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# Collision Avoidance Analysis for Lane Changing and Merging

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# Collision Avoidance Analysis for Lane Changing and Merging

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**April, 1999.**

# Collision Avoidance Analysis for Lane Changing and Merging[1]

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## Abstract:

One of the riskiest maneuvers that a driver has to perform in a conventional highway system is to merge into the traffic and/or to perform a lane changing maneuver. Lane changing/merging collisions are responsible for one-tenth of all crash-caused traffic delays often resulting in congestion. Traffic delays and congestion, in general, increases travel time and has a negative economic impact.

In this paper, we analyze the kinematics of the vehicles involved in a lane changing/merging maneuver, and study the conditions under which lane changing/merging crashes can be avoided. That is, given a particular lane change/merge scenario, we calculate the minimum longitudinal spacing which the vehicles involved should initially have so that no collision, of any type, takes place during the maneuver.

Simulations of a number of examples of lane changing maneuvers are used in order to demonstrate the results. These results together with appropriate sensors and equipment on board of vehicles could be used to assess the safety of lane changing maneuvers and provide warnings or take evasive actions to avoid collision.

**Keywords:** Lane changing, lane merging, crash avoidance, minimum safety spacing (MSS).

## Executive Summary:

In this paper, we examine the problem of safe lane changing and merging maneuvers in highway systems. By analyzing the kinematics of the vehicles involved in a lane changing or merging scenario, we present a general algorithm to calculate whether a particular lane changing/merging maneuver is safe, i.e., free of collisions. Moreover, we present a general algorithm for calculating the Minimum longitudinal Safety Spacings (MSS), that is, given a particular lane change/merging scenario, we calculate the minimum longitudinal spacings that the vehicles should initially have so that no collision takes place during the lane changing/merging maneuver. We, then, examine special cases of lane changing/merging scenarios. The databases generated by simulation results together with appropriate sensors and equipment on board of vehicles could be used to assess the safety of lane changing maneuvers and provide warnings or take evasive actions to avoid collision.

**Table of Contents:**

## List of Figures:

## List of Tables:

# 1. Introduction:

The inter-vehicle spacing or headway affects both safety and highway capacity. For collision free vehicle following, the spacing should be large enough in order to guarantee no collisions during all possible vehicle maneuvers. Lane changing/merging accidents consist of various types of vehicle collisions, such as rear-end collisions, single vehicle road departure accidents, side-wipe, and angle collisions. Lane changing/merging collisions constituted about 4.0% of all police-reported collisions in 1991, and accounted for about 0.5% of all fatalities [1].

Although the lane change crash problem is small relative to other types of crashes and does not account for a high percentage of traffic fatalities, this crash type is responsible for one-tenth of all crash-caused traffic delays often resulting into congestion. Traffic delays and congestion in general increases travel time and has a negative economic impact [8].

In practice, the possibility of merging collisions can be reduced by adjusting relative velocities and increasing the longitudinal inter-vehicles' spacing. Since roadway capacity is proportional to vehicle speed and inversely proportional to longitudinal inter-vehicle spacing, a large reduction in speed or a large increase in spacing leads to a low capacity highway system. For a high capacity highway system, the headway setting should be as small as possible. Since safety cannot be easily traded off, the choice of Minimum Safety Spacing (MSS) between vehicles for a collision free environment is important both from safety and capacity point of view.

In this work, we examine the problem of safe lane changing and merging maneuvers in highway systems. By analyzing the kinematics of the vehicles involved in a lane changing or merging scenario, we present a general algorithm to calculate whether a particular lane changing/merging maneuver is safe, i.e., free of collisions. Moreover, we present a general algorithm for calculating the Minimum longitudinal Safety Spacings (MSS), that is, given a particular lane change/merging scenario, we calculate the minimum longitudinal spacings that the vehicles should initially have so that no collision takes place during the lane changing/merging maneuver. We, then, examine special cases of lane changing/merging scenarios. For the cases where the merging vehicle moves with either constant longitudinal velocity [5] or acceleration, we explicitly calculate the MSSs and we show that the regions in the initial longitudinal spacing/ relative longitudinal velocity plane can be divided into safe and unsafe regions; once the vehicles start the lane changing/merging maneuver within the safe regions then collision-free maneuver is guaranteed. We finally analyze the switching longitudinal acceleration case [6], i.e., the case where the merging vehicle initially accelerates/decelerates with constant longitudinal acceleration in order to create enough spacing for the lane changing/merging maneuver and then it switches to another constant longitudinal acceleration/deceleration in order to adjust its velocity with the velocity in the destination lane. For this case we show that the results obtained for the constant acceleration case can be used in order to decide whether a particular lane changing/merging scenario is collision-free or not. In particular, we use the method of isoclines [7] to show that the lane changing/merging maneuver can be modeled by isoclines in the initial longitudinal spacing/ relative longitudinal velocity plane. If the isocline that corresponds to a particular lane changing/merging scenario ends in the safe region, then the maneuver is collision-free; even in the case where the initial vehicles' configuration is in the unsafe region, the lane changing/merging maneuver is safe as soon as the isocline leads in the safe region.

Due to the similarities between the lane changing and the merging problem, in this work we will consider and analyze only the lane changing problem in a two-lane highway. The results of this work can be easily extended/modified for the case of merging. Another interesting extention of this work could be for the three-or-more lane highways, where the merging vehicle intends to make a lane change into the middle lane of the three lane highway.

## 2. Minimum Longitudinal Safety Spacing:

Let us consider a lane changing situation where vehicle $M$ in Figure 1 will move from its current position between vehicles $L_o$ and $F_o$ to a new position between vehicles $L_d$ and $F_d$ in the neighboring lane. We refer to vehicles $L_d$, $F_d$, $L_o$, $F_o$ and $M$ as the leading vehicle in the destination lane, following vehicle in the destination lane, leading vehicle in the originating lane, following vehicle in the originating lane, and the vehicle which must perform the lane-changing (which will be called thereafter the merging vehicle), respectively.
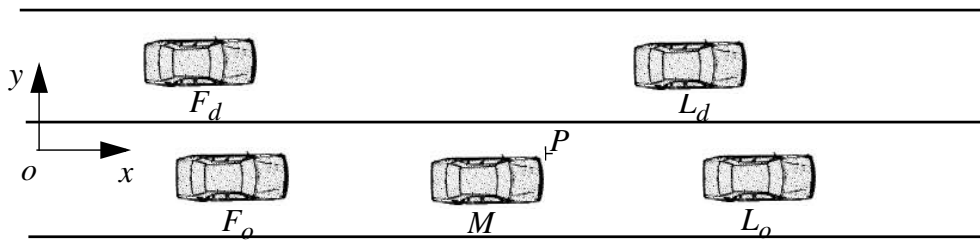


**Fig 1: Pre-lane changing configuration showing position of merging vehicle $M$.**

Without loss of generality, we assume that the merging vehicle, $M$, starts the lane-changing maneuver at t=0. This maneuver consists of two parts. Initially $M$ adjusts its longitudinal velocity and spacing for a time-interval $t_{adj}$, and then applies lateral acceleration to merge to the destination lane. In other words, $t_{adj}$ is the required time for the merging vehicle to adjust its longitudinal position and velocity before it starts merging to the destination lane.

Hereafter, the longitudinal acceleration/deceleration, the longitudinal velocity, the longitudinal position, and the lateral position of vehicle $i$ will be denoted by $a_i(t)$, $v_i(t)$, $x_i(t)$, and $y_i(t)$, respectively, where $i \in \{L_d, F_d, L_o, F_o, M\}$. Precisely speaking, $x_i(t)$ and $y_i(t)$ are, respectively, the longitudinal and lateral distances between the upper right corner of the vehicle $i$ (e.g. denoted by "P" for merging vehicle in Fig. 1) and an arbitrary origin (denoted by "O" in Fig. 1).

With the exception of the merging vehicle, the lateral acceleration of all other vehicles is assumed to be zero.

In our analysis, we assume a simple but realistic model for the lateral acceleration $a_{lat}(t)$ of the

merging vehicle that is used to complete the lane change maneuver. It can be modeled as a Sine function of time [2], and is given as follows:

$$a_{lat}(t) = \begin{cases} \dfrac{2\pi H}{t_{lat}^2} \times \sin\left(\dfrac{2\pi}{t_{lat}}(t - t_{adj})\right) & t_{adj} \leq t \leq t_{lat} + t_{adj} \\ \\ 0 & Otherwise \end{cases}$$

(1)

In (1) H is the total lateral displacement for the merging vehicle, $t_{adj}$ is the time elapse before lateral acceleration applies, and $t_{lat}$ is the total time, after $t_{adj}$, needed to complete the lane change. It should be noted that the lateral acceleration $a_{lat}(t)$, according to equation(1), would be positive within the first half of the lateral displacement, i.e. $t<(t_{lat}/2)+t_{adj}$, and negative in the second half.

Given $a_{lat}(t)$, the lateral velocity and position of the upper right corner of the merging vehicle M (denoted by *"p"* in Fig. 1) are given by the equations,

$$v_{lat}(t) = \begin{cases} \dfrac{-H}{t_{lat}} \times \cos\left(\dfrac{2\pi}{t_{lat}}(t - t_{adj})\right) + \dfrac{H}{t_{lat}} & t_{adj} \leq t \leq t_{lat} + t_{adj} \\ \\ 0 & Otherwise \end{cases}$$

(2)

and,

$$y_{lat}(t) = \begin{cases} H & t \geq t_{lat} + t_{adj} \\ \\ \dfrac{-H}{2\pi} \times \sin\left(\dfrac{2\pi}{t_{lat}}(t - t_{adj})\right) + \dfrac{H}{t_{lat}}(t - t_{adj}) & t_{adj} \leq t \leq t_{lat} + t_{adj} \\ \\ 0 & Otherwise \end{cases}$$

(3)

A "lane change crash" occurs when the merging vehicle M attempts to change its lane and strikes or is struck by a vehicle in the adjacent lane. The model (1) considered is an accurate model for many simple lane change/merge maneuvers during which more than two-thirds of lane change/ merge crashes occur [8].

The objective of this section is to use the simple lane change model described above and the longitudinal acceleration profiles of the five vehicles in figure 1, to find the initial minimum longitudinal spacing between M and each of the other vehicles such that during a specified time-interval [0,T], no collision, of any type, occurs. The length of the time interval T denotes the time under consideration.

In the following subsections we develop analytical expressions for the minimum longitudinal spacing between the merging vehicle M and all other vehicles involved, in order to guarantee a

lane change without collision.


## 2.1. Minimum Longitudinal Safety Spacing between *M* and $L_d$ :

Let's consider the vehicles *M* and $L_d$ during a lane change/merge maneuver as shown in figure 2. The merging vehicle, *M*, starts the lane-changing maneuver at t=0 by adjusting its longitudinal position and velocity, and then applying lateral acceleration at t=$t_{adj}$ according to (1). The type of collision between *M* and $L_d$ could be of angle, side-wipe, or rear-end collision.

Let S denote the initial lateral distance between the upper side of the merging vehicle and the lower side of the vehicle $L_d$. Since the leading vehicle $L_d$ remains in the destination lane, an angle and/or a side-wipe collision may occur as *M* passes the line LS in figure 2; LS is the tangent to the lower side of the leading vehicle $L_d$. The upper-right corner of *M* is the first point of the merging vehicle which passes the line LS at the point C.
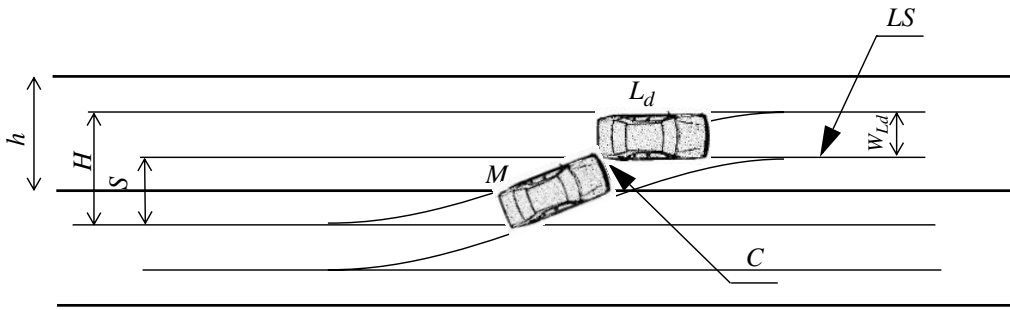


**Fig 2: The marginal collision point between the merging vehicle M and the leading vehicle $L_d$.**

It should be noted that, since the lateral acceleration of the leading vehicle $L_d$ is zero, the lateral position of $L_d$, $y_{Ld}$, is constant.

