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Elias N. Houstis
Purdue University, enh@cs.purdue.edu

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E. N. Houstis
Computer Science
Purdue University

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E. N. Houstis

Abstract. Collocation methods based on piecewise Hermite cubic polynomials are applied to linear elliptic problems subject to Dirichlet and Neumann boundary conditions on rectangular domains. A priori estimates are obtained for the error of approximation.

Introduction. We consider the problem of approximating linear elliptic boundary value problems subject to Dirichlet and Neumann boundary conditions on rectangular domains. The methods used and analyzed in this paper are collocation on lines and two-dimensional collocation based on piecewise cubic Hermite polynomials.

The method of collocation on lines has been applied by Yartsev [9], [10], [11] for solving elliptic and biharmonic type problems using trigonometric polynomials. More recently, bicubic splines have been used in [1], [7] to obtain a second order collocation method for solving a linear self adjoint elliptic problem with Dirichlet boundary conditions. The two-dimensional collocation scheme studied in this paper has been experimentally applied in [5], [6] for the solution of linear elliptic problems on curved domains and found more efficient than standard finite differences and other finite element methods.

In the case of linear self adjoint elliptic equations with Dirichlet boundary conditions and the same two-dimensional collocation method, Prenter and Russell [8] derived optimal estimates by assuming the existence of the collocation approximation and uniform bounds on partial derivatives of the approximating error. Our analysis is applied to more general elliptic boundary value problems and it is free of such assumptions.

1. Collocation on lines. We consider the linear elliptic problem

$$(1.1)\qquad Lu \equiv aD_x^2u + 2bD_xD_yu + cD_y^2u + dD_xu + eD_yu + fu = g \text{ in } \Omega \equiv (0,1)\times(0,1)$$

subject to boundary conditions

$$(1.2)\qquad Bu \equiv \alpha\frac{\partial u}{\partial n} + \beta u = 0 \quad \text{on } \partial\Omega = \text{boundary of } \Omega$$

with either $\alpha \equiv 0$ or $\beta \equiv 0$ but not both and $f \leq 0$.

Throughout we assume that (1.1), (1.2) has a solution. We denote this solution by u. Let $\Delta_x \equiv \{0 = x_0 < x_1 < \ldots < x_N = 1\}$ be a partition of [0,1], $\mathbb{P}_3$ the set of cubic polynomials and $\mathbb{P}_{3,\Delta_x}$ the set of piecewise cubic polynomials with respect to Δ_x.

We seek an approximate solution of (1.1), (1.2) in the form

$$(1.3)\qquad u_{\Delta_x}(x,y) = \sum_{i=1}^{2N+2} \phi_i(y)B_i(x)$$

where $\{B_i(x)\}_{i=1}^{2N+2}$ is a set of basis functions for the space of piecewise Hermite cubic polynomials $H^3_{\Delta_x} \equiv \mathbb{P}_{3,\Delta_x} \cap C^1$ and $Bu_{\Delta_x} \equiv 0$ at $x = 0$ and 1.

We choose a set of "collocation" points

$$(1.4)\qquad \xi_{i,\ell} \equiv \frac{x_i + x_{i+1}}{2} + \rho_\ell\frac{x_{i+1} - x_i}{2}, \quad i=0,\ldots,N-1 \quad \ell = 1,2$$

where $\{\rho_\ell\}_1^2$ are the roots of second degree Legendre polynomials with respect to the interval [-1,1].

Using the collocation method on lines we shall determine the ϕ_i functions by the system of second order differential equations

(1.5) $\{Lu_{\Delta_x}(x,y) - f(x,y)\}|_{x = \xi_{i,\ell}} = 0,$

for $\ell = 1,2$; $i = 0,1,\ldots,N-1$ with boundary conditions

(1.6) $$\{\alpha \frac{\partial u_{\Delta_x}}{\partial n} + \beta u_{\Delta_x}\} (x = \xi_{i,\ell}, 0) = 0,$$

(1.7) $$\{\alpha \frac{\partial u_{\Delta_x}}{\partial n} + \beta u_{\Delta_x}\} (x = \xi_{i,\ell}, 1) = 0.$$

We denote by $Lu_H \equiv \{Lu_H$ with $u_H \in H^3_{\Delta_x}\}$ and define the interpolation operator

$$Q_{\Delta_x} : L^2(\Omega) \to Lu_H$$

such that for $f \in L^2(\Omega)$

$$\{Q_{\Delta_x} f - f\}_{x = \xi_{i,\ell}} = 0$$

for $\ell = 1,2$; $i = 0,1,\ldots,N-1$.

