# Color constancy. I. Basic theory of two-stage linear recovery of spectral descriptions for lights and surfaces

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Changing a scene's illuminant causes the chromatic properties of reflected lights to change. This change in the lights from surfaces provides spectral information about surface reflectances and illuminants. We examine conditions under which these properties may be recovered by using bilinear models. Necessary conditions that follow from comparing the number of equations and the number of unknowns in the recovery procedure are not sufficient for unique recovery. Necessary and sufficient conditions follow from demanding a one-to-one relationship between quantum catch data and sets of lit surfaces. We present an algorithm for determining whether spectral descriptions of lights and surfaces can be recovered uniquely from reflected lights.

## 1. INTRODUCTION

Since the work of Young[1] and Helmholtz,[2] much work in color vision has been guided by three-dimensional models of chromatic processing that work in parallel over the visual field. These models envisage the visual system as a cascade of chromatic transformations that are applied to photoreceptoral quantum catches; each such transformation is represented linearly under appropriate stimulus conditions. Measurement of the dependence of these transformations on the visual system's state of adaptation and on temporal and spatial parameters of stimuli provides the framework for standard accounts of color detectability and appearance.[3]

Color researchers have sought to extend the standard model so that it applies better to normal, everyday viewing situations, in which chromatic mechanisms operate primarily on lights reflected from the surfaces of objects. Motivated by Land's and earlier work, this research has focused on color constancy, the stability of surface color appearance under varying illumination conditions.[4–12] A recent approach to color constancy uses empirically based linear models of illuminant and surface spectral properties to link physical variables describing surfaces and illumination to the operation of photoreceptoral mechanisms.[13–18] The stability of surface color appearance is equated with the recovery of surfaces' reflectance functions by the visual system: these functions describe surface color properties and do not change when the illuminant is changed.[19–29]

In this paper we use empirically based linear models of illuminants and surfaces to analyze the change in reflected lights that is caused by changing the illumination of a set of surfaces. A trichromatic visual system that is sensitive to such chromatic change can recover descriptors of surface reflectance functions and so exhibit color constancy, under natural viewing conditions.

### A. Single-View Theory

Previous theoretical work on color constancy dwelt on the problem of estimating surface and illuminant spectral properties in the situation in which a set of surfaces is viewed under a single unknown illuminant. Maloney[24] and Maloney and Wandell[25] established a general result for this single-view situation: a trichromatic system that views surfaces under an unknown illuminant can recover two reflectance descriptors per surface. It is desirable to recover three or more descriptors per surface, however, because neither color percepts[30,31] nor surface spectral properties[13,16] are described adequately by two-dimensional models.

One may recover three reflectance descriptors for each surface if one assumes that the visual system engages four or more photoreceptoral types in the task,[24,25] supplementing a trichromatic system with rods or anomalous long-wavelength photoreceptors, for instance.[32–35] There is, however, no evidence that such additional information is used independently to represent surface color under normal photopic viewing conditions.

A prominent assumption on scenes that provides three descriptors for each surface reflectance is the gray-world assumption: the space-averaged reflected light bears the spectral properties of the unknown illumination.[23,26,36] Under this assumption, a mechanism sensitive to the space-averaged light reaching the eyes is, in effect, looking at the illuminant and can determine its chromatic properties and those of the surfaces. Yet this assumption is not generally met by surfaces in a scene. Furthermore, there is no reason to believe that the stability of surface color appearance in normal viewing suffers when the assumption is not met.[37]

Highlights on two or more distinctly colored surfaces also provide information on the illuminant.[26,38–41] Yet human mechanisms of color constancy work in the absence of highlights, as in Land's demonstrations with Mondrians, and appear to ignore the information from highlights about illumination chromatic properties in assigning surface color.[42] Other sources of information that can be used to determine three color descriptors per surface include chromatic aberration and interreflection[27–29]; their relevance for human color processing is not yet clear.

Theoretical work with the single-view case shows that

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the computational problem of color constancy is soluble by a trichromatic system only if one of several unappealing assumptions is made. In the absence of such assumptions, the computational problem of determining three or more reflectance descriptors per surface from a single view is insoluble.[24,25]

### B. Two or More Views

The starting point of our analysis is the observation that the stability of surface color appearance is not an issue unless the chromatic properties of illumination change. One is thus led to ask whether the lights from a set of surfaces, viewed under two or more illuminants in turn, as in Land's demonstrations with Mondrians,[11,12] let the visual system estimate accurately both unknown surface reflectance properties and unknown illuminant spectral properties. Indeed, the chromatic change in reflected lights that is caused by changing the illumination of a set of surfaces provides abundant information concerning surface and illuminant properties.

That systematic changes in chromaticity are informative is most easily seen by analogy to structure from motion.[43–50] Suppose that points with fixed positions on a rigid body, say the eight corners of a cube, are projected onto a flat display as points of light. A single view of these points is ambiguous in two ways: (1) the three-dimensional configuration of these points cannot be determined from a single view and (2) the visual system perceives no three-dimensional structure. If one now makes these points move by, for instance, rotating the cube, then the three-dimensional configuration of these points may be determined and an observer perceives three-dimensional structure.

In the case of a Mondrian display lit by a single illuminant, a trichromatic system can recover, at best, a two-dimensional description of surface reflectances: this is the import of the research by Maloney and Wandell.[25] If the light is changed to provide further views of the Mondrian, then a three-dimensional description of each surface can be determined.[51,52] Land's demonstrations of color constancy, in which a Mondrian is illuminated sequentially by a variety of lights,[11,12] suggest that the visual system interprets such chromatic motion readily.

### C. Overview

In this paper we analyze the use of chromatic data, obtained under changing illumination, to determine stable surface color descriptors. The change in illumination can occur temporally or spatially or by some combination of the two. Our aim is to state the conditions on a bilinear model, which links illuminants, reflectances, and photoreceptors, so that it recovers uniquely illuminants and reflectances for any choice of number of photoreceptoral types, illuminant- and reflectance-model dimensions, number of views, and number of viewed surfaces.

The recovery procedures examined here are two-stage linear schemes, which are like those of Maloney and Wandell[25] and D'Zmura.[52] The procedure of Maloney and Wandell, for instance, uses quantum catch data first to determine a description of an illuminant spectral power distribution and then uses this information, in the second stage, to recover descriptions of surface reflectance functions.[25,53] D'Zmura's[52] scheme works to recover first the reflectances and then the illuminants. For two-stage linear recovery to be possible, it is necessary that the number of photoreceptoral types equal or exceed either the dimension of the illumination model or the dimension of the reflectance model. Our analysis here focuses on color constancy problems that meet this restriction.

The simplest criterion for recovery, namely, that the number of equations in the recovery procedure compare favorably with the number of unknowns to be recovered, makes unique recovery feasible but does not guarantee it. This fact motivates our examination of necessary and sufficient conditions for unique recovery. We derive from these a model check algorithm that provides a practical test of whether a particular bilinear model can be used to recover uniquely spectral descriptions from chromatic change. For bilinear models that pass such a test, we can be certain that the recovery procedure will work flawlessly when presented data that fall within the scope of those models.

In a first companion paper we present particular bilinear models that meet the criteria for unique recovery presented here.[54] We thus provide results on whether it is possible for a two-stage linear procedure to recover illuminant and reflectance descriptors for a particular choice of number of photoreceptors, illuminant- and reflectance-model dimensions, number of views, and number of viewed surfaces. In that paper are compiled the results of checking the function of a variety of bilinear models for dichromatic, trichromatic, and tetrachromatic visual systems.

Our starting point here is the construction of bilinear models that link illuminants, reflectances, and photoreceptors.[18,52,55] Such models have been featured in many prior treatments of the problem of color constancy.[20–26] Bilinear models are crucial to our analysis, and in Section 2 of this paper we develop them quite generally.

In Section 3 we analyze a two-stage linear algorithm (cf. Ref. 52) for recovering reflectance and illuminant descriptors from quantum catch data. The algorithm provides, at best, unique recovery up to an arbitrary positive scalar. This scalar expresses a well-known ambiguity: multiplying the intensity of a Mondrian's illuminant by some positive real number and dividing all surface reflectance functions by the same number has no effect on reflected lights. By comparing the number of equations and the number of unknowns in the recovery algorithm, we derive a criterion for recovery to be feasible. The feasibility criterion is a necessary condition for recovery. It generalizes the rule, proposed by Maloney and Wandell,[25] that the number of photoreceptors must exceed the number of descriptors to be recovered per reflectance.

In Section 4 we examine both necessary and sufficient conditions for recovery to work. An example involving dichromatic visual systems shows that meeting the feasibility criterion is not enough: an excess of equations over unknowns in the recovery procedure is no guarantee that the recovery works to provide reflectance and illuminant descriptors.

This leads to the heart of the paper (Section 5), in which we present a practical way to test whether a given bilinear model provides unique recovery. This model check algorithm provides a test of a wide variety of bilinear models. In particular, it allows us to test whether a trichromatic visual system can recover three descriptors per reflectance from two views.[52] The algorithm provides a sufficient test of bilinear model function. If a particular bilinear model passes the test, then its recovery of reflectance and illuminant descriptors is unique. On the other hand, if a particular model fails the test, we can draw no conclusions concerning unique recovery without further analysis.

In Section 6 we examine the applicability of the model check algorithm. Although it has a wide scope and is numerically tractable, there are cases of interest that escape its net (e.g., that in which a tetrachromatic visual system attempts to recover four descriptors per illuminant and three descriptors per reflectance[53]). Many of these we deal with by hand, on a case-by-case basis, in the companion paper.[54]

## 2. BILINEAR MODELS

Throughout this series of papers we assume that an accurate bilinear model is known to the visual system. In this section we review the construction of such a model from its illuminant, reflectance, and photoreceptor constituents and introduce notation.

A bilinear model is built of three elements (see Fig. 1): A, a linear, finite-dimensional model for the spectral power distributions of illuminants met in a particular environment; B, a linear, finite-dimensional model for the spectral reflectance functions of surfaces met in that environment; and C, the spectral sensitivities of the visual system's photoreceptors. The models for illumination and for reflectance comprise finite sets of basis functions that are combined linearly to form approximations to particular illuminant spectral power distributions or surface-reflectance functions, respectively. The coefficients in such an expansion are a finite set of descriptors for the particular function of wavelength. A bilinear model, which arises from given photoreceptoral spectral sensitivities and a choice of models for illumination and reflectance, describes completely all quantum catches that arise from the environment: the bilinear model maps illuminant and reflectance descriptors to observed quantum-catch data. In the following specification of a bilinear model we introduce a number of symbols; these and others are listed in Table 1. Throughout, we use boldface symbols for vectors and matrices.

