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A self-consistent collisionless distribution function for the relativistic analogue of the force-free Harris sheet is presented. This distribution function is the relativistic generalization of the distribution function for the non-relativistic collisionless force-free Harris sheet recently found by Harrison and Neukirch [Phys. Rev. Lett. 102, 135003 (2009)] as it has the same dependence on the particle energy and canonical momenta. We present a detailed calculation which shows that the proposed distribution function generates the required current density profile (and thus magnetic field profile) in a frame of reference in which the electric potential vanishes identically. The connection between the parameters of the distribution function and the macroscopic parameters such as the current sheet thickness is discussed.

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I. INTRODUCTION

Force-free plasma equilibria, i.e. plasma equilibria for which the current density is aligned with the magnetic field lines, are of great importance in both astrophysical and laboratory plasmas, in particular for modelling low-β systems. Within the framework of magnetohydrodynamics (MHD) a large number of analytical force-free equilibria are known.[1–3]

For collisionless plasmas, with equilibria being solutions of the time-independent Vlasov-Maxwell (VM) equations, the situation is completely different. So far only a small number of one-dimensional (1D) collisionless force-free plasma equilibria is known, of which most belong to the class of linear force-free fields[4–8]. So far, analytical self-consistent distribution functions have been found only for one example of non-linear force-free fields, the force-free Harris sheet[9–12]. Finding self-consistent force-free collisionless equilibria is difficult, because one is dealing with an inverse problem, i.e. find a solution of the Vlasov equation for a given magnetic field and electric current system (for a discussion of the problem see Ref. 10).

The equilibria mentioned in the previous paragraph have all been found for the non-relativistic regime. It is the aim of this paper to investigate whether it is possible to generalize the distribution function found for the force-free Harris sheet into the relativistic regime. We define the relativistic force-free condition as $J^\mu F_{\mu\nu} = 0$, where J^μ is the four-current density and $F_{\mu\nu}$ the electromagnetic four-tensor. If a frame of reference exists in which the electric field vanishes the relativistic force-free condition is identical with the non-relativistic force-free condition $\mathbf{J} \times \mathbf{B} = \mathbf{0}$ in this frame of reference.

Collisionless equilibria of the type we are trying to calculate in this paper could be of importance for investigations of physical processes like instabilities or magnetic reconnection in relativistic plasmas (see e.g. Refs 13–21). In order to find the relativistic generalization of the collisionless distribution function for the force-free Harris sheet, we shall use the relativistic version[22–24] of the distribution function of the normal Harris sheet[25] as a guide, together with the non-relativistic distribution function for the force-free Harris sheet.[9,11]

The paper is structured as follows: section II summarizes the mathematical framework of the non-relativistic force-free Harris sheet[9]. Section III then uses this as a basis for determining the distribution function for the relativistic force-free Harris sheet, and Section IV we present a discussion and our conclusions.

II. THE NON-RELATIVISTIC FORCE-FREE HARRIS SHEET

In a 1D VM equilibria with translational symmetry, it is assumed that all plasma variables depend only on one spatial coordinate, here taken to be z and that the magnetic flux density has components B_x and B_y. The magnetic flux density components can be written in terms of the vector potential $\mathbf{A} = (A_x, A_y, A_z)$ where,

$$B_x = -\frac{\mathrm{d}A_y}{\mathrm{d}z}, \tag{1}$$

$$B_y = \frac{\mathrm{d}A_x}{\mathrm{d}z}, \tag{2}$$

and the electric field is the gradient of the electric potential ϕ,

$$\mathbf{E} = -\nabla\phi = -\frac{\mathrm{d}\phi}{\mathrm{d}z}\hat{\mathbf{z}}. \tag{3}$$

These relations automatically satisfy Faraday's law $\nabla \times \mathbf{E} = \mathbf{0}$ and Gauss' law for the magnetic flux density $\nabla \cdot \mathbf{B} = 0$. Due to the symmetries of the system (time and spatial independence of x and y) the three obvious constants of motion for each particle species are the Hamiltonian or particle energy for each species s,

$$H_s = \frac{1}{2}m_s(v_x^2 + v_y^2 + v_z^2) + q_s\phi, \tag{4}$$

and the canonical momentum in the x and y directions respectfully,

$$p_{xs} = m_s v_x + q_s A_x, \tag{5}$$

$$p_{ys} = m_s v_y + q_s A_y, \tag{6}$$

where m_s and q_s are the mass and charge of each species. Here, we consider a plasma composed of two species of equal and opposite charge but of differing mass, that is, electrons and protons. All positive functions f_s satisfying the appropriate conditions for the existence of the velocity moments and depending only on the constants of motion, $f_s = f_s(H_s, p_{xs}, p_{ys})$ solve the steady-state Vlasov equation. For a quasi-neutral plasma in force balance, Ampère's law can be written as

$$\frac{\mathrm{d}^2 A_x}{\mathrm{d}z^2} = -\mu_0\frac{\partial P_{zz}}{\partial A_x}, \tag{7}$$

$$\frac{\mathrm{d}^2 A_y}{\mathrm{d}z^2} = -\mu_0\frac{\partial P_{zz}}{\partial A_y}, \tag{8}$$

where $P_{zz}(A_x, A_y)$ is the zz-component of the plasma pressure tensor,

$$P_{zz} = \sum_s \int_{-\infty}^{\infty} m_s v_z^2 f_s \mathrm{d}^3 v. \tag{9}$$

