Collision and Symmetry Breaking in the Transition to Strange Nonchaotic Attractors

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Strange nonchaotic attractors (SNAs) can be created due to the collision of an invariant curve with itself. This novel "homoclinic" transition to SNAs occurs in quasiperiodically driven maps which derive from the discrete Schrödinger equation for a particle in a quasiperiodic potential. In the classical dynamics, there is a transition from torus attractors to SNAs, which, in the quantum system, is manifest as the localization transition. This equivalence provides new insight into a variety of properties of SNAs, including its fractal measure. Further, there is a *symmetry breaking* associated with the creation of SNAs which rigorously shows that the Lyapunov exponent is nonpositive. We show that these characteristics associated with the appearance of SNA are robust and occur in a large class of systems.

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The unexpected—and fascinating—connection between strange nonchaotic dynamics [1] and localization phenomena [2,3] brings together two current strands of research in nonlinear dynamics and condensed matter physics. The former describes temporal dynamics converging on a fractal attractor on which the largest Lyapunov exponent is nonpositive [1] while the latter involves *exponentially* decaying wave functions. Recent work [4] has shown that the fluctuations in the exponentially decaying localized wave function are fractal, and this appears in the classical problem as an attractor with fractal measure. Here we exploit this relationship further to understand the mechanism for the transition to strange nonchaotic attractor (SNA), which is a subject of continuing interest [5].

In this Letter, we show that the transition to SNAs has two unusual and general features. First, SNAs can be created by the homoclinic collision of invariant curves with themselves. Second, the bifurcation to SNAs, when occurring such that the largest nontrivial Lyapunov exponent passes through zero, is accompanied by a symmetry breaking. These features provide us with a novel way to characterize and quantify the transition to SNA. Furthermore, by considering a variety of quasiperiodic maps, we demonstrate that these aspects of the SNA transition are generic.

The quasiperiodically forced dynamical system under investigation here is the Harper map [3],

$$x_{n+1} = f(x_n, \phi_n) \equiv -[x_n - E + 2\epsilon \cos 2\pi\phi_n]^{-1}, \tag{1}$$

with the rigid-rotation dynamics $\phi_n = n\omega + \phi_0$ giving quasiperiodic driving for irrational ω. This map is obtained from the Harper equation [6],

$$\psi_{n+1} + \psi_{n-1} + 2\epsilon \cos[2\pi(n\omega + \phi_0)]\psi_n = E\psi_n, \tag{2}$$

under the transformation $x_n = \psi_{n-1}/\psi_n$. Note that the lattice site index of the quantum problem is the time (or iteration) index in the classical problem. The Harper equation is a discrete Schrödinger equation for a particle in a periodic potential on a lattice. The wave function at site n of the lattice is ψ_n, and E is the energy eigenvalue. The parameters ϵ, ω, and ϕ_0 determine, respectively, the strength, periodicity, and phase (relative to the underlying lattice) of the potential. For irrational ω (usually taken to be the golden mean, $(\sqrt{5} - 1)/2$), the period of the potential is incommensurate with the periodicity of the lattice. For the classical map, both ϵ and E are important parameters, but the quantum problem is meaningful only when E is an eigenvalue of the system, so we limit our discussion of the classical system to these special values of E. However, as we discuss below, this restriction can be lifted when we consider perturbations of the map which are not related to the eigenvalue problem. For most of our work we set $E = 0$ which is an eigenvalue.

The Harper equation [6] is paradigmatic in the study of localization phenomena in quasiperiodic systems [7], exhibiting a localization transition at $\epsilon = 1$. For $\epsilon < 1$, all eigenstates are extended and hence are characterized by an infinite localization length, while for $\epsilon > 1$, eigenstates are localized with localization length $\gamma^{-1} = \ln\epsilon$. As we discuss below, the fact that the Lyapunov exponent of the Harper equation is known exactly is crucial in establishing the existence of SNA in the Harper map.