### A. Illuminants

An $m$-dimensional linear model for describing illuminants comprises $m$ orthogonal basis functions $\{A_1(\lambda), \ldots, A_i(\lambda), \ldots, A_m(\lambda)\}$ that are combined linearly to approximate any particular illuminant spectral power distribution $A(\lambda)$. In Fig. 1A are illustrated the three CIE-standard daylight basis functions of Judd *et al.*[14] that are used to approximate the various phases of daylight; these have been transformed linearly to provide three functions, sampled at 10-nm intervals, that are orthonormal on the interval 400 nm–700 nm of visible wavelengths. These three functions, obtained from a principal components analysis of a set of 622 daylight samples, provide excellent approximations to actual daylight illumination spectral power distributions.[3,14]

The accuracy of an illumination model is determined by how well the model approximates the spectral properties of lights met in a particular environment. The approximation $\hat{A}(\lambda)$ to illuminant $A(\lambda)$ is determined by projecting the illuminant $A(\lambda)$ onto the subspace of illuminants spanned by the model's basis functions:

$$A(\lambda) \cong \hat{A}(\lambda) = \sum_{i=1}^{m} a_i A_i(\lambda), \tag{1}$$

where

$$a_i = \int A(\lambda) A_i(\lambda) d\lambda \quad \text{for } i = 1, \ldots, m. \tag{2}$$

The $m$ coefficients $a_i$ are the model's descriptors of illuminant $A(\lambda)$. The accuracy of their description depends on the difference between illuminant $A(\lambda)$ and the model's approximation $\hat{A}(\lambda)$. Elsewhere we consider departures of illuminants (and reflectance functions) from their finite-

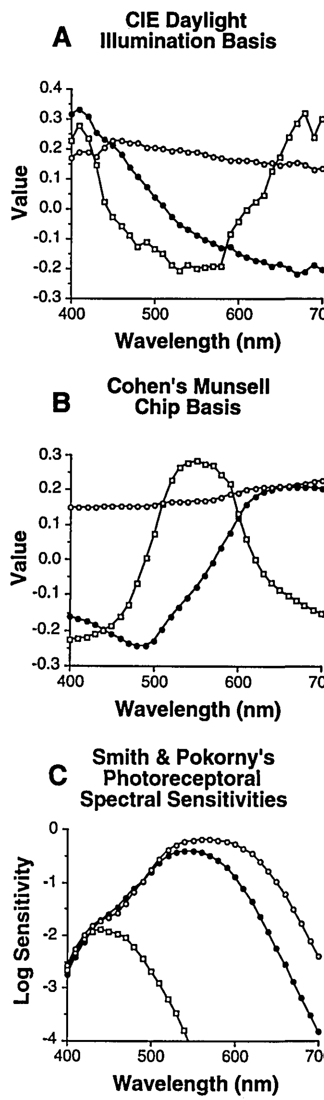


Fig. 1. Linear components of a bilinear model: an example. A, Model for illumination represented by three basis functions that describes well the phases of daylight[14]; B, three-dimensional model for surface reflectance that describes well Munsell chips[13]; C, linear model for human trichromatic photoreception.[56] See text for discussion.

**Table 1. List of Symbols**

| | |
|---|---|
| Bilinear model parameters | |
| $p$ | Number of photoreceptor types |
| $m$ | Illuminant model dimension |
| $n$ | Reflectance model dimension |
| $v$ | Number of views |
| $s$ | Number of surfaces |
| $d_v$ | Defect in views $m - v$ |
| $d_s$ | Defect in surfaces $n - s$ |
| $d_m$ | Defect in illumination $p - m$ |
| $d_n$ | Defect in reflectance $p - n$ |
| Functions of wavelength | |
| $A(\lambda)$ | Illuminant spectral power distribution |
| $A_i(\lambda)$ | $i$th illuminant model basis function, $i = 1,\ldots,m$ |
| $L(\lambda)$ | Reflected light |
| $Q_k(\lambda)$ | $k$th photoreceptor spectral sensitivity, $k = 1,\ldots,p$ |
| $R(\lambda)$ | Reflectance function |
| $R_j(\lambda)$ | $j$th reflectance model basis function, $j = 1,\ldots,n$ |
| Descriptors, vectors, and matrices | |
| $a_i, \mathbf{a}_w, \mathbf{a}, \mathbf{A}, \mathbf{z}, \mathbf{Z}$ | Illuminant descriptors |
| $\mathbf{B}_j$ | Bilinear model matrices |
| $\mathbf{C}_j$ | Block-diagonal bilinear model matrices diag$[\mathbf{B}_j \ldots \mathbf{B}_j]$ |
| $\mathbf{\Gamma}_{ij}$ | Gamma matrices $\mathbf{B}_i^{-1}\mathbf{B}_j$ |
| $\mathbf{d}_t, \mathbf{D}$ | Quantum catch data |
| $e_{ij}, \mathbf{e}, \mathbf{E}$ | Variables relating two sets of reflectances |
| $\mathbf{F}$ | Recovery matrix |
| $\mathbf{G}_{ij}$ | Block-diagonal gamma matrices diag$[\mathbf{\Gamma}_{ij} \ldots \mathbf{\Gamma}_{ij}]$ |
| $\mathbf{I}$ | Identity matrix |
| $r_j, \mathbf{r}_t, r_{tj}, \mathbf{R}$ | Reflectance descriptors |
| $\rho_{jt}, \boldsymbol{\rho}_j, \boldsymbol{\rho}, \mathbf{P}$ | Inverse of the matrix of reflectance descriptors |
| $\mathbf{w}_{1-j,k}, \mathbf{W}_{1-j}$ | Differences of linear combinations of gamma matrices |
| Miscellany | |
| E, U | Number of equations and number of monomial unknowns, respectively, provided by the model check algorithm |
| Q, D | Number $svp$ of quantum catch data and number $sn + vm$ of unknown descriptors to be recovered |

dimensional models[57]; here we assume that illuminants fall within the subspace spanned by the model basis functions. The model's approximation to $A(\lambda)$ is then exact, so that

$$A(\lambda) = \sum_{i=1}^{m} a_i A_i(\lambda). \qquad (3)$$

### B. Reflectances

An $n$-dimensional linear model for describing surface reflectance functions comprises $n$ orthogonal basis functions $\{R_1(\lambda), \ldots, R_j(\lambda), \ldots, R_n(\lambda)\}$ that are combined linearly to approximate a particular surface reflectance function $R(\lambda)$. The functions shown in Fig. 1B are the first three surface reflectance basis functions determined by Cohen[13]; the original functions have been orthonormalized. These three functions account for 99.18% of the variance among a set of 433 Munsell-chip reflectance functions.[13]

A reflectance function $R(\lambda)$ that falls within the subspace spanned by the model's basis functions is given by the following linear combination:

$$R(\lambda) = \sum_{j=1}^{n} r_j R_j(\lambda), \qquad (4)$$

where

$$r_j = \int R(\lambda) R_j(\lambda) \mathrm{d}\lambda \qquad \text{for } j = 1, \ldots, n. \qquad (5)$$

The $n$ numbers $r_j$ are the descriptors of the surface reflectance function $R(\lambda)$. Because these descriptors depend only on surface reflectance properties, they do not change when the illuminant changes: surface-reflectance descriptors are color constant.

### C. Quantum Catch Data

A light $L(\lambda)$ that is reflected from a surface with reflectance $R(\lambda)$, viewed under illuminant $A(\lambda)$, is the product of the two functions and has the following expansion:

$$L(\lambda) = A(\lambda)R(\lambda) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i r_j A_i(\lambda) R_j(\lambda). \qquad (6)$$

A $p$-chromatic visual system comprising photoreceptors with $p$ linearly independent spectral sensitivities $\{Q_1(\lambda), \ldots, Q_k(\lambda), \ldots, Q_p(\lambda)\}$ responds to the reflected light $L(\lambda)$ by producing $p$ quantum catches $q_k, k = 1, \ldots, p$. An exemplary set of photoreceptoral spectral sensitivity functions for $p = 3$, those of Smith and Pokorny,[56] is shown in Fig. 1C. Quantum catches $q_k$ are given by the integral of the product of the $k$th photoreceptoral mechanism's spectral sensitivity $Q_k(\lambda)$ and the light $L(\lambda)$; using Eq. (6), one obtains the following expressions for the $k$th quantum catch:

$$q_k = \int Q_k(\lambda) L(\lambda) \mathrm{d}\lambda = \int Q_k(\lambda) \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} a_i r_j A_i(\lambda) R_j(\lambda) \right\} \mathrm{d}\lambda. \qquad (7)$$

To bring the $m$ illuminant descriptors $a_i$ and the $n$ reflectance descriptors $r_j$ outside the integral in Eq. (7), we find it convenient to define the bilinear model matrices $\mathbf{B}_j$, with entries

$$(\mathbf{B}_j)_{ki} = \int Q_k(\lambda) A_i(\lambda) R_j(\lambda) \mathrm{d}\lambda, \qquad j = 1, \ldots, n. \qquad (8)$$

The rows of $\mathbf{B}_j$ are indexed by $k$, which runs over the $p$ photoreceptoral spectral sensitivities, and its columns are indexed by $i$, which runs over the $m$ illuminant-model basis functions. The $n$ matrices $\mathbf{B}_j$, one for each reflectance basis function, are thus of matrix dimension $p \times m$.

Using Eqs. (7) and (8), we express the dependence of quantum catches on reflectance and illuminant descriptors in terms of the components of the matrices $\mathbf{B}_j$:

$$q_k = \sum_{j=1}^{n} \sum_{i=1}^{m} r_j (\mathbf{B}_j)_{ki} a_i. \qquad (9)$$

Holding the illuminant descriptors $a_i$ constant in Eq. (9), we can see that the quantum catches vary linearly with the reflectance descriptors $r_j$. Likewise, the quantum catches vary linearly with the illuminant descriptors when the reflectance descriptors are held constant. This bilinearity continues to hold when we express the quantum catches that arise from many surfaces, such as those in a Mondrian display, that are viewed under two or more illuminants.

### D. Multiple Views of Multiple Surfaces

We now extend bilinear models to treat multiple views of multiple surfaces. Suppose that a Mondrian comprising $s$ surfaces with distinct (i.e., linearly independent) reflectance functions is lit, in turn, by $v$ illuminants. Each surface provides $p$ quantum catches, so that the total number of quantum-catch data from $v$ views of $s$ surfaces is $svp$. Let us introduce indices $t$ and $w$, which run over the number $s$ of surfaces and number $v$ of views, respectively. Then the quantum catch $q_{twk}$ of the $k$th photoreceptoral type produced by the $t$th surface viewed under the $w$th light is related to the $n$ reflectance descriptors $r_{tj}$, for $j = 1, \ldots, n$, and the $m$ illuminant descriptors $a_{wi}$, for $i = 1, \ldots, m$, in the following generalization of Eq. (9):

$$q_{twk} = \sum_{j=1}^{n} \sum_{i=1}^{m} r_{tj}(\mathbf{B}_j)_{ki} a_{wi}. \tag{10}$$

Note that, by identifying the quantum catches according to their surface of origin, we have assumed correspondence, namely, the ability of the visual system to keep track of surfaces as they are seen under different illuminants.

