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Collocation solutions for the time fractional telegraph equation using cubic B-spline finite elements

Orkun Tasbozan and Alaattin Esen

Abstract. In this study, we investigate numerical solutions of the fractional telegraph equation with the aid of cubic B-spline collocation method. The fractional derivatives have been considered in the Caputo forms. The $L1$ and $L2$ formulae are used to discretize the Caputo fractional derivative with respect to time. Some examples have been given for determining the accuracy of the regarded method. Obtained numerical results are compared with exact solutions arising in the literature and the error norms L_2 and L_∞ have been computed. In addition, graphical representations of numerical results are given. The obtained results show that the considered method is effective and applicable for obtaining the numerical results of nonlinear fractional partial differential equations (FPDEs).

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1 Introduction

Fractional calculus that means arbitrary order differentiation and integration is one of the most important subject in mathematics. The interest to these subject grows day by day due to its long history and huge amount of application area [9]. However, scientists expressed that the most suitable way for describing the complex and nonlinear events in nature is using fractional order derivative and integral. They are generally considered to model the problems in biology, fluid mechanics, diffusion and etc [13]. There are huge amount of studies that contain numerical and analytical solution procedures for FPDEs [1–3, 5, 6, 8, 12, 14, 15].

The finite element method (FEM) is a well known method generally used for acquiring the solution of integer order PDEs. However, in this study, the considered method is employed to acquire the approximate solution of fractional telegraph equations. FEM depends on dividing the domain region of the regarded problem into an equivalent system of finite elements with relevant nodes and to choose the most suitable element type to model most closely the physical behavior. So the huge problem turns into solvable small problems with the help of FEM. These considered small nodes must be smallest to get useful results and large enough to increase computational difficulty [7].The cubic B-spline FEM is employed for getting the numerical solutions of the fractional telegraph equation where $L1$ and $L2$ formulae are used to discretize the fractional derivative as utilized in [2]. The various forms of time fractional telegraph equations have been solved by many authors. For instance, Wei et. al. [16] have utilized fully discrete local discontinuous Galerkin method for solving the fractional telegraph equation and Hosseini et. al. [4] have employed radial basis functions for obtaining the numerical results of fractional telegraph equation.

In this study, we will take the following the fractional telegraph equations into consideration as models

$$\frac{\partial^\mu}{\partial t^\mu}U(x,t) + s_1\frac{\partial^{\mu-1}}{\partial t^{\mu-1}}U(x,t) + s_2U(x,t) - s_3\frac{\partial^2}{\partial x^2}U(x,t) = f_1(x,t) \tag{1.1}$$

and

$$\frac{\partial^\mu}{\partial t^\mu}U(x,t) + \frac{\partial^{\mu-1}}{\partial t^{\mu-1}}U(x,t) + \lambda\frac{\partial}{\partial x}U(x,t) - \frac{\partial^2}{\partial x^2}U(x,t) = f_2(x,t) \tag{1.2}$$

where

$$\frac{\partial^\mu U(x,t)}{\partial t^\mu} = \frac{1}{\Gamma(2-\mu)}\int_0^t (t-\tau)^{1-\mu}\frac{\partial^2 U(x,\tau)}{\partial \tau^2}d\tau \quad ,1<\mu<2$$

and

$$\frac{\partial^{\mu-1} U(x,t)}{\partial t^{\mu-1}} = \frac{1}{\Gamma(2-\mu)}\int_0^t (t-\tau)^{1-\mu}\frac{\partial U(x,\tau)}{\partial \tau}d\tau \quad ,0<\mu-1<1$$

where fractional derivatives are by means of in the Caputo type [6, 10] and s_1, s_2, s_3, λ are constants. For the fractional telegraph equation, the boundary conditions of the (1.1) and (1.2) are considered as

$$U(a,t) = h_1(t)\ , \quad U(b,t) = h_2(t), \quad t \in [0,\infty], x \in [a,b] \tag{1.3}$$