Of the two Lyapunov exponents for the Harper equation, that corresponding to the ϕ dynamics is 0, while the other can be easily calculated as

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} y_i, \tag{3}$$

where y_i is the "stretch exponent" defined through

$$\begin{aligned} y_i &= \ln|f'(x_i)| = \ln x_{i+1}^2 \\ &= -2\ln|x_i - E + 2\epsilon \cos 2\pi\phi_i|. \end{aligned} \tag{4}$$

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It is easy to see that in the localized state,

$$\lambda = -2\gamma\,, \tag{5}$$

and, therefore, the localized wave function of the Harper equation corresponds to an attractor with negative Lyapunov exponent for the Harper map.

The second important point in establishing the existence of SNA in the Harper equation stems from the fact that the fluctuations about the localized wave function in the Harper equation are fractal. This result, based on renormalization studies [4] of the Harper equation, suggests that the corresponding attractor in the Harper map has a fractal measure and hence is an SNA. Furthermore, a perturbative argument starting from the strong coupling limit provides a rigorous proof for the existence of SNA for $E = 0$ [3], making the Harper mapping one of the few systems where the existence of SNA is well established.

We now discuss the scenario for the formation of SNAs in this system when $E = 0$. For $\epsilon < 1$, the phase space is foliated by invariant curves, each parametrized by the initial conditions. It is important to note that for $\epsilon < 1$ there are no attractors in the system since all the curves are neutrally stable. However, at $\epsilon = 1$, trajectories converge on an attractor. The convergence is power law and hence the Lyapunov exponent is zero: We can characterize this via a power-law exponent

$$\beta = \lim_{N\to\infty} \frac{1}{\ln N} \ln \prod_{i=0}^{N-1} \exp y_i\,, \tag{6}$$

the transition from a family of invariant tori to an attractor being signaled by a nonzero value of β.

The transition from an invariant curve to the attractor can be described as a collision phenomenon as we discuss below. For $\epsilon < 1$, the invariant curves have two branches [see Fig. 1(a)] deriving from the fact that for $\epsilon = 0$, the map does not have a period-1 fixed point for real x but has instead a period-2 orbit. As $\epsilon \to 1$, the two branches approach each other and *collide* at $\epsilon = 1$, the point of collision being a singularity. Since the dynamics in ϕ is ergodic, the collision occurs at a dense set of points. Furthermore, this happens for each invariant curve, and in effect all invariant curves approach each other and collide at $\epsilon = 1$, forming an attractor [see Fig. 1(b)]. We quantify this collision by demonstrating that as $\epsilon \to 1$, the distance d between the two branches goes to zero as a power law.

When the quasiperiodic forcing frequency ω is the golden mean ratio, the distance between the two branches of an invariant curve can be calculated by first noting that a point (x_i, ϕ_i) and its successive Fibonacci iterates, $(x_{i+F_k}, \phi_{i+F_k})$, where F_k is the kth Fibonacci number, are closely spaced in ϕ [8]. If the two branches of the invariant curve are labeled C (for central) and N (for noncentral) [see Fig. 1(a)], the sequence of Fibonacci iterates follows the symbolic coding *CCNCCNCCNCCN*... or *NNCNNCNNCNNC*.... This follows from the fact that

