in w . **(iv)** Without leasing, $V(w, s)$ is strictly concave in w for $w \in \text{int}\{w : d(w, s) = 0\}$. **(v)** Under Assumption 1, V is strictly increasing in s .

The proofs of part (i)-(iii) of the proposition are relatively standard. Part (iii) however only states that the value function is concave, not strictly concave. The value function turns out to be linear in net worth when dividends are paid. The value function may also be linear in net worth on some intervals where no dividends are paid, due to the substitution between leased and owned capital. All our proofs below hence rely on weak concavity only. Nevertheless we can show that without leasing, the value function is strictly concave where no dividends are paid (see part (iv) of the proposition). Finally, we note that Assumption 1 is only required for part (v) of the proposition.

Consider the first order conditions of the firm financing problem in equations (1)-(5). Denote the multipliers on the constraints (2), (3), (4), and (5) by μ , $\Pi(s, s')\mu(s')$, $\Pi(s, s')\lambda(s')$, and $\bar{\nu}_l$.¹¹ Let ν_d and $\underline{\nu}_l$ be the multipliers on the constraint $d \geq 0$ and $k'_l \geq 0$. The first order conditions are

$$\mu = 1 + \nu_d \tag{6}$$

$$\mu = \sum_{s' \in S} \Pi(s, s') \{ \mu(s') [A(s')f'(k') + (1 - \delta)] + \lambda(s')\theta\varphi(1 - \delta) \} + \bar{\nu}_l\varphi \tag{7}$$

$$(1 - u_l)\mu = \sum_{s' \in S} \Pi(s, s') \{ \mu(s')(1 - \delta) + \lambda(s')\theta(1 - \delta) \} + \bar{\nu}_l - \underline{\nu}_l \tag{8}$$

$$\mu(s') = \beta V_w(w'(s'), s'), \quad \forall s' \in S, \tag{9}$$

$$\mu = \mu(s')R + \lambda(s')R, \quad \forall s' \in S, \tag{10}$$

where we have assumed that the constraints $k' \geq 0$ and $w'(s') \geq 0$, $\forall s' \in S$, are slack as shown in Lemma 1 below. The envelope condition is

$$V_w(w, s) = \mu.$$

Since we assume that the marginal product of capital is unbounded as capital goes to zero, investment is strictly positive. Because the borrower's ability to issue promises against capital is limited, this in turn implies that the borrower's net worth is positive in all states in the next period, as the next lemma shows.

Lemma 1 *Under Assumption 2, investment and net worth in all states are strictly positive, $k' > 0$ and $w'(s') > 0$, $\forall s' \in S$.*

We start by characterizing the firm's payout policy. The firm's dividend policy is very intuitive: there is a state contingent cutoff level of net worth $\bar{w}(s)$, $\forall s \in S$, above which the firm pays dividends. Moreover, whenever the firm has net worth w exceeding the cutoff $\bar{w}(s)$, paying dividends in the amount $w - \bar{w}(s)$ is optimal. All firms with net

¹¹Note that we scale some of the multipliers by $\Pi(s, s')$ to simplify the notation.

worth w exceeding the cutoff $\bar{w}(s)$ in a given state s , choose the same level of capital. Finally, the investment policy is unique in terms of the choice of capital k' . The following proposition summarizes the characterization of firms' payout policy:

Proposition 2 (Dividend policy) *There is a state contingent cutoff level of net worth, above which the marginal value of net worth is one and the firm pays dividends: (i) $\forall s \in S$, $\exists \bar{w}(s)$ such that, $\forall w \geq \bar{w}(s)$, $\mu(w, s) = 1$. (ii) For $\forall w \geq \bar{w}(s)$,*

$$[d_o(w, s), k'_o(w, s), k'_{l,o}(w, s), w'_o(s'), b_o(s')] = [w - \bar{w}(s), \bar{k}'_o(s), \bar{k}'_{l,o}(s), \bar{w}'_o(s'), \bar{b}_o(s')]$$

where $\bar{x}_o \equiv [0, \bar{k}'_o(s), \bar{k}'_{l,o}(s)\bar{w}'_o(s'), \bar{b}_o(s')]$ attains $V(\bar{w}(s), s)$. Indeed, $k'_o(w, s)$ is unique for all w and s . (iii) Without leasing, the optimal policy x_o is unique.

3.2 User cost of capital

This section provides definitions for the user cost of intangible capital, purchased physical capital, and leased capital, extending Jorgenson (1963)'s definition to our model with collateral constraints. Lucas and Prescott (1971), Abel (1983), and Abel and Eberly (1996) define the user cost of capital for models with adjustment costs. The definitions clarify the main economic intuition behind our results and allow a very simple characterization of the leasing decision.

Our definition of the user cost of physical capital which is purchased u_p is

$$u_p \equiv R^{-1}(r + \delta) + \sum_{s' \in S} \Pi(s, s') \frac{\lambda(s')}{\mu} (1 - \theta)(1 - \delta)$$

where $\lambda(s')$ is the Kuhn-Tucker multiplier on the state s' collateral constraint. Note that the user cost of purchased physical capital has two components. The first component is simply the Jorgensonian user cost of capital, paid in advance. The second component captures the additional cost of internal funds, which command a premium due to the collateral constraints. Indeed, $(1 - \theta)(1 - \delta)$ is the fraction of capital that needs to be financed internally, because the borrower cannot credibly pledge that amount to lenders. Similarly, we define the user cost of intangible capital u_i as $u_i \equiv R^{-1}(r + \delta) + \sum_{s' \in S} \Pi(s, s') \frac{\lambda(s')}{\mu} (1 - \delta)$. The only difference is that all of physical capital needs to be financed with internal funds and hence the second term involves fraction $1 - \delta$ rather than only a fraction $1 - \theta$ of that amount.

Using our definitions of the user cost of purchased physical and intangible capital, and (10), we can rewrite the first order condition for investment, equation (7), as

$$\varphi u_p + (1 - \varphi) u_i = \sum_{s' \in S} \Pi(s, s') \frac{\mu(s')}{\mu} A(s') f'(k') + \bar{v}_l \varphi.$$

Optimal investment equates the weighted average of the user cost of physical and intangible capital with the expected marginal product of capital.

The user cost of physical capital can also be written in a *weighted average cost of capital* form as

$$u_p = 1 - \left[R^{-1}\theta + \sum_{s' \in S} \Pi(s, s') \frac{\mu(s')}{\mu} (1 - \theta) \right] (1 - \delta),$$

where the fraction of physical capital that can be financed with external funds, θ , is discounted at R , while the fraction of physical capital that has to be financed with internal funds, $1 - \theta$, is discounted at $(\sum_{s' \in S} \Pi(s, s') \mu(s') / \mu)^{-1}$, which strictly exceeds R as long as $\lambda(s') > 0$, for some $s' \in S$.