It proves useful to recast Eq. (10) in matrix form. We introduce $p$-dimensional data vectors $\mathbf{d}_{tw} = [q_{tw1} \ldots q_{twp}]^T$ ($T$ denotes tranpose), and $m$-dimensional vectors of illuminant descriptors $\mathbf{a}_w = [a_{w1} \ldots a_{wm}]^T$; Eq. (10) becomes

$$\mathbf{d}_{tw} = \sum_{j=1}^{n} r_{tj} \mathbf{B}_j \mathbf{a}_w, \tag{11}$$

for $t = 1, \ldots, s$ and $w = 1, \ldots, v$. A variant of Eq. (11) is encountered below in the discussion of recovery. If we define the $pv$-dimensional vectors $\mathbf{d}_t = [\mathbf{d}_{t1}{}^T \ldots \mathbf{d}_{tv}{}^T]^T$, the $mv$-dimensional vector $\mathbf{a} = [\mathbf{a}_1{}^T \ldots \mathbf{a}_v{}^T]^T$, and the $pv \times mv$ block-diagonal matrices

$$\mathbf{C}_j = \mathrm{diag}[\mathbf{B}_j, \ldots, \mathbf{B}_j], \tag{12}$$

in which each of the $v$ blocks along the diagonal is $\mathbf{B}_j$, Eq. (11) takes on the form (cf. Ref. 52)

$$\mathbf{d}_t = \sum_{j=1}^{n} r_{tj} \mathbf{C}_j \mathbf{a} \qquad t = 1, \ldots, s. \tag{13}$$

### E. Feasibility Condition

The number of quantum catches is $svp$, and this must equal or exceed the number of unknown descriptors to be recovered [Eq. (10)]. There are $sn$ total descriptors for the surface reflectances and $vm$ descriptors for the illuminants. Taking into account the inevitable ambiguity of scale, we see that unless

$$svp \geq sn + vm - 1, \tag{14}$$

unique linear recovery of reflectance and illumination descriptors is impossible. In Section 3 we show that this feasibility condition simplifies considerably for two-stage linear recovery procedures.

## 3. TWO-STAGE LINEAR RECOVERY PROCEDURES

The following analysis of recovery rests on a geometric intuition similar to that introduced by Maloney[24] and Maloney and Wandell.[25] These authors represent the response of a trichromatic visual system to the light from a single surface by a three-dimensional vector that resides in a three-dimensional space of quantum catches. Lights from two (or more) surfaces provide two (or more) vectors, and if the surfaces have distinct reflectances that are described by a two-dimensional model of reflectance, these vectors lie in a plane through the origin: a two-dimensional subspace. Maloney and Wandell showed that the orientation of this plane in the space of quantum catches is determined by the chromatic properties of the illuminant and that the (two-dimensional) position within the plane of each surface's vector of quantum catches is determined by its reflectance properties.

Suppose now that three surfaces are lit first by one light and then by another to provide two views. The quantum catches from a single surface form a six-dimensional vector that resides in a six-dimensional space of quantum catch pairs. Lights from three (or more) surfaces provide three (or more) of these six-dimensional data vectors. If the surfaces have distinct reflectances that are described by a three-dimensional model of reflectance, then these vectors define a three-dimensional subspace within the space of quantum catch pairs. The orientation of this subspace is determined by the chromatic properties of the pair of illuminants, each of which is constrained to vary in three dimensions, and the (three-dimensional) position of each surface's vector of quantum catches within the subspace is determined by its (three-dimensional) reflectance properties.[52]

We use Eq. (13) to develop this and similar arguments more formally. For expository purposes, we treat first the problem of recovering descriptors in cases where $p = m$ and $s = n$ (Subsection 3.A). The recovery is unique, at best, up to an arbitrary positive scalar. We continue by determining the dependence of the number of equations and the number of unknowns on bilinear model parameters (Subsection 3.B). A necessary condition follows from the requirement that the number of equations equal or exceed the number of unknowns in the recovery procedure. This condition generalizes to multiple views Maloney and Wandell's[25] rule that the number of photoreceptoral types must exceed the dimension of the reflectance model. We then augment the recovery procedure to handle rectangular bilinear models met in problems where $p > m$, thus extending its scope to problems where $p \geq m$ (Subsection 3.C). Finally, in Subsection 3.D we discuss transposition: interchanging the roles of surfaces and illuminants appropriately in a working recovery procedure leads to another working procedure.

### A. Two-Stage Linear Recovery with Square Bilinear Model Matrices

Problems of the form $(p\,m\,n\,v\,s) = (p\,p\,n\,v\,n)$ provide a convenient starting point for our analysis. These include two of special interest, namely, that where $(p\,m\,n\,v\,s) = (3\,3\,2\,1\,2)$ studied by Maloney[24] and Maloney and Wandell[25] and the problem $(3\,3\,3\,2\,3)$ examined by D'Zmura.[52] In both cases, the $p \times m$ bilinear model matrices $\mathbf{B}_j$, $j = 1, \ldots, n$, are square, so that the multiple-view model matrices $\mathbf{C}_j$ of Eq. (12) are also square, as is the matrix $\mathbf{R}$ formed by the reflectance descriptors $r_{tj}$.

Assume that in Eq. (13) the data vectors $\mathbf{d}_t$ from the $s$ surfaces, where $s = n$, span an $n$-dimensional subspace in $\mathbb{R}^{pv}$. For each fixed vector of illuminant descriptors $\mathbf{a}$, the $n$ vectors $\mathbf{C}_j\mathbf{a}, j = 1, \ldots, n$, provide a basis for that subspace. The reflectance descriptors $r_{tj}$ thus express the data vectors in terms of that basis. These reflectance descriptors can be obtained by assuming that (1) the $n \times n$ square matrix $\mathbf{R}$ of reflectance descriptors (with entries $r_{tj}$) has an inverse $\mathbf{P}$ (with entries $\rho_{jt}$) and (2) that each of the $pv \times pv$ square matrices $\mathbf{C}_j$, for $j = 1, \ldots, n$, is nonsingular. Equation (13) then may be rewritten in the following form[52]:

$$\sum_{t=1}^{n} \rho_{jt} \mathbf{C}_j^{-1} \mathbf{d}_t = \mathbf{a} \qquad \text{for } j = 1, \ldots, n. \tag{15}$$

We combine each surface's quantum-catch data to form the $pv \times n$ data matrix

$$\mathbf{D} = [\mathbf{d}_1 \ldots \mathbf{d}_n], \tag{16}$$

and define the $n$ inverse reflectance column vectors

$$\boldsymbol{\rho}_j = [\rho_{j1} \ldots \rho_{jn}]^T; \tag{17}$$

then system (15) entails

$$\mathbf{C}_j^{-1} \mathbf{D} \boldsymbol{\rho}_j = \mathbf{a} \qquad \text{for } j = 1, \ldots, n. \tag{18}$$

In terms of the $pv \times n$ matrices

$$\mathbf{F}_j = \mathbf{C}_j^{-1} \mathbf{D} \qquad \text{for } j = 1, \ldots, n, \tag{19}$$

one has

$$\mathbf{F}_1 \boldsymbol{\rho}_1 = \ldots = \mathbf{F}_n \boldsymbol{\rho}_n = \mathbf{a}, \tag{20}$$

and by taking differences in Eq. (20), one finds that

$$\mathbf{F}_1 \boldsymbol{\rho}_1 - \mathbf{F}_2 \boldsymbol{\rho}_2 = \ldots = \mathbf{F}_1 \boldsymbol{\rho}_1 - \mathbf{F}_n \boldsymbol{\rho}_n = \mathbf{0}. \tag{21}$$

These $n - 1$ matrix equations can be combined to form a homogeneous system as follows:

$$\begin{bmatrix} \mathbf{F}_1 & -\mathbf{F}_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{F}_1 & \mathbf{0} & -\mathbf{F}_3 & & \mathbf{0} \\ \vdots & & & \ddots & \\ \mathbf{F}_1 & \mathbf{0} & & & -\mathbf{F}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \vdots \\ \boldsymbol{\rho}_n \end{bmatrix} = \mathbf{0}, \tag{22}$$

which can be written more compactly as

$$\mathbf{F} \boldsymbol{\rho} = \mathbf{0}, \tag{23}$$

where $\mathbf{F}$ is the partitioned $(n - 1)pv \times n^2$ matrix appearing in Eq. (22) and the $n^2$-dimensional column vector $\rho$ is given by

$$\boldsymbol{\rho} = [\boldsymbol{\rho}_1^T \ldots \boldsymbol{\rho}_n^T]^T. \tag{24}$$

Unique recovery of the descriptors $\boldsymbol{\rho}$, which lie in the kernel of $\mathbf{F}$, will obtain from a singular value decomposition of $\mathbf{F}$[58] if and only if the kernel of $\mathbf{F}$ is one dimensional:

$$\dim[\ker(\mathbf{F})] = 1. \tag{25}$$

The first stage of the recovery procedure thus uses a singular value decomposition of $\mathbf{F}$ to return the descriptors $\rho_{jt}$ and, by matrix inversion, the reflectance descriptors $r_{tj}$. The procedure's second stage returns the vector $\mathbf{a}$ containing the illuminant descriptors by application of any one of the identities of Eq. (20).

### B. Condition $pv > n$

When a problem's parameters are chosen so that $p = m$ and $n = s$, as above, the general feasibility condition [inequality (14)] becomes

$$pnv \geq n^2 + pv - 1 \tag{26a}$$

or, equivalently,

$$pv(n - 1) \geq n^2 - 1. \tag{26b}$$

Note that the left-hand side is the number of rows of the recovery matrix $\mathbf{F}$ [Eq. (23)], while the right-hand side is the number of columns, minus one: for $\mathbf{F}$ to have a kernel of dimension one, it is necessary that the number of its rows equal or exceed the number of its columns, minus one. Canceling the positive common factor $n - 1$ from both sides of inequality (26b), we find that

$$pv \geq n + 1 \tag{27a}$$

or, more simply,

$$pv > n. \tag{27b}$$

When the number $v$ of views is taken to be one, inequality (27b) becomes $p > n$, which is precisely the rule introduced by Maloney and Wandell[25] in their analysis of one-view problems.

The condition $pv > n$ for unique recovery has a simple geometric significance: if the $pv$-dimensional vectors $\mathbf{C}_j\mathbf{a}, j = 1, \ldots, n$, are to provide a basis for an $n$-dimensional subspace of $\mathbb{R}^{pv}$ that is fixed uniquely by $\mathbf{a}$, it must occur that the subspace is proper, i.e., $pv > n$.

This is an appropriate place to caution the reader again that, while inequalities (14) and (27b) provide necessary conditions for unique recovery of descriptors, they are far from being sufficient: their purpose is to distinguish problems $(p\,m\,n\,v\,s)$ in which unique recovery is feasible from those problems in which it is not.

### C. Rectangular Bilinear Model Matrices ($p > m$)

The two-stage algorithm developed under the restriction $p = m$ can be extended to handle problems in which the number $p$ of photoreceptoral types exceeds the dimension $m$ of the illuminant model. If $p > m$, then the $p \times m$ bilinear model matrices $\mathbf{B}_j$ are rectangular and so do not possess inverses. One can circumvent this problem by supplementing the matrices $\mathbf{B}_j$ with additional entries to make them square, invertible matrices $\mathbf{B}_j^*$.