The magnetic flux density components of the force-free Harris sheet are given by

$$B_x = B_0 \tanh(z/\lambda), \tag{10}$$

$$B_y = \frac{B_0}{\cosh(z/\lambda)}. \tag{11}$$

The $x-$component of the magnetic flux density has the same spatial structure as the Harris sheet, but whereas the Harris sheet is kept in force balance by pressure gradients, the force-free Harris sheet maintains force balance via a magnetic shear $y-$component.

Assuming that the pressure takes the form $P_{zz}(A_x, A_y) = P_1(A_x) + P_2(A_y)$, equations (7) and (8) combined with the force-free condition ($B_x^2 + B_y^2 = \text{const.}$) gives the condition for force balance:

$$\left(\frac{\mathrm{d}A_x}{\mathrm{d}z}\right)^2 + 2\mu_0 P_1(A_x) = 2\mu_0 P_{01}, \tag{12}$$

$$\left(\frac{\mathrm{d}A_y}{\mathrm{d}z}\right)^2 + 2\mu_0 P_2(A_y) = 2\mu_0 P_{02}, \tag{13}$$

where P_{01} and P_{02} are constants. Solving these equations for $P_1(A_x)$ and $P_2(A_y)$ the plasma pressure for the force-free Harris sheet is

$$P_{zz} = \frac{B_0^2}{2\mu_0}\left[\frac{1}{2}\cos\left(\frac{2A_x}{B_0\lambda}\right) + \exp\left(\frac{2A_y}{B_0\lambda}\right)\right] + P_{03}, \tag{14}$$

where P_{03} is a constant. Since the pressure is the sum of two independent functions that are a function of A_x and A_y respectively, the distribution function is assumed to be of the form

$$f_s = \exp(-\beta_s H_s)[g_{1s}(p_{xs}) + g_{2s}(p_{ys})], \tag{15}$$

where the reciprocal thermal energy of species s is $\beta_s = (k_B T_s)^{-1}$. Applying Channell's fourier transform method[5], echoed in Harrison and Neukirch[9], the plasma pressure integral can be solved for the distribution function f_s. Therefore, a collisionless distribution function for the force-free Harris sheet is

$$\begin{aligned} f_s = \frac{n_{0,s}}{v_{th,s}^3} &\exp(-\beta_s H_s)[a_s \cos(\beta_s u_{xs} p_{xs}) \\ &+ \exp(\beta_s u_{ys} p_{ys}) + b_s], \end{aligned} \tag{16}$$

where $v_{th,s} = (m_s\beta_s)^{-1/2}$ is the thermal speed of species s and u_{xs}, u_{ys}, a_s and b_s are constants.

III. RELATIVISTIC FORCE-FREE HARRIS SHEET

If the thermal energy of the plasma, k_BT, approaches or exceeds the rest energy, mc^2, a non-relativistic treatment is no longer sufficient to describe the system. In a relativistic framework, P^ν are the components of the canonical momentum four-vector $\mathbf{P} = \mathbf{p} + q_s\mathbf{A}$, where: $\mathbf{p} = m\mathbf{w} = (E/c, p^1, p^2, p^3)$ is the four-momentum; $\mathbf{w} = (w^0, w^1, w^2, w^3)$ is the four-velocity; and $\mathbf{A} = (\phi/c, A^1, A^2, A^3)$ is the four-potential. The Hamiltonian or particle energy for each species s, H_s corresponds to the speed of light times the zeroth component of the canonical momentum four-vector $P^0c = E + q_s\phi$. The subsequent canonical components are now functions of the four-velocity, $P^i = m_sw^i + q_sA^i$. A relativistic analogue of the force-free Harris sheet distribution function can be written as

$$\begin{aligned} f_s = f_{s0} \exp{(-\beta_s P^0 c)}[a_s \cos{(\beta_s u_{xs} P^1)} \\ + \exp{(\beta_s u_{ys} P^2)} + b_s], \end{aligned} \tag{17}$$

where $f_{s0} = n_{0s}m_s\beta_s/(4\pi c)$ and n_{0s} is the mean particle density. Details of the normalisation calculation are given in appendix A. Using the relation $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ we can recast the distribution function as

$$\begin{aligned} f_s &= f_{1s}(w^0, w^2) + f_{2s}(w^0, w^1) + f_{3s}(w^0, w^1) + f_{4s}(w^0) \\ &= f_{s0}[c_{1s} \exp{(-c\beta_s m_s(w^0 - u_{ys}w^2/c))} \\ &+ c_{2s} \exp{(-c\beta_s m_s(w^0 + iu_{xs}w^1/c))} \\ &+ c_{3s} \exp{(-c\beta_s m_s(w^0 - iu_{xs}w^1/c))} \\ &+ c_{4s} \exp{(-c\beta_s m_s w^0)}], \end{aligned} \tag{18}$$