Let $t_C$+$t_{adj}$ be the time-instant at which the upper-right corner of the merging vehicle is at the point C in figure 2. The type of collision which may take place at or after this time-instant is angle, side-wipe or rear-end collision. An angle or a side-wipe collision may occur during or after the upper-right corner of *M* passes the point C, i.e., at or after the time-instant $t_C$+$t_{adj}$. A rear-end collision may occur after the vehicle *M* accomplishes the lane changing maneuver, i.e., after the merging vehicle has completely merged. According to equation (3), the time-instant $t_C$+$t_{adj}$ can be found by solving the following equation for t=$t_C$+$t_{adj}$,

$$y_{lat}(t) \ = \ S \ = \ y_{Ld} - w_{Ld} \tag{4}$$

where $w_{Ld}$ is the width of the leading vehicle $L_d$. The algorithm for solving for $t_C$+$t_{adj}$ is shown in Appendix A.

By taking all types of collisions mentioned above into account, the condition for no collision between $M$ and $L_d$ is given by:

$$x_M(t) < x_{Ld}(t) - l_{Ld} - w_M \times Sin(\theta(t)) \qquad \forall t \in [t_C + t_{adj}, T] \qquad \textbf{(5)}$$

where $l_{Ld}$ is the length of the leading vehicle $L_d$, $w_M$ is the width of the merging vehicle $M$, and $\theta$ is the angle between the tangent of the path at point $y_{lat}(t)$ and the horizontal axis.

The last term in equation (5), $w_M \times Sin(\theta(t))$, is to prevent any angle collision between any point on the front bumper of the merging vehicle $M$ and the lower-left corner of leading vehicle $L_d$ in the time interval $[t_C+t_{adj}, t_{lat}+t_{adj}]$. From the definition of $\theta$, it can be seen that

$$\tan(\theta(t)) = \frac{\partial y_{lat}(t)}{\partial x_M(t)} = \frac{\partial y_{lat}(t)/\partial t}{\partial x_M(t)/\partial t} = \frac{v_{lat}(t)}{v_M(t)} \qquad \textbf{(6)}$$

Equation (6) indicates that the value of *tan(θ(t))* and consequently *Sin(θ(t))* can be evaluated at each time instant based on the lateral and longitudinal velocity of the merging vehicle. In Appendix B we present a graphical description of the linear approximation we used in (5). The error of this approximation decreases as the merging path becomes smoother, i.e. as $t_{lat}$ increases.

The maximum value of *θ(t)* and consequently the maximum value of *Sin(θ(t))* in equation(5) will be at the time instant $t=t_C+t_{adj}$. (Note that $t_C+t_{adj}>(t_{lat}/2)+t_{adj}$, i.e. $a_{lat}(t_c+t_{adj})<0$). Let's define

$$l_{L1} = l_{Ld} + w_M \times \underset{t}{Max}(Sin\theta(t)) \equiv l_{Ld} + w_M \times Sin(\theta(t_C + t_{adj})),$$ then equation (5) can be simplified as follows:

$$x_M(t) < x_{Ld}(t) - l_{L1} \qquad \forall t \in [t_C + t_{adj}, T] \qquad \textbf{(7)}$$

Let *Sr(t)* be the longitudinal spacing between point *P* of vehicle *M* and the rear end (bumper) of vehicle $L_d$. (Note that $y_{Ld}(t)=0$). That is:

$$Sr(t) = x_{Ld}(t) - l_{L1} - x_M(t) \qquad \forall t \in [t_C + t_{adj}, T] \qquad \textbf{(8)}$$

As long as the spacing Sr is greater than zero, i.e. Sr(t)>0 for $t \geq t_C+t_{adj}$, no collision will occur during the lane-changing maneuver. Based on (8), we can rewrite equation (5) as:

$$Sr(t) = \left(Sr(0) + \int_0^t \int_0^\lambda (a_{Ld}(\tau) - a_M(\tau))d\tau d\lambda + (v_{Ld}(0) - v_M(0))t\right) > 0 \qquad \textbf{(9)}$$

$$\forall t \in [t_C + t_{adj}, T]$$

where $Sr(0) = x_{Ld}(0) - l_{Ld} - x_M(0)$. Our objective is to find the initial minimum value of Sr(0) which guarantees no collision between the leading vehicle $L_d$ and the merging vehicle $M$. The minimum value of Sr(0) is the minimum initial longitudinal relative spacing between $L_d$ and $M$, for collision free vehicle merging and is denoted by MSS($L_d$,M). It is calculated using (8) as follows,

$$MSS(Ld, M) = \underset{t}{Max}\left(\int_0^t \int_0^\lambda (a_M(\tau) - a_{Ld}(\tau))d\tau d\lambda + (v_M(0) - v_{Ld}(0))t\right)$$

$$\forall t \in [t_C + t_{adj}, T]$$

(10)

From equation (8) it is clear that the minimum initial longitudinal relative spacing between $L_d$ and $M$, MSS(Ld,M), depends on the relative longitudinal acceleration, the relative initial longitudinal velocity between the two vehicles, as well as the time interval $t_C+t_{adj}$. This time interval depends on the lateral distance S, the lateral time $t_{lat}$, and the adjustment time $t_{adj}$.

## 2.2. Minimum Longitudinal Safety Spacing between M and $F_d$:

Now, let us consider the merging vehicle, $M$, and the following vehicle in the destination lane, $F_d$, during a lane changing/merging maneuver as shown in figure 3. The merging vehicle, $M$, starts its merging maneuver at t=0, and its lateral movement at $t=t_{adj}$. The type of possible collision between $M$ and $F_d$ could be of angle, side-wipe, and rear-end collision.

Since the following vehicle $F_d$ is in the destination lane and has zero lateral motion, an angle and/or a side-wipe collision may occur during or after the vehicle $M$ passes the line LS in figure 3, where the line LS is the tangent to the lower side of the following vehicle $F_d$.

We define the point C as the intersection between the upper-left corner of the vehicle $M$ and the line LS; obviously the point C is the marginal point that a collision between the two vehicles could occur.
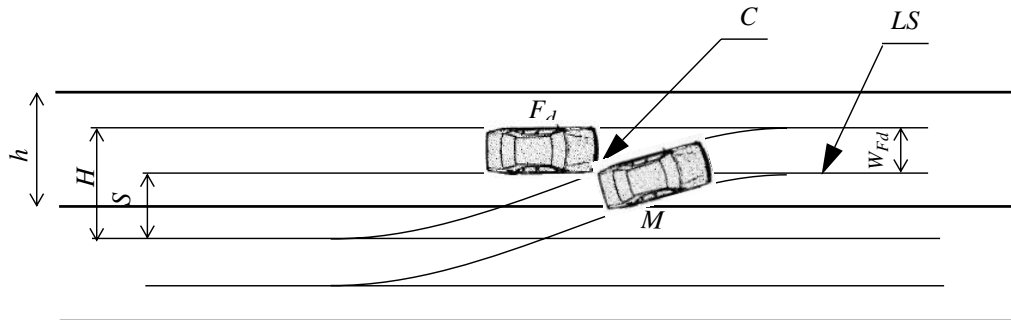


**Fig 3: The marginal collision point between the merging vehicle M and the following vehicle $F_d$.**

In this case we need to find the coordinates of the point C as well as the time-instant at which the upper-left corner of the merging vehicle passes this point. Since equation (3) describes the motion of the upper-right corner of the merging vehicle, the coordinates of the point C cannot be calculated directly from equation (3). In order to overcome this difficulty we apply first order approximation (tangent to the vehicle's path) to calculate the lateral position of the other corners of the merging vehicle as follows:

$$y_{upper-left}(t) \cong y_{lat}(t) - l_M \times Sin(\theta(t))$$
$$y_{lower-right}(t) \cong y_{lat}(t) - w_M \times Cos(\theta(t)) \qquad\qquad (11)$$
$$y_{lower-left}(t) \cong y_{lat}(t) - (l_M \times Sin(\theta(t)) + w_M \times Cos(\theta(t)))$$

where $\theta(t)$ is the angle between the tangent of the path at point $y_{lat}(t)$ and the horizontal axis, and $l_M$, $w_M$ are the length and width of the merging vehicle, respectively.

Let $t_C+t_{adj}$ be the time-instant at which the upper-left corner of the merging vehicle is at the point C in figure 3. The type of collision which may take place during or after this time-instant are angle, side-wipe and rear-end collision. An angle or a side-wipe collision may occur during or after the upper-left corner of *M* passes the point C. Rear-end collision may happen after *M* accomplishes its lane change maneuver. Using equation (11), the time-instant $t_C+t_{adj}$ can be found by solving the following equation:

$$y_{lat}(t) - l_M \times Sin(\theta(t)) \cong S = y_{Fd} - w_{Fd} \qquad\qquad (12)$$

Where $w_{Fd}$ is the width of the following vehicle $F_d$. Using equation (6), equation (12) can be rewritten as,

$$y_{lat}(t) - l_M \times \frac{v_{lat}(t)}{\sqrt{v_{lat}^2(t) + v_M^2(t)}} \cong S \qquad\qquad (13)$$

The above equation can be solved numerically using equations (2) and (3); In Appendix A, we provide an algorithm for calculating $t_C+t_{adj}$.

Considering all possible types of collision, the condition for collision avoidance between *M* and $F_d$ would be:

$$x_{Fd}(t) < x_M(t) - l_M \times Cos(\theta(t)) \qquad \forall t \in [t_C + t_{adj}, T] \qquad\qquad (14)$$

It should be noted that the maximum value of *Cos(θ(t))*, in the time internal [$t_C+t_{adj}$,T], will be at or after the time instant t=$t_{lat}+t_{adj}$, where the value of $\theta$ is minimum and is equal to zero. Therefore, equation (14) can be simplified as follows:

$$x_{Fd}(t) < x_M(t) - l_M \qquad \forall t \in [t_C + t_{adj}, T] \tag{15}$$

The above approximation results in a conservative condition for no collision condition during the time interval $t_C + t_{adj} \leq t \leq t_{lat} + t_{adj}$, i.e. before $M$ completes its lane changing maneuver.

The longitudinal spacing between the rear of the vehicle $M$ and the front of the vehicle $F_d$ is given by:

$$Sr(t) = x_M(t) - l_M - x_{Fd}(t) \qquad \forall t \in [t_C + t_{adj}, T] \tag{16}$$

As long as the longitudinal spacing in (16) is positive, i.e. Sr(t)>0 for $t \geq t_C + t_{adj}$, no collision occurs. Based on (15), we can rewrite equation (16) as follows:

$$Sr(t) = \left( Sr(0) + \int_0^t \int_0^\lambda (a_M(\tau) - a_{Fd}(\tau)) d\tau d\lambda + (v_M(0) - v_{Fd}(0))t \right) > 0 \tag{17}$$

$$\forall t \in [t_C + t_{adj}, T]$$

where $Sr(0) = x_M(0) - l_M - x_{Fd}(0)$. Our objective is to find the initial minimum value of Sr(0) which guarantees no collision between the merging vehicle $M$ and the following vehicle $F_d$. The minimum value of Sr(0) is the minimum initial longitudinal relative spacing between $F_d$ and $M$, for collision free vehicle merging and is denoted by MSS(M,F$_d$). It is calculated using (17) as follows,

$$MSS(M, Fd) = \underset{t}{Max} \left( \int_0^t \int_0^\lambda (a_{Fd}(\tau) - a_M(\tau)) d\tau d\lambda + (v_{Fd}(0) - v_M(0))t \right) \tag{18}$$

$$\forall t \in [t_C + t_{adj}, T]$$

From equation (17) it is clear that the minimum initial longitudinal relative spacing between $F_d$ and $M$, MSS(M,Fd), depends on the relative longitudinal acceleration, the relative initial longitudinal velocity between the two vehicles, as well as the time interval $t_C + t_{adj}$. This time interval depends on the lateral distance S, the lateral time $t_{lat}$, and the adjustment time $t_{adj}$.