and the initial conditions as

$$U(x,0) = g_1(x)\ , \quad U_t(x,0) = g_2(x), \quad x \in [a,b]. \tag{1.4}$$

In order to acquire finite element schemes for solving the telegraph equations, we will also discretize the fractional derivatives using the $L1$ and $L2$ formulae respectively

$$\frac{\partial^{\mu-1} f(t)}{\partial t^{\mu-1}}|_{t_m} = \frac{(\Delta t)^{1-\mu}}{\Gamma(3-\mu)} \sum_{k=0}^{m-1} b_k^{\mu} \left[f(t_{m-k}) - f(t_{m-1-k}) \right]$$

and

$$\frac{\partial^{\mu} f(t)}{\partial t^{\mu}}|_{t_m} = \frac{(\Delta t)^{-\mu}}{\Gamma(3-\mu)} \sum_{k=0}^{m-1} b_k^{\mu} \left[f(t_{m-k}) - 2f(t_{m-1-k}) + f(t_{m-2-k}) \right]$$

where

$$b_k^{\mu} = (k+1)^{2-\mu} - k^{2-\mu}.$$

The rest of paper is regarded as follows. In Section 2 the cubic B-spline collocation method for Eqs. (1.1) and (1.2) is expressed. Also to present the accuracy and capability of the considered method are devoted in Section 3 as numerical experiments. In Section 4 the conclusion is denoted.

2 Finite Element Collocation Method

First of all B-spline base functions are described. For this goal, we debate that the interval $[a,b]$ of the equations is partitioned into N finite elements of at an equal rate equal length by the nodal points x_m, $m = 0,1,2,...,N$ such that $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ where $h = x_{m+1} - x_m$. The cubic B-splines $\psi_m(x)$, $(m = -1(1)N+1)$, at the knots x_m are defined over the interval $[a,b]$ by [11]

$$\psi_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3, \quad x \in I_{m-2}, \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, x \in I_{m-1}, \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, x \in I_m, \\ (x_{m+2} - x)^3, \quad x \in I_{m+1}, \\ 0, \quad \text{otherwise} \end{cases}$$

where I_k denotes the interval $[x_k, x_{k+1}]$.

The set of splines $\{\psi_{-1}(x), \psi_0(x), \ldots, \psi_N(x), \psi_{N+1}(x)\}$ forms a basis for the functions defined over $[a, b]$. Therefore, we have the right to write a solution $U_N(x,t)$ of the cubic B-splines trial functions as follows

$$U_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\psi_m(x) \tag{2.1}$$

where $\delta_m(t)$'s are unknown element parameters which are time dependent quantities to be determined from the initial, boundary and cubic B-spline collocation conditions. Since each cubic B-spline covers four consecutive elements, each element $[x_m, x_{m+1}]$ is covered by four different cubic B-splines. During the solution process of this problem, the finite elements are going to be determined with the interval $[x_m, x_{m+1}]$ and the element nodal points x_m, x_{m+1}. Using the nodal values U_m, $U_m^{'}$ and $U_m^{''}$ given in terms of the parameter $\delta_m(t)$

$$\begin{aligned} U_m &= U(x_m,t) = \delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t), \\ U_m' &= U'(x_m,t) = \tfrac{3}{h}(-\delta_{m-1}(t) + \delta_{m+1}(t)), \\ U_m'' &= U''(x_m,t) = \tfrac{6}{h^2}(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)), \end{aligned} \tag{2.2}$$

the variation of $U_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ is given by

$$U_N(x,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\psi_j(x).$$

Firstly, if we substitute the global approximation (2.1) and its necessary derivatives (2.2) into Eq. (1.1), we obtain the following set of the ordinary differential equations:

$$\begin{aligned} &\left(\ddot{\delta}_{m-1}(t) + 4\ddot{\delta}_m(t) + \ddot{\delta}_{m+1}(t)\right) + s_1\left(\dot{\delta}_{m-1}(t) + 4\dot{\delta}_m(t) + \dot{\delta}_{m+1}(t)\right) \\ &+ s_2\left(\delta_{m-1}(t) + 4\delta_m(t) + \delta_{m+1}(t)\right) - s_3\frac{6}{h^2}\left(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)\right) \\ &= f_1(x,t) \end{aligned} \tag{2.3}$$