Using the definitions of the user cost of physical capital above and (10), the first order condition with respect to leased capital, (8), simplifies to

$$u_l = u_p - \bar{v}_l / \mu + \underline{v}_l / \mu. \tag{11}$$

The decision between purchasing capital and leasing reduces to a straight comparison of the user costs. If the user cost of leasing exceeds the user cost of purchased capital, then $\underline{v}_l > 0$ and the borrower purchases all capital. If the reverse is true, $\bar{v}_l > 0$ and all capital is leased. When $u_l = u_p$, the borrower is indifferent between leasing and purchasing capital at the margin.

4 Leasing and the capital structure

This section analyzes the dynamic leasing decision in detail. We focus on the deterministic case to highlight the economic intuition and facilitate explicit characterization. The analysis is rendered easier in this case by the fact that the collateral constraint binds throughout. We first analyze the capital structure decision without leasing, and show that higher tangibility and higher collateralizability are equivalent in that case. We then study the dynamic choice between leasing and secured financing. Finally, we show that leasing enables firms to grow faster.

4.1 Capital structure without leasing

When there is no leasing, higher tangibility and higher collateralizability are equivalent. Thus, firms which operate in industries with more intangible capital are more constrained and constrained for longer, all else equal.

Proposition 3 (Tangibility and collateralizability) *Without leasing, a higher fraction of physical capital φ is equivalent to a higher fraction θ that can be collateralized.*

This result is immediate as without leasing, φ and θ affect only (4) and only the product of the two matters.

The dynamics of firm financing in this case are as follows. As long as net worth is below a cutoff \bar{w} , firms pay no dividends and accumulate net worth over time which allows them to increase the amount of capital they deploy. Once net worth reaches \bar{w} , dividends are positive and firms no longer grow.

Proposition 4 (Deterministic capital structure dynamics) *For $w \leq \bar{w}$, no dividends are paid and capital is strictly increasing in w and over time. For $w > \bar{w}$, dividends are strictly positive and capital is constant at a level \bar{k}' .*

4.2 Dynamic choice between leasing and secured financing

When leasing is an option, firms have to choose a leasing policy in addition to the investment, financing and payout policy. The following assumption ensures that the monitoring cost are such that leasing is neither dominated nor dominating:

Assumption 3 *Leasing is neither dominated nor dominating, that is,*

$$R^{-1}(1 - \theta)(1 - \delta) > m > (R^{-1} - \beta)(1 - \theta)(1 - \delta).$$

The left most expression and the right most expression are the opportunity costs of the additional down payment requirement when purchasing capital, which depend on the borrower's discount rate. The additional down payment requirement is $R^{-1}(1 - \theta)(1 - \delta)$ which is recovered the next period. If the borrower is very constrained, the recovered funds are not valued at all, which yields the expression on the left. If the borrower is least constrained, the recovered funds are valued at β , the discount factor of the borrower, and the opportunity cost is only the wedge between the funds discounted at the lenders' discount rate and the borrower's discount rate.

When firms can lease capital, the financing dynamics are as follows: when firms have low net worth, they lease all the physical capital and purchase only the intangible capital. Over time, firms accumulate net worth and increase their total capital. When they reach a certain net worth threshold, they start to substitute owned capital for leased capital, continuing to accumulate net worth. Once firms own all their physical and intangible capital, they further accumulate net worth and increase the capital stock until they start to pay dividends. At that point, capital stays constant.

Proposition 5 (Deterministic capital structure dynamics with leasing) *Suppose Assumption 3 holds. For $w \leq \bar{w}$, no dividends are paid and capital is increasing in w and over time. For $w > \bar{w}$, dividends are strictly positive and capital is constant at a level \bar{k}' . There exist $\underline{w}_l < \bar{w}_l < \bar{w}$, such that for $w \leq \underline{w}_l$ all physical capital is leased and for $w < \bar{w}_l$ some capital is leased.*

This result extends the static model of Eisfeldt and Rampini (2009) to a dynamic environment.

4.3 Leasing and firm growth

Leasing allows constrained firms to grow faster. To see this note that the minimum amount of internal funds required to purchase one unit of capital is $1 - R^{-1}\theta\varphi(1 - \delta)$, since the borrower can borrow against fraction θ of the resale value of physical capital, which is fraction φ of capital. The minimum amount of internal funds required when physical capital is leased is $1 - \varphi + u_l\varphi$, since the borrower has to finance all intangible capital with internal funds $(1 - \varphi)$ and pay the leasing fee on physical capital up front $(u_l\varphi)$. Per unit of internal funds, the borrower can hence buy capital in the amount of one over these minimum amounts of internal funds. Under Assumption 3, leasing allows higher leverage, that is, $1/(1 - \varphi + u_l\varphi) > 1/(1 - R^{-1}\theta\varphi(1 - \delta))$. Thus, leasing allows firms to deploy more capital and hence to grow faster.

Corollary 1 (Leasing and firm growth) *Leasing enables firms to grow faster.*

Figure 3 illustrates the net worth dynamics with and without leasing. The figure displays the transition function between current net worth w and net worth in the next period w' . The dashed line describes the transition with leasing. For low values of current net worth it lies strictly above the solid line which describes the transition without leasing. For these values of net worth the borrower chooses to lease at least some of its physical capital.

While we focus on the deterministic case for the analysis of leasing here, the same economic intuition carries over to the general stochastic case. However, the analysis of the general case has to proceed numerically.

5 Risk management and the capital structure

One advantage of our model is that borrowers have access to complete markets, subject to the collateral constraints due to limited enforcement. This is useful because it allows

an explicit analysis of risk management. Thus, we are able to provide a unified analysis of optimal firm policies in terms of financing, investment, leasing, and risk management. Our model hence extends the work on risk management of Froot, Scharfstein, and Stein (1993) to a fully dynamic model of firm financing subject to financial constraints in the case of a standard neoclassical production function.

We analyze the case of independent productivity shocks here. This allows us to study the firms' hedging policy explicitly, as investment opportunities do not vary with independent shocks, in the sense that, all else equal, the expected productivity of capital does not vary with the current realization of the state s . More generally, both cash flows and investment opportunities vary, and the correlation between the two obviously affects the desirability of hedging, as in Froot, Scharfstein, and Stein (1993).

In our model, we do not explicitly take a stand on whether the productivity shocks are firm specific or aggregate. Since all states are observable, as the only friction considered is limited enforcement, our analysis applies either way. Hedging in this section can hence be interpreted either as using for example loan commitments to hedge idiosyncratic shocks to a firm's net worth or as using traded assets to hedge aggregate shocks which affect firms' cash flows.¹²

When productivity is independent across time, that is, $\Pi(s, s') = \Pi(s')$, $\forall s, s' \in S$, the state s is no longer a state variable. This implies that the value of net worth across states is ordered as follows:

Proposition 6 (Value of internal funds and collateral constraints) *The marginal value of net worth is (weakly) decreasing in the state s' , and the multipliers on the collateral constraints are (weakly) increasing in the state s' , that is, $\forall s', s'_+ \in S$ such that $s'_+ > s'$, $\mu(s') \geq \mu(s'_+)$ and $\lambda(s') \leq \lambda(s'_+)$.*

Thus, the marginal value of net worth is higher in states with low cash flows due to low realizations of productivity. We can now show that complete hedging is never optimal.