The boundary value problem (1.5) - (1.7) can be written equivalently as

(1.8) $$Lu_{\Delta_x} = Q_{\Delta_x} f \quad \text{in } \Omega$$

with u_{Δ_x} satisfying the boundary conditions (1.6), (1.7).

2. Error analysis. In this section, we derive a priori bounds for the collocation on lines procedure; first for the boundary value problem (1.1), (1.2) in the L^∞-norm and second for a self-adjoint elliptic problem in the L^2-norm. Let $W^{k,p}(\Omega)$ be the Sobolev space of functions having L^p-derivatives of order k on Ω. If X is a normed space and $\psi:[0,1] \to X$, define

$$||\psi||_{L^2(X;0,1)} = \int_0^1 ||\psi(y)||_X^2 dy, \quad ||\psi||_{L^\infty(X;0,1)} = \sup_{0 \le y \le 1} ||\psi(y)||_X$$

THEOREM 2.1. Assume the solution u of (1.1), (1.2) is in $L^\infty(W^{6,\infty}; 0,1)$.

If (i) the Green's function $G(x,y;\xi,\eta)$ for (1.1), (1.2) exists,

(ii) $\|D_\xi^{\,i} G\|_{L^\infty(\Omega\times\Omega)} \le K$, $i = 0, \ldots, n$, considering $G(x,y;\cdot,\eta)$ as an element of $C^{(n)}[0,x] \times C^{(n)}[x,1]$,

and

(iii) the coefficients of L are in $L^\infty(C^{(2+n)}; 0,1)$,

then

(a) the collocation on lines approximation u_{Δ_x} exists,

and

(b) for the error of approximation we have

$$\|u - u_{\Delta_x}\|_{L^\infty(\Omega)} \le C(\Delta x)^{2+\min(n,2)} \tag{2.1}$$

where $\Delta x = \max|x_{i+1}-x_i|$ and C is a generic constant independent of Δx.

Proof. The existence and uniqueness of u_{Δ_x} for each $y\varepsilon[0,1]$ is a direct consequence of Theorem (3.1) in [2]. Furthermore, from the same theorem we obtain an estimate for the error and its derivatives:

$$\max_{(x,y)\varepsilon\Omega} | D_x^{\,i}(u(x,y) - u_{\Delta_x}(x,y)) | \le C(\Delta x)^2, \quad i = 0, 1, 2 \tag{2.2}$$

where C is a constant independent of Δx.

Also, provided the coefficients of the operator are in $C^{(2+n)}([0,1])$ as functions of x, Lemma 4.1 in [2] implies that there exists constant C independent of Δx such that

$$\max_{(x,y)\varepsilon\Omega} | D_x^{2+i}(Lu - Lu_{\Delta_x})(x,y) | \le C \text{ for } i = 0, \ldots, n. \tag{2.3}$$

Let's denote by $r \equiv Lu - Lu_{\Delta_x}$. Since r vanishes at $\xi_{i,\ell}$ $i = 0, \ldots, N-1$; $\ell = 1,2$ then

(2.4) $\quad r = r[\xi_{i1}, \xi_{i2}, x;y](x - \xi_{i1})(x - \xi_{i2})$

in the subinterval $[x_i, x_{i+1}]$, where $f[x_0,\ldots x_n;y]$ is the nth divided difference of f on the points $x_0,\ldots,x_n$.

Assumption (i) implies that (1.1), (1.2) is uniquely solvable and

$$(2.5)\qquad (u - u_{\Delta_x})(x,y) = \iint_{\Omega} G(x,y;\xi,\eta)(f - Q_{\Delta}f)(\xi,\eta)d\xi d\eta$$

$$= \iint_{\Omega} G(x,y;\xi,\eta)(Lu - Lu_{\Delta_x})(\xi,\eta)d\xi d\eta .$$

By expanding in Taylor series the function

$$F(x,y;\xi,\eta) \equiv G(x,y;\xi,\eta)r[\xi_{i1}, \xi_{i2}, \xi\eta]$$

inside the interval $[x_i, x_{i+1}]$ we obtain

$$F(x,y;\xi,\eta) = t(x,y;\xi,\eta) + \frac{1}{n!}\, D_{\xi}^{n}F(x,y;\tilde{\sigma}(\xi),\eta)(\xi - \sigma)^{n}$$

where t is a polynomial of degree at most n - 1 with respect to ξ and $\tilde{\sigma}(\xi) \in (x_i,x_{i+1})$.