where $\ddot{}$ denotes μ^{th} and $\dot{}$ denotes $(\mu-1)^{th}$ fractional derivative with respect to time. If time parameters $\delta_m(t)$'s, $\dot{\delta}_m(t)$'s and $\ddot{\delta}_m(t)$'s in Eq. (2.3) are discretized by the Crank-Nicolson formula, $L1$ formula and $L2$ formula, respectively:

$$\delta = \frac{1}{2}(\delta^n + \delta^{n+1}), \tag{2.4}$$

$$\dot{\delta} = \frac{d^{\mu-1}\delta}{dt^{\mu-1}} = \frac{(\Delta t)^{1-\mu}}{\Gamma(3-\mu)} \sum_{k=0}^{n-1} \left[(k+1)^{2-\mu} - k^{2-\mu}\right] \left[\delta^{n-k} - \delta^{n-k-1}\right], \tag{2.5}$$

and

$$\ddot{\delta} = \frac{d^{\mu}\delta}{dt^{\mu}} = \frac{(\Delta t)^{-\mu}}{\Gamma(3-\mu)} \sum_{k=0}^{n-1} \left[(k+1)^{2-\mu} - k^{2-\mu}\right] \left[\delta^{n-k} - 2\delta^{n-k-1} + \delta^{n-k-2}\right], \tag{2.6}$$

we attain relating parameters $\delta_m^{n+1}(t)$

$$\begin{aligned}
&(1 + s_1\Delta t + s_2h^2\alpha - 6s_3\alpha)\delta_{m-1}^{n+1} + (4 + 4s_1\Delta t + 4s_2h^2\alpha + 12s_3\alpha)\delta_m^{n+1} \\
&\quad + (1 + s_1\Delta t + s_2h^2\alpha - 6s_3\alpha)\delta_{m+1}^{n+1} = (2 + s_1\Delta t - s_2h^2\alpha + 6s_3\alpha)\delta_{m-1}^{n} \\
&\quad + (8 + 4s_1\Delta t - 4s_2h^2\alpha - 12s_3\alpha)\delta_m^{n} + (2 + s_1\Delta t - s_2h^2\alpha + 6s_3\alpha)\delta_{m+1}^{n} \\
&\quad - \sum_{k=1}^{n}[(k+1)^{2-\mu} - k^{2-\mu}][(\delta_{m-1}^{n-k+1} - 2\delta_{m-1}^{n-k} + \delta_{m-1}^{n-k-1}) \\
&\quad + 4(\delta_m^{n-k+1} - 2\delta_m^{n-k} + \delta_m^{n-k-1}) + (\delta_{m+1}^{n-k+1} - 2\delta_{m+1}^{n-k} + \delta_{m+1}^{n-k-1})] \\
&\quad - (\delta_{m-1}^{n-1} + 4\delta_m^{n-1} + \delta_{m+1}^{n-1}) + 2h^2\alpha f_1(x_m, t_n) \\
&\quad - s_1\Delta t\sum_{k=1}^{n}[(k+1)^{2-\mu} - k^{2-\mu}][(\delta_{m-1}^{n-k+1} - \delta_{m-1}^{n-k}) + 4(\delta_m^{n-k+1} - \delta_m^{n-k}) \\
&\quad + (\delta_{m+1}^{n-k+1} - \delta_{m+1}^{n-k})]
\end{aligned} \tag{2.7}$$

where

$$\alpha = \frac{(\Delta t)^{\mu}\Gamma(3-\mu)}{2h^2}.$$

If we substitute (2.1) and its derivatives (2.2) into Eq. (1.2), we attain the following set:

$$\begin{array}{l}
\left(\ddot{\delta}_{m-1}(t) + 4\ddot{\delta}_m(t) + \ddot{\delta}_{m+1}(t)\right) + \left(\dot{\delta}_{m-1}(t) + 4\dot{\delta}_m(t) + \dot{\delta}_{m+1}(t)\right) \\
+\frac{3\lambda}{h}\left(-\delta_{m-1}(t) + \delta_{m+1}(t)\right) - \frac{6}{h^2}\left(\delta_{m-1}(t) - 2\delta_m(t) + \delta_{m+1}(t)\right) = f_2(x,t)
\end{array}. \tag{2.8}$$