where

$$c_{1s} = \exp{(-\beta_s q_s(\phi - u_{ys}A^2))}, \tag{19}$$

$$c_{2s} = \frac{a_s}{2} \exp{(-\beta_s q_s(\phi + iu_{xs}A^1))}, \tag{20}$$

$$c_{3s} = \frac{a_s}{2} \exp{(-\beta_s q_s(\phi - iu_{xs}A^1))}, \tag{21}$$

$$c_{4s} = b_s \exp{(-\beta_s q_s \phi)}. \tag{22}$$

In the non-relativistic scenario the $zz-$component of the plasma pressure tensor is the key plasma parameter characterising the system; in the relativistic case it is the energy-momentum tensor that is key. The energy-momentum tensor has components $T^{\alpha\beta} =$

$(T^{\alpha\beta})_{plasma} + (T^{\alpha\beta})_{em}$, incorporating contributions to the energy and momentum from the kinetic and electromagnetic behaviour of the plasma. In this context, a relativistic force-free system corresponds to $T^{\alpha\beta}{}_{,\beta} = -J^{\beta}F_{\alpha\beta} = 0$, where $F_{\alpha\beta}$ are the components of the Faraday tensor and $\mathbf{J} = (c\rho, J^1, J^2, J^3)$ is the four-current. In a frame where $\phi = 0$, $\mathbf{B} = (B^1, B^2, 0)$ and $\mathbf{J} = (c\rho, J^1, J^2, 0)$, the only non-zero component of the Lorentz force is $J_1 B^2 - J_2 B^1 = 0$.

Neglecting viscosity and heat conduction the 33-component of $(T^{\alpha\beta})_{plasma}$, which we will refer to as the plasma pressure P, is given by

$$P = c\sum_s m_s \int \mathrm{d}^4 w\ (w^3)^2 f_s(\mathbf{w})\ \delta(w_\nu w^\nu - c^2) \tag{23}$$

$$= c\sum_s (P_{1s} + P_{2s} + P_{3s} + P_{4s}). \tag{24}$$

Note that $(T^{\alpha\beta})_{plasma} = p^\alpha N^\beta$, where $\mathbf{N} = n\mathbf{u}$ is the number flux four-vector and n is the number density. To solve the first pressure integral P_{1s} we will perform a coordinate transformation defined by the transformation matrix

$$[\Lambda^{\bar\beta}_{\alpha}] = \begin{bmatrix} \gamma_{1s} & 0 & -u_{ys}\gamma_{1s}/c & 0 \\ 0 & 1 & 0 & 0 \\ -u_{ys}\gamma_{1s}/c & 0 & \gamma_{1s} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

where $\gamma_{1s} = (1 - u_{ys}^2/c^2)^{-1/2}$, $w^{\bar\beta} = \Lambda^{\bar\beta}_{\alpha} w^\alpha$ and $w^\alpha = \Lambda^{\alpha}_{\bar\beta} w^{\bar\beta}$ such that $\Lambda^{\bar\beta}_{\alpha}\Lambda^{\alpha}_{\bar\beta} = 1$. Note that this coordinate transformation, and following transformations, are used as a means to evaluate the integral and do not physically correspond to a Lorentz boost. The Jacobian of the system is simply $J = 1$ therefore, $\mathrm{d}w^{\bar\beta}$=$\mathrm{d}w^\alpha$ and $f_{1s}(w^0, w^2) = f_{1s}(w^{\bar 0})$. Making the change of variables yields

$$P_{1s} = m_s \int \mathrm{d}^4\bar{w}\ (w^{\bar 3})^2 f_{1s}(w^{\bar 0})\ \delta(w_\nu w^\nu - c^2), \tag{26}$$

which can be written as

$$P_{1s} = m_s \int \mathrm{d}^3\bar{w}\ \frac{(w^{\bar 3})^2}{w^{\bar 0}} f_{1s}(w^{\bar 0}), \tag{27}$$

where

$$f_{1s} = f_{s0} c_{1s} \exp\left(-c\beta_s m_s w^{\bar 0}/\gamma_{1s}\right). \tag{28}$$

A convenient way of expressing $w^{\bar 0}$ can be obtained using the inner product of the four-velocity with itself, $\mathbf{w}\cdot\mathbf{w} = g_{\mu\nu} w^\nu w_\mu$ which gives $w^{\bar 0} = \sqrt{c^2 + (w)^2}$. Note the metric used here is defined as $g_{\mu\nu}$ =diag$(+1, -1, -1, -1)$. The integral can now be easily evaluated by changing to spherical coordinates, making use of the Jüttner transformation ($w/c = \sinh x$) and using the known integral[26]

$$\int_0^\infty \sinh^{2\nu} x \ \exp\left(-\beta \cosh x\right) \mathrm{d}x \\ = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^\nu \Gamma(\nu + \tfrac{1}{2}) K_\nu(\beta), \tag{29}$$