Assuming that the $m$ columns of the rectangular matrices $\mathbf{B}_j$ are linearly independent, one can find $d_m = p - m$ further columns for each matrix that span the unique subspace orthogonal to that spanned by the existing columns. The parameter $d_m$ is the defect in illumination that specifies the number of columns $\mathbf{b}^*$ that are adjoined to each matrix $\mathbf{B}_j$ to transform it from a $p \times m$ matrix to a square, invertible $p \times p$ matrix $\mathbf{B}_j^*$:

$$\mathbf{B}_j^* = [\mathbf{B}_j \mathbf{b}_{j,1}^* \ldots \mathbf{b}_{j,d_m}^*]. \tag{28}$$

If the rectangular matrices $\mathbf{B}_j$ are replaced by the square matrices $\mathbf{B}_j^*$, the two-stage recovery procedure of Subsection 3.A works to make recovery possible when $p > m$. The vectors of illuminant descriptors that are recovered have zeros in the entries that correspond to the last $p - m$ illuminant dimensions.

### D. Bilinear Model Transposition

The foregoing analysis of $p$-chromatic systems that use $m$- and $n$-dimensional models for illuminants and reflectances, respectively, to recover descriptors from $v$ views of $s$ surfaces applies equally well to a $p$-chromatic system that uses $n$- and $m$-dimensional models for illuminants and reflectances, respectively, to recover descriptors from $s$ views of $v$ surfaces. Each surface provides a view of the illuminant(s). This transposition is well suited to the analysis of the recovery of spectral descriptions from chromatic change in the case of many views of few surfaces.

We define the transposed bilinear model matrices $\mathbf{B}_i'$, for $i = 1, \ldots, m$, with entries

$$(\mathbf{B}_i')_{kj} = \int Q_k(\lambda) A_i(\lambda) R_j(\lambda) \mathrm{d}\lambda \tag{29}$$

[cf. Eq. (8)] and use these to express the dependence of quantum catches on illuminant and reflectance descriptors [cf. Eq. (10)]:

$$q_{wtk} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{wi} (\mathbf{B}_i')_{kj} r_{tj}. \tag{30}$$

This system of equations is of precisely the same form as that of Eq. (10), except that the roles of surfaces and illuminants have been interchanged: the dimensions $n$ and $m$ reverse their roles, as do the number of surfaces $s$ and the number of views $v$. The two-stage recovery procedure can be applied to this system under the conditions that (1) the illuminant descriptors $a_{wi}$ form a square invertible matrix ($v = m$) and (2) the (possibly augmented) transposed model matrices are invertible ($p \geq n$).

Feasibility condition (14) is invariant under the simultaneous interchange of $v$ and $s$, and $m$ and $n$. When $p \geq n$ and $v = m$, one obtains the analog to inequality (27b):

$$ps > m. \tag{31}$$

The condition $p \geq n$ arises from the fact that the transposed bilinear model matrices must be invertible for the two-stage recovery procedure to function. We call the parameter $d_n = p - n$ the defect in reflectance.

## 4. THE FEASIBILITY CONDITION DOES NOT GUARANTEE RECOVERY

That a favorable comparison of number of data to number of unknowns [inequality (14)] proves necessary but not sufficient for recovery to be unique should come as no surprise. Inequalities (14), (27b), and (31) involve merely the dimensional parameters $(p\, m\, n\, v\, s)$ of a recovery problem. They speak neither to the structure of a model's matrices $\mathbf{B}_j$ (or $\mathbf{C}_j$) nor to the structure of recovery matrices **F**, both of which are critical to actual recovery.

In this section we work toward formulating necessary and sufficient conditions that guarantee the unique recovery of spectral descriptors for feasible problems with parameters $(p\, m\, n\, v\, s)$ that fulfill $s = n$ (or dually $v = m$). The general formulation is nonlinear and is the subject of Section 5. Here we content ourselves with special cases that point the way to those more general considerations.

In Subsection 4.A we develop necessary and sufficient conditions for the problem $(p\, m\, n\, v\, s) = (3\,3\,3\,3\,3)$ of recovering descriptors from three views of three surfaces. This provides an example of a special class of problems, namely, those for which $p = m = v$ and $s = n$; and, as shown in Subsections 4.A and 4.B, linear methods suffice. In Subsection 4.C, we apply these methods to the problem $(p\, m\, n\, v\, s) = (2\,2\,2\,2\,2)$, in which a dichromatic visual system is presented two views of two surfaces and seeks to recover two descriptors per reflectance and per illuminant. We show that this dichromatic problem leads to catastrophic failure, despite the fact that the necessary condition expressed by inequalities (14), (27b), and (31) is fulfilled. This dichromatic example shows in dramatic fashion that the necessary conditions deduced by comparing equations and unknowns in a recovery procedure are not sufficient to guarantee recovery. In Subsection 4.D we discuss the variety of ways in which a bilinear model may fail.

### A. Toward Necessary and Sufficient Conditions

If different sets of lit surfaces give rise to identical quantum catch data, there is no hope of recovering unique spectral descriptions of lights and surfaces from quantum-catch data. A bilinear model must provide a one-to-one relationship between sets of lit surfaces and quantum-catch data if recovery is to be unique.

Consider the quantum catch data from two sets of three surfaces, in which each set is represented by three linearly independent vectors of reflectance descriptors (see Fig. 2).

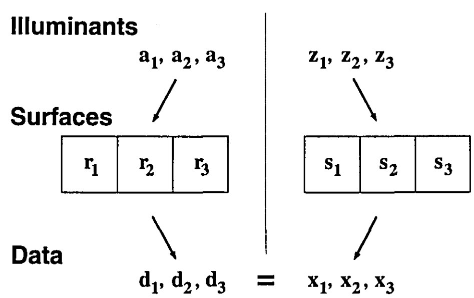


Fig. 2. Formulation of necessary and sufficient conditions for the unique recovery of illuminant and reflectance descriptors from quantum catch data, up to an arbitrary positive scalar: an example provided by the case $(p\, m\, n\, v\, s) = (3\,3\,3\,3\,3)$. The data $\mathbf{d}_t$ from the three surfaces $\mathbf{r}_t$, $t = 1, 2, 3$, when viewed sequentially under the three illuminants $\mathbf{a}_w$, $w = 1, 2, 3$, must be equal to the data $\mathbf{x}_t$ from the surfaces $\mathbf{s}_t$ viewed under the illuminants $\mathbf{z}_w$, if and only if the illuminants are identical up to a single scalar [$\mathbf{z}_w = \lambda \mathbf{a}_w$ for $w = 1, 2, 3$] and the reflectances are identical up to the reciprocal scalar [$\mathbf{s}_t = (1/\lambda)\mathbf{r}_t$ for $t = 1, 2, 3$]. See text for discussion.

Suppose that the first set is lit by three distinct illuminants $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, in turn, and that the second set is lit by three distinct illuminants $\mathbf{z}_1, \mathbf{z}_2$, and $\mathbf{z}_3$. The subspace spanned by the data vectors $\mathbf{d}_t, t = 1, 2, 3$, from the first set of surfaces either is identical to the subspace spanned by the data vectors $\mathbf{x}_t, t = 1, 2, 3$, from the second set, or it differs. Let us suppose that these subspaces are, in fact, identical three-dimensional subspaces of the nine-dimensional space of quantum catch data vectors. The only way that these subspaces can be identical, if the bilinear model used in the recovery procedure is to provide unique descriptors up to an arbitrary positive scalar, is for $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ to be related to $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, respectively, by a single scale factor: triples of illuminants must be in one-to-one correspondence with data subspaces.

The positions within the subspace of the data vectors from the surfaces with vectors of reflectance descriptors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, lit by $\mathbf{a}$, are either identical to the surfaces lit by $\mathbf{z}$ with reflectance vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ or not. If the data vectors are identical, then the linear algorithms of Section 3 provide unique reflectance descriptors, up to an arbitrary positive scalar, only if the position of a surface's data vector within a particular subspace is determined by the surface's reflectance descriptors.

Stacking the illuminant vectors $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ to form a nine-dimensional vector $\mathbf{z}$ and stacking the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ to form a vector $\mathbf{a}$, we find that the illuminant triple $\mathbf{a}$ generates a data subspace spanned by the three data vectors $\mathbf{d}_t$ from the surfaces $\mathbf{r}_t$,

$$\mathbf{d}_t = \sum_{j=1}^{3} r_{tj}\mathbf{C}_j\mathbf{a}, \tag{32}$$

and that the illuminant triple $\mathbf{z}$ generates a subspace spanned by the three data vectors $\mathbf{x}_t$ from the surfaces $\mathbf{s}_t$,

$$\mathbf{x}_t = \sum_{j=1}^{3} s_{tj}\mathbf{C}_j\mathbf{z}, \tag{33}$$

for $t = 1, 2, 3$ [see Eq. (13) and Fig. 2].

Suppose now that the triples of data vectors are identical, so that

$$\mathbf{d}_t = \mathbf{x}_t = \sum_{i=1}^{3} s_{ti}\mathbf{C}_i\mathbf{z} = \sum_{j=1}^{3} r_{tj}\mathbf{C}_j\mathbf{a} \qquad \text{for } t = 1, 2, 3. \tag{34}$$

If the matrix $\mathbf{S}$ of reflectance descriptors $s_{ij}$ is invertible, Eq. (34) yields

$$\mathbf{C}_i\mathbf{z} = \sum_{j=1}^{3} e_{ij}\mathbf{C}_j\mathbf{a} \qquad \text{for } i = 1, 2, n = 3, \tag{35}$$

where

$$e_{ij} = \sum_{t=1}^{3} \sigma_{it} r_{tj} \qquad \text{for } i = 1, 2, n = 3, \tag{36}$$

in terms of the elements $\sigma_{it}$ composing the matrix $\mathbf{S}^{-1}$. The $3 \times 3$ matrix with entries $e_{ij}$ must be a multiple of the identity matrix if reflectance descriptors are to correspond uniquely to data vector positions within the illumination-defined subspace.

We conclude that the recovery procedure of Section 3 will work, given data from three views of three surfaces with linearly independent vectors of reflectance descriptors, if and only if Eq. (35) has just the trivial scaling solutions

$$e_{11} = e_{22} = e_{33}, \qquad e_{12} = e_{21} = e_{13} = e_{31} = e_{23} = e_{32} = 0. \tag{37}$$

To check the function of a bilinear model, namely, to determine whether the only solution to Eq. (35) is the scaling solution [Eqs. (37)], define $\mathbf{G}_{ij} = \mathbf{C}_i^{-1}\mathbf{C}_j$ (the bilinear model matrices $\mathbf{B}_j$ must be invertible), so that

$$\mathbf{z} = \sum_{j=1}^{3} e_{1j}\mathbf{G}_{1j}\mathbf{a} = \sum_{j=1}^{3} e_{2j}\mathbf{G}_{2j}\mathbf{a} = \sum_{j=1}^{3} e_{3j}\mathbf{G}_{3j}\mathbf{a}. \tag{38}$$

The three blocks along the diagonals of the $\mathbf{G}_{ij}$ are identical, so that Eq. (38), relating vectors in $\mathbb{R}^9$, may be recast to relate $3 \times 3$ matrices $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ and $\mathbf{Z} = [\mathbf{z}_1 \quad \mathbf{z}_2 \quad \mathbf{z}_3]$ through the $3 \times 3$ submatrices $\mathbf{\Gamma}_{ij}$ of the $\mathbf{G}_{ij}$:

$$\mathbf{Z} = \sum_{j=1}^{3} e_{1j}\mathbf{\Gamma}_{1j}\mathbf{A} = \sum_{j=1}^{3} e_{2j}\mathbf{\Gamma}_{2j}\mathbf{A} = \sum_{j=1}^{3} e_{3j}\mathbf{\Gamma}_{3j}\mathbf{A}. \tag{39}$$

The gamma matrices figure prominently in the following analyses. They are defined by

$$\mathbf{\Gamma}_{ij} = \mathbf{B}_i^{-1}\mathbf{B}_j \tag{40}$$

and have the properties

$$\mathbf{\Gamma}_{ij} = \mathbf{\Gamma}_{ji}^{-1}, \tag{41}$$

$$\mathbf{\Gamma}_{ii} = \mathbf{I}, \tag{42}$$

the identity matrix.