If parameters $\delta_m(t)$'s and $\dot{\delta}_m(t)$'s are discretized by the (2.4)-(2.6) formulas, we attain relating parameters $\delta_m^{n+1}(t)$

$$\begin{aligned}
&(1+\Delta t-3\lambda h\alpha-6\alpha)\delta_{m-1}^{n+1}+(4+4\Delta t+12\alpha)\delta_m^{n+1}\\
&\quad+(1+\Delta t+3\lambda h\alpha-6\alpha)\delta_{m+1}^{n+1}=(2+\Delta t+3\lambda h\alpha+6\alpha)\delta_{m-1}^{n}\\
&\quad+(8+4\Delta t-12\alpha)\delta_m^{n}+(2+\Delta t-3\lambda h\alpha+6\alpha)\delta_{m+1}^{n}\\
&\quad-\sum_{k=1}^{n}[(k+1)^{2-\mu}-k^{2-\mu}][(\delta_{m-1}^{n-k+1}-2\delta_{m-1}^{n-k}+\delta_{m-1}^{n-k-1})\\
&\quad+4(\delta_{m}^{n-k+1}-2\delta_{m}^{n-k}+\delta_{m}^{n-k-1})+(\delta_{m+1}^{n-k+1}-2\delta_{m+1}^{n-k}+\delta_{m+1}^{n-k-1})]\\
&\quad-(\delta_{m-1}^{n-1}+4\delta_m^{n-1}+\delta_{m+1}^{n-1})+2h^2\alpha f_2(x_m,t_n)\\
&\quad-\Delta t\sum_{k=1}^{n}[(k+1)^{2-\mu}-k^{2-\mu}][(\delta_{m-1}^{n-k+1}-\delta_{m-1}^{n-k})+4(\delta_{m}^{n-k+1}-\delta_{m}^{n-k})\\
&\quad+(\delta_{m+1}^{n-k+1}-\delta_{m+1}^{n-k})]
\end{aligned}\tag{2.9}$$

where

$$\alpha=\frac{(\Delta t)^{\mu}\Gamma(3-\mu)}{2h^2}.$$

Note that the both of iterative system (2.7) and (2.9) consist of δ_m^{-1} will be observed when $n=0$ or $k=n$. So, we use the initial condition to approximate δ_m^{-1} as $\delta_m^{-1}=\delta_m^{0}-\Delta t g_2(x)$. The both of iterative system (2.7) and (2.9) consist of $N+1$ linear equations including $N+3$ unknown parameters $(\delta_{-1},\ldots,\delta_{N+1})^T$. To obtain a unique solution to these systems, we need two additional constraints. These are obtained from the boundary conditions and their second derivatives and then are used to eliminate δ_{-1} and δ_{N+1} from the system (2.7) and (2.9) as follows:

$$\begin{aligned}
\delta_{-1}(t)&=-4\delta_0(t)-\delta_1(t)+U(x_0,t),\\
\delta_{N+1}(t)&=-4\delta_N(t)-\delta_{N-1}(t)+U(x_N,t).
\end{aligned}$$

Then, these systems of equations become a matrix equation with the $N+1$ unknowns $\mathbf{c}=(\delta_0,\ldots,\delta_N)^T$ in the form

$$\mathbf{A}\mathbf{c}^{n+1}=\mathbf{B}\mathbf{c}^{n}.$$

The initial vector $\mathbf{c}^0=(\delta_0,\ldots,\delta_N)^T$ is determined from the initial and boundary conditions. So the approximation (2.1) can be rewritten for the initial condition as

$$U_N(x,0)=\sum_{m=-1}^{N+1}\delta_m(0)\psi_m(x)$$

where the $\delta_m(0)$'s are unknown parameters. We require the initial numerical approximation $U_N(x,0)$ satisfy the following conditions:

$$U_N(x,0) = U(x_m,0), m = 0,1,...,N$$
$$(U_N)_{xx}(a,0) = g_1''(a), (U_N)_{xx}(b,0) = g_1''(b).$$