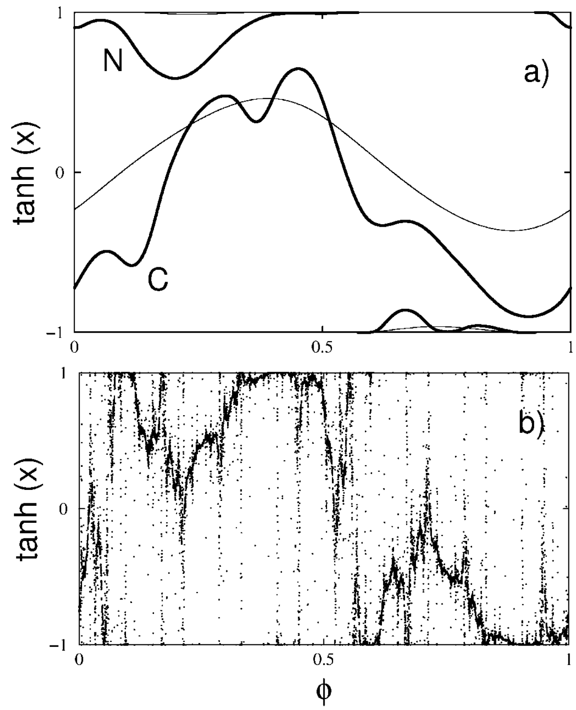

FIG. 1. (a) Invariant curves for the Harper map with $E = 0$ at $\epsilon = 0.3$ (thin line) and $\epsilon = 0.65$ (thick line) with initial conditions $x_0 = 0.5$ and $\phi_0 = 0.4$. [We plot $\tanh(x)$ rather than x on the ordinate so that the entire range of x can be depicted.] For different initial conditions we will get different curves. As ϵ is increased, the two branches of a given curve approach each other and eventually collide as $\epsilon \to 1$ as indeed do all the other curves as well. The central and noncentral branches of these curves are marked C and N, respectively. (b) The SNA that is born at $\epsilon = 1$.

the Fibonacci numbers are successively even, odd, odd, even, odd, odd,.... Thus, if k is chosen appropriately, such that F_k is even and F_{k+1} is odd (or vice versa),

$$d_k(i) = |x_{i+F_k} - x_{i+F_{k+1}}| \tag{7}$$

measures the approximate vertical distance between the curves at (x_i, ϕ_i). Minimizing this distance along the invariant curve, we find that the closest approach of the two branches decreases as a power,

$$d = \min[\lim_{k\to\infty} d_k(i)] \sim (1 - \epsilon)^{\delta}. \tag{8}$$

Our results, given in Fig. 2, provide a *quantitative* characterization of the transition to SNA in this system.

For eigenvalues other than $E = 0$, the scenario for SNA formation may be different. When the eigenvalue E is at the band edge, the SNAs appear to be formed via the fractalization route, namely, by gradually wrinkling and forming a fractal [9]. The reason for this difference can be traced to the simple fact that unlike the $E = 0$ case, below $\epsilon = 1$ the invariant curve for the minimum eigenvalue has a single branch which originates from a fixed point for $\epsilon = 0$.

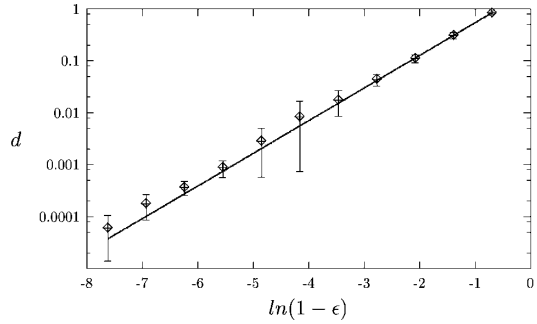

FIG. 2. The minimum vertical distance d between the branches of invariant curves in the Harper map as a function of the parameter $(1 - \epsilon)$, for the eigenvalue $E = 0$ and for the Fibonacci iterates (see the text) F_k and F_{k+1} with $k = 23$. Since this quantity depends on the particular curve chosen, an average is taken over 12 different invariant curves, which gives the average behavior and the variance. The resulting exponent [the solid line is a fit to a power law; see Eq. (8)] is $\delta \approx 1.4$.

The self-collision of invariant curves to form SNAs is a general mechanism. Consider a family of maps,

$$x_{i+1} = -[x_i + \alpha x_i^\nu + 2\epsilon \cos 2\pi \phi_i]^{-1},$$
$$\phi_{i+1} = \phi_i + \omega, \qquad \text{mod } 1, \tag{9}$$

which bear no relation to an eigenvalue problem. For ν an odd integer, the above map is invertible and hence does not have any chaotic attractors. Numerical results for $\nu = 3$ show that in these perturbed maps, an SNA is also born after the attractor collides with itself. Similar results are obtained for other polynomial or sinusoidal perturbations.