Assuming that the three illuminants are distinct, namely, that they are described by linearly independent vectors of descriptors, then we can multiply the matrices of the system [Eq. (39)] on the right-hand side by the inverse $\mathbf{A}^{-1}$ of the matrix $\mathbf{A}$:

$$\mathbf{Z}\mathbf{A}^{-1} = \sum_{j=1}^{3} e_{1j}\mathbf{\Gamma}_{1j} = \sum_{j=1}^{3} e_{2j}\mathbf{\Gamma}_{2j} = \sum_{j=1}^{3} e_{3j}\mathbf{\Gamma}_{3j}. \tag{43}$$

If the scaling solution [Eqs. (37)] is the only solution to this system of equations for a particular choice of bilinear model, then the leftmost matrix in Eq. (43) can only be a multiple of the identity: a particular three-dimensional data subspace determines the corresponding illuminants up to an arbitrary positive scalar.

To investigate the space of solutions to the equations [Eq. (43)] for a particular bilinear model, form the two differences

$$\sum_{j=1}^{3} e_{1j}\mathbf{\Gamma}_{1j} - \sum_{j=1}^{3} e_{2j}\mathbf{\Gamma}_{2j} = \mathbf{0},$$
$$\sum_{j=1}^{3} e_{1j}\mathbf{\Gamma}_{1j} - \sum_{j=1}^{3} e_{3j}\mathbf{\Gamma}_{3j} = \mathbf{0}. \tag{44}$$

Continue by forming, of each matrix $\mathbf{\Gamma}_{ij}$, a nine-dimensional column vector $\gamma_{ij}$ that lists the columns of $\mathbf{\Gamma}_{ij}$ in order, and use these vectors to construct the $9 \times 3$ matrices

$$\mathbf{L}_i = [\gamma_{i1} \quad \gamma_{i2} \quad \gamma_{i3}] \qquad \text{for } i = 1, 2, 3. \tag{45}$$

Define the nine-dimensional column vector

$$\mathbf{e} = [\mathbf{e}_1{}^T \quad \mathbf{e}_2{}^T \quad \mathbf{e}_3{}^T]^T, \qquad \mathbf{e}_i = [e_{i1} \quad e_{i2} \quad e_{i3}]^T. \tag{46}$$

Finally, define the partitioned $18 \times 9$ model check matrix $\mathbf{L}$ as follows:

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & -\mathbf{L}_2 & \mathbf{0} \\ \mathbf{L}_1 & \mathbf{0} & -\mathbf{L}_3 \end{bmatrix}, \tag{47}$$

where $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ are the $9 \times 3$ matrices given in Eq. (45).

The system of equations (47) can then be written compactly as

$$\mathbf{Le} = \mathbf{0}. \tag{48}$$

If the system [Eq. (48)] has just a one-parameter family of solutions, those solutions are the scaling solutions. The necessary and sufficient condition for none but scaling solutions is that the rank of the matrix $\mathbf{L}$ be exactly eight or, equivalently, that the null space of $\mathbf{L}$ be one dimensional.

To review, the necessary and sufficient conditions for a bilinear model to work in the case where $(p\,m\,n\,v\,s) = (3\,3\,3\,3\,3)$ are that (1) the $3 \times 3$ bilinear model matrices $\mathbf{B}_j$, for $j = 1, 2, 3$, are nonsingular and that (2) rank $(\mathbf{L}) = 8$; the trivial scaling solution [Eqs. (37)] will lie within the one-dimensional kernel of the model check matrix $\mathbf{L}$. Recovery of three descriptors each for three reflectances and three illuminants is unique by a bilinear model that passes this model check, as long as the three vectors of descriptors for the reflectances and those for the illuminants are linearly independent.

### B. Necessary and Sufficient Conditions When $p = m = v, n = s$

The preceding analysis generalizes readily to a necessary and sufficient test of a bilinear model in cases in which (1) the number $p$ of photoreceptoral types, the dimension $m$ of the illuminant model, and the number $v$ of views are equal and (2) the number $s$ of surfaces with linearly independent vectors of reflectance descriptors is equal to the dimension $n$ of the reflectance model.

In cases where $p = m = v$, the system of equations given in Eq. (39) generalizes to the system

$$\begin{aligned}[\mathbf{z}_1 \ldots \mathbf{z}_v] &= \left(\sum_{j=1}^{n} e_{1j}\boldsymbol{\Gamma}_{1j}\right)[\mathbf{a}_1 \ldots \mathbf{a}_v] = \ldots \\ &= \left(\sum_{j=1}^{n} e_{nj}\boldsymbol{\Gamma}_{nj}\right)[\mathbf{a}_1 \ldots \mathbf{a}_v]. \end{aligned} \tag{49}$$

The matrices $\mathbf{Z} = [\mathbf{z}_1 \ldots \mathbf{z}_v]$ and $\mathbf{A} = [\mathbf{a}_1 \ldots \mathbf{a}_v]$ are square, invertible $p \times p$ matrices, as are the gamma matrices. Multiplying Eq. (49) on the right-hand side by $\mathbf{A}^{-1}$ and taking differences between linear combinations of gamma matrices as before [cf. Eq. (44)], we find that

$$\left(\sum_{h=1}^{n} e_{1h}\boldsymbol{\Gamma}_{1h} - \sum_{h=1}^{n} e_{jh}\boldsymbol{\Gamma}_{jh}\right) = \mathbf{0} \qquad \text{for } j = 2, \ldots, n. \tag{50}$$

This system provides $(n - 1)$ blocks of $pm$ linear equations in the $n^2$ variables $e_{ij}$. It can be written in the compact form

$$\mathbf{Le} = \mathbf{0}, \tag{51}$$

by an easy generalization of the constructions that led from Eq. (44) to Eq. (48). In Eq. (51), the vector $\mathbf{e} = [e_{11} \quad e_{12} \ldots e_{nn}]^T$ is $n^2$ dimensional, and the $(n - 1)p^2 \times n^2$ matrix $\mathbf{L}$ has the form

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_1 & -\mathbf{L}_2 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{L}_1 & \mathbf{0} & -\mathbf{L}_3 & & \mathbf{0} \\ \vdots & & & \ddots & \\ \mathbf{L}_1 & \mathbf{0} & & & -\mathbf{L}_n \end{bmatrix}, \tag{52}$$

where each block $\mathbf{L}_j$, $j = 1, \ldots, n$, is $p^2 \times n$.

For a bilinear model with $p = m = v$ and $n = s$ to be capable of recovering reflectance and illuminant descriptors uniquely from quantum catch data, it is necessary and sufficient that the kernel of the matrix $\mathbf{L}$ be one dimensional. In this case, the kernel contains only the scaling solutions

$$\begin{aligned} e_{11} - e_{jj} &= 0 \qquad \text{for } j = 2, \ldots, n, \\ e_{ij} &= 0 \qquad \text{for } i \neq j. \end{aligned} \tag{53}$$

### C. Failure of $(p\,m\,n\,v\,s) = (2\,2\,2\,2\,2)$

We now analyze the system of Eq. (50) for the dichromatic case $(p\,m\,n\,v\,s) = (2\,2\,2\,2\,2)$. Although this case provides a reflectance-recovery procedure with a first stage involving four equations in three unknowns [cf. inequality (26b)], it is readily shown that recovery is never possible. This example shows that satisfying the feasibility condition, which follows from comparing numbers of equations and unknowns in the recovery procedure, is not sufficient for recovery to work.

Taking as our starting point the system of Eq. (50), we show the failure of (2 2 2 2 2) by applying the Cayley–Hamilton theorem, which states that a matrix satisfies its own characteristic equation.[59] The system of Eq. (50), specialized to the case at hand, has the form

$$e_{11}\boldsymbol{\Gamma}_{11} + e_{12}\boldsymbol{\Gamma}_{12} - e_{21}\boldsymbol{\Gamma}_{21} - e_{22}\boldsymbol{\Gamma}_{22} = \mathbf{0}. \tag{54}$$

The gamma matrices are $2 \times 2$ invertible matrices. Noting that $\boldsymbol{\Gamma}_{11} = \boldsymbol{\Gamma}_{22} = \mathbf{I}$, the $2 \times 2$ identity matrix [Eq. (42)], and that $\boldsymbol{\Gamma}_{21} = \boldsymbol{\Gamma}_{12}{}^{-1}$ [Eq. (41)], one has

$$(e_{11} - e_{22})\mathbf{I} + e_{12}\boldsymbol{\Gamma}_{12} - e_{21}\boldsymbol{\Gamma}_{12}{}^{-1} = \mathbf{0}. \tag{55}$$

Multiplying this equation through by $\boldsymbol{\Gamma}_{12}$ and reordering produces

$$-e_{21}\mathbf{I} + (e_{11} - e_{22})\boldsymbol{\Gamma}_{12} + e_{12}\boldsymbol{\Gamma}_{12}{}^2 = \mathbf{0}. \tag{56}$$

If $\lambda_1$ and $\lambda_2$ are the nonzero but not necessarily distinct eigenvalues of the $2 \times 2$ invertible matrix $\boldsymbol{\Gamma}_{12}$, then, by the Cayley–Hamilton theorem,

$$(\boldsymbol{\Gamma}_{12} - \lambda_1\mathbf{I})(\boldsymbol{\Gamma}_{12} - \lambda_2\mathbf{I}) = \mathbf{0}, \tag{57}$$

which leads to

$$\lambda_1\lambda_2\mathbf{I} - (\lambda_1 + \lambda_2)\boldsymbol{\Gamma}_{12} + \boldsymbol{\Gamma}_{12}{}^2 = \mathbf{0}. \tag{58}$$

Comparing Eqs. (56) and (58) term by term shows that the dichromatic system possesses the following infinite set of solutions:

$$\begin{aligned} e_{12} &= c, \\ -e_{21} &= c\lambda_1\lambda_2 = c \det(\boldsymbol{\Gamma}_{12}), \\ e_{22} - e_{11} &= c(\lambda_1 + \lambda_2) = c \operatorname{trace}(\boldsymbol{\Gamma}_{12}), \end{aligned} \tag{59}$$

where $c$ is an arbitrary constant.