Using the these conditions leads to the form

$$W\mathbf{c}^0 = \mathbf{b}$$

where

$$W = \begin{bmatrix} 6 & 0 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & & & \ddots & & \\ & & & & 1 & 4 & 1 \\ & & & & & 0 & 6 \end{bmatrix}$$

and

$$\mathbf{b} = (U(x_0,0) - \tfrac{h^2}{6}g_1''(a), U(x_1,0), \ldots, U(x_{N-1},0), U(x_N,0) - \tfrac{h^2}{6}g_1''(b))^T.$$

3 Numerical Solutions

In this part the collocation method is employed the get the numerical solutions of fractional telegraph equations.

Problem 1: Firstly, we regard the Eq. (1.1) for $s_1 = 1$, $s_2 = 1$, $s_3 = \pi$ and

$$f_1(x,t) = \left(\frac{6t^{3-\mu}}{\Gamma(4-\mu)} + \frac{6t^{4-\mu}}{\Gamma(5-\mu)}\right)\sin^2 x + t^3\sin^2 x - 2\pi t^3(\cos^2 x - \sin^2 x)$$

with conditions

$$U(0,t) = 0, U(1,t) = t^3\sin^2 1,$$

$$U(x,0) = 0, U_t(x,0) = 0.$$

The analytical solution of the equation is assigned by [4]

$$U(x,t) = t^3\sin^2 x.$$

Table 1: Error norms of problem 1 for $\Delta t = 0.001$ and $t_f = 1$.

N	$L_2 \times 10^3$ norm				
	$\mu = 1.1$	$\mu = 1.3$	$\mu = 1.5$	$\mu = 1.7$	$\mu = 1.9$
20	0.195038	0.184330	0.159778	0.123919	0.110397
40	0.081814	0.078898	0.075070	0.072421	0.083052
50	0.075933	0.072714	0.069677	0.068773	0.080012
N	$L_\infty \times 10^3$ norm				
	$\mu = 1.1$	$\mu = 1.3$	$\mu = 1.5$	$\mu = 1.7$	$\mu = 1.9$
20	0.428937	0.402022	0.340983	0.238881	0.174542
40	0.116103	0.111652	0.109535	0.113099	0.133398
50	0.117608	0.112542	0.109224	0.110952	0.128583

L_2 and L_∞ obtained for equation for different value of N, μ, $\Delta t = 0.001$ and $t_f = 1$ is indicated in Table 1. As it is obviously in sight from Table 1, the numerical and exact results acquired by the method are in concordance with each other because of the decreasing values of the L_2 and L_∞. The value of N increases, the acquired numerical solutions become more true. In Table 2, we evaluate error norms for $N = 50$, $t_f = 1$, different values of $\Delta t, \mu$. Table 2 denotes when the value of Δt decreases, the accuracy of numerical solutions increases. We see these from the decreasing values of L_2 and L_∞. Figure

Table 2: Error norms of problem 1 for $N = 50$, $t_f = 1$, different values of μ, Δt.

Δt	$L_2 \times 10^3$ norm				
	$\mu = 1.1$	$\mu = 1.3$	$\mu = 1.5$	$\mu = 1.7$	$\mu = 1.9$
0.01	0.751346	0.704953	0.670194	0.673262	0.781459
0.005	0.373164	0.350671	0.333347	0.333591	0.388860
0.0005	0.044837	0.043045	0.040440	0.037918	0.042398
Δt	$L_\infty \times 10^3$ norm				
	$\mu = 1.1$	$\mu = 1.3$	$\mu = 1.5$	$\mu = 1.7$	$\mu = 1.9$
0.01	1.209530	1.148807	1.101893	1.105711	1.241761
0.005	0.602371	0.572466	0.548471	0.549184	0.620010
0.0005	0.083125	0.070534	0.058772	0.057179	0.068173

1 shows the graphs of the analytical (lines) and the numerical solutions for $N = 50$, $\mu = 1.5$ and $\Delta t = 0.001$, different values of t. The figures represents that the numerical and analytical results are compatible with each other.