Theorem 1 (Optimality of incomplete hedging) *Incomplete hedging is optimal, that is, $\exists s', \hat{s}' \in S$, such that $w'(s') \neq w'(\hat{s}')$. Moreover, the borrower never hedges the highest state, that is, is always borrowing constrained against the highest state, $\lambda(\bar{s}') > 0$ where $\bar{s}' = \max\{s' : s' \in S\}$. The borrower hedges a lower interval of states, $[\underline{s}', \dots, s'_h]$, where $\underline{s}' = \min\{s' : s' \in S\}$, if at all.*

¹²See Rampini and Viswanathan (2008) for an interpretation of our state contingent financing in terms of loan commitments.

The intuition for this result is the following. Complete hedging would imply that the marginal value of net worth is equalized across all states next period. But hedging involves conserving net worth in a state-contingent way at a return R . Given the borrower's relative impatience, it can never be optimal to save in this state contingent way for all states next period. This implies the optimality of incomplete hedging.

The second aspect in the theorem is that, since the marginal value of net worth is higher in states with low cash flow realizations, it is optimal to hedge the net worth in these states, if it is optimal to hedge at all. Firms' optimal hedging policy implicitly ensures a minimum level of net worth in all states next period. Note that as emphasized by Rampini and Viswanathan (2008), if a firm is sufficiently constrained, it may be optimal not to hedge at all!

When firms' productivity follows a general Markov process, both net worth and investment opportunities vary, and the hedging policy needs to take the shortfall between financing needs and available funds across states into account. Numerical analysis of interesting special cases is possible.

We emphasize that our explicit dynamic model of collateral constraints due to limited enforcement is essential for this result. If the borrower's ability to pledge were not limited, then the borrower would always want to pledge more against high net worth states next period to equate net worth across all states. However, in our model the ability to credibly pledge to pay is limited and there is an opportunity cost to pledging to pay in high net worth states next period, since such pledges are also required for financing current investment. This opportunity cost implies that the borrower never chooses to fully hedge net worth shocks.

6 Conclusion

We argue that collateral determines the capital structure. We provide a dynamic agency based model of the capital structure subject to collateral constraints due to limited enforcement. In the model firms require both tangible and intangible capital, and the fraction of tangible assets required is a key determinant of leverage and the dynamics of firm financing.

Firms' ability to lease capital is explicitly taken into account in contrast to previous dynamic models of firm financing and investment with financial constraints. The extent to which firms lease is determined by firms' financial condition, and more constrained firms lease more. We show that leasing enables firms to grow faster. Using definitions of the user cost of purchased tangible capital and leased capital, the leasing decision reduces

to a simple comparison of these user costs. The user cost of purchased physical capital moreover has a weighted average cost of capital representation.

The model has implications for risk management. Specifically, we prove the optimality of incomplete hedging. It is never optimal for the borrower to hedge to the point that the marginal value of internal funds is equal across all states.

We also provide stylized empirical facts which highlight the importance of tangibility as a determinant of the capital structure in the data. Firm leverage changes substantially with the fraction of assets which is tangible. Moreover, the lack of tangible assets largely explains why some firms have low leverage, and hence addresses the “zero debt puzzle.” Leased capital is quantitatively important and further reduces the fraction of firms with low leverage.

We conclude that the tangibility of assets and firms’ ability to lease capital are critical ingredients for studies of the capital structure. Calibrated versions of our model and empirical work are required to assess the extent to which our model of collateralized financing only is able to capture key features of the data. The simple form of the optimal contract in our dynamic agency based capital structure model may facilitate the empirical implementation, which has remained a challenge for other such agency based models. Moreover, due to its simplicity, our model may also prove to be a useful framework to address other theoretical questions in dynamic corporate finance.

Appendix

Proof of Proposition 1. The proposition is proved in Lemma 2-6 below.

Lemma 2 $\Gamma(w, s)$ is convex, given (w, s) , and convex in w and monotone in the sense that $w \leq \hat{w}$ implies $\Gamma(w, s) \subseteq \Gamma(\hat{w}, s)$.

Proof of Lemma 2. Suppose $x, \hat{x} \in \Gamma(w, s)$. For $\phi \in (0, 1)$, let $x_\phi \equiv \phi x + (1 - \phi)\hat{x}$. Then $x_\phi \in \Gamma(w, s)$ since equations (2), (4), and (5), as well as the right hand side of equation (3) are linear and, since f is concave,

$$\begin{aligned} A(s')f(k'_\phi) + (k'_\phi - k'_{l,\phi})(1 - \delta) &\geq \phi[A(s')f(k') + (k' - k'_l)(1 - \delta)] \\ &\quad + (1 - \phi)[A(s')f(\hat{k}') + (\hat{k}' - \hat{k}'_l)(1 - \delta)]. \end{aligned}$$

Let $x \in \Gamma(w, s)$ and $\hat{x} \in \Gamma(\hat{w}, s)$. For $\phi \in (0, 1)$, let $x_\phi \equiv \phi x + (1 - \phi)\hat{x}$. Since equations (3), (4), and (5) do not involve w and \hat{w} , respectively, and $\Gamma(w, s)$ is convex given w , x_ϕ satisfies these equations. Moreover, since x and \hat{x} satisfy equation (2) at w and \hat{w} , respectively, and equation (2) is linear in x and w , x_ϕ satisfies the equation at w_ϕ . Thus, $x_\phi \in \Gamma(\phi w + (1 - \phi)\hat{w}, s)$. In this sense, $\Gamma(w, s)$ is convex in w .

If $w \leq \hat{w}$, then $\Gamma(w, s) \subseteq \Gamma(\hat{w}, s)$. \square

Lemma 3 The operator T satisfies Blackwell's sufficient conditions for a contraction and has a unique fixed point V .

Proof of Lemma 3. Suppose $g(w, s) \geq f(w, s)$, for all $(w, s) \in \mathbf{R}_+ \times S$. Then, for any $x \in \Gamma(w, s)$,

$$(Tg)(w, s) \geq d + \beta \sum_{s' \in S} \Pi(s, s')g(w'(s'), s') \geq d + \beta \sum_{s' \in S} \Pi(s, s')f(w'(s'), s').$$

Hence,

$$(Tg)(w, s) \geq \max_{x \in \Gamma(w, s)} d + \beta \sum_{s' \in S} \Pi(s, s')f(w'(s'), s') = (Tf)(w, s)$$

for all $(w, s) \in \mathbf{R}_+ \times S$. Thus, T satisfies monotonicity.

Operator T satisfies discounting since

$$T(f + a)(w, s) \geq \max_{x \in \Gamma(w, s)} d + \beta \sum_{s' \in S} \Pi(s, s')(f + a)(w'(s'), s') = (Tf)(w, s) + \beta a.$$

Therefore, operator T is a contraction and, by the contraction mapping theorem, has a unique fixed point V . \square

Lemma 4 V is strictly increasing and concave in w .