where K is the modified Bessel Function of the second kind. Therefore

$$P_{1s} = \frac{k_{1s} n_{0s}}{\beta_s c} \exp\left(-\beta_s q_s (\phi - u_{ys} A^2)\right), \tag{30}$$

where $k_{1s} = \gamma_{1s}^2 K_2(\Lambda_{1s})$ and $\Lambda_{is} = m_s c^2 \beta_s / \gamma_{is}$. Similarly P_{2s} and P_{3s} can be evaluated using the following coordinate transformation

$$[\Delta_\alpha^{\bar{\beta}}] = \begin{bmatrix} \gamma_{2s} & \pm i u_{xs} \gamma_{2s}/c & 0 & 0 \\ \pm i u_{xs} \gamma_{2s}/c & \gamma_{2s} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

where $\gamma_{2s} = (1 + u_{xs}^2/c^2)^{-1/2}$, $w^{\bar{\beta}} = \Delta_\alpha^{\bar{\beta}} w^\alpha$ and $w^\alpha = \Delta_{\bar{\beta}}^{\alpha} w^{\bar{\beta}}$ such that $\Delta_\alpha^{\bar{\beta}} \Delta_{\bar{\beta}}^{\alpha} = 1$. The Jacobian is again $J = 1$, with $f_{2s}(w^0, w^1) = f_{2s}(w^{\bar{0}})$ and $f_{3s}(w^0, w^1) = f_{3s}(w^{\bar{0}})$. This yields

$$P_{2s} = \frac{k_{2s} n_{0s}}{2\beta_s c} \exp\left(-\beta_s q_s (\phi + i u_{xs} A^1)\right), \tag{32}$$

$$P_{3s} = \frac{k_{2s} n_{0s}}{2\beta_s c} \exp\left(-\beta_s q_s (\phi - i u_{xs} A^1)\right), \tag{33}$$

where $k_{2s} = a_s \gamma_{2s}^2 K_2(\Lambda_{2s})$. The final pressure integral P_{4s} can be trivially evaluated, without any need for a coordinate transformation, using Eq. (29) yielding

$$P_{4s} = \frac{k_{3s} n_{0s}}{\beta_s c} \exp\left(-\beta_s q_s \phi\right), \tag{34}$$

where $k_{3s} = b_s K_2(\Lambda_{3s})$ and $\Lambda_{3s} = m_s c^2 \beta_s$. The total plasma pressure is then

$$P = \sum_s \frac{n_{0s} k_{1s}}{\beta_s} \exp\left(-\beta_s q_s \phi\right) [\exp\left(\beta_s q_s u_{ys} A^2\right) \\ + (k_{2s}/k_{1s}) \cos\left(\beta_s q_s u_{xs} A^1\right) + k_{3s}/k_{1s}]. \tag{35}$$

The charge density ρ is given by (see Eq.(B5)),

$$\rho = -\frac{\partial P}{\partial \phi} = \sum_s q_s \exp\left(-\beta_s q_s \phi\right) N_s(A^1, A^2), \tag{36}$$

where,

$$N_s(A^1, A^2) = n_{0s}k_{1s}[\exp(\beta_s q_s u_{ys} A^2) \\ +(k_{2s}/k_{1s})\cos(\beta_s q_s u_{xs} A^1) \\ +k_{3s}/k_{1s}]. \quad (37)$$

A charge neutral plasma requires that $\rho = 0$, hence

$$\phi = \frac{1}{e(\beta_i + \beta_e)} \ln\left(\frac{N_i}{N_e}\right). \quad (38)$$

The condition of vanishing electric field is satisfied by $N_e = N_i$, which is true if

$$\beta_i u_{yi} = -\beta_e u_{ye}, \quad (39)$$

$$\beta_i |u_{xi}| = \beta_e |u_{xe}|, \quad (40)$$

$$n_{0i}k_{1i} = n_{0e}k_{1e} = n_0, \quad (41)$$

$$k_{2i}/k_{1i} = k_{2e}/k_{1e} = a, \quad (42)$$

$$k_{3i}/k_{1i} = k_{3e}/k_{1e} = b. \quad (43)$$

As a result the plasma pressure is given by

$$P = \frac{(\beta_i + \beta_e)}{\beta_i \beta_e} n_0[\exp(-\beta_e e u_{ye} A^2) \\ +a\cos(\beta_e e u_{xe} A^1) + b]. \quad (44)$$

The $x-$ and $y-$ components of the current density are calculated using Eqs. (B6) and (B7) (see Appendix B)

$$J^1 = \frac{\partial P}{\partial A^1} = -\frac{(\beta_i + \beta_e)}{\beta_i} n_0 a e u_{xe} \sin(\beta_e e u_{xe} A^1), \quad (45)$$

$$J^2 = \frac{\partial P}{\partial A^2} = -\frac{(\beta_i + \beta_e)}{\beta_i} n_0 e u_{ye} \exp(-\beta_e e u_{ye} A^2). \quad (46)$$