Problem 2: Secondly, we assume for $s_1 = 1$, $s_2 = 1$ and $s_3 = 1$ in the equation (1.1)

$$f_1(x,t) = \left(\frac{t^{1-\mu}}{\Gamma(2-\mu)} + \frac{t^{2-\mu}}{\Gamma(3-\mu)} - 4tx^2 - 4x^3 + 3t + 7x \right) e^{-x^2}.$$

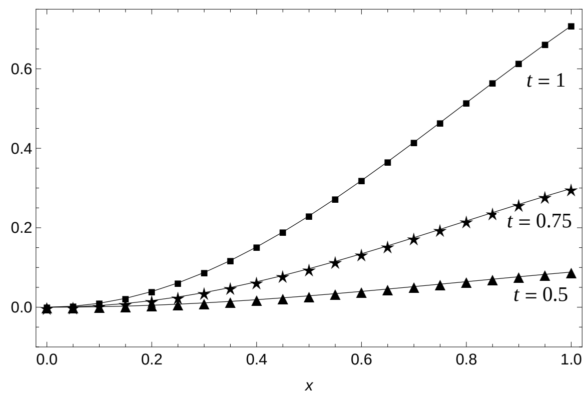

Figure 1: The curves of the analytical(lines) and numerical solutions of problem 1 for $\mu = 1.5$, $N = 50$, $\Delta t = 0.001$ and different values of t.

with conditions

$$U(0,t) = t\ , \quad U(1,t) = (t+1)e^{-1},$$

$$U(x,0) = xe^{-x^2}\ , \quad U_t(x,0) = e^{-x^2}.$$

The exact solution of the equation is acquired by [4]

$$U(x,t) = (t+x)e^{-x^2}.$$

Error norms L_2 and L_∞ obtained for problem 1 for different values of N, μ, $\Delta t = 0.0005$ and $t_f = 1$ is given in Table 3. The table represents that all the analytical and numerical are compatible with each other for the the decreasing values of the error norms L_2 and L_∞. It is determined that when the number of division increases, the accuracy of obtained numerical results decreases as seen from the values of L_2 and L_∞ error norms. In Table 4, L_2

Table 3: Error norms of the problem 2 for $\Delta t = 0.0005$ and $t_f = 1$.

N	$\mu = 1.1$		$\mu = 1.3$		$\mu = 1.5$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
10	0.917961	1.327237	0.922233	1.333627	0.614392	0.909724
15	0.439288	0.633841	0.430411	0.621802	0.109623	0.185629
20	0.271346	0.390449	0.257806	0.372020	0.106419	0.167584

and L_∞ error norms are expressed for changing values of Δt, μ, $N = 20$ and $t_f = 1$. When the Table 4 examined it is clearly seen that as the value of Δt decreases, accuracy of the obtained results increased. The values of L_2 and L_∞ error norms proves this situation.

Table 4: Error norms of the problem 2 with $N = 20$, $t_f = 1$, different values of Δt, μ.

Δt	$\mu = 1.1$		$\mu = 1.3$		$\mu = 1.5$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.01	1.319930	1.854672	1.048109	1.484929	1.076646	1.581044
0.005	0.768010	1.084079	0.627719	0.893166	0.729204	1.063442
0.0005	0.271346	0.390449	0.257806	0.372020	0.167584	0.646934

Figures 2 shows the graphs of the exact (denoted by lines) solutions and the numerical solutions for $N = 40$, $\mu = 1.5$ and $\Delta t = 0.0005$ at $t = 0.5$, $t = 0.75$ and $t = 1.0$. The graphical illustrations represents that the numerical and analytical results are compatible with each other.