Proof of Lemma 4. Let $x_o \in \Gamma(w, s)$ and $\hat{x}_o \in \Gamma(\hat{w}, s)$ attain $(Tf)(w, s)$ and $(Tf)(\hat{w}, s)$, respectively. Suppose f is increasing in w and suppose $w \leq \hat{w}$. Then,

$$(Tf)(\hat{w}, s) = \hat{d}_o + \beta \sum_{s' \in S} \Pi(s, s') f(\hat{w}'_o(s'), s') \geq d + \beta \sum_{s' \in S} \Pi(s, s') f(w'(s'), s').$$

Hence,

$$(Tf)(\hat{w}, s) \geq \max_{x \in \Gamma(w, s)} d + \beta \sum_{s' \in S} \Pi(s, s') f(w'(s'), s') = (Tf)(w, s),$$

that is, Tf is increasing in w . Moreover, suppose $w < \hat{w}$, then

$$(Tf)(\hat{w}, s) \geq (\hat{w} - w) + d_o + \beta \sum_{s' \in S} \Pi(s, s') f(w'_o(s'), s') > (Tf)(w, s),$$

implying that Tf is strictly increasing. Hence, T maps increasing functions into strictly increasing functions, which implies that V is strictly increasing.

Suppose f is concave. Then, for $\phi \in (0, 1)$, let $x_{o,\phi} \equiv \phi x_o + (1 - \phi)\hat{x}_o$ and $w_\phi \equiv \phi w + (1 - \phi)\hat{w}$, we have

$$\begin{aligned} (Tf)(w_\phi, s) &\geq d_{o,\phi} + \beta \sum_{s' \in S} \Pi(s, s') f(w'_{o,\phi}(s'), s') \\ &\geq \phi \left[d_o + \beta \sum_{s' \in S} \Pi(s, s') f(w'_o(s'), s') \right] + (1 - \phi) \left[\hat{d}_o + \beta \sum_{s' \in S} \Pi(s, s') f(\hat{w}'_o(s'), s') \right] \\ &= \phi(Tf)(w, s) + (1 - \phi)(Tf)(\hat{w}, s). \end{aligned}$$

Thus, Tf is concave, and T maps concave functions into concave functions, which implies that V is concave. \square

Lemma 5 *Without leasing, $V(w, s)$ is strictly concave in w for $w \in \text{int}\{w : d(w, s) = 0\}$.*

Proof of Lemma 5. Without leasing, k'_t is set to zero throughout and all the prior results continue to hold. Suppose $w, \hat{w} \in \text{int}\{w : d(w, s) = 0\}$, $\hat{w} > w$. There must exist some state s^t_* , where $s^t = \{s_0, s_1, \dots, s_t\}$, which has strictly positive probability and in which the capital stock choice at \hat{w} is different from the choice at w , i.e., $\hat{k}'(s^t_*) \neq k'(s^t_*)$. Suppose instead that $\hat{k}(s^t) = k(s^t)$, $\forall s^t \in S^t, t = 0, 1, \dots$. Then there must exist some state s^{t*} with strictly positive probability in which $\hat{d}_o(s^{t*}) > d_o(s^{t*})$ and for which borrowing is not constrained along the path of s^{t*} . Reducing $\hat{d}_o(s^{t*})$ by η and paying out the present value at time 0 instead changes the objective by $(R^{-t} - \beta^t)(\hat{d}_o(s^t) - d_o(s^t)) > 0$, contradicting the optimality of $d(\hat{w}, s) = 0$.

Assume, without loss of generality, that $\hat{k}'_o(s'_*) \neq k''_o(s'_*)$, for some $s'_* \in S$. Rewrite the Bellman equation as

$$V(w, s) = \max_{\substack{x \in \Gamma(w, s), \\ x'(s') \in \Gamma(w'(s'), s'), \forall s' \in S}} d + \beta \sum_{s' \in S} \Pi(s, s') \left\{ d'(s') + \beta \sum_{s'' \in S} \Pi(s', s'') V(w''(s''), s'') \right\}$$

and note the convexity of the constraint set. Using the fact that $\hat{k}_o''(s'_*) \neq k_o''(s'_*)$, that V is concave and strictly increasing, and that $f(k)$ is strictly concave, we have, for $\phi \in (0, 1)$, and denoting $x_{o,\phi} = \phi x_o + (1 - \phi)\hat{x}_o$ and analogously for other variables,

$$\begin{aligned} V(w_\phi, s) &> d_{o,\phi} + \beta \sum_{s' \in S} \Pi(s, s') \left\{ d'_{o,\phi}(s') + \beta \sum_{s'' \in S} \Pi(s', s'') V(w''_{o,\phi}(s''), s'') \right\} \\ &\geq \phi V(w, s) + (1 - \phi) V(\hat{w}, s). \end{aligned}$$

The first (strict) inequality is due to the fact that for s'' following s'_* equation (3) is slack and hence a net worth $w''(s'') > w''_{o,\phi}(s'')$ is feasible. The second inequality is due to concavity of V . \square

Lemma 6 *Under Assumption 1, V is strictly increasing in s .*

Proof of Lemma 6. Let $S = \{s_1, \dots, s_n\}$, with $s_{i-1} < s_i$, $\forall i = 2, \dots, n$ and $N = \{1, \dots, n\}$. Define the step function on the unit interval $b : [0, 1] \rightarrow \mathbb{R}$ as

$$b(\omega) = \sum_{i=1}^n b(s'_i) \mathbf{1}_{B_i}(\omega), \quad \forall \omega \in [0, 1],$$

where $\mathbf{1}$ is the indicator function, $B_1 = [0, \Pi(s, s'_1)]$, and

$$B_i = \left(\sum_{j=1}^{i-1} \Pi(s, s'_j), \sum_{j=1}^i \Pi(s, s'_j) \right], \quad i = 2, \dots, n.$$

For \hat{s} , define \hat{B}_i , $\forall i \in N$, analogously. Let $B_{ij} = B_i \cap \hat{B}_j$, $\forall i, j \in N$, of which at most $2n - 1$ are non-empty. Then, we can define the step function $\hat{b} : [0, 1] \rightarrow \mathbb{R}$ as

$$\hat{b}(\omega) = \sum_{j=1}^n \sum_{i=1}^n b(s'_i) \mathbf{1}_{B_{ij}}(\omega), \quad \forall \omega \in [0, 1].$$

We can then define the stochastic debt policy for \hat{B}_j , $\forall j \in N$, with positive Lebesgue measure ($\lambda(\hat{B}_j) > 0$), as $\hat{b}(s'_i|s'_j) = b(s'_i)$ with conditional probability $\pi(s'_i|s'_j) = \lambda(B_{ij})/\lambda(\hat{B}_j)$. Moreover, this implies a stochastic net worth

$$\begin{aligned} \hat{w}'(s'_i|s'_j) &= A(s'_j)f(k') + (k' - k'_i)(1 - \delta) - R\hat{b}(s'_i|s'_j) \\ &\geq A(s'_i)f(k') + (k' - k'_i)(1 - \delta) - R\hat{b}(s'_i) = w'(s_i), \quad a.e., \end{aligned}$$

with strict inequality when $i < j$, since under Assumption 1, $\lambda(B_{ij}) = 0$ whenever $i > j$. Moreover, $\hat{w}'(s'_i|s'_j) > w'(s_i)$ with positive probability given Assumption 1.