The original system of Eq. (54) expresses the conditions under which the eight quantum catch data provided by two views of two surfaces to a dichromatic system can be generated by distinct combinations of reflectances and illuminants. Were the scaling solution $e_{11} - e_{22} = e_{12} = e_{21} = 0$ the only solution to Eq. (54) [obtained by setting $c = 0$ in Eq. (59)], then we would be assured that, except for an arbitrary scale, only one combination of two illuminants and two surfaces could produce a particular set of data. The existence of other solutions [$c \neq 0$ in Eq. (59)], however, shows that there are infinitely many distinct combinations of two illuminants and two surfaces, not related by scaling, that can give rise to any received set of eight quantum catch data. No recovery procedure can *ever* work for the problem $(p\,m\,n\,v\,s) = (2\,2\,2\,2\,2)$.

In the companion paper we present the other half of a surprising result: although dichromats cannot recover two descriptors per reflectance when presented two views of two surfaces, a dichromat can recover *three* descriptors per reflectance when presented two views of three surfaces.[54] This suggests that the natural dimension for dichromatic surface color representation may be three.

### D. Varieties of Failure

The failure of the dichromatic case $(p\,m\,n\,v\,s) = (2\,2\,2\,2\,2)$ is total: no matter which illuminants and reflectances provide a set of quantum catch data, they will never be recovered uniquely if bilinear models with parameters $(2\,2\,2\,2\,2)$ are used. It is also possible, in principle, for the failure of recovery to be partial: the color constancy algorithm works most of the time to provide the correct descriptors but fails to provide a unique solution (up to an arbitrary positive scalar) under special classes of lights. Thus recovery would function properly for most illuminants but would occasionally break down. An instance of partial failure would not necessarily be picked up in simulations of recovery involving a less than exhaustive set of test cases (e.g., those of D'Zmura[52]). In the companion paper we report several such cases of partial failure that satisfy the $pv > n$ rule.[54] Our chief concern here is with identifying models that never fail.

## 5. SUFFICIENT TESTS OF UNIQUE RECOVERY

Our aim is to distinguish color constancy algorithms that (1) recover perfectly reflectance and illuminant descriptors when provided adequate data, from algorithms that (2) work imperfectly, suffering partial failure, and to distinguish these from algorithms that (3) fail totally. Because the comparison of the number of data and the number of unknowns [inequality (14)] does not provide a sufficient condition for recovery to work, we must conduct further analysis along the lines of the previous section.

Our special concern lies with the case in which a trichromatic visual system is provided two views of three surfaces and attempts to recover three descriptors per surface and per illuminant,[52] and in Subsection 5.A we commence with an analysis of this case. The conditions that a bilinear model with parameters $(p\,m\,n\,v\,s) = (3\,3\,3\,2\,3)$ must meet, if it is to provide for the unique recovery of reflectance and illuminant descriptors, are similar to those in the three-view case of Subsection 4.A but are more intricate. These conditions lead to a numerically tractable, sufficient test of whether a given bilinear model permits the unique recovery of three descriptors per reflectance and illuminant from two views of three or more surfaces.

The sufficient conditions on a bilinear model in the case where $(p\,m\,n\,v\,s) = (3\,3\,3\,2\,3)$ are but one instance of a more general set of results. Our first step is to generalize the conditions for unique recovery for (3 3 3 2 3) to cases in which bilinear model matrices are square and nonsingular ($p = m$; Subsection 5.B.1). We present a general algorithm for checking whether a particular bilinear model represented by square matrices permits the unique recovery of illuminant and reflectance descriptors. Our second step is to generalize the conditions to cases in which the bilinear model matrices are rectangular ($p \geq m$; Subsection 5.B.2) and to sketch the model check algorithm for these cases, leaving details of its derivation to Appendix A.

### A. Two Views of Three Surfaces: $(p\,m\,n\,v\,s) = (3\,3\,3\,2\,3)$

Two is the minimum number of views required for a trichromatic visual system to recover three-dimensional descriptions of illuminants and reflectances from three surfaces.[25,52] The conditions on a bilinear model for unique recovery from two views follow from an argument generalizing that developed above in the three-view case. It leads to a set of 45 equations in 36 unknowns that can guarantee unique recovery by the model if the set of equations provides exactly 36 independent equations.

Suppose that a first set of three distinct surfaces is lit by illuminants $\mathbf{a}_1$ and $\mathbf{a}_2$ and that a second set is lit by $\mathbf{z}_1$ and $\mathbf{z}_2$. If the subspaces spanned by the quantum catch data from the two sets are identical, then the two pairs of illuminants must be identical, up to scaling, for a particular bilinear model to work. The basic equations [Eq. (39)] now involve matrices of illuminant descriptors that are not invertible:

$$[\mathbf{z}_1 \quad \mathbf{z}_2 \quad \mathbf{0}] = \sum_{j=1}^{3} e_{1j}\boldsymbol{\Gamma}_{1j}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \sum_{j=1}^{3} e_{2j}\boldsymbol{\Gamma}_{2j}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \sum_{j=1}^{3} e_{3j}\boldsymbol{\Gamma}_{3j}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}]. \tag{60}$$

We cannot rid ourselves of the matrix of illuminants $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}]$ by inversion, as in the three-view case. Yet if we form the two differences

$$\left(\sum_{j=1}^{3} e_{1j}\boldsymbol{\Gamma}_{1j} - \sum_{j=1}^{3} e_{2j}\boldsymbol{\Gamma}_{2j}\right)[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \mathbf{0},$$
$$\left(\sum_{j=1}^{3} e_{1j}\boldsymbol{\Gamma}_{1j} - \sum_{j=1}^{3} e_{3j}\boldsymbol{\Gamma}_{3j}\right)[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \mathbf{0}, \tag{61}$$

sufficient conditions on the bilinear model, expressed through the gamma matrices, become apparent. By Eqs. (61), each row of the two difference matrices $\mathbf{W}_{1-2}$ and $\mathbf{W}_{1-3}$, where

$$\mathbf{W}_{1-2} = \left(\sum_{j=1}^{3} e_{1j}\boldsymbol{\Gamma}_{1j} - \sum_{j=1}^{3} e_{2j}\boldsymbol{\Gamma}_{2j}\right),$$
$$\mathbf{W}_{1-3} = \left(\sum_{j=1}^{3} e_{1j}\boldsymbol{\Gamma}_{1j} - \sum_{j=1}^{3} e_{3j}\boldsymbol{\Gamma}_{3j}\right), \tag{62}$$

must be orthogonal to the subspace spanned by illuminants $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{W}_{1\text{-}2}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \mathbf{W}_{1\text{-}3}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \mathbf{0}$$

or

$$\begin{bmatrix} \mathbf{w}_{1\text{-}2,1} \\ \mathbf{w}_{1\text{-}2,2} \\ \mathbf{w}_{1\text{-}2,3} \\ \mathbf{w}_{1\text{-}3,1} \\ \mathbf{w}_{1\text{-}3,2} \\ \mathbf{w}_{1\text{-}3,3} \end{bmatrix} [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{0}] = \mathbf{0}, \tag{63}$$

where $\mathbf{w}_{1-j,k}$ is the $k$th row of the difference matrix $\mathbf{W}_{1-j}$. This means that the six row vectors of Eq. (63) are collinear. This condition can be expressed by stating that each pairwise cross product between these six three-dimensional vectors must vanish:

$$\mathbf{w}_{1-j,k} \times \mathbf{w}_{1-g,h} = \mathbf{0} \qquad \text{for } j, g = 2, 3, \text{ and } k, h = 1, 2, 3. \tag{64}$$

There are 15 independent cross products, each of which provides three scalar equations (three $2 \times 2$ determinants), namely,

$$w_{1-j,k2}w_{1-g,h3} - w_{1-j,k3}w_{1-g,h2} = 0,$$
$$w_{1-j,k3}w_{1-g,h1} - w_{1-j,k1}w_{1-g,h3} = 0,$$
$$w_{1-j,k1}w_{1-g,h2} - w_{1-j,k2}w_{1-g,h1} = 0, \tag{65}$$

for $j, g = 2, 3$ and $k, h = 1, 2, 3$, for a total of 45 equations. These equations are quadratic in the variables $e_{ij}$ of Eqs. (62).

For unique recovery, we desire that there be no way that the rows of the differences [Eq. (63)] can be mutually perpendicular to any two linearly independent vectors of illuminant descriptors. Accordingly, we derive a set of homogeneous equations in the quadratic combinations of the variables $e_{ij}$ that must have no nontrivial solution if the bilinear model is to function. Yet we do not examine solutions to the quadratic equations in terms of the variables $e_{ij}$, although to do so would provide necessary and sufficient conditions for unique recovery. Rather, by taking each of the quadratic unknowns as a distinct variable, we examine solutions to the linear equations that arise. While such linearization provides a test that is sufficient but not necessary, the test is eminently feasible. If, for a particular bilinear model, the only solution to the linearized equations is $\mathbf{0}$, then the illuminants $\mathbf{a}_1$ and $\mathbf{a}_2$ are related to $\mathbf{z}_1$ and $\mathbf{z}_2$, respectively, by a single scale factor, and the reflectances $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are related to $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ by the reciprocal of that scale factor.

To find the coefficients determined by the bilinear model, expressed through the matrices $\mathbf{\Gamma}_{ij}$ with entries $(\mathbf{\Gamma}_{ij})_{kl}$, on the quadratic combinations of the variables $e_{ij}$, set $(x, y) = (2, 3)$, $(3, 1)$, or $(1, 2)$ to cycle through the determinants of Eqs. (65) and use the definition of the difference matrices [Eqs. (62)] to produce

$$\sum_{q,r=1}^{3} \{[e_{1q}(\mathbf{\Gamma}_{1q})_{kx} - e_{jq}(\mathbf{\Gamma}_{jq})_{kx}][e_{1r}(\mathbf{\Gamma}_{1r})_{hy} - e_{gr}(\mathbf{\Gamma}_{gr})_{hy}] - [e_{1q}(\mathbf{\Gamma}_{1q})_{ky} - e_{jq}(\mathbf{\Gamma}_{jq})_{ky}][e_{1r}(\mathbf{\Gamma}_{1r})_{hx} - e_{gr}(\mathbf{\Gamma}_{gr})_{hx}]\} = 0, \tag{66}$$

for $j, g = 2, 3$ and $k, h = 1, 2, 3$. Each of these equations involves quadratic combinations of the variables $e_{ij}$. The number of quadratic unknowns in the complete set of 45 equations is the number of distinct quadratic combinations of the variables $e_{ij}$. Note, however, that the variables $e_{11}$, $e_{22}$, and $e_{33}$ are coefficients on the identity matrices $\mathbf{\Gamma}_{11}$, $\mathbf{\Gamma}_{22}$, and $\mathbf{\Gamma}_{33}$ [Eq. (42)]. When one takes the differences between equations [Eqs. (61)], the variable $e_{11}$ only ever appears in combination with $e_{22}$ and $e_{33}$: the three variables $e_{11}$, $e_{22}$, and $e_{33}$ form only two independent factors $(e_{11} - e_{22})$ and $(e_{11} - e_{33})$ in the present set of equations. The number of distinct quadratic unknowns in the 45 equations is thus the number of distinct quadratic combinations of $(e_{11} - e_{22})$, $(e_{11} - e_{33})$, $e_{12}$, $e_{21}$, $e_{13}$, $e_{31}$, $e_{23}$, and $e_{32}$. That number one finds by allocating two indistinguishable balls among eight cells, and the number of distinct ways to do this,[60] is given by the binomial coefficient

$$\binom{8 + 2 - 1}{2} = \binom{9}{2} = 36.$$

Sufficient conditions for a trichromatic bilinear model to provide unique recovery of three-dimensional illuminants and reflectances from two views of three surfaces are (1) that the bilinear model matrices $\mathbf{B}_j$ be nonsingular and (2) that the 45 homogeneous equations [Eq. (66)] in the 36 distinct quadratic combinations of the eight variables $(e_{11} - e_{22})$, $(e_{11} - e_{33})$, $e_{12}$, $e_{21}$, $e_{13}$, $e_{31}$, $e_{23}$, and $e_{32}$ provide a model check matrix with full rank of 36. Note that this collection of 36 monomial unknowns includes, in particular, $(e_{11} - e_{22})^2$, $(e_{11} - e_{33})^2$, $e_{12}^2$, $e_{13}^2$, $e_{21}^2$, $e_{23}^2$, $e_{31}^2$, and $e_{32}^2$. Thus, in finding that the 36 quadratic unknowns are zero in the above set of equations, one finds that $e_{11} - e_{22} = e_{11} - e_{33} = 0$ and $e_{12} = e_{21} = e_{13} = e_{31} = e_{23} = e_{32} = 0$. Going back to the original equations [Eq. (60)], this means that the two-stage linear color constancy algorithm works perfectly to recover descriptors when provided adequate data.