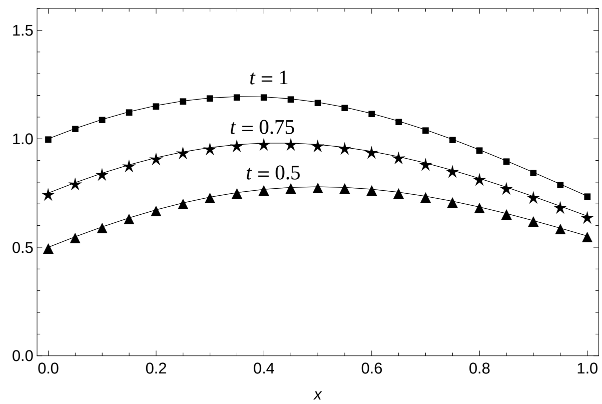

Figure 2: The curves of the analytical(lines) and numerical solutions of problem 2 for $\mu = 1.5$, $N = 40$, $\Delta t = 0.0005$ and different values of t.

Problem 3: The equation (1.2) for $\lambda = 0$ is considered

$$f_2(x,t) = \frac{6t^{3-\mu}}{\Gamma(4-\mu)} \sin(2\pi x) + \frac{6t^{4-\mu}}{\Gamma(5-\mu)} \sin(2\pi x) + 4\pi^2 t^3 \sin(2\pi x).$$

with conditions

$$U(0,t) = 0\ , \quad U(1,t) = 0,$$
$$U(x,0) = 0\ , \quad U_t(x,0) = 0.$$

The analytical solution of the equation is denoted by [16]

$$U(x,t) = t^3 \sin(2\pi x).$$

The comparison with error norms of [16] and error norms L_2 and L_∞ obtained for problem 3 for values of μ, N, Δt and $t_f = 1$ is given in Table 5. L_2

and L_∞ error norms which are given in Table 5 implies that analytical and numerical solutions are compatible with each other. When the number of division increases, the obtained numerical results become more accurate. In addition, numerical solutions which are obtained by this method are better than method by in [16].

Table 5: The comparison of [16] with numerical solutions of the problem 3 with $t_f = 1$ at difference μ, N and Δt.

N	Δt	$\mu = 1.1$		$\mu = 1.5$		$\mu = 1.9$	
		$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
80	0.00025	0.067770	0.095841	0.054139	0.076565	0.028931	0.040914
100	0.00025	0.049615	0.070167	0.057464	0.081267	0.075559	0.106857
100	0.0002	0.002040	0.002885	0.006280	0.008881	0.023155	0.032746
[16]		0.107022	0.466428	0.106908	0.466393	0.107050	0.466334

Figures 3 shows the graphs of the exact (denoted by lines) solutions and the numerical solutions for $N = 100$, $\mu = 1.5$ and $\Delta t = 0.0002$ at $t = 0.5$(triangles), $t = 0.75$(stars) and $t = 1.0$(squares). Figure shows that both numerical and exact results are analogous.

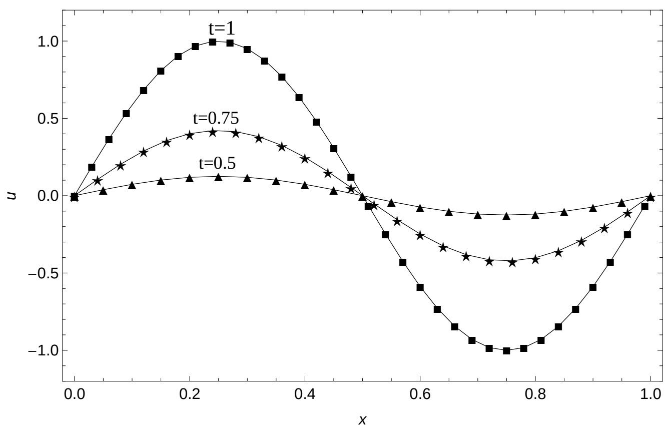

Figure 3: The curves of the analytical(lines) and numerical solutions of problem 3 for $\mu = 1.5$, $N = 100$, $\Delta t = 0.0002$ and different values of t.