Now suppose $\hat{s} > s$ and $f(w, \hat{s}) \geq f(w, s)$, $\forall w \in \mathbb{R}_+$. Let x_o attain the $(Tf)(w, s)$. Then

$$\begin{aligned} (Tf)(w, \hat{s}) &\geq d_o + \beta \sum_{\hat{s}' \in S} \Pi(\hat{s}, \hat{s}') \sum_{s' \in S} \pi(s'|\hat{s}') f(\hat{w}'_o(s'|\hat{s}'), \hat{s}') \\ &> d_o + \beta \sum_{s' \in S} \Pi(s, s') f(w'_o(s'), s') = (Tf)(w, s). \end{aligned}$$

Thus, T maps increasing functions into strictly increasing functions, implying that V is strictly increasing in s . \square

Proof of Lemma 1. We first show that if $k' > 0$, then $w'(s') > 0$, $\forall s' \in S$. Note that (3) holds with equality. Using (4) we conclude

$$w'(s') = A(s')f(k') + (k' - k'_i)(1 - \delta) - Rb(s') \geq A(s')f(k') + ((k' - k'_i) - \theta(\varphi k' - k'_i))(1 - \delta) > 0.$$

To show that $k' > 0$, note that (7) and (10) imply that

$$\mu(1 - R^{-1}\theta\varphi(1 - \delta)) \geq \sum_{s' \in S} \Pi(s, s')\mu(s') [A(s')f'(k') + (1 - \theta\varphi)(1 - \delta)]. \quad (12)$$

Suppose that $\mu = 1$. Then $k' > 0$ since $\mu(s') \geq \beta V_w(w'(s'), s') \geq \beta$ and hence the right hand side goes to $+\infty$ as $k' \rightarrow 0$, a contradiction. Suppose instead that $\mu > 1$ and hence $d = 0$. For k' sufficiently small, $\exists \hat{s}' \in S$, such that $\mu(\hat{s}') = R^{-1}\mu$. But then

$$\begin{aligned} 0 &\geq \sum_{s' \in S \setminus \hat{s}} \Pi(s, s')\mu(s') [A(s')f'(k') + (1 - \theta\varphi)(1 - \delta)] \\ &\quad + \{ \Pi(s, \hat{s}')R^{-1}[A(\hat{s}')f'(k') + (1 - \theta\varphi)(1 - \delta)] - (1 - R^{-1}\theta\varphi(1 - \delta)) \} \mu. \end{aligned}$$

where the first term is positive and the second term goes to $+\infty$ as $k' \rightarrow 0$, a contradiction. \square

Proof of Proposition 2. Part (i): By the envelope condition, $\mu(w, s) = V_w(w, s)$. By Lemma 4, V is concave in w and hence $\mu(w, s)$ is decreasing in w . The first order condition (6) implies that $\mu(w, s) \geq 1$. If $d(\hat{w}, s) > 0$, then $\mu(\hat{w}, s) = 1$ and $\mu(w, s) = 1$ for all $w \geq \hat{w}$. Let $\bar{w}(s) = \inf\{w : d(w, s) > 0\}$.

Part (ii): Suppose $w > \hat{w} \geq \bar{w}(s)$ and let \hat{x}_o attain $V(\hat{w}, s)$. Since $V_w(w, s) = 1$ for $w \geq \bar{w}(s)$, $V(w, s) = V(\hat{w}, s) + \int_{\hat{w}}^w dv$. The choice $x_o = [w - \hat{w} + \hat{d}_o, \hat{k}'_o, \hat{k}'_{i,o}, \hat{w}'_o(s'), \hat{b}'_o(s')]$ attains $V(w, s)$ and thus is weakly optimal.

The optimal choice \hat{x}_o is unique in terms of the capital stock \hat{k}'_o . To see this, suppose instead that \hat{x}_o and \tilde{x}_o both attain $V(\hat{w}, s)$, but $\hat{k}'_o \neq \tilde{k}'_o$. Recalling that $\Gamma(\hat{w}, s)$ is convex and noting that

$$\begin{aligned} A(s')f(k'_{o,\phi}) + (k'_{o,\phi} - k'_{i,o,\phi})(1 - \delta) &> \phi[A(s')f(\hat{k}'_o) + (\hat{k}'_o - \hat{k}'_{i,o})(1 - \delta)] \\ &\quad + (1 - \phi)[A(s')f(\tilde{k}'_o) + (\tilde{k}'_o - \tilde{k}'_{i,o})(1 - \delta)], \end{aligned}$$

where $x_{o,\phi}$ is defined as usual, we conclude that at $x_{o,\phi}$, (3) is slack, and hence there exists a feasible choice that attains a strictly higher value, a contradiction. Indeed, $x_o(w, s)$ is unique in terms of $k'_o(w, s)$, for all w and s .

Now take $w > \hat{w}$ and let x_o attain $V(w, s)$. By part (i) of Proposition 1, $x_{o,\phi} \in \Gamma(w_\phi, s)$. Moreover, if $k'_o \neq \hat{k}'_o$, then there exists a feasible choice such that $V(w_\phi) > \phi V(w, s) + (1 - \phi)V(\hat{w}, s)$ contradicting the linearity of V . Thus, $k'_o(w, s) = \bar{k}'_o(s)$, $\forall w \geq \bar{w}(s)$.

Part (iii): We now show that without leasing the optimal policy is unique also in terms of state-contingent net worth, state-contingent borrowing, and the dividend. Define $\hat{S}^0 = \{s' : \hat{w}'_o(s') < \bar{w}(s')\}$ and $\hat{S}^+ = S \setminus \hat{S}^0$. Then $\forall s' \in \hat{S}^0$, $\hat{w}'_o(s')$ is unique. To see this suppose instead that there is a \tilde{x}_o with $\tilde{w}'_o(s') \neq \hat{w}'_o(s')$ for some $s' \in \hat{S}$ that also attains $V(\hat{w}, s)$. Then a convex combination $x_{o,\phi}$ is feasible and attains a strictly higher value due to strict concavity of $V(w, s')$ for $w < \bar{w}(s')$ (part (iv) of Proposition 1). For the alternative optimal policy \tilde{x}_o define \tilde{S}^0 and \tilde{S}^+ analogously to \hat{S}^0 and \hat{S}^+ . By above, $\tilde{S}^0 \supseteq \hat{S}^0$. For any $s' \in \tilde{S}^+$, $\tilde{w}'_o(s') \geq \bar{w}(s')$. For suppose instead that $\tilde{w}'_o(s') < \bar{w}(s')$, then by strict concavity of V for $w < \bar{w}(s')$ a convex combination would again constitute a feasible improvement. Thus, $\tilde{S}^+ \supseteq \hat{S}^+$ and as a consequence $S^+ \equiv \tilde{S}^+ = \hat{S}^+$ and $S^0 \equiv \tilde{S}^0 = \hat{S}^0$. For $s' \in S^0$, $b_o(s')$ is uniquely determined by (3). For $s' \in S^+$, equation (4) holds with equality and determines $b_o(s')$ uniquely, and $w'_o(s')$ is then uniquely determined by (3). Hence, the optimal policy is unique. Moreover, the policy determined by part (ii) (with $k'_{i,o}(w, s)$ set to 0) is the unique optimal policy for $w > \bar{w}(s)$. \square