The corresponding, self-consistent vector potential can be obtained via Ampère's Law,

$$\frac{\mathrm{d}^2 A^1}{\mathrm{d}z^2} = -\mu_0 J^1, \quad (47)$$

$$\frac{\mathrm{d}^2 A^2}{\mathrm{d}z^2} = -\mu_0 J^2. \quad (48)$$

The resulting vector potential components can be written as,

$$A^1 = \alpha_1 \arctan(\exp(z/\lambda_1)), \quad (49)$$

$$A^2 = \alpha_2 \ln(\cosh^2(z/\lambda_2)), \quad (50)$$

and the magnetic flux density components are

$$B^1 = -(2\alpha_2/\lambda_2)\tanh(z/\lambda_2), \tag{51}$$

$$B^2 = \frac{\alpha_1/(2\lambda_1)}{\cosh(z/\lambda_1)}, \tag{52}$$

where

$$\alpha_1 = 4/(\beta_e e u_{xe}), \tag{53}$$

$$\alpha_2 = 1/(\beta_e e u_{ye}), \tag{54}$$

$$\lambda_1 = \left(\frac{\beta_i}{\beta_e(\beta_i+\beta_e)a\mu_0 n_0 e^2 u_{xe}^2}\right)^{1/2}, \tag{55}$$

$$\lambda_2 = \left(\frac{2\beta_i}{\beta_e(\beta_i+\beta_e)\mu_0 n_0 e^2 u_{ye}^2}\right)^{1/2}. \tag{56}$$

For a force-free system we require $J_1B^2 - J_2B^1 = 0$. This condition is satisfied provided $4\alpha_2 = \pm\alpha_1$ ($\alpha = |\alpha_1|$) and $\lambda_1 = \lambda_2 = \lambda$ which implies $u_{xe} = -u_{ye}$ and $a = 1/2$. Therefore, the relativistic force-free Harris sheet is

$$B^1 = B_0\tanh(z/\lambda), \tag{57}$$

$$B^2 = \frac{B_0}{\cosh(z/\lambda)}, \tag{58}$$

where $B_0 = \alpha/(2\lambda)$. The relationship between the microscopic and macroscopic parameters of the equilibria can be deduced by comparing Eq. (14) to Eq. (44). This yields

$$\frac{B_0^2}{2\mu_0} = n_0\frac{(\beta_i+\beta_e)}{\beta_e\beta_i}, \tag{59}$$

$$a = \frac{1}{2}, \tag{60}$$

$$P_{03} = n_0\frac{(\beta_i+\beta_e)}{\beta_e\beta_i}b, \tag{61}$$

$$\frac{2}{|B_0|\lambda} = e\beta_e|u_{xe}| = e\beta_i|u_{xi}|, \tag{62}$$

$$\frac{2}{B_0\lambda} = -e\beta_e u_{ye} = e\beta_i u_{yi}, \tag{63}$$

where we have assumed that λ is positive and B_0 can be negative. Following Neukirch *et al.*[11] the connection with the original Harris sheet can be made using Eq. (59) and Eq. (63) to obtain an expression for λ,

$$\lambda = \left(\frac{2(\beta_e+\beta_i)}{\mu_0 e^2\beta_e\beta_i n_0(u_{yi}-u_{ye})^2}\right)^{1/2} \tag{64}$$

Therefore, the relativistic force-free Harris sheet is equivalent to the non-relativistic force-free Harris sheet[9–12] (and consistent with the relativistic Harris sheet[22]) where we can now formally access the relativistic regime, $m_s c^2 \beta_s \ll 1$.

When $m_s c^2 \beta_s \gg 1$ the relativistic solution reduces to its non-relativistic counterpart where,

$$n_0 = n_{0s}\gamma_{1s}^2 K_2(\Lambda_{1s}) \approx \alpha_1 \exp\left(\frac{m_s\beta_s u_{ys}^2}{2}\right), \tag{65}$$

$$\begin{aligned} a &= \frac{a_s\gamma_{2s}^2 K_2(\Lambda_{2s})}{\gamma_{1s}^2 K_2(\Lambda_{1s})} \\ &\approx \alpha_2 \exp\left(-\frac{m_s\beta_s}{2}(u_{xs}^2 + u_{ys}^2)\right), \end{aligned} \tag{66}$$

$$b = \frac{b_s K_2(\Lambda_{3s})}{\gamma_{1s}^2 K_2(\Lambda_{1s})} \approx \alpha_3 \exp\left(-\frac{m_s\beta_s u_{ys}^2}{2}\right), \tag{67}$$

and

$$\begin{aligned} \alpha_1 &= n_{0s}(1 + u_{ys}^2/c^2)(1 + u_{ys}^2/(4c^2)) \\ &\quad \times \left(\frac{\pi}{2m_s\beta_s c^2}\right)^{1/2} \exp\left(-m_s\beta_s c^2\right), \end{aligned} \tag{68}$$

$$\alpha_2 = a_s\frac{(1 - u_{xs}^2/c^2)}{(1 + u_{ys}^2/c^2)}\frac{(1 - u_{xs}^2/(4c^2))}{(1 + u_{ys}^2/(4c^2))}, \tag{69}$$