## 2.3. Minimum Longitudinal Safety Spacing between $M$ and $L_o$:

Consider now the case of the merging vehicle, $M$, and the leading vehicle in the originating lane, $L_o$, during the lane changing maneuver shown in figure 4. The vehicle $M$ starts its merging maneuver at t=0, and its lateral movement at t=$t_{adj}$. The type of collision between $M$ and $L_o$ could be of angle, or rear-end collision, but not side-wipe collision.

Since the leading vehicle $L_o$ remains in the originating lane, an angle collision may occur before the lower-right corner of $M$ passes the line LS at point C in figure 4.
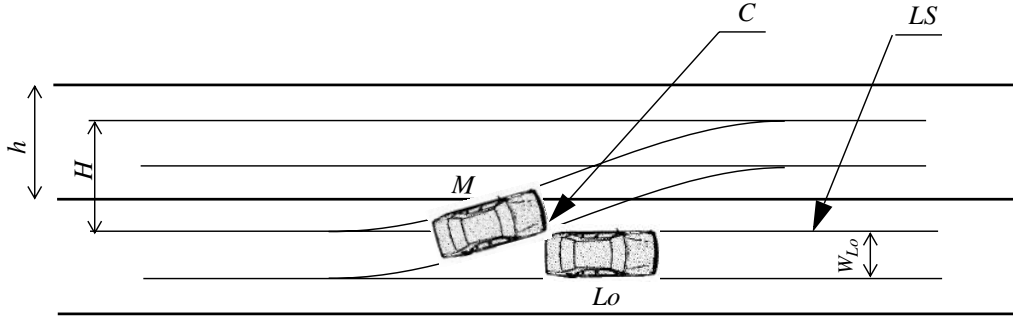


**Fig 4: The marginal collision point between the merging vehicle M and the leading vehicle $L_o$.**

Since the lateral acceleration of the leading vehicle $L_o$ is zero, the lateral position of $L_o$, $y_{Lo}$, is constant.

Let $t_C+t_{adj}$ be the time-instant at which the lower-right corner of the merging vehicle is at the point C as shown in figure 4. A rear-end collision may happen before $M$ starts its lane change maneuver at $t_{adj}$. An angle collision may occur after the longitudinal adjustment time, $t_{adj}$, and before the lower-right corner of $M$ passes the point C. Considering the first order approximation in (11), the time-instant $t_C+t_{adj}$ can be found by solving the following equation.

$$y_{lat}(t) - w_M \times Cos(\theta(t)) \cong S = y_{Lo} \tag{19}$$

where $w_M$ is the width of the merging vehicle $M$, Using (6), equation (19) can be rewritten as:

$$y_{lat}(t) - w_M \times \frac{v_M(t)}{\sqrt{v_{lat}^2(t) + v_M^2(t)}} \cong S \tag{20}$$

The above equation can be easily solved numerically using equations (2) and (3); In Appendix A, we present an algorithm for calculating $t_C+t_{adj}$.

Considering all types of collision, the condition for collision avoidance between $M$ and $L_o$ would be:

$$x_M(t) < x_{Lo}(t) - l_{Lo} - w_M \times Sin(\theta(t)) \qquad \forall t \in [0, t_C + t_{adj}] \tag{21}$$

where $l_{Lo}$ is the length of the leading vehicle $L_o$, and $\theta$ is the angle between the tangent of the path at point $y_{lat}(t)$ and the horizontal axis.

The last term in equation (21), $w_M \times Sin(\theta(t))$, is to prevent any angle collision between any point on the rear bumper of the leading vehicle $L_o$ and the lower-right corner of the merging vehicle $M$ in the time interval [$t_{adj}$, $t_C+t_{adj}$]. It should be noted that the maximum value of $\theta$ and consequently the maximum value of $Sin(\theta(t))$ in equation (21) will be at the time instant $t=t_C+t_{adj}$. Let's define $l_{L1}=l_{Lo}+Max(Sin(\theta(t)))$, then equation (21) can be rewritten as follows:

$$x_M(t) < x_{Lo}(t) - l_{L1} \qquad \forall t \in [0, t_C + t_{adj}] \tag{22}$$

According to (22), the longitudinal spacing between the front of the vehicle $M$ and the rear of vehicle $L_o$ would be:

$$Sr(t) = x_{Lo}(t) - l_{L1} - x_M(t) \qquad \forall t \in [0, t_C + t_{adj}] \tag{23}$$

As long as the spacing Sr(t) is greater than zero, i.e. Sr(t)>0 for $t \leq t_C+t_{adj}$, no collision occurs. According to (22), we can rewrite equation (23) as:

$$Sr(t) = \left( Sr(0) + \int_0^t \int_0^\lambda (a_{Lo}(\tau) - a_M(\tau))d\tau d\lambda + (v_{Lo}(0) - v_M(0))t \right) > 0 \tag{24}$$

$$\forall t \in [0, t_C + t_{adj}]$$

Our objective is to find the initial minimum value of Sr(0) which guarantees no collision between the leading vehicle $L_o$ and the merging vehicle $M$. The minimum value of Sr(0) is the minimum initial longitudinal relative spacing between $L_o$ and $M$, for collision free vehicle merging and is denoted by MSS(Lo,M).

$$MSS(Lo, M) = \underset{t}{Max}\left\{ \left( \int_0^t \int_0^\lambda (a_M(\tau) - a_{Lo}(\tau))d\tau d\lambda + (v_M(0) - v_{Lo}(0))t \right), 0 \right\} \tag{25}$$

$$\forall t \in [0, t_C + t_{adj}]$$

Equation (25) indicates that the minimum initial longitudinal relative spacing between $M$ and $L_o$, MSS(Lo,M), depends on the relative longitudinal acceleration, the relative initial velocity between the two vehicles, as well as the time interval $t_C+t_{adj}$. This time interval depends on the values of the lateral position $y_{Lo}$ of the leading vehicle in the originating line, the lateral time $t_{lat}$, and the adjustment time $t_{adj}$.

## 2.4. Minimum Longitudinal Safety Spacing between $M$ and $F_o$:

We consider the merging vehicle $M$, and the following vehicle $F_o$ in the originating lane, during

a lane change/merge maneuver as shown in figure 5. The merging vehicle, *M*, starts its merging maneuver at t=0, and its lateral movement at t=$t_{adj}$. The types of possible collision between *M* and $F_o$ could be of angle, and rear-end collision, but not side-wipe collision.

Since the following vehicle $F_o$ is in the originating lane and has zero lateral motion, an angle and/or a rear-end collision may occur during or before the vehicle *M* passes the line LS in figure 5, where LS is the tangent to the upper side of the following vehicle $F_o$.

We define the point C as the intersection between the lower-left corner of the vehicle *M* and the line LS; obviously the point C is the marginal point that a collision between two vehicles may occur.

Since the following vehicle in originating lane, $F_o$, has no lateral acceleration, the lateral position of $y_{Fo}$ is constant.
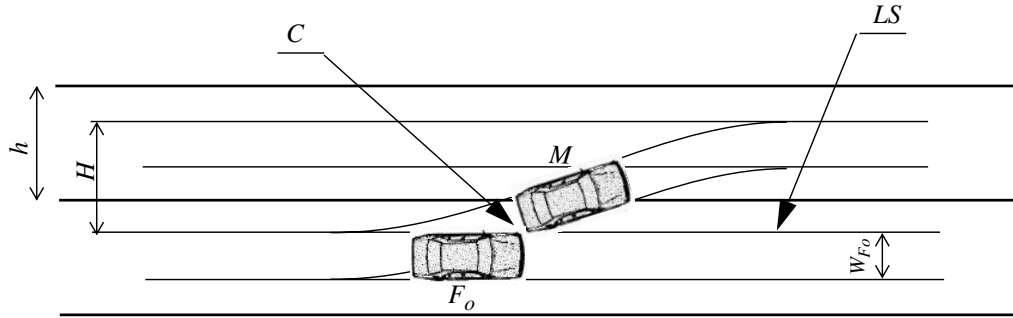


**Fig 5: The last possible collision point between the merging vehicle M and the following vehicle Fo.**

Let $t_C+t_{adj}$ be the time-instant at which the lower-left corner of the merging vehicle is at the point C in figure 5. A rear-end collision may take place before *M* starts its lane change maneuver at $t_{adj}$. An angle collision may occur after the time instant $t_{adj}$ and before the lower-left corner of *M* passes C. Using the first order approximation in (11), the time-instant $t_C+t_{adj}$ can be found by solving the following equation.

$$y_{lat}(t) - (l_M \times Sin(\theta(t)) + w_M \times Cos(\theta(t))) \cong S = y_{Fo} \tag{26}$$

Using (6), equation (23) can be rewritten as follows:

$$y_{lat}(t) - \left( l_M \times \frac{v_{lat}(t)}{\sqrt{v_{lat}^2(t) + v_M^2(t)}} + w_M \times \frac{v_M(t)}{\sqrt{v_{lat}^2(t) + v_M^2(t)}} \right) \cong S \tag{27}$$

The above equation can be easily solved numerically using equations (2) and (3); In Appendix A, we provide an algorithm for calculating $t_C+t_{adj}$.

Considering all possible types of collision, the condition for no collision between $M$ and $F_o$ is given by:

$$x_{Fo}(t) < x_M(t) - l_M \times Cos(\theta(t)) \qquad \forall t \in [0, t_C + t_{adj}] \tag{28}$$

It should be noted that the maximum value of $Cos(\theta(t))$, in the time internal $[0, t_C+t_{adj}]$, will be before the time instant $t=t_{adj}$, where the value of $\theta$ is minimum and is equal to zero. Therefore, equation (28) can be simplified as follows:

$$x_{Fo}(t) < x_M(t) - l_M \qquad \forall t \in [0, t_C + t_{adj}] \tag{29}$$

Equation(29) results in a conservative condition for no collision during the time interval $[t_{adj}, t_C+t_{adj}]$, i.e. after $M$ starts its lane changing maneuver. According to (29), the longitudinal spacing between the rear of the vehicle $M$ and the front of the vehicle $F_o$ is given by:

$$Sr(t) = x_M(t) - l_M - x_{Fo}(t) \qquad \forall t \in [0, t_C + t_{adj}] \tag{30}$$

As long as the longitudinal spacing in (30) is greater than zero, i.e. $Sr(t) > 0$ for $t \leq t_C + t_{adj}$, no collision occurs. Based on (30), we can rewrite equation (25) as follows:

$$Sr(t) = \left( Sr(0) + \int_0^t \int_0^\lambda (a_M(\tau) - a_{Fo}(\tau)) d\tau d\lambda + (v_M(0) - v_{Fo}(0))t \right) > 0 \tag{31}$$
$$\forall t \in [0, t_C + t_{adj}]$$

Our objective here is to find the initial minimum value of Sr(0) which guarantees no collision between the following vehicle $F_o$ and the merging vehicle $M$. The minimum value of Sr(0) is the minimum initial longitudinal relative spacing between $M$ and $F_o$, for collision free vehicle merging and is denoted by MSS(M,Fo). It is calculated using (31) as follows;

$$MSS(M, Fo) = \underset{t}{Max} \left\{ \left( \int_0^t \int_0^\lambda (a_{Fo}(\tau) - a_M(\tau)) d\tau d\lambda + (v_{Fo}(0) - v_M(0))t \right), 0 \right\} \tag{32}$$
$$\forall t \in [0, t_C + t_{adj}]$$

From equation (32) it is clear that the minimum initial longitudinal relative spacing between $M$ and $F_o$, MSS(M,Fo), depends on the relative longitudinal acceleration, the relative initial longitudinal velocity between two vehicles, as well as the time interval $t_C+t_{adj}$. This time interval depends on the lateral position $y_{Fo}$, lateral time $t_{lat}$, and adjustment $t_{adj}$.