Proof of Proposition 4. Denote with a prime variables which in the stochastic case were a function of the state tomorrow, that is, w' , b' , μ' , and λ' . We first characterize a steady state. From (9) and the envelope condition we have $\mu' = \beta\mu$. Then (10) implies $\lambda' = (R^{-1} - \beta)\mu > 0$, that is, the borrower is constrained in the steady state, and (7) can be written as

$$1 - [R^{-1}\theta\varphi + \beta(1 - \theta\varphi)](1 - \delta) = \beta A' f'(k')$$

which implicitly defines \bar{k}' , the steady state value of capital. Denoting steady state variables with a bar, using (4) and (3) at equality, we have $\bar{b} = R^{-1}\theta\varphi\bar{k}'(1 - \delta)$ and the cum-dividend net worth in the steady state $\bar{w}_{cum} = A' f(\bar{k}') + \bar{k}'(1 - \theta\varphi)(1 - \delta)$. Dividends in the steady state are

$$\begin{aligned} \bar{d} &= A' f(\bar{k}') - \bar{k}'(1 - [R^{-1}\theta\varphi + (1 - \theta\varphi)](1 - \delta)) \\ &> A' f(\bar{k}') - \beta^{-1}\bar{k}'(1 - [R^{-1}\theta\varphi + \beta(1 - \theta\varphi)](1 - \delta)) \\ &= \int_0^{\bar{k}'} \{A' f'(k') - \beta^{-1}(1 - [R^{-1}\theta\varphi + \beta(1 - \theta\varphi)](1 - \delta))\} dk' > 0 \end{aligned}$$

and hence $\bar{\mu} = 1$. The lowest level of net worth for which \bar{k}' is feasible is $\bar{w} \equiv \bar{w}_{cum} - \bar{d}$, and \bar{w} is the ex-dividend net worth in the steady state. Thus, for $w < \bar{w}$, $k' < \bar{k}'$. Using the first order conditions and the envelope condition we have

$$\frac{V_w(w)}{V_w(w')} = \beta \frac{\mu}{\mu'} = \beta \frac{A' f'(k') + (1 - \theta\varphi)(1 - \delta)}{1 - R^{-1}\theta\varphi(1 - \delta)}.$$

Note that the right hand side equals 1 at \bar{k}' and is decreasing in k' . Thus, if $k' < (>) \bar{k}'$, $V_w(w) > (<) V_w(w')$ and $w < (>) w'$. Since $k' < \bar{k}'$ for $w < \bar{w}$, $w < w'$ and w increases over time. If $w > \bar{w}$, then either $d > 0$ (and $V_w(w) = 1$) or $d = 0$ and $k' > \bar{k}'$. In the first case, concavity and the fact that $V_w(w') \geq 1$ imply $V_w(w') = 1$ and hence $k' = \bar{k}'$. In the second case, $w > w'$, but simply saving w at R would result in higher net worth and hence $k' > \bar{k}'$ cannot be optimal. \square

Proof of Proposition 5. Consider the optimal policy without leasing from Proposition 4. The user cost of physical capital at \bar{w} is $\bar{u}_p = R^{-1}(r+\delta) + (R^{-1} - \beta)(1-\theta)(1-\delta) < u_l$ under Assumption 3. Thus, there is no leasing at \bar{w} and the solution is as before as long as w is sufficiently high. Recall that as w decreases μ'/μ decreases and hence λ'/μ increases. Note also that under Assumption 2, as w goes to zero, k' and μ'/μ go to zero and hence λ'/μ goes to R^{-1} and u_p goes to $R^{-1}(r+\delta) + R^{-1}(1-\theta)(1-\delta) > u_l$ given Assumption 3. When $\lambda'/\mu = m/((1-\theta)(1-\delta))$, $u_l = u_p$ and (7) simplifies to

$$1 - R^{-1}\theta\varphi(1-\delta) = \left(R^{-1} - \frac{m}{(1-\theta)(1-\delta)} \right) [Af'(k') + (1-\theta\varphi)(1-\delta)],$$

which defines \underline{k}' . At $\bar{w}_l \equiv (1 - R^{-1}\theta\varphi(1-\delta))\underline{k}'$ all the physical capital is owned and at $\underline{w}_l \equiv (1 - \varphi + u_l\varphi)\underline{k}'$ all the physical capital is leased. For $w \in [\underline{w}_l, \bar{w}_l]$, leased capital is

$$k'_l = \frac{(1 - R^{-1}\theta\varphi(1-\delta))\underline{k}' - w}{1 - R^{-1}\theta(1-\delta) - u_l}$$

which is linear and decreasing in w . Moreover, w' is linearly decreasing in k'_l and hence linearly increasing in w . For $w < \underline{w}_l$, $k' = w/(1-\varphi+u_l\varphi)$ and $w' = Af'(k') + k'(1-\varphi)(1-\delta)$. \square

Proof of Proposition 6. If $w(s') \leq w(s'_+)$, then $\mu(s') \geq \mu(s'_+)$ by concavity. Moreover, $\mu(s') + \lambda(s') = \mu(s'_+) + \lambda(s'_+)$, so $\lambda(s') \leq \lambda(s'_+)$. Suppose instead that $w(s') > w(s'_+)$. Then $\lambda(s') = 0$ since otherwise net worth in state s' could not be larger than in state s'_+ . But then $\mu(s') = \mu(s'_+) + \lambda(s'_+)$, implying $\mu(s'_+) \leq \mu(s')$ or $w(s') \leq w(s'_+)$ using concavity, a contradiction. \square

Proof of Theorem 1. Suppose that $\lambda(s') = 0$, $\forall s' \in S$. Then (9), (10), and the envelope condition imply that $V_w(w) = \mu = \mu(s')R = V_w(w'(s'))\beta R < V_w(w'(s'))$ and, by concavity, $w > w'(s')$, $\forall s' \in S$.

If $d = 0$, then saving the entire net worth w at R would imply net worth $Rw > w'(s')$ in all states next period and hence attain a higher value of the objective, contradicting optimality.