### B. General Model Check Algorithm

The foregoing check of a particular bilinear model in the case where $p = m = 3$, $n = s = 3$ and $v = 2$ leads directly to a general method for checking the efficacy of bilinear models in cases where $p = m = v + d_v$ and $n = s$. We call the nonnegative number $d_v$ the defect in views, which tells us how many fewer views than illuminant model dimensions are provided: $d_v = m - v$. In the case where $(p\,m\,n\,v\,s) = (3\,3\,3\,3\,3)$ of Section 4, in which the defect in views $d_v = 0$, the check of a bilinear model involves equations linear in the $e_{ij}$. In the case where $(p\,m\,n\,v\,s) = (3\,3\,3\,2\,3)$, in which the defect $d_v = 1$, the check involves equations quadratic in the $e_{ij}$. In general, the degree of the polynomials in the $e_{ij}$ is equal to $d_v + 1$. In Subsection 5.B.1 we determine the equations and monomial unknowns involved in a linear, sufficient check of a particular bilinear model in the case of square bilinear model matrices, where $p = m = v + d_v$ and $n = s$. We generalize this derivation for square bilinear model matrices $(p = m)$ to the case in which the bilinear model matrices are rectangular $(p \geq m)$ in Subsection 5.B.2.

#### 1. *Square Bilinear Model Matrices $(p = m)$*

The strategy behind the check, as in Subsection 5.A, is to generate a set of homogeneous equations in the distinct combinations of the variables $(e_{11} - e_{22}), \ldots, (e_{11} - e_{nn})$, and $e_{ij}$, for $i \neq j$, of degree $d_v + 1$, that depend solely on the bilinear model matrix entries. If these linear equations provide a model check matrix that has full rank, then the bilinear model provides unique recovery of illuminant and reflectance descriptors from quantum catch data.

If the number of photoreceptors $p$ equals the dimension $m$ of the illuminant model, then the bilinear model matrices $\mathbf{B}_j$ are $p \times p$ square matrices, as are the gamma matrices $\mathbf{\Gamma}_{ij} = \mathbf{B}_i^{-1}\mathbf{B}_j$. If the number of surfaces $s$ equals the dimension $n$ of the reflectance model, then Eq. (60) assumes the more general form

$$\left[\mathbf{z}_1 \ldots \mathbf{z}_v \; \underbrace{\mathbf{0} \ldots \mathbf{0}}_{d_v}\right] = \left(\sum_{j=1}^{n} e_{1j}\mathbf{\Gamma}_{1j}\right)\left[\mathbf{a}_1 \ldots \mathbf{a}_v \; \underbrace{\mathbf{0} \ldots \mathbf{0}}_{d_v}\right] = \ldots = \left(\sum_{j=1}^{n} e_{nj}\mathbf{\Gamma}_{nj}\right)\left[\mathbf{a}_1 \ldots \mathbf{a}_v \; \underbrace{\mathbf{0} \ldots \mathbf{0}}_{d_v}\right]. \quad (67)$$

As in the preceding section, we define the difference matrices

$$\mathbf{W}_{1-j} = \left(\sum_{q=1}^{n} e_{1q}\mathbf{\Gamma}_{1q} - \sum_{q=1}^{n} e_{jq}\mathbf{\Gamma}_{jq}\right) \qquad \text{for } j = 2, \ldots, n, \quad (68)$$

in terms of which we have

$$\mathbf{W}_{1-j}\left[\mathbf{a}_1 \ldots \mathbf{a}_v \; \underbrace{\mathbf{0} \ldots \mathbf{0}}_{d_v}\right] = \mathbf{0} \qquad \text{for } j = 2, \ldots, n. \quad (69)$$

This implies that the rows $\mathbf{w}_{1-j,k}$ of the $n - 1$ matrices $\mathbf{W}_{1-j}$ all lie in the $d_v$-dimensional subspace orthogonal to that spanned by the $v$ linearly independent vectors of illuminant descriptors:

$$\begin{bmatrix} \mathbf{w}_{1-2,1} \\ \vdots \\ \mathbf{w}_{1-2,p} \\ \cdots \\ \vdots \\ \cdots \\ \mathbf{w}_{1-n,1} \\ \vdots \\ \mathbf{w}_{1-n,p} \end{bmatrix}\left[\mathbf{a}_1 \ldots \mathbf{a}_v \; \underbrace{\mathbf{0} \ldots \mathbf{0}}_{d_v}\right] = \mathbf{0}. \quad (70)$$

To determine the conditions, expressed in terms of the variables $e_{ij}$ and the entries of the gamma matrices, under which the rows all lie in a $d_v$-dimensional subspace, we form all possible determinants of order $d_v + 1$ within the $(n - 1)p \times p$ matrix of rows and set each of these to zero. Each such determinant provides a single equation in monomial unknowns of degree $d_v + 1$ in the variables $e_{ij}$.

The total number of equations E is the number of distinct determinants of order $d_v + 1$ that may be formed within the $(n - 1)p \times p$ matrix of rows and is given in terms of products of binomial coefficients:

$$\underset{(p=m=v+d_v,\, n=s)}{\text{Number of Equations}} = \mathsf{E} = \binom{(n-1)p}{d_v + 1}\binom{p}{d_v + 1} = \binom{np - m}{m - v + 1}\binom{m}{m - v + 1}. \quad (71)$$

To determine the total number of monomial unknowns, note again that the variable $e_{11}$ only ever appears in the form $(e_{11} - e_{jj})$, so that the total number of linear variables is $n^2 - 1$. The total number U of monomial unknowns of degree $d_v + 1$ is then found by allocating $d_v + 1$ balls among $n^2 - 1$ cells:

$$\underset{(p=m=v+d_v,\, n=s)}{\text{Number of Unknowns}} = \mathsf{U} = \binom{(n^2 - 1) + (d_v + 1) - 1}{d_v + 1} = \binom{n^2 + m - v - 1}{m - v + 1}. \quad (72)$$

As in the preceding section, we form a homogeneous set of E linear equations in the U monomial unknowns of degree $d_v + 1$ in the variables $(e_{11} - e_{jj})$, for $j = 2, \ldots, n$, and $e_{ij}$, for $i \neq j$. If the model check matrix expressing these equations for a particular bilinear model has a rank equal to the number of unknowns, then its kernel has dimension zero, which implies that the only solution to the system of Eq. (67) is the scaling solution

$$e_{11} - e_{jj} = 0 \qquad \text{for } j = 2, \ldots, n,$$
$$e_{ij} = 0 \qquad \text{for } i \neq j. \quad (73)$$

In this event, the descriptors can be recovered uniquely, up to an arbitrary positive scalar.

These conditions on a particular bilinear model are expressed for the case in which $p = m = v + d_v$ and $n = s$. A bilinear model with $p$ photoreceptoral types that passes this check must also work if it is provided further photoreceptoral types [one can simply ignore the data from the additional photoreceptoral types(s)]. Likewise, a visual system provided $v$ views with a bilinear model that fulfills this check must also pass the check if it is provided further views [one can ignore the data from the additional view(s)]. Finally, a bilinear model with an $n$-dimensional reflectance model component that fulfills this check must also work in the case where $n < s$ [one can ignore the data from the additional, linearly dependent surface(s)]. To summarize these entailments,

$$(p\,m\,n\,v\,s) \Rightarrow (p + 1\,m\,n\,v\,s), \quad (74)$$

$$(p\,m\,n\,v\,s) \Rightarrow (p\,m\,n\,v + 1\,s), \quad (75)$$

$$(p\,m\,n\,v\,s) \Rightarrow (p\,m\,n\,v\,s + 1). \quad (76)$$

*2. Rectangular Bilinear Model Matrices ($p > m$)*

The algorithm for checking square bilinear models discussed above fails to cover cases in which the number $p$ of photoreceptoral types exceeds the dimension $m$ of the illuminant model. The bilinear model matrices are not invertible in the latter case, so that the direct formulation of conditions on a model in terms of linear combinations of gamma matrices fails. However, we are still able to adhere to the strategy of showing that a bilinear model must admit of only the scaling solution [Eqs. (73)] if two sets of surfaces under two respective sets of illuminant(s) provide the same quantum catch data. As suggested by the recovery procedure in cases where $p > m$ (Subsection 3.C), the technique is to supplement the singular bilinear model matrices $\mathbf{B}_j$ with additional entries to make them square, invertible matrices $\mathbf{B}_j{}^*$. We find that the necessary and sufficient conditions on the original bilinear model involve a check of whether a set of homogeneous equations of degree $d_v + 1$ in the underlying variables $e_{ij}$ has only the trivial solution. We then linearize these equations by investigating the system's solutions in the space of the monomial unknowns, thus providing a practical sufficient test. We present the derivation of the model check algorithm for rectangular bilinear model matrices in Appendix A.

The sufficient test of unique recovery provided by the model check algorithm is expressed for problems with parameters that satisfy $p \geq m \geq v$ and $n = s$. If a bilinear model with such parameters passes this test, the entailments of relations (74)–(76) follow, as does the further entailment

$$(m > v), \qquad (p\, m\, n\, v\, s) \Rightarrow (p\, m-1\, n\, v\, s)\,. \tag{77}$$

Clearly, a recovery procedure that can produce $m$ descriptors per illuminant can recover $m - 1$ descriptors per illuminant (ignore one of the descriptors). One more entailment embodies transposition. If the model check algorithm shows that a particular bilinear model provides unique recovery in the case $(p\, m\, n\, v\, s)$, then model transposition shows that the transposed model will provide unique recovery for the problem $(p\, n\, m\, s\, v)$:

$$(p\, m\, n\, v\, s) \Rightarrow (p\, n\, m\, s\, v)\,. \tag{78}$$

## 6. APPLICABILITY OF THE MODEL CHECK ALGORITHM

We have described above a model check algorithm, which provides a sufficient test of whether a particular bilinear model, with parameters that satisfy $p \geq m \geq v$ and $n = s$, always allows the recovery of illuminant and reflectance descriptors from quantum catch data. The sufficient test checks (1) whether the bilinear model matrices are invertible and (2) whether a suitably constructed model check matrix of dimension $\mathsf{E} \times \mathsf{U}$ that is determined by the bilinear model matrix entries has full rank. A model passing the check is distinguished by the fact that quantum catch data are related uniquely to sets of lit surfaces (Fig. 2).