A more fundamental characteristic of this route to SNAs is a dynamical symmetry breaking. Although the dynamics is nontrivial for the variable x, the Lyapunov exponent is *exactly* zero for $\epsilon < 1$. To understand this from a dynamical point of view, we first note that for finite times along a trajectory, the local expansion and contraction rates vary. It turns out that a meaningful way to understand the role of the parameter ϵ is to study the return map for the stretch exponents,

$$y_{i+1} = -2 \ln|\text{sgn}(x_i) \exp(y_i/2) - E + 2\epsilon \cos 2\pi \phi_i|. \tag{10}$$

Shown in Fig. 3(a) is the above map for $E = 0$ and $\epsilon = 0.5$. There is a reflection symmetry evident, namely, $(x, y \rightarrow -y, -x)$ although this symmetry is not easy to see directly in the mapping, Eq. (10) itself owing to the quasiperiodic nature of the dynamical equations. However, as a consequence of the symmetry, the positive and the negative terms cancel exactly in Eq. (3), giving a zero Lyapunov exponent. All finite sums of the stretch exponents, namely, the finite-time Lyapunov exponents [10] also share the same symmetry features.

This symmetry is maintained for $0 < \epsilon \leq 1$, above which this symmetry is broken [Fig. 3(b)]. When the negative stretch exponents exceed the positive ones, the Lyapunov exponent λ therefore becomes negative; coupled with the fact that the attractor has a dense set of singularities [3], this rigorously confirms the existence of strange nonchaotic dynamics.

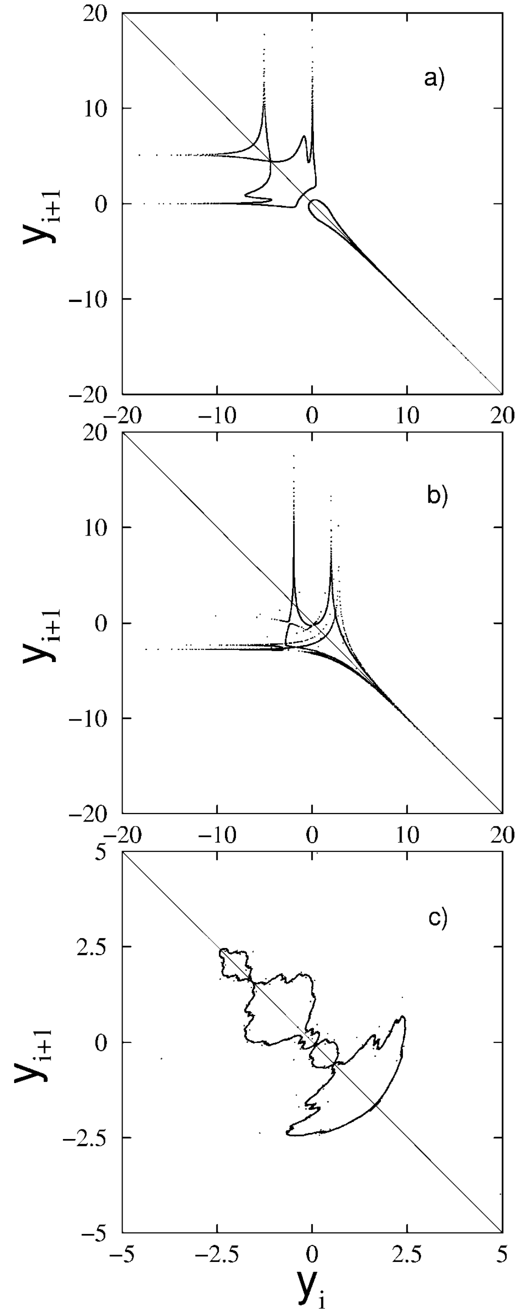

FIG. 3. The return map for the stretch exponents, Eq. (10) (a): for $E = 0$, $\epsilon = 0.5$, with initial conditions, $x_0 = 0.2$, $\phi_0 = 0.7$; (b) for $E = 0$, $\epsilon = 2$; and (c) at the band edge, $E = E_m = -2.597\,515\,185\ldots$, $\epsilon = 1$.