Suppose $d > 0$ and hence $w > \bar{w}$ as defined in Proposition 2. That proposition also implies that $V(w)$ can be attained by the same optimal policy as at \bar{w} except that $d = w - \bar{w}$. Since $V_w(w'(s')) > 1$, we conclude that $w'(s') < \bar{w}$. But then paying out $d = w - \bar{w}$ as before and saving \bar{w} at R raises net worth in all states next period and hence improves the value of the objective, a contradiction.

Hence, $\exists s' \in S$ such that $\lambda(s') > 0$, and, since $\lambda(s')$ is increasing in s' by Proposition 6, $\lambda(\bar{s}') > 0$ where $\bar{s}' = \max\{s' : s' \in S\}$. If $\lambda(s') > 0, \forall s'$, then $w'(s') = A(s')f(k') + k'(1 - \theta\varphi)(1 - \delta) - k'_l(1 - \theta)(1 - \delta)$ and hence $w'(s') \neq w'(\hat{s}')$, $s \neq \hat{s}'$. If $\lambda(\hat{s}') = 0$ for some \hat{s}' , then $\mu(\hat{s}') = \mu(\bar{s}') + \lambda(\bar{s}') > \mu(\bar{s}')$ and $w'(\hat{s}') < w'(\bar{s}')$.

Suppose $\lambda(s') = 0$ for some $s' \in S$. For any $s'_- < s'$, $\mu(s'_-) \geq \mu(s')$ by Proposition 6, and $\mu(s'_-) \leq \mu(s'_-) + \lambda(s'_-) = \mu(s')$, implying $\mu(s'_-) = \mu(s')$. Thus, the borrower hedges all states below $s'_h = \max\{s' : \lambda(s') = 0\}$. Note that the set may be empty, that is, the borrower may not hedge at all. \square

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Table 1: Tangible assets and liabilities

This table reports balance sheet data from the Flow of Funds Accounts of the United States for the Second Quarter 2008 [Federal Reserve Statistical Release Z.1], Tables B.100, B.102, and B.103. Data are in trillions of dollars.

Panel A: Balance Sheet of Households (and Nonprofit Organizations)

Assets		Liabilities	
Tangible assets	\$26.1	Total liabilities	\$14.5
Real estate (households)	\$19.4	Mortgages (households)	\$10.6
Real estate (nonprofit org.)	\$2.4	Mortgages (nonprofit org.)	\$0.3
Consumer durables	\$4.1	Consumer credit	\$2.6
Financial assets	\$44.3	Net Worth	\$56.0
Total	\$70.5	Total	\$70.5

Panel B: Balance Sheet of (Nonfinancial) Corporate Businesses

Assets		Liabilities	
Tangible assets	\$14.9	Total liabilities	\$12.9
Real estate	\$9.0		
Equipment and software	\$4.0		
Inventories	\$1.9		
Financial assets	\$14.1	Net Worth	\$16.2
Total	\$29.0	Total	\$29.0

Panel C: Balance Sheet of Noncorporate Businesses

Assets		Liabilities	
Tangible assets	\$7.8	Total liabilities	\$5.2
Real estate	\$7.2	Mortgages	\$2.6
Equipment and software	\$0.5		
Financial assets	\$3.2	Net Worth	\$5.9
Total	\$11.0	Total	\$11.0

Panel D: "Net" Balance Sheet of Households

Assets		Liabilities	
Tangible assets	\$48.6		
Real estate	\$38.0		
Consumer durables	\$4.1		
Equipment and software	\$4.5		
...	...	Net Worth	\$56.0
Total	\$56.0	Total	\$56.0

Table 2: Tangible assets and (debt) leverage

Tangibility: Property, Plant, and Equipment – Total (Net) (Item #8) divided by Assets; Assets: Assets – Total (Item #6) plus Price – Close (Item #24) times Common Shares Outstanding (Item #25) minus Common Equity – Total (Item #60) minus Deferred taxes (Item #74); Leverage: Long-Term Debt – Total (Item #9) divided by Assets. Annual firm level Compustat data for 2007 are used excluding financial firms.

Tangibility and Debt Leverage

Tangibility quartile	Quartile cutoff (%)	Leverage (%)		Low leverage firms (%) (leverage \leq 10%)
		median	mean	
1	6.3	7.4	10.8	58.3
2	14.3	9.8	14.0	50.4
3	32.2	12.4	15.5	40.6
4	n.a.	22.6	24.2	14.9

Table 3: Tangible assets and debt, rental, and true leverage

True Tangibility: Property, Plant, and Equipment – Total (Net) (Item #8) plus 10 times Rental Expense (#47) divided by True Assets; True Assets: Assets – Total (Item #6) plus Price – Close (Item #24) times Common Shares Outstanding (Item #25) minus Common Equity – Total (Item #60) minus Deferred taxes (Item #74) plus 10 times Rental Expense (#47); Debt Leverage: Long-Term Debt – Total (Item #9) divided by True Assets; Rental Leverage: 10 times Rental Expense (#47) divided by True Assets; True Leverage: Debt Leverage plus Rental Leverage. Annual firm level Compustat data for 2007 are used excluding financial firms.

Tangibility and Leverage

True tangibility quartile	Quartile cutoff (%)	Leverage (%)						Low leverage firms (%) (leverage \leq 10%)		
		Debt leverage		Rental leverage		True leverage		Debt leverage	Rental leverage	True leverage
		median	mean	median	mean	median	mean			
1	13.2	6.5	10.4	3.7	4.2	11.4	14.6	61.7	97.7	46.0
2	24.1	9.8	12.9	6.9	8.1	18.4	21.0	50.1	68.2	16.1
3	40.1	13.1	14.8	8.0	10.5	24.2	25.3	41.7	60.6	12.0
4	n.a.	18.4	20.4	7.2	13.8	32.3	34.2	24.4	57.3	3.7

Table 4: Leverage and size revisited

True Book Assets: Assets – Total (Item #6) plus 10 times Rental Expense (#47); Debt Leverage: Long-Term Debt – Total (Item #9) divided by True Book Assets; Rental Leverage: 10 times Rental Expense (#47) divided by True Book Assets; True Leverage: Debt Leverage plus Rental Leverage. Annual firm level Compustat data for 2007 are used excluding financial firms.

Debt and Rental Leverage across Size (True Book Assets) Deciles

Decile	Size deciles									
	1	2	3	4	5	6	7	8	9	10
Median debt leverage	6.0	7.3	7.4	14.1	19.5	22.6	20.6	20.2	21.6	17.8
Median rental leverage	21.8	14.6	10.8	11.1	11.2	9.1	9.7	9.1	7.8	7.3
Median true leverage	30.6	24.2	21.0	28.8	36.4	37.7	33.4	36.6	31.7	26.3

Figure 1: Leverage vs. size revisited

Debt leverage (dashed), rental leverage (dash dotted), and true leverage (solid) across size deciles for Compustat firms.