In all the expressions for MSS, the value of MSS depends on the value of $t_{adj}$, the time elapsed before lateral acceleration is applied. During this time the merging vehicle $M$ adjusts its

longitudinal position and velocity before it starts merging.

# 3. Special Cases and Simulations:

In this section we consider specific profiles for the longitudinal acceleration of the vehicles involved in lane changing maneuvers in order to derive closed form expressions for MSS.

We consider the constant longitudinal velocity for all vehicles in Figure 1 except for *M*, which can be considered as the steady state condition before the lane changing/merging scenario starts. In the following subsections we will come up with the minimum longitudinal safety spacing MSS with respect to relative velocity between the merging vehicle *M* and each of the other vehicles in Figure 1 by considering two cases:

In the first case, the merging vehicle *M* performs the merging scenario with constant longitudinal velocity. Obviously, the velocity of the merging vehicle will remain the same as the velocity of the vehicles in the originating lane, i.e. the velocity before starting its maneuver. In the second case, the merging vehicle *M* applies a constant longitudinal acceleration/deceleration in order to reach the velocity of the vehicles in the destination lane after the specific time $t_{long}+t_{adj}$.

In each case, and based on the relative longitudinal velocity and position between *M* and the other vehicles, the merging vehicle *M* can determine whether the merging scenario is safe or unsafe before the merging maneuver starts.

### 3.1. Constant Longitudinal Velocity:
This is the case where all five vehicles are moving with constant longitudinal velocity, i.e. the longitudinal acceleration for all vehicles is zero, $a_i(t)=0$ $\forall t \in [0, T]$ and for all $i \in \{L_d, F_d, L_o, F_o, M\}$.

*Minimum Longitudinal Safety Spacing between M and $L_d$:*

According to (9), the condition for collision avoidance between *M* and $L_d$ with constant longitudinal velocity is:

$$Sr(t) = (Sr(0) + (v_{Ld} - v_M)t) > 0 \qquad \forall t \in [t_C + t_{adj}, T] \qquad \textbf{(33)}$$

The minimum initial longitudinal safety spacing, MSS(Ld,M) is:

$$MSS(Ld, M) = \underset{t}{Max}((v_M - v_{Ld})t) \qquad \forall t \in [t_C + t_{adj}, T] \qquad \textbf{(34)}$$

Since the relative longitudinal velocity, $v_M - v_{Ld}$ is constant, equation (34) can be easily rewritten as follows:

$$MSS(Ld, M) = \begin{cases} (v_M - v_{Ld}) \times T & v_M - v_{Ld} \geq 0 \\ (v_M - v_{Ld}) \times (t_C + t_{adj}) & Otherwise \end{cases} \qquad (35)$$

***Minimum Longitudinal Safety Spacing between M and $F_d$:***

Equation (17) provides the condition for collision avoidance between $M$ and $F_d$. In the case of constant longitudinal velocity, this condition is as follows:

$$Sr(t) = (Sr(0) + (v_M - v_{Fd})t) > 0 \qquad \forall t \in [t_C + t_{adj}, T] \qquad (36)$$

The minimum initial longitudinal safety spacing, MSS(M,Fd) will be:

$$MSS(M, Fd) = \underset{t}{Max}((v_{Fd} - v_M)t) \qquad \forall t \in [t_C + t_{adj}, T] \qquad (37)$$

or, equivalently,

$$MSS(M, Fd) = \begin{cases} (v_{Fd} - v_M) \times T & v_{Fd} - v_M \geq 0 \\ (v_{Fd} - v_M) \times (t_C + t_{adj}) & Otherwise \end{cases} \qquad (38)$$

***Minimum Longitudinal Safety Spacing between M and $L_o$:***

According to (24), the condition for collision avoidance between $M$ and $L_o$ with constant longitudinal velocity is as follows:

$$Sr(t) = (Sr(0) + (v_{Lo} - v_M)t) > 0 \qquad \forall t \in [0, t_C + t_{adj}] \qquad (39)$$

The minimum initial longitudinal safety spacing, MSS(Lo,M) will be:

$$MSS(Lo, M) = \underset{t}{Max}\{((v_M - v_{Lo})t), 0\} \qquad \forall t \in [0, t_C + t_{adj}] \qquad (40)$$

or, equivalently,

$$MSS(Lo, M) = \begin{cases} (v_M - v_{Lo}) \times (t_C + t_{adj}) & v_M - v_{Lo} \geq 0 \\ 0 & Otherwise \end{cases} \qquad \text{(41)}$$

### *Minimum Longitudinal Safety Spacing between M and $F_o$:*

According to Equation (31), the condition for collision avoidance between $M$ and $F_o$ with constant longitudinal velocity is as follows:

$$Sr(t) = (Sr(0) + (v_M - v_{Fo})t) > 0 \qquad\qquad \forall t \in [0, t_C + t_{adj}] \qquad \text{(42)}$$

The minimum initial longitudinal safety spacing, MSS(M,Fo) will be:

$$MSS(M, Fo) = \underset{t}{Max}\{((v_{Fo} - v_M)t), 0\} \qquad\qquad \forall t \in [0, t_C + t_{adj}] \qquad \text{(43)}$$

or, equivalently,

$$MSS(M, Fo) = \begin{cases} (v_{Fo} - v_M) \times (t_C + t_{adj}) & v_{Fo} - v_M \geq 0 \\ 0 & Otherwise \end{cases} \qquad \text{(44)}$$

### *Simulation results:*

The validity of equations (35), (38), (41), and (44) was evaluated using a series of simulations. In these simulations, the time T was set equal to 50 sec, the adjustment time $t_{adj}$ equal to 0 sec, the lateral time, in equation (1), $t_{lat}$, equal to 5 sec, and the lateral displacement H to 12 feet. Figures 6, 7, 8, and 9 show the initial relative longitudinal spacings versus the relative longitudinal velocity between $M$ and the other four vehicles involved in the lane-changing maneuver. The solid lines (which will be called thereafter *safety margins*) in these figures represent the margins between safe and unsafe lane changing regions.

For positive relative velocity, $v_M$-$v_{Ld}$, in figure 6, the safety margin is a line with tangent equal to T=50. For negative relative velocity, it is approximately a line with tangent equal to 2.8 which is the value of $t_C$+$t_{adj}$ in equation (4). Thus for constant longitudinal velocity, the safety margin consists of two lines passing through the origin with different tangents. This conclusion complies with equation (35).

A similar situation appears in figures 7 and 8 for the spacings between $M$ and $F_d$, and $M$ and $L_o$, i.e. the safety margins consist of two lines with different tangents which are given by equations (38) and (41). In the case of the spacing between $M$ and $F_o$ in figure 9, one of the components of the safety margin is not a straight line; the reason for this is due to the approximation we made in

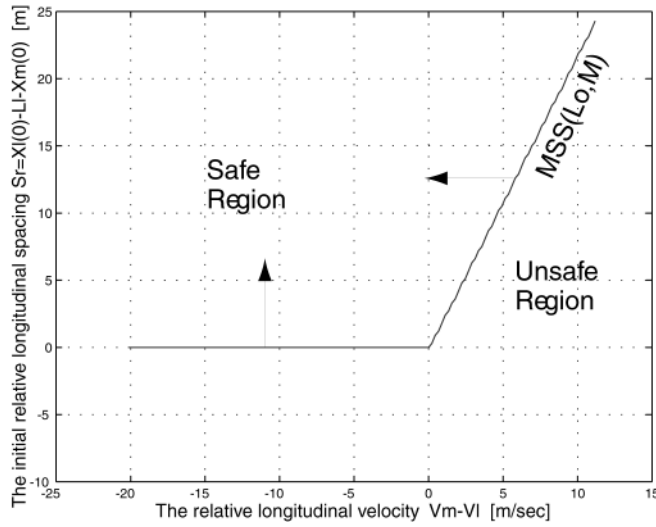equation (27). The more smoother the lane changing is, the more accurate is the approximation in (27).
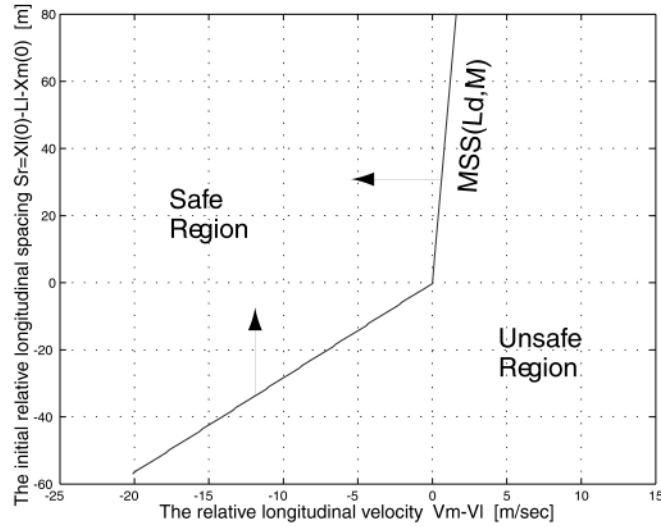


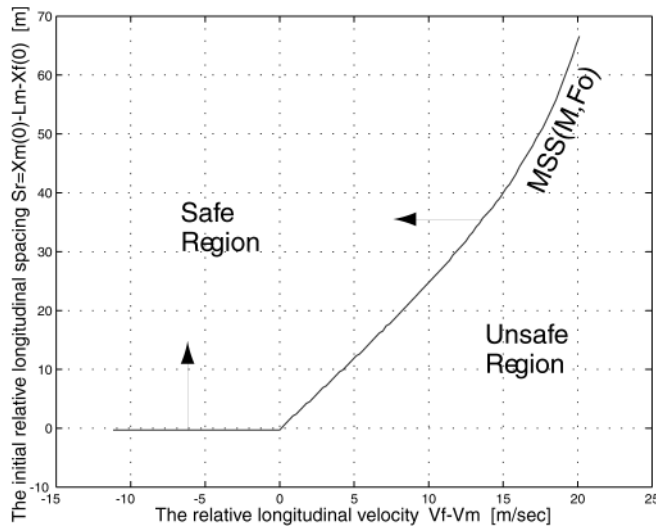**Fig 6: The collision region between M and $L_d$, with constant velocity.**
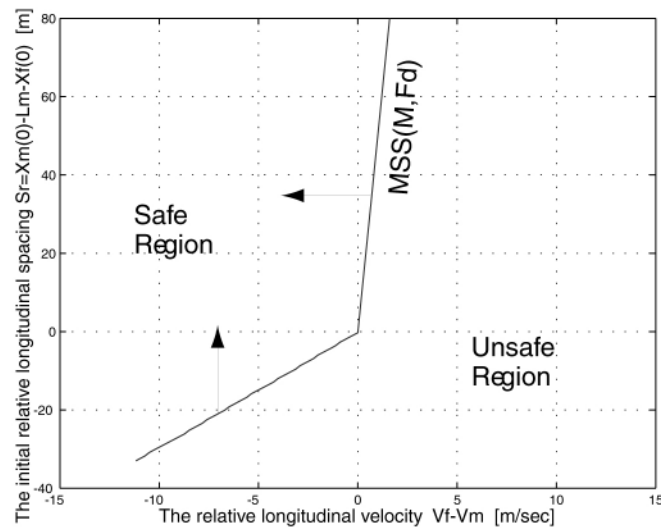


**Fig 7: The collision region between M and $F_d$, with constant velocity.**

**Fig 8: The collision region between M and L$_o$, with constant velocity.**



**Fig 9: The collision region between M and F$_o$, with constant velocity.**

### 3.2. Switching Longitudinal Acceleration:

In this subsection we will examine the acceleration profile proposed in [6] for lane changing. According to [6] all vehicles in both lanes, except *M*, are moving with constant longitudinal velocity, i.e. $a_i(t)=0$ for i∈{$L_d$, $F_d$, $L_o$, $F_o$}, while the longitudinal acceleration profile of the merging vehicle, *M*, is the one plotted in figure 10. More precisely, the merging vehicle initially accelerates/decelerates with constant longitudinal acceleration $a_{adj}$, in order to create enough spacing with the rest four vehicles; at the time-instant $t_{adj}$, the merging vehicle starts merging

and it switches its longitudinal acceleration to $a_M$; the merging vehicle continues to accelerate with acceleration $a_M$, until its velocity becomes equal to the velocity of the vehicles in the destination lane. The time-instant $t_{adj}+t_{long}$ denotes the time-instant at which the velocity of the merging vehicle is equal to the one of the vehicles in the originating lane. After this time-instant the merging vehicle's acceleration becomes zero. Note that in the extreme case where $t_{adj}=0$, the merging vehicle moves with constant acceleration.