Symmetry breaking appears to be operative in a large class of systems, including the mapping where SNAs were first shown to exist [1], viz. $x_{n+1} = 2\epsilon \cos 2\pi \phi_n \tanh x_n$, and similar systems where the transition to SNA is via the blowout bifurcation [11]. In all these instances, the largest Lyapunov exponent goes through zero when the SNA is born.

When the eigenvalue E differs from 0, say at the band edge, the attractor in the localized state is also a SNA which is born at $\epsilon = 1$, with zero Lyapunov exponent. Again [see Fig. 3(c)] there is the symmetry in the return map for the stretch exponents which is broken for $\epsilon > 1$.

In summary, our work shows that the fractal measure of the trajectory has its origin in the homoclinic collisions of an invariant curve with itself. This characterization of the transition to SNAs can be quantified and may serve as a useful scenario for the appearance of SNAs in a variety of nonlinear dissipative systems.

Furthermore, we demonstrate that the transition from an invariant curve to a SNA proceeds via a symmetry breaking. A zero value for the Lyapunov exponent of a system can arise in a number of ways, and the present instance, namely, the exact cancellation of expanding and contracting terms is very special. (There is similar symmetry breaking at all period-doubling bifurcations in such systems as well, but these points are of measure zero.) It is conceivable that there are more complex symmetries in other systems which similarly lead to a zero value for the Lyapunov exponent. The significance of this symmetry and its breaking in the corresponding quantum problem may be an important question in characterizing the localization transition itself.

There are numerous lattice models exhibiting localization in aperiodic potentials [12], including the quantum kicked rotor [13,14]. The corresponding derived aperiodic mappings are worthy of further study and might well extend the subject of SNA to systems beyond quasiperiodically driven maps. In addition, there are interesting open questions regarding localization and its absence in quasiperiodic potentials with discrete steps [15]. It is conceivable that this type of mapping of the quantum problem onto the classical problem may provide better understanding of localization phenomena.

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[1] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Physica (Amsterdam) **13D**, 261 (1984).
[2] A. Bondeson, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. **55**, 2103 (1985).
[3] J. A. Ketoja and I. I. Satija, Physica (Amsterdam) **109D**, 70 (1997).
[4] J. A. Ketoja and I. I. Satija, Phys. Rev. Lett. **75**, 2762 (1995).
[5] A. Prasad and R. Ramaswamy (to be published).
[6] P. G. Harper, Proc. Phys. Soc. London A **68**, 874 (1955).
[7] See, e.g., B. Souillard, Phys. Rep. **103**, 41 (1984); S. Ostlund and R. Pandit, Phys. Rev. B **29**, 1394 (1984); S. Aubry and G. André, Ann. Isr. Phys. Soc. **3**, 133 (1980).
[8] The Fibonacci numbers are recursively defined through $F_{k+1} = F_k + F_{k-1}$ with $F_{-1} = 0$, $F_0 = 1$, and $\lim_{k\to\infty} F_k/F_{k+1} \to (\sqrt{5} - 1)/2$, the golden mean ratio. If the driving frequency is some other irrational number, then it will be necessary to consider different sequences of iterations, which will depend on the successive rational approximants to the frequency.
[9] K. Kaneko, Prog. Theor. Phys. **71**, 140 (1994); T. Nishikawa and K. Kaneko, Phys. Rev. E **54**, 6114 (1996).
[10] H. D. I. Abarbanel, R. Brown, and M. B. Kennel, J. Nonlinear Sci. **1**, 175 (1991).
[11] T. Yalçinkaya and Y. C. Lai, Phys. Rev. Lett. **77**, 5040 (1996).
[12] For a review of such models, see J. B. Sokoloff, Phys. Rep. **126**, 189 (1985).
[13] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).
[14] I. I. Satija, B. Sundaram, and J. Ketoja, Phys. Rev. E (to be published).
[15] J. Ketoja and I. I. Satija, Phys. Rev. B **59**, 9174 (1999).