$$\alpha_3 = b_s\frac{(1 - u_{ys}^2/(4c^2))}{(1 + u_{ys}^2/c^2)}. \tag{70}$$

This is consistent with Neukirch *et al.*[11]. In the ultra-relativistic regime $m_s c^2 \beta_s \ll 1$ and

$$n_0 \approx \frac{2n_{0s}}{(m_s c^2\beta_s)^2}(1 - u_{ys}^2/c^2)^{-2}, \tag{71}$$

$$a \approx a_s\left(\frac{1 - u_{ys}^2/c^2}{1 + u_{xs}^2/c^2}\right)^2 = \frac{1}{2}, \tag{72}$$

$$b \approx b_s(1 - u_{ys}^2/c^2)^2. \tag{73}$$

The general solution is constrained by the condition

$$a = \frac{a_s\gamma_{2s}^2 K_2(\Lambda_{2s})}{\gamma_{1s}^2 K_2(\Lambda_{1s})} = \frac{1}{2}, \tag{74}$$

where a_s is a free parameter. This condition dictates the allowable values of $u_{xs} = -u_{ys} = u_s$ for a given $m_s c^2\beta_s$, Fig. 1 exhibits the relationship between u_s/c and $m_s c^2\beta_s$ for $a_s = 1$, giving a hint of the general dependence for arbitrary a_s.

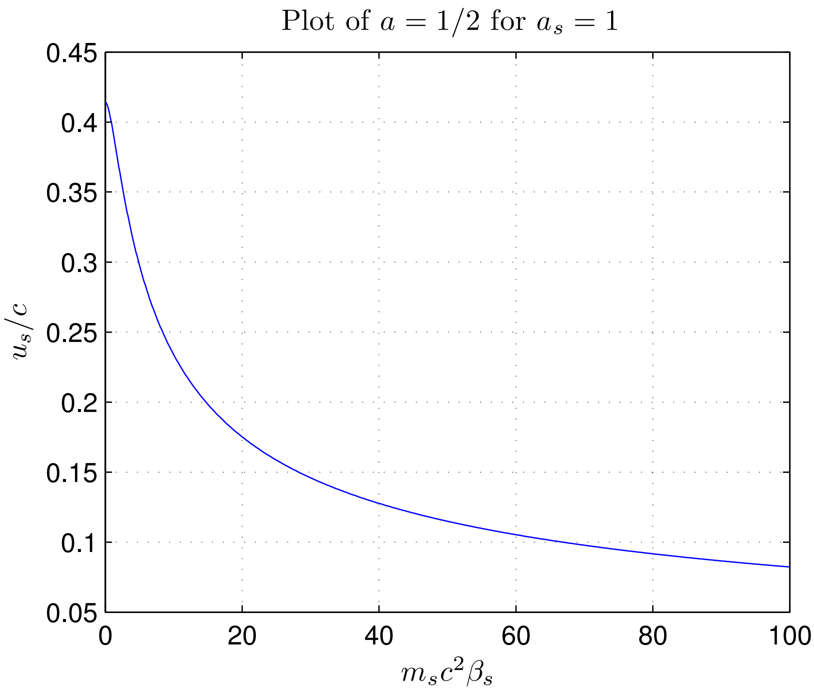

FIG. 1. Plot of u_s/c against $m_s c^2 \beta_s$ for $a_s = 1$, exhibiting the relationship $a = 1/2$ that constrains the relativistic force-free Harris sheet.

IV. SUMMARY AND CONCLUSIONS

Recently the first nonlinear force-free, non-relativistic VM equilibrium for the force-free Harris sheet was reported by Harrison and Neukirch[9]. If the thermal energy of the plasma approaches or exceeds its rest energy, a non-relativistic treatment is no longer sufficient and a relativistic analogue must be sought. This paper has presented a collisionless distribution function for the relativistic force-free Harris sheet. Mirroring the non-relativistic solution[9–11], where the properties of the pressure tensor were exploited, the energy-momentum tensor $T^{\alpha\beta}$ was wielded in an equivalent role[27], allowing the calculation of the equilibrium. In our calculation, we restrict ourselves to a frame where the electric potential vanishes ($\phi = 0$), $\mathbf{B} = (B^1, B^2, 0)$, and $\mathbf{J} = (c\rho, J^1, J^2, 0)$, where the only non-zero component of the Lorentz force is $J_1 B^2 - J_2 B^1 = 0$. The condition of vanishing electric field is true only in one frame of reference; in a moving frame ϕ is non-zero. The resulting relativistic force-free Harris sheet is identical to its non-relativistic counterpart but now the relativistic regime where $m_s c^2 \beta_s \ll 1$, can be formally accessed.

Alternative methods of studying relativistic VM equilibria have also been developed[28–34].

Suzuki explored possible equilibria by describing the deviation of the distribution function from a Maxwell-Jüttner distribution using orthogonal polynomial series[28]. Using this novel method a new, two dimensional equilibrium was reported. Kocharovsky and co-workers use the method of invariants of particle motion to find exact solutions of the VM system for arbitrary particle energy distributions[29–31]. There technique allows for the description of multicomponent plasmas, that may be relativistic or not, for a general magnetic geometry[29].