In the following discussions and in order to get some insight about the problem, we will first analyze the case where $t_{adj}=0$ and then the more general case $t_{adj}>0$.
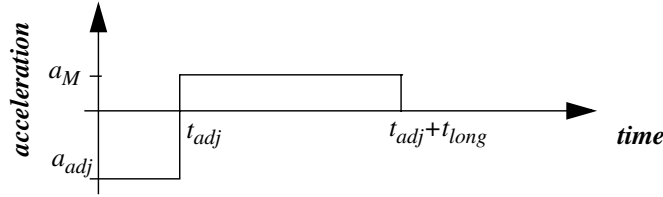


**Fig 10: The switching longitudinal accelerating profile of merging vehicle, *M*.**

***Minimum Longitudinal Safety Spacing between M and $L_d$:***

*I) $t_{adj}=0$:*
In this case, the velocity of all vehicles except *M* is constant while the velocity of the vehicle *M* will become equal to the velocity of $L_d$ at the time instant $t_{long}$ and remain constant thereafter. Therefore, the value of the longitudinal acceleration of the merging vehicle, $a_M$, will be as follows:

$$a_M = \begin{cases} \dfrac{v_{Ld} - v_M(0)}{t_{long}} & t \le t_{long} \\ \\ 0 & Otherwise \end{cases} \tag{45}$$

Any accident between the two vehicles may occur while or after the vehicle *M* passes the point C in figure 2 and before its velocity is equal to that of $L_d$ at $t_{long}$. According to equation (9) the condition for collision avoidance between *M* and $L_d$ is:

$$Sr(t) = \left( Sr(0) - a_M \frac{t^2}{2} + (v_{Ld} - v_M(0))t \right) > 0 \qquad \forall t \in [t_C, t_{long}] \tag{46}$$

Substituting (45) in (46), we obtain:

$$Sr(t) = \left( Sr(0) + (v_{Ld} - v_M(0)) \times \left( t - \frac{t^2}{2t_{long}} \right) \right) > 0 \qquad \forall t \in [t_C, t_{long}] \qquad \text{(47)}$$

Therefore, the minimum initial longitudinal safety spacing, MSS(Ld,M), will be:

$$MSS(Ld, M) = \underset{t}{Max} \left( (v_M(0) - v_{Ld}) \times \left( t - \frac{t^2}{2t_{long}} \right) \right) \qquad \forall t \in [t_C, t_{long}] \qquad \text{(48)}$$

Considering different values of initial relative longitudinal velocity, $v_M(0) - v_{Ld}$, equation (48) can be solved analytically. If the value of $v_M(0)-v_{Ld}$ is positive, the function in (48) will be a concave function and has a maximum value at $t=t_{long}$. If the value is negative, the function is a convex function and the maximum value will be at the boundary $t=t_C$. Therefore equation (48) can be rewritten as following:

$$MSS(Ld, M) = \begin{cases} (v_M(0)-v_{Ld}) \times t_{long}/2 & v_M(0) - v_{Ld} \geq 0 \\ (v_M(0)-v_{Ld}) \times t_C & Otherwise \end{cases} \qquad \text{(49)}$$

The validity of equation (49) was checked by a simulation program. In the simulation, the time T was set equal to 50 sec, the adjustment time $t_{adj}$ equal to 0 sec, the time $t_{lat}$ equal to 5 sec, the lateral displacement H equal to 12 feet, and $t_{long}$ equal to 10 sec. The safe and unsafe region is shown in figure 11.

Figure 11 shows the initial relative longitudinal distance versus the relative longitudinal velocity between $M$ and $L_d$. The solid curve is the safety margin between safe and unsafe regions. For positive relative velocities, $v_M(0)-v_{Ld}$, the safety margin corresponds to a line with tangent equal to $t_{long}/2=5$ while for negative relative velocities, it is a line with tangent equal to 2.8 which is the value of $t_C+t_{adj}$ in equation (4).

Comparison between figures 6 and 11 indicates that the safe region has been expanded. Therefore the switching longitudinal acceleration policy with $t_{adj}=0$ is more reliable than the constant longitudinal velocity policy for the case of vehicles $M$ and $L_d$.
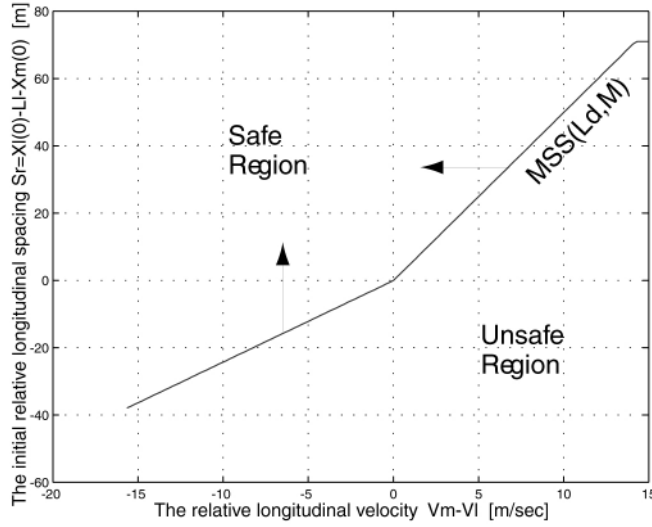
**Fig 11: The collision region between M and L$_d$, with switching longitudinal acceleration.**

*II) t$_{adj}$>0:*

It can be seen, in the case I, that even if initially the two vehicles' relative spacing and velocity is at the unsafe region it may happen that no collision occurs due to the switching acceleration policy. The initial relative spacing and velocity between the vehicles *M* and *L$_d$*, defines a starting point in figure 11. If this point happens to be in the safe area, there is no need to apply any adjustment acceleration, a$_{adj}$ to the merging vehicle, since the lane changing will be safe. If the initial relative spacing and velocity of the two vehicles belong to the unsafe region, then we have to apply the switching acceleration policy, in order to achieve appropriate relative spacing and velocity before the merging vehicle starts merging. In other words, we want to move from the unsafe region into the safe region and then start merging. Let's now define the following state space variables:

$$x_1 = x_{Ld} - x_M - l_{L1}$$
$$x_2 = v_M - v_{Ld}$$

(50)

By differentiating the above variables with respect to time, it is easy to see that the following equations are valid,

$$\dot{x}_1 = \dot{x}_{Ld} - \dot{x}_M = v_{Ld} - v_M = -x_2$$
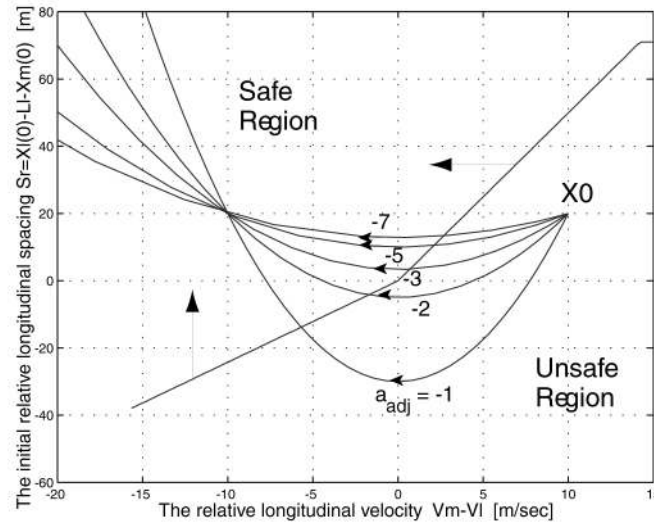$$\dot{x}_2 = \dot{v}_M = a_{adj}$$

(51)

Using the technique of isoclines [7], we can see that equation (51) can be rewritten as,

$$\frac{dx_1}{dx_2} = \frac{-x_2}{a_{adj}} \qquad\qquad (52)$$

Solving the differential equation (52), we obtain the isoclines,

$$x_1 = -\frac{x_2^2}{2a_{adj}} + c \qquad\qquad (53)$$

Here the constant $c$ is the integration constant which depends on the initial values, $x_1(0)$ and $x_2(0)$.



**Fig 12: Applying $a_{adj}$ to move from unsafe area into safe area.**

Figure 12 shows the isoclines that correspond to different values of $a_{adj}$ for the simulation in figure 11. The initial point has been chosen to be in the unsafe region. Applying negative $a_{adj}$, it is possible to move into the safe region in order to start merging maneuver. The larger is the absolute value of $a_{adj}$, the faster we move into the safe region. The minimum value of $t_{adj}$ for each $a_{adj}$ is determined by the point of intersection between the corresponding isocline curve of $a_{adj}$ and the safety margin in figure 12.

It should be noted that, $a_{adj}$ is limited by the acceleration/braking capabilities of the vehicle. Moreover, in order to maintain passenger comfort, the $|a_{adj}|$ must be less than a value say $a_{comf}$ which defines the maximum acceleration that maintains passenger comfort [6]. In addition, large $|a_{adj}|$ results in larger "shock wave" propagating down the originating lane as well as larger minimum initial longitudinal relative spacing $Sr(0)$, which in turn reduces the throughput of the highway. Therefore a trade-off between safety (large accelerations for faster movement from

unsafe region into the safe one) and passenger comfort as well as the highway throughput has to be made when we design the lane-changing scenario.

In the above discussion we assumed that we can apply $a_{adj}$ for a period of $t_{adj}$ seconds; it may not always be feasible because the vehicle's velocity may exceed its limits or, even worse, we may have the unrealistic situation where the velocity of the vehicle becomes negative. In this case the acceleration profile of figure 10 cannot be applied; however, one can use the modified acceleration profile shown in figure 13.
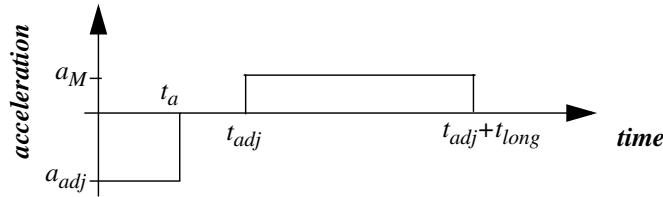


**Fig 13: The modified switching longitudinal accelerating profile of merging vehicle, *M*.**

In figure 13, the velocity of merging vehicle, *M*, remains constant in the interval $[t_a, t_{adj}]$. Constant velocity will help the vehicle to create enough relative spacing in order to enter the safe region.