From the orthogonality of the Legendre polynomials we derive the relation

$$(2.6)\qquad (u-u_{\Delta_x})(x,y) = \frac{1}{n!}\int_0^1 \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} D_{\xi}^{n}F(x,y;\tilde{\sigma}(\xi),\eta)(\xi-\sigma)^{n} \prod_{\ell=1}^{2}(\xi-\xi_{i\ell})d\xi d\eta$$

for $n \le 2$.

From the definition of $F(x,y;\xi,\eta)$ we obtain that

$$(2.7)\qquad D_{\xi}^{n}F(x,y;\xi,\eta) = \frac{1}{(n+2)!}\sum_{s=0}^{n}\binom{n+2}{2+s}(D_{\xi}^{n-s}G(x,y;\cdot,\eta))(\xi)(D_{\xi}^{2+s}r)(\theta_{\xi,s},\eta)$$

with $\theta_{\xi,s} \in (x_i,x_{i+1})$.

If $x \notin (x_i,x_{i+1})$ then from the assumption (ii), relations (2.6), (2.7) and the inequality (2.3)

$$\|D_\xi^{2+s} r\|_{L^\infty(\Omega)} \le C \qquad \text{for } s = 0, \ldots, n$$

so we obtain

$$(2.8) \quad \frac{1}{n!}\left| \int_0^1 \sum_{\substack{j=0 \\ j\neq i}}^{N-1} \int_{x_j}^{x_{j+1}} D_\xi^n F(x,y;\tilde{\sigma}(\xi),\eta)(\xi-\sigma)^n \prod_{\ell=1}^{2} (\xi-\xi_{i\ell})\, d\xi\, d\eta \right|$$

$$\le C\ (\Delta x)^{2+\min(n,2)}$$

On the other hand if $x \in (x_i, x_{i+1})$ then $G(x,y;\cdot,\eta) \in W^{1,\infty}$ so that

$$(2.9) \quad \left|\int_0^1 \int_{x_i}^{x_{i+1}} D_\xi F(x,y;\tilde{\sigma}(\xi),\eta)(\xi-\sigma) \prod_{\ell=1}^{2} (\xi-\xi_{i\ell})\, d\xi\, d\eta\right| \le C(\Delta x)^4.$$

Conclusion b) follows from (2.8) - (2.9). This concludes the proof of Theorem 2.1.

Next, we consider the self-adjoint linear elliptic problem

$$(2.10) \quad Lu \equiv -D_x(pD_x u) - D_y(qD_y u) + cu = f \text{ in } \Omega = (0,1) \times (0,1)$$

subject to boundary conditions

$$(2.11) \quad B(u) = 0 \qquad \text{on } \partial\Omega .$$

Notice that the assumptions (i) and (ii) of Theorem 2.1 are satisfied for equation (2.10) with boundary conditions (2.11), see reference [3, p.123].

<u>THEOREM 2.2</u>. Assume the solution u of (2.10), (2.11) is in $L^2(W^{5,2};\ 0,1)$ and the coefficients of L are in $L^\infty(C^{(2+n)};\ 0,1)$. Then

(α) for each $y \in (0,1)$ the collocation on lines approximation u_{Δ_x} exists in the space $H^3_{\Delta_x}$

and

(β) for the error of approximation we have

(2.12) $$||u-u_{\Delta_x}||_{W^{1,2}(\Omega)} \le C(\Delta x)^3$$

where C is independent of Δx.

Proof. The existence and uniqueness of the collocation on lines approximation u_{Δ_x} follows from arguments similar to those in the proof of Theorem 2.1.

In order to derive the error estimate (2.12) we define $F(x,y) \equiv (u-u_{\Delta_x})r[\xi_{i1}, \xi_{i2}, x;y]$ and expand it in Taylor series inside the interval $[x_i, x_{i+1}]$ to obtain

(2.13) $$F(x,y) = t(x,y) + D_x F(\tilde{\sigma}(x),y)(x-\sigma) ,$$

where $t(x,y)$ is constant polynomial with respect to x and $\tilde{\sigma}(x) \in (x_i, x_{i+1})$.