Relativistic equilibria have also been studied extensively within the context of plasma pinches and electron beams used in fusion and laboratory plasmas[35–42]. In these investigations an electron beam, described by a prescribed distribution function, is embedded in a background plasma permeated by a background magnetic field. The resulting electromagnetic fields are self-consistently calculated yielding stable Vlasov equilibria. A variety of tailored seed distribution functions have been investigated such as monoenergetic distributions[35,36]; warm beams (i.e. a drifting Maxwellian)[37,38]; and helical (force-free) beams (i.e. a combination of axial and azimuthal configurations)[38–42].

The work presented here lays the critical bedrock for further investigations of plasma phenomena such as waves, instabilities and magnetic reconnection. In particular the distribution function reported here can be used as the initial conditions for numerical investigations of magnetic reconnection in relativistic, collisionless plasmas.

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Appendix A: Normalisation Constant, f_{s0}

Consider the distribution function,

$$\begin{aligned} f_s &= f_{1s}(w^0, w^2) + f_{2s}(w^0, w^1) + f_{3s}(w^0, w^1) + f_{4s}(w^0) \\ &= f_{s0}[c_{1s} \exp\left(-c\beta_s m_s (w^0 - u_{ys} w^2/c)\right) \\ &+ c_{2s} \exp\left(-c\beta_s m_s (w^0 + i u_{xs} w^1/c)\right) \\ &+ c_{3s} \exp\left(-c\beta_s m_s (w^0 - i u_{xs} w^1/c)\right) \\ &+ c_{4s} \exp\left(-c\beta_s m_s w^0\right)], \end{aligned} \tag{A1}$$

where

$$c_{1s} = \exp\left(-\beta_s q_s (\phi - u_{ys} A^2)\right), \tag{A2}$$

$$c_{2s} = \frac{a_s}{2} \exp\left(-\beta_s q_s (\phi + i u_{xs} A^1)\right), \tag{A3}$$

$$c_{3s} = \frac{a_s}{2} \exp\left(-\beta_s q_s (\phi - i u_{xs} A^1)\right), \tag{A4}$$

$$c_{4s} = b_s \exp\left(-\beta_s q_s \phi\right). \tag{A5}$$

The four-current density $J^\mu = (c\rho, J^1, J^2, J^3)$ is defined as

$$J^\mu = c \sum_s q_s \int \mathrm{d}^4 w \; w^\mu f_s \; \delta(w_\nu w^\nu - c^2). \tag{A6}$$

To evaluate the normalisation constant f_{s0}, we make use of the zeroth component of the four-current (the charge density $\rho = \sum_s q_s n_s$) to calculate the charge number density,

$$n_{0s}\tilde{n} = \int \mathrm{d}^4 w \; w^0 f_s \; \delta(w_\nu w^\nu - c^2) \tag{A7}$$

$$= n_{1s} + n_{2s} + n_{3s} + n_{4s}, \tag{A8}$$

where n_{0s} is the mean particle density. The first charge number density integral can be written as

$$\begin{aligned} n_{1s} &= \int \mathrm{d}^4 w \; w^0 f_{1s}(w^0, w^2) \; \delta(w_\nu w^\nu - c^2) \\ &= \int \mathrm{d}^3 w \; f_{1s}(w^0, w^2). \end{aligned} \tag{A9}$$

To solve the first density integral a coordinate transformation defined by Eq.(25) is performed,

$$n_{1s} = \gamma_{1s} \int \mathrm{d}^3 \bar{w} \; f_{1s}(w^{\bar{0}}). \tag{A10}$$

Recall that the new set of variables are denoted by an overbar, where

$$f_{1s} = f_{s0} c_{1s} \exp\left(-c\beta_s m_s w^{\bar{0}}/\gamma_1\right). \tag{A11}$$

As detailed in section III, the inner product of the four-velocity with itself can be used to obtain $w^{\bar{0}} = \sqrt{c^2 + (w)^2}$. The integral can now be simply evaluated after changing to spherical coordinates, making use of the Jüttner transformation, $w/c = \sinh x$, and using the known definite integral[26]

$$\int_0^\infty \exp\left(-\beta \cosh x\right) \sinh \gamma x \sinh x \mathrm{d}x = \frac{\gamma}{\beta} K_\gamma(\beta), \tag{A12}$$

where K is the modified Bessel function of the second kind. Therefore the first integral becomes

$$n_{1s} = \frac{4\pi c^3 f_{s0} \gamma_{1s} K_2(\Lambda_{1s})}{\Lambda_{1s}} \exp\left(-\beta_s q_s (\phi - u_{ys} A^2)\right). \tag{A13}$$