In the case the modified acceleration profile of figure 13 is used, we have that for the time-interval $[t_a, t_{adj}]$, the state space equations can be rewritten as:

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = 0$$

(54)

Which results in,

$$x_2 = c = x_2(t_a)$$
$$x_1 = (-x_2(t_a))(t - t_a) + x_1(t_a) \qquad \forall t \in [t_a, t_{adj}]$$

(55)

If we intend to increase the relative spacing, $x_1$, we have to make sure that $x_2(t_a)$ has a negative value (negative relative acceleration). In other words, we have to make sure that, at $t=t_a$, we are in the left half plane in figure 11 and then we can set $a_{adj}$ equal to zero in order to move into the safe region. Figure 14 shows the trajectory of moving from the unsafe into the safe region.
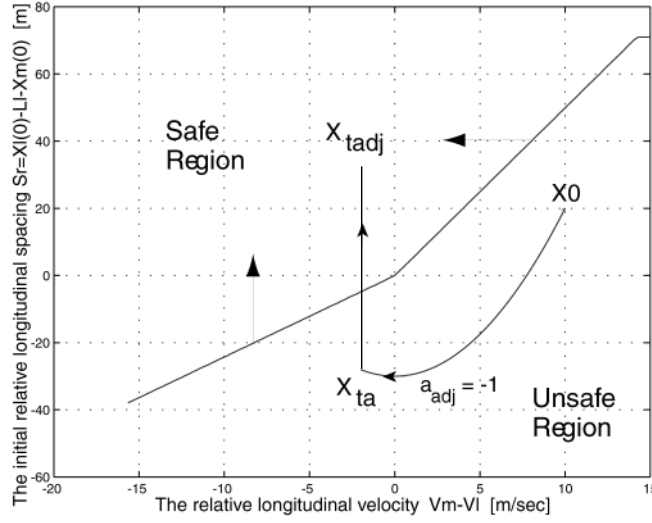
**Fig 14: Applying modified a_adj to move from unsafe area into safe area.**


*Minimum Longitudinal Safety Spacing between M and $F_d$:*

*I) $t_{adj}=0$:*

Here, we assume that the velocity of all vehicles except *M* is constant while the velocity of *M* will be increasing/decreasing until it becomes equal to the velocity of $F_d$ at $t=t_{long}$. Therefore the value of $a_M$ will be:

$$a_M = \begin{cases} \dfrac{v_{Fd} - v_M(0)}{t_{long}} & t \le t_{long} \\ 0 & Otherwise \end{cases} \tag{56}$$

Any accident between the two vehicles may occur while or after *M* passes the point C in figure 3 and before the velocity of the merging vehicle becomes equal to that of $F_d$. According to equation (17) the condition for collision avoidance between *M* and $F_d$ is:

$$Sr(t) = \left( Sr(0) + a_M \frac{t^2}{2} + (v_M(0) - v_{Fd})t \right) > 0 \qquad \forall t \in [t_C, t_{long}] \tag{57}$$

Substituting (56) in (57), we obtain:

$$Sr(t) = \left( Sr(0) + (v_M(0) - v_{Fd}) \times \left( t - \frac{t^2}{2t_{long}} \right) \right) > 0 \qquad \forall t \in [t_C, t_{long}] \tag{58}$$

The minimum initial longitudinal safety spacing, MSS(M,Fd) is:

$$MSS(M, Fd) = \underset{t}{Max}\left((v_{Fd} - v_M(0)) \times \left(t - \frac{t^2}{2t_{long}}\right)\right) \qquad \forall t \in [t_C, t_{long}] \qquad \text{(59)}$$

If the value of $v_{Fd} - v_M(0)$ is positive, the function in equation (59) is a concave function and has a maximum value at $t = t_{long}$. If the value is negative, the function is a convex function and the maximum value will be at the boundary $t = t_C$. Therefore equation (59) can be rewritten as follows:

$$MSS(M, Fd) = \begin{cases} (v_{Fd} - v_M(0)) \times t_{long}/2 & v_{Fd} - v_M(0) \geq 0 \\ (v_{Fd} - v_M(0)) \times t_C & Otherwise \end{cases} \qquad \text{(60)}$$

We use simulations to demonstrate the above results. In simulations, T was set equal to 50 sec, the adjustment time $t_{adj}$ equal to 0 sec, $t_{lat}$ equal to 5 sec, the lateral displacement H equal to 12 feet, and $t_{long}$ equal to 10 sec. The safe and unsafe regions are shown in figure 15.

Figure 15 shows the initial relative longitudinal distance versus the relative longitudinal velocity between M and $F_d$. For initial positive relative velocities, $v_{Fd}-v_M(0)$, the safety margin is a line with tangent equal to $t_{long}/2=5$. For negative relative velocities, it is a line with tangent equal to 2.95 which is the value of $t_C+t_{adj}$ in equation (12).
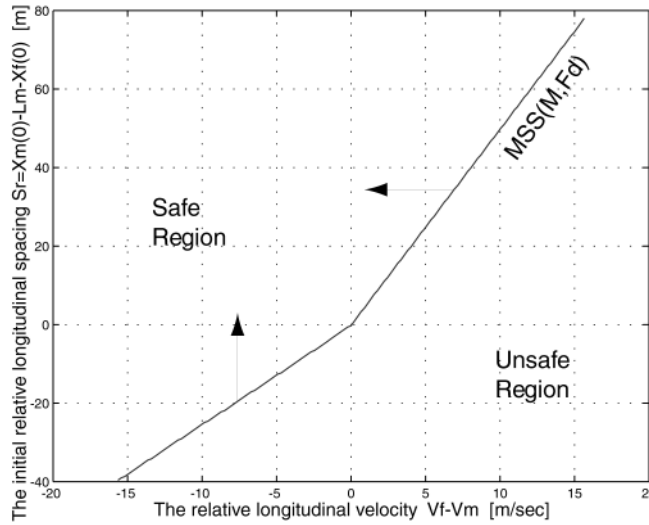


**Fig 15: The collision region between F$_d$ and M, with switching longitudinal acceleration.**

Comparison between figure 7 and 15 indicates that the safe region has been expanded here, too.

*II) $t_{adj}>0$:*
Similar to the case of vehicles $F_d$ and M, we define the state variables as following:
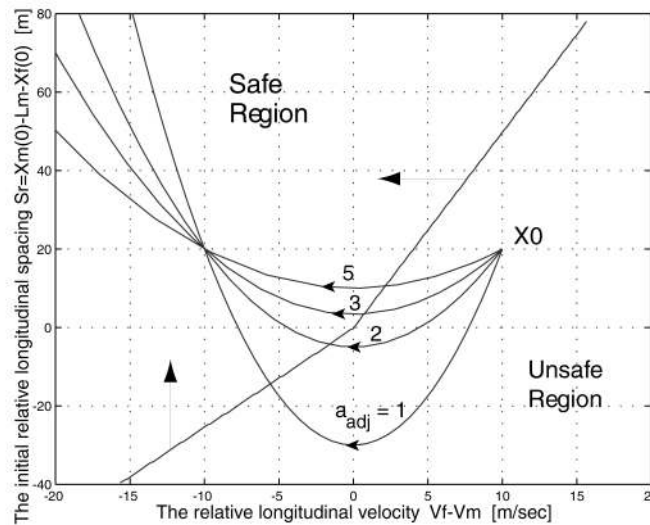
$$x_1 = x_M - x_{Fd} - l_M$$
$$x_2 = v_{Fd} - v_M$$

$(61)$

and we differentiate them with respect to time, to obtain,

$$\dot{x}_1 = \dot{x}_M - \dot{x}_{Fd} = v_M - v_{Fd} = -x_2$$
$$\dot{x}_2 = -\dot{v}_M = -a_{adj}$$

$(62)$

Therefore the isoclines are as following,

$$x_1 = \frac{x_2^2}{2a_{adj}} + c$$

$(63)$

Here $c$ is a constant that depends on the initial values, $x_1(0)$ and $x_2(0)$.



**Fig 16: Applying $a_{adj}$ to move from unsafe area into safe area.**

Figure 16 shows the isoclines that correspond to different values of $a_{adj}$ for the simulation in figure 15. The initial point has been chosen to be in the unsafe region. Applying positive $a_{adj}$, it is possible to move into the safe region in order to start the merging maneuver. The larger is the value of $a_{adj}$, the faster we move into the safe region. The minimum value of $t_{adj}$ for each $a_{adj}$ is determined by the point of intersection between the isocline that corresponds to the $a_{adj}$ and the safety margin.

The value of $v_M$ cannot exceed an upper bound, which is determined by the capabilities of the

vehicle and passenger's comfort. Similar to the case of vehicles $L_d$ and $M$, we may use the modified profile in figure 13, in the case where the acceleration profile of figure 10 requires velocities that exceed the aforementioned upper bound. In this case the state space equations for the interval $[t_a, t_{adj}]$ become,

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = 0$$

$$(64)$$

And the isoclines are given by:

$$x_2 = c = x_2(t_a)$$
$$x_1 = (-x_2(t_a))(t - t_a) + x_1(t_a) \qquad \forall t \in [t_a, t_{adj}]$$

$$(65)$$
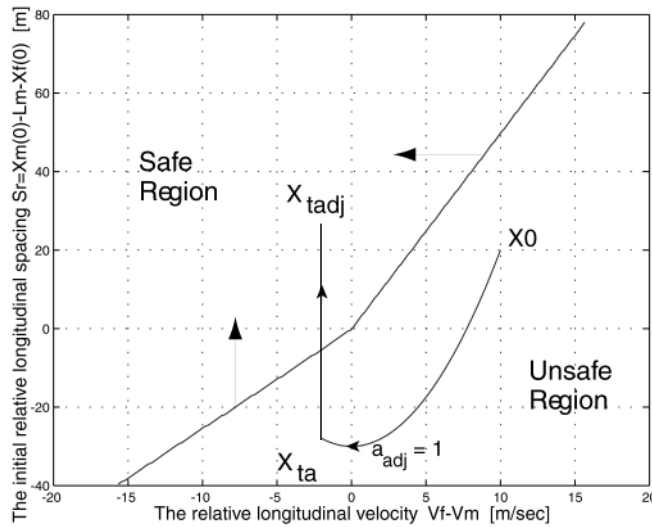


**Fig 17: Applying modified $a_{adj}$ to move from unsafe area into safe area.**

Again, we have to make sure that we are in the left half plane, $x_2 < 0$, and then we may set $a_{adj}=0$. Figure 17 shows the trajectory of moving from the unsafe region into the safe region.

### *Minimum Longitudinal Safety Spacing between M and $L_o$:*
### *I) $t_{adj}=0$:*
The velocity of all vehicles except $M$ is assumed to be constant while the velocity of $M$ will become equal to the velocity of the destination lane after the time-instant $t_{long}$ and remain constant thereafter, i.e. we assume $v_M(t)=v_{Ld}=v_{Fd}$ for $t$ in the interval $[t_{long},T]$. Therefore, the

value of $a_M$ will be:

$$a_M = \begin{cases} \dfrac{v_{Lo} - v_M(0)}{t_{long}} & t \le t_{long} \\ 0 & Otherwise \end{cases} \tag{66}$$

Any collision between the two vehicles may occur before the vehicle $M$ passes the point C in figure 4. Without loss of generality we also assume that $t_{long} > t_C$. According to equation (24) the condition for collision avoidance between $M$ and $L_o$ is:

$$Sr(t) = \left( Sr(0) - a_M \dfrac{t^2}{2} + (v_{Lo} - v_M(0))t \right) > 0 \qquad \forall t \in [0, t_C] \tag{67}$$

Substituting (66) in (67), we obtain:

$$Sr(t) = \left( Sr(0) + (v_{Lo} - v_M(0)) \times \left( t - \dfrac{t^2}{2t_{long}} \right) + (v_{Lo} - v_{Ld}) \times \dfrac{t^2}{2t_{long}} \right) > 0 \tag{68}$$

$$\forall t \in [0, t_C]$$

It should be noted that the last term in equation (68) is independent of $v_M$, but depends on the difference between the velocity of the vehicles in the originating and destination lanes. The minimum initial longitudinal safety spacing, MSS(Lo,M) will be:

$$MSS(Lo, M) = \underset{t}{Max}\left\{ \left( (v_M(0) - v_{Lo}) \times \left( t - \dfrac{t^2}{2t_{long}} \right) + (v_{Ld} - v_{Lo}) \times \dfrac{t^2}{2t_{long}} \right), 0 \right\} \tag{69}$$

$$\forall t \in [0, t_C]$$

In order to solve equation (69) analytically, we define the following two new variables:

$$\alpha = \dfrac{v_{Lo} - v_M(0)}{v_{Ld} - v_M(0)} \tag{70}$$

$$t_{max} = \alpha \times t_{long}$$

The above transformation is applicable provided that $v_{Ld} - v_M(0)$ is not zero. If $v_{Ld} - v_M(0)$ is zero, the value of $a_M$ is zero, too. In this case, we can apply the results for the case of constant velocity, (subsection 3.1).