Problem 4: We finally consider the equation (1.2) for $\lambda = 0.1$

$$f_2(x,t) = \frac{6t^{3-\mu}}{\Gamma(4-\mu)} \cos x + \frac{6t^{4-\mu}}{\Gamma(5-\mu)} \cos x + t^3(\cos x - 0.1 \sin x).$$

with conditions

$$U(0,t) = t^3 \ , \quad U(2\pi, t) = t^3,$$

$$U(x,0)=0\ ,\quad U_t(x,0)=0.$$

The analytical solution of the equation is evaluated by [16]

$$U(x,t)=t^3\cos(x).$$

Error norms L_2 and L_∞ obtained for problem 4 for values of μ, N, $\Delta t = 0.0005$ and $t_f = 1$ is given in Table 6. L_2 and L_∞ error norms indicates that, the analytical and numerical are compatible with each other. When the number of nodes increases, accuracy of numerical results increases. This situation can be seen from the L_2 and L_∞ error norms. In Table 7, we

Table 6: Error norms of the problem 4 with $t_f = 1$ and $\Delta t = 0.0005$ at difference μ and N.

N	$\mu = 1.4$		$\mu = 1.6$		$\mu = 1.8$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
50	6.439927	8.618383	4.825867	6.618786	2.677085	3.709226
100	1.692563	2.405105	1.336086	1.841166	1.056597	1.007581
150	0.946625	1.099383	0.878070	0.838264	0.976906	0.689826

examined the L_2 and L_∞ for different values of μ, Δt, $N = 150$ and $t_f = 1$. It is clearly seen that the accuracy of the numerical results increases when the value of Δt decreases. The L_2 and L_∞ error norms proves this event.

Table 7: Error norms of problem 4 with $N = 150$ and $t_f = 1$ for different values of μ and Δt.

Δt	$\mu = 1.4$		$\mu = 1.6$		$\mu = 1.8$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.005	8.225758	5.867495	8.974139	6.258915	11.406016	7.730185
0.001	1.590317	1.143651	1.651409	1.188126	2.044400	1.427477
0.0005	0.946625	1.099383	0.878070	0.838264	0.976906	0.689826

The analytical (denoted by lines) solutions and the numerical solutions for $N = 150$, $\mu = 1.8$ and $\Delta t = 0.0005$ at $t =$ are represented in Figure 4. The graphical illustration expresses that all the numerical and analytical results are compatible with each other.

4 Conclusion

In the present study, cubic B-spline finite element methods are employed to acquire the numerical solutions of time fractional telegraph equation. All

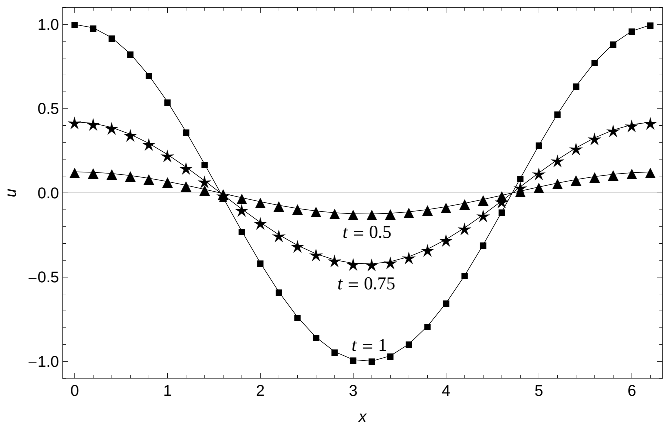

Figure 4: The curves of the analytical(lines) and numerical solutions of problem 4 for $\mu = 1.8$, $N = 300$ and $\Delta t = 0.0005$ at t.

the fractional derivatives are in Caputo type. The $L1$ and $L2$ formulae are used to discretize the fractional derivative. The obtained results indicate the accuracy and applicability of the presented method. Some graphical and table representations are given to compare the numerical results.

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Orkun Tasbozan

Department of Mathematics
Hatay Mustafa Kemal University
Hatay
Turkey

E-mail: `otasbozan@mku.edu.tr`

Alaattin Esen

Inonu University
Malatya
Turkey

E-mail: `alaattin.esen@inonu.edu.tr`