The remaining density integrals can be evaluated in a similar fashion using Eq. (A12). Whereas n_{2s} and n_{3s} make us of the coordinate transformation defined by Eq. (31), this is not required for evaluating n_{4s}. Therefore,

$$\begin{aligned}\frac{n_{0s}\tilde{n}}{f_{s0}} = \frac{4\pi c}{m_s \beta_s} &\exp\left(-\beta_s q_s \phi\right)[\gamma_{1s}^2 K_2(\Lambda_{1s}) \exp\left(\beta_s q_s u_{ys} A^2\right) \\ &+ a_s \gamma_{2s}^2 K_2(\Lambda_{2s}) \cos\left(\beta_s q_s u_{xs} A^1\right) \\ &+ b_s K_2(\Lambda_{3s})].\end{aligned} \tag{A14}$$

By letting

$$\begin{aligned}\tilde{n} = \exp\left(-\beta_s q_s \phi\right)&[\gamma_{1s}^2 K_2(\Lambda_{1s}) \exp\left(\beta_s q_s u_{ys} A^2\right) \\ &+ a_s \gamma_{2s}^2 K_2(\Lambda_{2s}) \cos\left(\beta_s q_s u_{xs} A^1\right) \\ &+ b_s K_2(\Lambda_{3s})],\end{aligned} \tag{A15}$$

we find that the normalisation constant is given by

$$f_{s0} = \frac{n_{0s} m_s \beta_s}{4\pi c}. \tag{A16}$$

Appendix B: $J^\gamma = \partial P/\partial A^\gamma$ Relations

This appendix details the derivation of the $J^\gamma = \partial P/\partial A^\gamma$ relations required for the calculation of the relativistic force-free Harris sheet following the work of Otto[27]. In the kinetic treatment outlined in this manuscript the $\delta\delta-$component of the energy-momentum tensor $((T^{\alpha\beta})_{plasma} = p^{\alpha}N^{\beta})$ is defined by the integral

$$P = mc\int \mathrm{d}^4 w\ (w^{\delta})^2 f\ \delta(w_{\nu}w^{\nu} - c^2). \tag{B1}$$

This is the plasma pressure. Note that for clarity we suspend momentarily the use of the particle species subscript s. For a distribution function $f = f(P^{\nu})$, we make the substitution $h = 2w^{\delta}$ (hence $\mathrm{d}w^{\delta} = \frac{1}{2}\mathrm{d}h$) in the subsequent calculations. Taking the derivative of the plasma pressure with respect to A^{γ} yields

$$\begin{aligned}
\frac{\partial P}{\partial A^{\gamma}} &= mc\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\mathrm{d}w^{\gamma}\mathrm{d}w^{\delta}\ (w^{\delta})^2\frac{\partial f}{\partial A^{\gamma}}\ \delta(w_{\nu}w^{\nu} - c^2)\\
&= \frac{mc}{8}\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\mathrm{d}w^{\gamma}\mathrm{d}h\ h^2\frac{\partial P^{\gamma}}{\partial A^{\gamma}}\frac{\partial f}{\partial P^{\gamma}}\ \delta(g(\mathbf{w}))\\
&= \frac{qc}{8}\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\mathrm{d}w^{\gamma}\mathrm{d}h\ h^2\frac{\partial f}{\partial w^{\gamma}}\ \delta(g(\mathbf{w}))\\
&= \frac{qc}{2}\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\mathrm{d}w^{\gamma}(g(\mathbf{w}) + h^2/4)\frac{\partial f}{\partial w^{\gamma}},
\end{aligned} \tag{B2}$$

where $g(\mathbf{w})$ is defined as

$$g(\mathbf{w}) = (w^{\alpha})^2 - (w^{\beta})^2 - (w^{\gamma})^2 - \frac{h^2}{4} - c^2. \tag{B3}$$

This can be simplified by integrating the w^{γ} integral by parts,

$$\begin{aligned}
\frac{\partial P}{\partial A^{\gamma}} &= \frac{qc}{2}\left([(w_{\nu}w^{\nu} + (w^{\delta})^2 - c^2)f]_{-\infty}^{+\infty} \pm 2\int w^{\gamma} f\mathrm{d}w^{\gamma}\right)\\
&\quad\times\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\\
&= \pm qc\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\mathrm{d}w^{\gamma}\ w^{\gamma} f\\
&= \pm qc\int \mathrm{d}w^{\delta}\ \delta(w_{\nu}w^{\nu} - c^2)\int \mathrm{d}w^{\alpha}\mathrm{d}w^{\beta}\mathrm{d}w^{\gamma}\ w^{\gamma} f\\
&= \pm qc\int \mathrm{d}^4 w\ w^{\gamma} f\ \delta(w_{\nu}w^{\nu} - c^2)\\
&= \pm J^{\gamma}.
\end{aligned} \tag{B4}$$

Note that $\int \mathrm{d}w^{\delta}\ \delta(w_{\nu}w^{\nu} - c^2) = 1$. Where the lower sign corresponds to the zeroth component case ($\gamma = 0$). Therefore

$$\rho = -\frac{\partial P}{\partial \phi}, \tag{B5}$$

$$J^1 = \frac{\partial P}{\partial A^1}, \tag{B6}$$

$$J^2 = \frac{\partial P}{\partial A^2}. \tag{B7}$$

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