Let us also define the following function,

$$S(t) = Max\left\{\left[(v_M(0) - v_{Lo}) \times \left(t - \frac{t^2}{2t_{long}}\right) + (v_{Ld} - v_{Lo}) \times \frac{t^2}{2t_{long}}\right], 0\right\}$$ (71)

From the definition of the function S(t) and equation (70), it can be easily seen that

$$MSS(Lo, M) = \underset{t}{Max}\{S(t)\} \qquad \forall t \in [0, t_C]$$ (72)

Using the above definitions we can analytically solve equation (69) as shown in Table 1.

**Table 1: Analytical values of MSS(L$_o$,M)**

| IF | & IF | & IF | THEN |
|---|---|---|---|
| $\alpha<0$ | – | $v_{Ld}<v_M(0)$ <br> $v_{Lo}>v_M(0)$ | MSS(Lo,M)= S(0) |
| " | – | $v_{Ld}>v_M(0)$ <br> $v_{Lo}<v_M(0)$ | MSS(Lo,M)=S(t$_C$) |
| $\alpha>0$ | $t_{max}>t_C$ | $v_{Ld}<v_M(0)$ <br> $v_{Lo}<v_M(0)$ | MSS(Lo,M)=S(t$_C$) |
| " | " | $v_{Ld}>v_M(0)$ <br> $v_{Lo}>v_M(0)$ | MSS(Lo,M)=S(0) |
| $\alpha>0$ | $t_{max}<t_C$ <br> $t_{max}>t_C/2$ | $v_{Ld}<v_M(0)$ <br> $v_{Lo}<v_M(0)$ | MSS(Lo,M)=S(t$_{max}$) |
| " | " | $v_{Ld}>v_M(0)$ <br> $v_{Lo}>v_M(0)$ | MSS(Lo,M)=S(0) |
| $\alpha>0$ | $t_{max}<t_C/2$ | $v_{Ld}<v_M(0)$ <br> $v_{Lo}<v_M(0)$ | MSS(Lo,M)=S(t$_{max}$) |
| " | " | $v_{Ld}>v_M(0)$ <br> $v_{Lo}>v_M(0)$ | MSS(Lo,M)=S(t$_C$) |

We performed simulations in order to verify our theoretical results. In the simulation, T was set equal to 50 sec, $t_{adj}$ equal to 0 sec, $t_{lat}$ equal to 5 sec, the lateral displacement H equal to 12 feet, and $t_{long}$ equal to 10 sec. The relative speed between the vehicles $L_d$ and $L_o$ was set equal to 20 and -20 miles/hour. The safe and unsafe regions are shown in figure 18.

Figure 18 shows the initial relative longitudinal spacing versus the relative longitudinal velocity between $M$ and $L_o$. It should be noted that the less is the relative velocity between the vehicles $L_d$ and $L_o$, the larger the safe region becomes.

Comparison between figures 8 and 18 indicates that the tangent of the safety margins remain almost the same, while there is a "horizontal shift" on the safety margins.
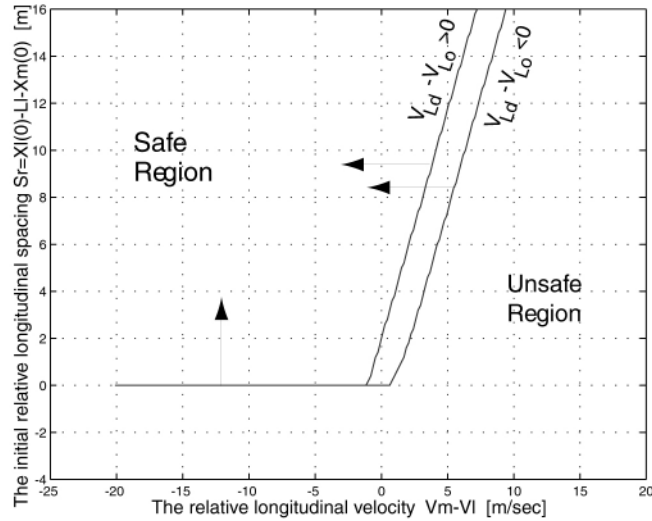


**Fig 18: The collision region between M and Ld, with switching longitudinal acceleration.**

## II) $t_{adj}>0$:
Similar to the previous cases we define the variables,

$$x_1 = x_{Lo} - x_M - l_{L1}$$
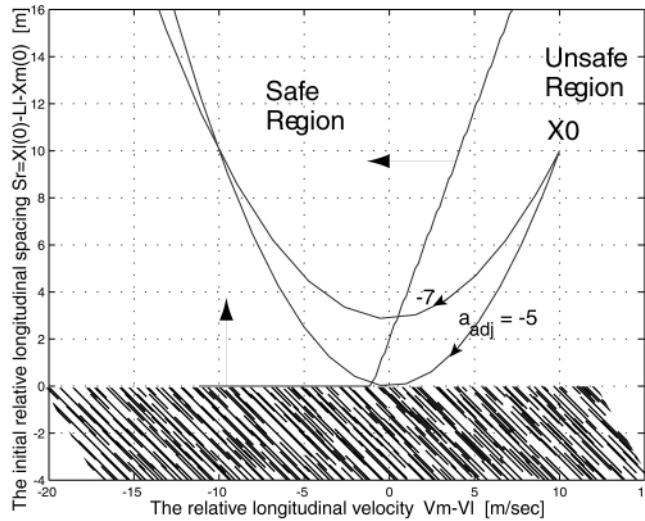$$x_2 = v_M - v_{Lo}$$

(73)

The state space equations are as follows,

$$\dot{x}_1 = \dot{x}_{Lo} - \dot{x}_M = v_{Lo} - v_M = -x_2$$
$$\dot{x}_2 = \dot{v}_M = a_{adj}$$

(74)

The isoclines are found to be,

$$x_1 = -\frac{x_2^2}{2a_{adj}} + c$$

(75)

where constant $c$ is a constant that depends on the initial values, $x_1(0)$ and $x_2(0)$.



**Fig 19: Applying $a_{adj}$ to move from the unsafe region into the safe region.**

Figure 19 shows the isoclines corresponding to various values of $a_{adj}$. The initial point has been chosen to be in the unsafe region. By applying $a_{adj}$, it is possible to move into the safe region in order to start the merging maneuver. It should be noted that only two values of $a_{adj}$, i.e. -5, -7, are acceptable here. The other values result in $Sr(t)<0$, for some $t$ in the interval $[0,t_{adj}]$, which cause collision between $M$ and $L_d$ (the hatches area in figure 19 corresponds to the negative values of $Sr(t)$ which is *infeasible region*). The minimum value of $t_{adj}$ for each $a_{adj}$ is chosen to be the time-instant the corresponding isocline curve intersects with the safety region.

Unfortunately in this case we can not easily apply the acceleration profile of figure 13. This is largely due to the existence of the infeasible region, and the small area bounded between the *MSS*s curve and the infeasible area. It should be noted that in the previous cases we use this area, where the relative longitudinal velocity between the merging vehicle $M$ and the vehicles in the destination lane is negative, to move into before adjusting the longitudinal acceleration zero. In the case of positive relative velocity between $L_d$ and $L_o$, i.e. $v_{Ld}-v_{Lo}>0$, it is unrealistic to exploit the modified switching acceleration profile, see figure 19.

*Minimum Longitudinal Safety Spacing between M and $F_o$:*
*I) $t_{adj}=0$:*
The velocity of all vehicles except $M$ is assumed to be constant while the velocity of $M$ will become equal to the velocity of the destination line at the time-instant $t_{long}$ and remain constant thereafter, i.e. we assume $v_M(t)=v_{Ld}=v_{Fd}$ for $t$ in the interval $[t_{long},T]$. Therefore the value of $a_M$

will be the same as that in equation (66).

Any collision between the two vehicles may occur before $M$ passes the point C in figure 5. Without loss of generality we also assume that $t_{long} > t_C$. According to equation (31) the condition for collision avoidance between $M$ and $F_o$ is:

$$Sr(t) = \left(Sr(0) + a_M\frac{t^2}{2} + (v_M(0) - v_{Fo})t\right) > 0 \qquad \forall t \in [0, t_C] \tag{76}$$

Substituting (66) in (76), we obtain:

$$Sr(t) = \left(Sr(0) + (v_M(0) - v_{Fo}) \times \left(t - \frac{t^2}{2t_{long}}\right) + (v_{Ld} - v_{Fo}) \times \frac{t^2}{2t_{long}}\right) > 0 \tag{77}$$
$$\forall t \in [0, t_C]$$

It should be noted that the last term in equation (77) is dependent on the difference between the velocity of originating and destination lane. The minimum initial longitudinal safety spacing, MSS(M,Fo) will be:

$$MSS(M, Fo) = \underset{t}{Max}\left\{\left((v_{Fo} - v_M(0)) \times \left(t - \frac{t^2}{2t_{long}}\right) + (v_{Fo} - v_{Ld}) \times \frac{t^2}{2t_{long}}\right), 0\right\} \tag{78}$$
$$\forall t \in [0, t_C]$$

In order to solve Equation (78) analytically, we define the following two new variables:

$$\alpha = \frac{v_M(0) - v_{Fo}}{v_M(0) - v_{Ld}} \tag{79}$$
$$t_{max} = \alpha \times t_{long}$$

It the relative velocity between two vehicles $M$ and $L_d$, $v_M(0)-v_{Ld}$, happens to be zero in equation (79), the value of $a_M$ will be zero, too. In this case, we can apply the results of subsection 3.2 for the case of constant velocity.

By defining the new variable S(t) as following:

$$S(t) = Max\left\{\left((v_{Fo} - v_M(0)) \times \left(t - \frac{t^2}{2t_{long}}\right) + (v_{Fo} - v_{Ld}) \times \frac{t^2}{2t_{long}}\right), 0\right\} \tag{80}$$

we can see that the solution of (79) is as summarized in Table 2.

**Table 2: Analytical values of MSS(M,F$_o$)**

| IF | & IF | & IF | THEN |
|---|---|---|---|
| $\alpha<0$ | – | $v_{Ld}<v_M(0)$ <br> $v_{Fo}>v_M(0)$ | MSS(M,Fo)=S(t$_C$) |
| " | – | $v_{Ld}>v_M(0)$ <br> $v_{Fo}<v_M(0)$ | MSS(M,Fo)=S(0) |
| $\alpha>0$ | t$_{max}$>t$_C$ | $v_{Ld}<v_M(0)$ <br> $v_{Fo}<v_M(0)$ | MSS(M,Fo)=S(0) |
| " | " | $v_{Ld}>v_M(0)$ <br> $v_{Fo}>v_M(0)$ | MSS(M,Fo)=S(t$_C$) |
| $\alpha>0$ | t$_{max}$<t$_C$ <br> t$_{max}$>t$_C$/2 | $v_{Ld}<v_M(0)$ <br> $v_{Fo}<v_M(0)$ | MSS(M,Fo)=S(0) |
| " | " | $v_{Ld}>v_M(0)$ <br> $v_{Fo}>v_M(0)$ | MSS(M,Fo)=S(t$_{max}$) |
| $\alpha>0$ | t$_{max}$<t$_C$/2 | $v_{Ld}<v_M(0)$ <br> $v_{Fo}<v_M(0)$ | MSS(M,Fo)=S(t$_C$) |
| " | " | $v_{Ld}>v_M(0)$ <br> $v_{Fo}>v_M(0)$ | MSS(M,Fo)=S(t$_{max}$) |

We use simulations to demonstrate the above results. In the simulations, the time T was set equal to 50 sec, t$_{adj}$ equal to 0 sec, t$_{lat}$ equal to 5 sec, the lateral displacement H equal to 12 feet, and t$_{long}$ to 10 sec. The relative velocity between the vehicles $L_d$ and $F_o$ was set equal to 20 and -20 miles/hour. The safe and unsafe regions are shown in figure 20.
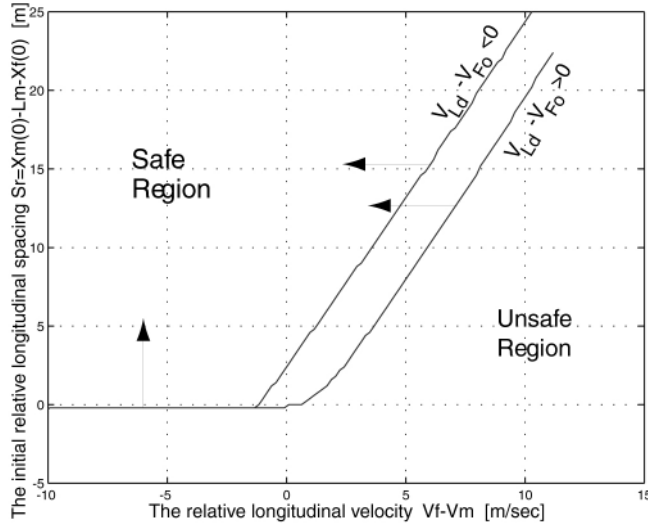
**Fig 20: The collision region between Fo and M, with switching acceleration.**

Figure 20 shows the initial relative longitudinal spacing versus the relative longitudinal velocity between $M$ and $F_o$. Notice that the larger is the relative velocity between the vehicles $L_d$ and $F_o$, the larger the safe regions become.