The relation (2.13) can be rewritten as

$$F(x,y) = t(x,y) + D_x(u-u_{\Delta_x})\, D_x^2 r(\theta(x),y)(x-\sigma) + (u-u_{\Delta_x})\, D_x^3 r(\theta(x),y)(x-\sigma).$$

From the ellipticity of L we get

$$||u-u_{\Delta_x}||^2_{W^{1,2}(\Omega)} \le \left|\int_\Omega (u-u_{\Delta_x})(Lu-Lu_{\Delta_x})dxdy\right| \le \int_0^1 \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (u-u_{\Delta_x})(Lu-Lu_{\Delta_x})dxdy.$$

Also from (2.3) and the choice of the collocation points we obtain

$$||u-u_{\Delta_x}||^2_{W^{1,2}(\Omega)} \le \int_0^1 \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} D_x F(\tilde{\sigma}(x),y)(x-\sigma) \prod_{\ell=1}^{2} (x-\xi_{i\ell})dxdy$$

$$\le C\{||u-u_{\Delta_x}||_{L^2(\Omega)} + ||D_x(u-u_{\Delta_x})||_{L^2(\Omega)}\}(\Delta x)^3$$

$$\le C\, ||u-u_{\Delta_x}||_{W^{1,2}(\Omega)}\, (\Delta x)^3$$

Therefore

$$||u-u_{\Delta_x}||_{W^{1,2}(\Omega)} \le C(\Delta x)^3$$

This concludes the proof of Theorem 2.2.

<u>THEOREM 2.3</u>. Assume the solution u of (2.10), (2.11) is in $L^2(W^{5,2};0,1)$. If (i) the coefficients of L are in $L^2(C^{(2+n)};0,1)$ and (ii) L is strongly coersive, then, for the error of approximation we have

$$(2.14) \quad ||u-u_{\Delta_x}||_{L^2(\Omega)} \leq C(\Delta x)^4$$

<u>Proof</u>. We let

$$\psi_H(x,y) \equiv (||u-u_{\Delta_x}||_{L^2(\Omega)})^{-1}(u(x,y)-u_{\Delta_x}(x,y))$$

and consider the problem of finding the unique solution of $L\phi_H = \psi_H$, $B(\phi_H) = 0$. From assumption (ii) we get

$$(2.15) \quad ||D_x^k D_y^j \phi_H||_{L^2} \leq K||\psi_H||_{L^2} \quad (0 \leq k + j \leq 2).$$

From the definition of ϕ_H and since L is self adjoint we obtain

$$||u-u_{\Delta_x}||_{L^2(\Omega)} = (\psi_H, u-u_{\Delta_x})$$

$$= \iint_\Omega \phi_H(Lu-Lu_{\Delta_x})dx \; .$$

Let $r \equiv Lu-Lu_{\Delta_x}$. On the interval $[x_i,x_{i+1}]$, $r(\cdot,y)$ vanishes at $\xi_{i,\ell}$, $\ell = 1,2$. Hence, $r = r[\xi_{i1}, \xi_{i2}, x;y](x - \xi_{i1})(x - \xi_{i2})$.

Consider

$$F(x,y) = \phi_H(x,y)r[\xi_{i1}, \xi_{i2}, x;y]$$

and expanding in Taylor's series inside (x_i,x_{i+1}) we obtain

$$F(x,y) = t(x,y) + (x-x_i)^2 D_x^2 F(\sigma_{x,i},y) \quad .$$

By orthogonality we get

$$(2.16) \quad ||u-u_{\Delta_x}||_{L^2(\Omega)} = \sum_{i=0}^{N-1} \int_0^1 \int_{x_i}^{x_{i+1}} \{ \prod_{\ell=1}^{2} (x-\xi_{i\ell}) \int_{x_i}^{x} D_x^2 F(s,y)(x-s)ds \} dxdy$$

where

$$D_x^2F(s,y) = D_x^2\phi_H(D_x^2r)(\theta_{s,i}) + D_x\phi_H(D_x^3r)(\theta_{s,i}) + \phi_H(D_x^4r)(\theta_{s,i}) \; .$$

From (2.3), (2.15), (2.16) we obtain the inequality

$$\|u-u_{\Delta_x}\|_{L^2(\Omega)} \le (\Delta x)^4 \sum_{i=0}^{N-1} \int_0^1 (\|D_x^2\phi_H D_x^2 r\|_{L^2(I_i)} + \|D_x\phi_H D_x^3 r\|_{L^2(I_i)} + \|D_x^2\phi_H D_x^4 r\|_{L^2(I_i)}) \le C(\Delta x)^4$$

This concludes the proof of Theorem 2.3.