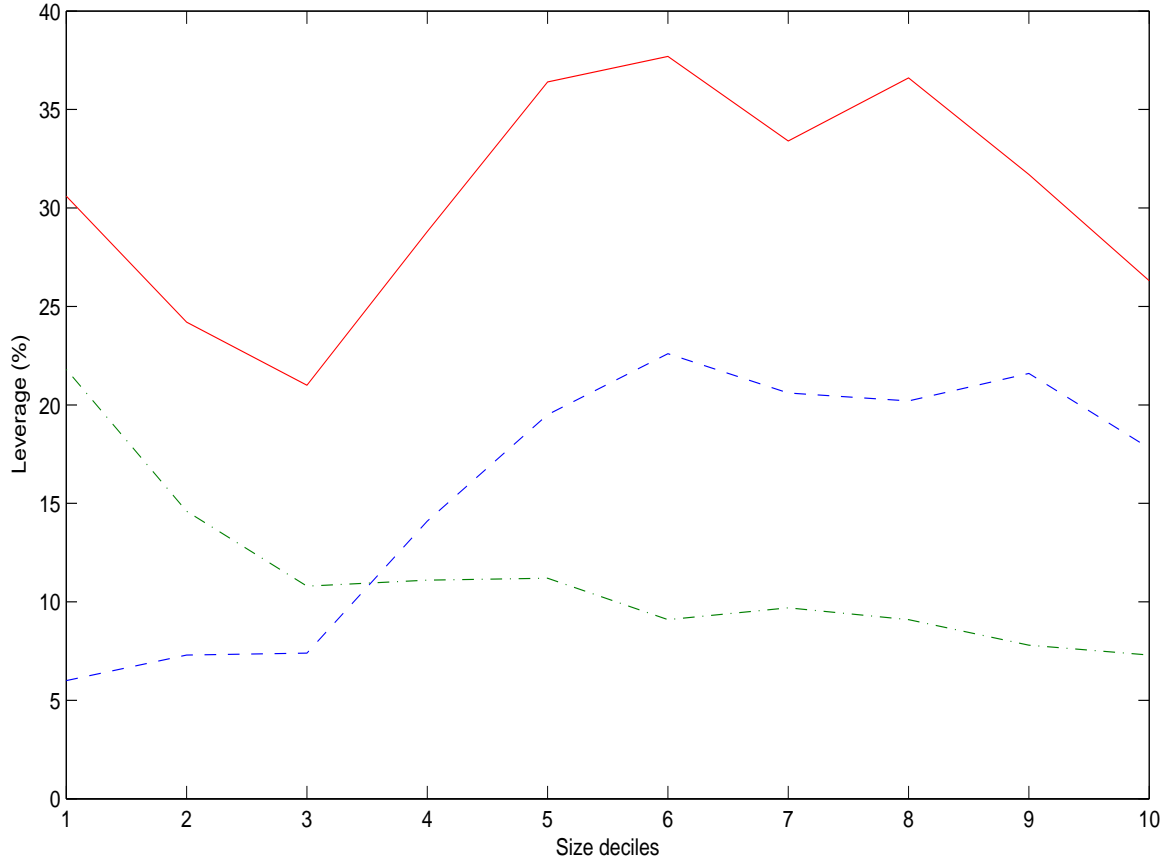


Figure 2: Investment and leasing policy

Total capital (k'), leased capital k'_l (dash dotted), and purchased capital $k' - k'_l$ (dashed) as a function of current net worth (w) in the deterministic case. The kinks and vertical lines are (from left to right) at \underline{w}_l , \bar{w}_l , \bar{w} , and \bar{w}_{cum} , respectively. The parameter values are: $\beta = 0.90$, $r = 0.05$, $\delta = 0.10$, $m = 0.03$, $\theta = 0.60$, $\varphi = 0.50$, $A' = 0.50$, and $\alpha = 0.333$.

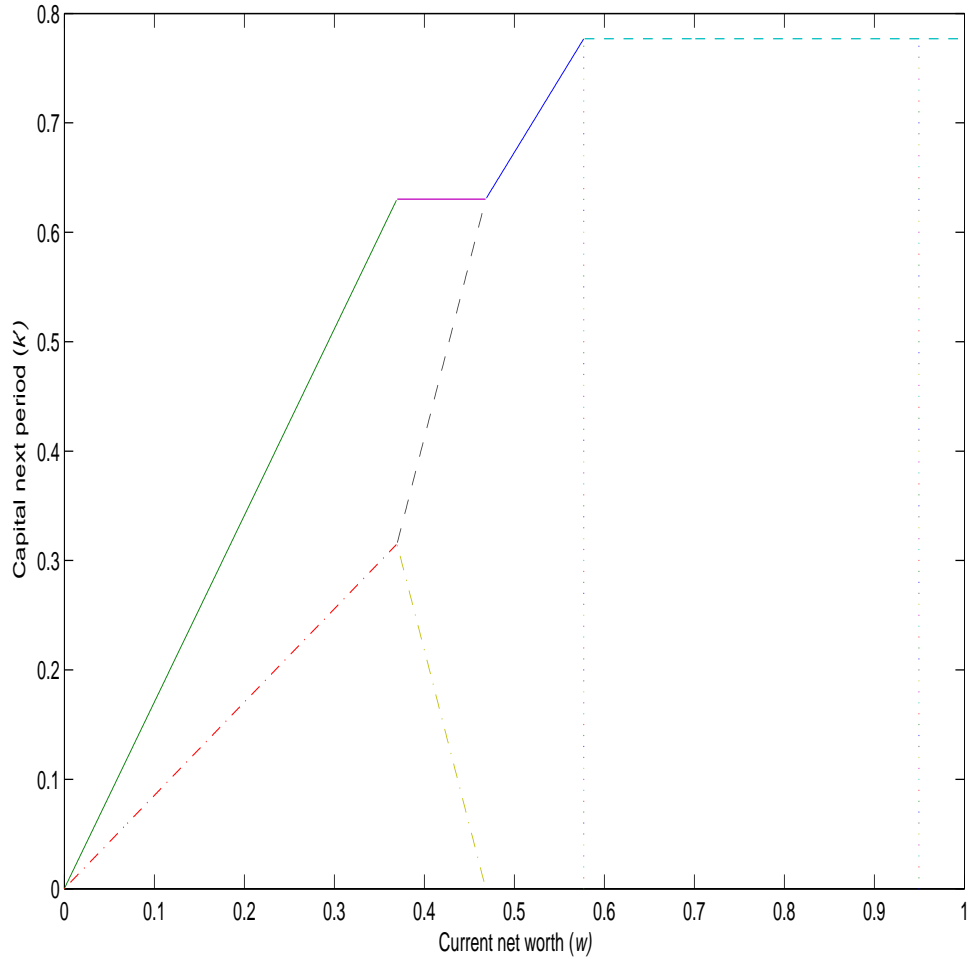


Figure 3: Evolution of net worth

Net worth next period (w') as a function of current net worth (w) in the deterministic case. The solid line describes the transition of net worth without leasing and the dashed line the transition with leasing. The kinks and vertical lines are (from left to right) at \underline{w}_l , \bar{w}_l , \bar{w} , and \bar{w}_{cum} , respectively. For the parameter values see the caption of Figure 2.