Comparison between figures 9 and 20 indicates that the tangent of the safety margins remain almost the same, while similar to subsection 3.3.c we observe a "horizontal shift" on the safety margins.

*II) $t_{adj} > 0$:*
Here, the state space variable can be defined as follows:
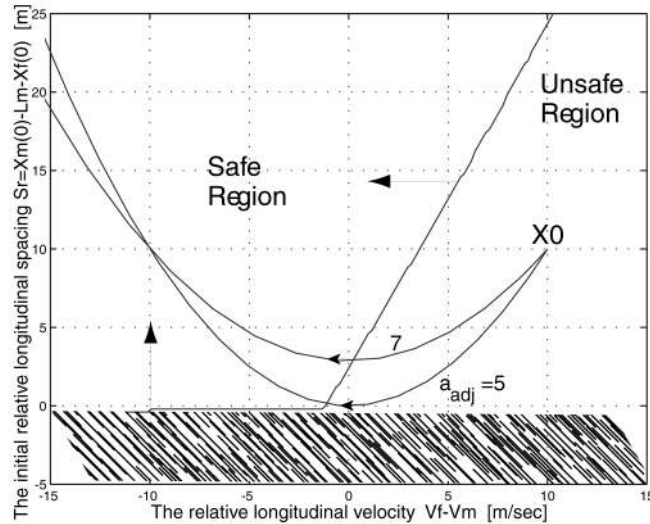
$$x_1 = x_M - x_{Fo} - l_M$$
$$x_2 = v_{Fo} - v_M \tag{81}$$

The state space equations are as follows,

$$\dot{x}_1 = \dot{x}_M - \dot{x}_{Fo} = v_M - v_{Fo} = -x_2$$
$$\dot{x}_2 = -\dot{v}_M = -a_{adj} \tag{82}$$

The isoclines corresponding to $a_{adj}$ are found to be,

$$x_1 = \frac{x_2^2}{2a_{adj}} + c \tag{83}$$

where the constant *c* depends on the initial values, $x_1(0)$ and $x_2(0)$.



**Fig 21: Applying $a_{adj}$ to move from the unsafe region into the safe area.**

Figure 21 shows the isoclines corresponding to various values $a_{adj}$. The initial point has been chosen to be in the unsafe region. By applying $a_{adj}$, it is possible to move into the safe region in order to start merging maneuver. It should be noted that only two values of $a_{adj}$, i.e. 5 and 7, are acceptable here. The other values result in $Sr(t)<0$ that is collision between *M* and $F_o$. The minimum acceptable value of $t_{adj}$ for each $a_{adj}$ is determined by the point of intersection between the corresponding isocline curve and the safety margin.

## 4. Conclusions:

In this paper, we analyzed the kinematics of the vehicles involved in a lane changing/merging maneuver, and studied the conditions under which lane changing/merging crashes can be avoided. That is, given a particular lane change/merge scenario, we calculated the minimum longitudinal spacing which the vehicles involved should initially have so that no collision, of any type, takes place during the maneuver.

Simulations of a number of examples were presented to demonstrate the results. We assumed in our simulations that all vehicles, involved in the lane changing/merging maneuver, were initially at steady state, i.e. their velocities were constant prior to the maneuver. Except for the merging vehicle, all vehicles keep their steady state velocity during the merging maneuver. Three different longitudinal acceleration scenarios - constant longitudinal velocity, switching

longitudinal acceleration, and modified switching longitudinal acceleration - were applied to the merging vehicle in order to determine the safe and unsafe region as well as the MSS between the merging vehicle and its surrounding vehicles. We observed that the switching scenario and the modified switching scenario expanded the safe region for lane changing. Furthermore, by considering the longitudinal adjustment acceleration for the merging vehicle, we studied the possibility of moving from the unsafe region into the safe region. Our results together with appropriate sensors and equipment on the board of vehicles could be used to assess the safety of lane changing maneuvers and provide warnings or take evasive actions to avoid collision.

## 5. References:

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[2] S.E. Shladover, et. al, "Automatic vehicle control development in the PATH program," *IEEE Transactions on Vehicular Technology*, Vol 40, PP. 114-130, 1991

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[4] A. Kanaris, P. Ioannou, F. Ho, " Spacing and capacity evaluations for different AHS concepts," Technical Report, University of Southern California, 1995.

[5] J.L. Bascunana, "Analysis of lane change collision avoidance," *Systems and Issues in ITS,* (SP-1106), 1997.

[6] A. Kanaris, E.B. Kosmatopoulos, and P.A. Ioannou, "Strategies and spacing requirements for lane changing and merging in automated highway systems," Technical Report, University of Southern California, 1997.

[7] P.A. Cook, *Nonlinear Dynamical Systems*, Prentice-Hall,1986

[8] J. D. Chovan, et al " Examination of Lane Change Crashes and Potential IVHS Countermeasures," National Highway Traffic Safety Administration, 1994.

# Appendixes:

## Appendix A:

### a. Calculation of $t_C+t_{adj}$:

Here we provide with an algorithm for calculating $t_C+t_{adj}$ in equation (4).

    1- **Set**           time t=0 ;
                     initial lateral position of the merging vehicle $y_{lat}(t)=0$;
    2- **Choose**    the sampling time dt;
                     adjustment and lateral time $t_{adj}$ and $t_{lat}$;
                     total lateral displacement H;
                     initial lateral position of the leading vehicle $y_{Ld}$;
                     width of the leading vehicle $w_{Ld}$;
                     the time T;
    3- **Set**           $t_C+t_{adj}$=Large Number;
    4- **While**     t<T            **DO:**
            **If**     $y_{lat}(t)<(y_{L1}-w_{L1})$**Then**
                    t=t+dt;
                    Calculate $y_{lat}(t)$ based on equation (3);
          **Else**    $t_C+t_{adj}$=t;
                    Break;
          **End**;
        **End;**

### b. Calculation of $t_C+t_{adj}$:

The algorithm for calculating $t_C+t_{adj}$ in equation (12) is as follows.

    1- **Set**           time t=0;
                     initial lateral position and velocity of the merging vehicle $y_{lat}(t)=0$
                     $v_{lat}(t)=0$;
    2- **Choose**    the sampling time dt;
                     adjustment and lateral time $t_{adj}$ and $t_{lat}$;
                     total lateral displacement H;
                     initial lateral position of the following vehicle $y_{Fd}$;
                     width of the following vehicle $w_{Fd}$;
                     the time under consideration T;
                     the longitudinal acceleration $a_M(t)$ and initial velocity $v_M(0)$;
    3- **Set**           $t_C+t_{adj}$= Large Number;
    4- **While**     t<T            **DO:**
            **If**     (left hand side of equation (13)) < $(y_{Fd}-w_{Fd})$**Then**

Calculate $v_M(t)$ as follows:

$$v_M(t + dt) = v_M(t) + \int_t^{(t+dt)} a_M(\lambda)d\lambda;$$

t=t+dt;
Calculate $y_{lat}(t)$ and $v_{lat}(t)$ based on equations (3) and (2), respectively;
**Else**    $t_C+t_{adj}=t$;
        Break;
**End**;
**End**;

## c. Calculation of $t_C+t_{adj}$:

The algorithm for calculating $t_C+t_{adj}$ in equation (19) is as follows.

1- **Set**        time t=0;
            initial lateral position and velocity of the merging vehicle $y_{lat}(t)=0$
            $v_{lat}(t)=0$;
2- **Choose**    the sampling time dt;
            adjustment and lateral time $t_{adj}$ and $t_{lat}$;
            total lateral displacement H;
            initial lateral position of the following vehicle $y_{Lo}$;
            width of the merging vehicle $w_M$;
            the time under consideration T;
            the longitudinal acceleration $a_M(t)$ and initial velocity $v_M(0)$;
3- **Set**        $t_C+t_{adj}$=Large Number;
4- **While**     t<T          **DO:**
      **If**    (left hand side of equation (19)) < $y_{Lo}$**Then**
         Calculate $v_M(t)$ as follows:

$$v_M(t + dt) = v_M(t) + \int_t^{(t+dt)} a_M(\lambda)d\lambda;$$

         t=t+dt;
         Calculate $y_{lat}(t)$ and $v_{lat}(t)$ based on equations (3) and (2), respectively;
      **Else**    $t_C+t_{adj}=t$;
            Break;
      **End**;
   **End**;

## d. Calculation of $t_C+t_{adj}$:

The algorithm for calculating $t_C+t_{adj}$ in equation (26) is as follows.

1- **Set**        time t=0;

initial lateral position and velocity of the merging vehicle $y_{lat}(t)=0$

$v_{lat}(t)=0$;

2- **Choose**   the sampling time dt;

adjustment and lateral time $t_{adj}$ and $t_{lat}$;

total lateral displacement H;

initial lateral position of the following vehicle $y_{Fo}$;

width and length of the merging vehicle $w_M$, $l_M$;

the time under consideration T;

the longitudinal acceleration $a_M(t)$ and initial velocity $v_M(0)$;

3- **Set**   $t_C+t_{adj}$=Large Number;

4- **While**   t<T   **DO:**

   **If**   (left hand side of equation (27)) < $y_{Fo}$**Then**

   Calculate $v_M(t)$ as follows:

$$v_M(t + dt) \ = \ v_M(t) + \int_t^{(t + dt)} a_M(\lambda)d\lambda;$$

   t=t+dt;

   Calculate $y_{lat}(t)$ and $v_{lat}(t)$ based on equations (3) and (2), respectively;

   **Else**   $t_C+t_{adj}$=t;

   Break;

   **End**;

**End;**


## Appendix B: First order approximation:

Figure B-1 shows the first order approximation (tangent to the vehicle's path) in order to calculate the lateral position of the other corners of the merging vehicle.
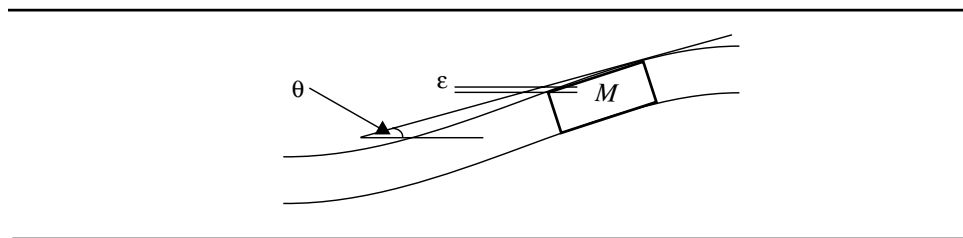


**Fig A-1: First order approximation (tangent to the vehicle's path).**

Where   $\theta$ is the angle between tangent of the path at the point $y_{lat}(t)$,

$\varepsilon$ is the error of upper-left corner approximation of the vehicle.

The error of approximation decreases as the merging path becomes smoother, i.e. $t_{lat}$ increases.