3. Two-dimensional collocation. In this section we consider the problem of approximating the solution of (1.1), (1.2) by a piecewise bicubic Hermite polynomial. Let $\Delta y = \{y_j\}_0^{M+1}$ be a partition of [0,1] in the y-direction and $\Delta y = \max_j |y_{j+1}-y_j|$.

Also, we denote by $\Delta \equiv \Delta_x \times \Delta_y$ a partition of Ω and by H_Δ^3 the vector space of all piecewise bicubic polynomials $p(x,y)$ with respect to Δ such that $D_x^\ell D_y^n p(x,y)$ is continuous on Ω for all $0 \le \ell, n \le 1$. The Gaussian points in the interval $[y_i, y_{i+1}]$ are

$$(3.1) \qquad \eta_{i,m} \equiv \frac{y_i + y_{i+1}}{2} + \rho_m \frac{y_{i+1} - y_i}{2} \; , \; i = 0,1,\ldots, M-1 \; , \; m = 1,2 \; .$$

Also, we define a two-dimensional analogue of the interpolation operator of Section 1 as the tensor product

$$Q_\Delta \equiv Q_{\Delta_x} \times Q_{\Delta_y} \equiv Q_{\Delta_x} Q_{\Delta_y} \, .$$

We seek an approximate solution of (1.1), (1.2) $u \in H^3_\Delta$ such that

(3.2) $\quad Lu_\Delta = Q_\Delta f$,

(3.3) $\quad B(u_\Delta) = 0$.

4. Error analysis of two-dimensional collocation. In this section we establish a priori bounds for the two-dimensional collocation scheme introduced in Section 3. The analysis can be easily made, if we realize that the system of linear equations (3.2), (3.3) is the one obtained by approximating ϕ_i's of (1.5)-(1.7) by elements in $H^3_{\Delta_y}$ and collocating at the Gaussian points (3.1).

THEOREM 4.1. Assume the solution u of (1.1), (1.2) is in $W^{6,\infty}(\Omega)$ and hypotheses (i), (ii) of Theorem 2.1. Further assume the coefficients of L are in $C^{(2+n)}(\Omega)$. Then,

(a) the system of linear equations (3.2), (3.3) has a unique solution, and

(b) for the error of approximation we have

(3.4) $\quad ||u-u_\Delta||_{L^\infty} \le C\{(\Delta y)^4 + (\Delta x)^{2+\min(n,2)}\}$

where C is a constant independent of Δx, Δy.

Proof. The existence and uniqueness of u_Δ is a consequence of Theorem 3.1 in [4]. Let $\{B_i(x)B_j(y)\}_{i=1}^{2N+2}\,{}_{j=1}^{2M+2}$ be a set of basis functions of the space H^3_Δ, then

$$u_\Delta = \sum_{i=1}^{2N+2} \sum_{j=1}^{2M+2} \alpha_{ij} B_i(x)B_j(y).$$

By the triangle inequality we get

(3.5) $$||u-u_\Delta||_{L^\infty(\Omega)} \le ||u-u_{\Delta_x}||_{L^\infty(\Omega)}$$

$$+ \max_{(x,y)\in\Omega} \left| \sum_{i=1}^{2N+2} (\phi_i(y) - \sum_{j=1}^{2M+2} \alpha_{ij} B_j(y)) B_i(x) \right| .$$

Also, from Theorem 4.1 in [4] we have that

(3.6) $$||\phi_i - \sum_{j=1}^{2M+2} \alpha_{ij} B_j||_{L^\infty([0,1])} \le C\,\Delta y^4 .$$

Finally, inequalities (3.5), (3.6) yield (3.4). This concludes the proof of Theorem 3.1.

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