For color constancy problems such as $(p\, m\, n\, v\, s) = (3\,3\,3\,3\,3)$, where the defect in views $d_v = m - v = 0$ and the monomial unknowns (of degree 1) are identical to the underlying variables $(e_{11} - e_{jj})$ for $j = 2, \ldots, n$, and $e_{ij}$, for $i \neq j$, it is both necessary and sufficient for a particular bilinear model to produce a model check matrix of full rank. For problems such as $(p\, m\, n\, v\, s) = (3\,3\,3\,2\,3)$, where $d_v > 0$, it is certainly sufficient for unique recovery that a particular bilinear model provide a model check matrix of full rank, but it is not, in general, necessary. Without further analysis, we deduce nothing from the fact that a particular bilinear model, with parameters satisfying $d_v > 0$, produces a model check matrix with a nontrivial kernel. The reason for this is that the vectors in the kernel of a model check matrix that fails to have full rank, each of which describes linear combinations of

$$\binom{n^2 + m - v + 1}{m - v + 1}$$

monomial unknowns of degree $d_v + 1$ in the underlying variables, need not be consistent with the equations that express the monomial unknowns' form in terms of the underlying variables.

### A. Trichromacy

The diagrams of Fig. 3 summarize the applicability of the model check algorithm to the testing of two-stage linear algorithms for recovering spectral descriptions from chromatic change in the case of trichromacy. Their horizontal axes mark the dimension $n$ of the reflectance model, taken equal to the number $s$ of surfaces, while their vertical axes mark the number $v$ of views.

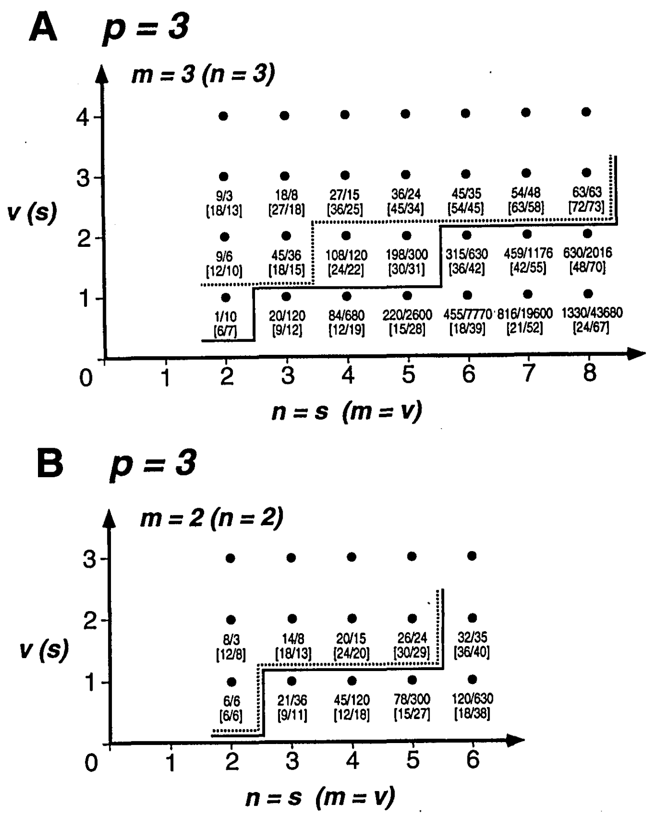


Fig. 3. Conditions for recovering spectral descriptions from chromatic change: trichromacy. A, Square problems where $p = m = 3$ or, by transposition, $p = n = 3$; B, rectangular problems where $p = 3$, $m = 2$ or, by transposition, $p = 3$, $n = 2$. In both diagrams, the horizontal axis marks the dimension $n$ of the reflectance model, taken equal to the number $s$ of surfaces, while the vertical axis marks the number $v$ of views. The solid lines in both diagrams that start at the lower left and work toward the upper right separate problems that satisfy the necessary condition $svp \geq sn + vm - 1$ [inequality (14)] from those that do not. The number of quantum catch data and the number of descriptors to be recovered are indicated for each case by the bracketed pair [Q/D] beneath the appropriate lattice point. Points that lie beneath and to the right of the solid lines, where $\mathsf{Q} < \mathsf{D} - 1$, fail the feasibility condition $svp \geq sn + vm - 1$ and so represent problems for which unique recovery is impossible. The dotted lines divide problems that satisfy the necessary condition for the test provided by the model check algorithm to be performed, namely, that $\mathsf{E} \geq \mathsf{U}$ [Eqs. (A10) below and (72)]. The pair E/U is shown directly beneath each point. In cases where $m > v$, such tests are sufficient; in problems where $m = v$, they are necessary and sufficient. By transposition, each problem $(p\, m\, n\, v\, s)$ also represents the transposed problem $(p\, n\, m\, s\, v)$, and the transposed parameters are indicated in parentheses at the tops of the diagrams and along their axes. See text for further discussion.

The solid lines in both diagrams (A: $p = 3$, $m = 3$ and B: $p = 3$, $m = 2$), which start at the lower left and work toward the upper right, divide feasible problems that satisfy the necessary condition $svp \geq sn + vm - 1$ [inequality (14)] from those problems for which recovery is impossible. The number $svp$ of quantum-catch data and the number $sn + vm$ of unknown descriptors are indicated for each problem by the bracketed pair [Q/D] beneath the appropriate lattice point. Points that lie beneath and to the right of the solid lines represent problems with parameters that fail to satisfy $pv > n$ [inequality (27b)], for which perfect recovery is impossible.

Problems for which the reflectance dimension $n$ is one (first column) are omitted; such surfaces are homochromatic. Likewise, problems for which the illumination dimension $m$ is one are omitted; such lights are homochromatic.

The dotted lines divide problems with parameters that satisfy the condition for the sufficient test provided by the model check algorithm to be performed, namely, that $\mathsf{E} \geq \mathsf{U}$ [Eqs. (71) and (72)]. The pair E/U is shown directly beneath each point. Points that lie above and to the left of the dotted lines represent bilinear models with parameters $(p\,m\,n\,v\,s)$ for which the model check algorithm provides a sufficient test. Points that lie below and to the right represent problems where $\mathsf{U} > \mathsf{E}$, for which the model check algorithm does not apply.

The bracketed parameters that label both panels of Fig. 3 refer to the parameters that are obtained by model transposition. All results that one finds concerning recovery for a problem with parameters $(p\,m\,n\,v\,s)$ obtain also for the problem with transposed parameters $(p\,n\,m\,s\,v)$. In particular, the bracketed pair [Q/D] is unaltered by model transposition, and the solid lines continue to divide problems that lead to feasible recoveries from those that do not. Likewise, the pair E/U refers in the transposed case to model check numbers of equations and unknowns, and the dotted lines divide problems to which the algorithm may usefully be applied from those to which it cannot.

Note that problems where $v > m$ are not considered (top rows of the panels in Fig. 3); views in excess of $m$ distinct views provide no further information and reduce to cases where $m = v$. Likewise, problems where $s > n$ are not considered; surfaces in excess of $n$ distinct ones required for forming an invertible matrix of reflectance descriptors are redundant.

All told, the necessary condition $pv > n$ [inequality (27b)] suggests that, when the dimension of the illumination model is three, a trichromatic visual system can recover from one view two, from two views five, and from three or more views eight descriptors per reflectance. Likewise, when the dimension of the illumination model is two, the necessary condition suggests that a trichromatic visual system can recover from one view two and from two or more views five descriptors per reflectance. Yet the sufficient test provided by the model check algorithm has a narrower scope. It speaks neither to the problem where $(p\,m\,n\,v\,s) = (3\,3\,2\,1\,2)$, considered by Maloney[24] and Maloney and Wandell[25] in their work on single-view theory, nor to the problems in which a trichromatic system attempts to recover from two views either four or five descriptors per reflectance.

### B. Dichromacy and Tetrachromacy

In Figs. 4 and 5 are presented similar diagrams for the recovery of spectral descriptions from chromatic change for dichromatic (Fig. 4) and tetrachromatic (Fig. 5) systems.

For dichromatic visual systems, the two problems $(p\,m\,n\,v\,s) = (2\,2\,2\,2\,2)$ and $(2\,2\,3\,2\,3)$ satisfy the necessary condition for recovery (solid lines). Furthermore, particular bilinear models with either of these parameter sets can be tested with the model-check algorithm (dotted lines). The diagram suggests that a dichromatic visual system might be able to recover perfectly three descriptors per reflectance from two views when the dimension of the illumination model is two, and this result is shown in the companion paper.[54] The problem $(2\,2\,3\,2\,3)$ and its transpose $(2\,3\,2\,3\,2)$ exhaust the two-stage linear recovery possibilities for dichromacy.

Figure 5 provides a partial listing of tetrachromatic cases; the diagrams are limited to problems for which the surface-related parameters $n$ and $s$ are less than or equal to eight. For problems in which $p = m = 4$ (Fig. 5A), where the defect in illumination is zero, an examination of necessary conditions suggests the feasibility of recovering three, seven, and more than eight reflectance descriptors per surface from one, two, and three or four views, respectively. The model check algorithm, however, provides no test of the one-view problems and operates only on problems concerning two views with up to four reflectance descriptors. The mismatch between the problems that meet the necessary condition and the problems testable by the model check algorithm diminishes when the dimension $m$ of the illumination model is set to three (Fig. 5B) and includes the problems for one view $(4\,3\,3\,1\,3)$ and for two views $(4\,3\,7\,2\,7)$.

### C. Limits through 6-Chromacy

In Fig. 6 are shown limits on the applicability of the model check algorithm and on the recovery of surface reflectance descriptors provided by the necessary condition $pv > n$ for dichromatic through 6-chromatic systems. At each point is shown in the format $N_s/N_n$ (1) the maximum number $N_s$ of reflectance descriptors per surface that may be recovered by a bilinear model that can be tested with the model check algorithm and (2) the maximum number $N_n$ of reflectance descriptors that may be recovered by a bilinear model that satisfies $pv > n$. These values are shown for the ranges of the permissible values $v$ of the number of views that are indicated in the legends at the right. The number $s$ of surfaces is taken equal to the dimension $n$ of the reflectance model.

The dashes for the problems $(p\,m\,n\,v\,s) = (2\,2\,n\,1\,n)$ indicate that for no value of $n$ can the model check algorithm be performed and for no value is the necessary condition met. The single dashes for the problems $(p\,p\,n\,1\,n)$, for $p = 3, 4, 5, 6$, indicate that for no value of $n$ can the model-check algorithm be performed.

### D. Positive Entailments

The scope of the model check algorithm is broadened considerably by the five entailments given in relations (74)–(78), all of which are of the following form: if the model check algorithm shows that a particular bi-

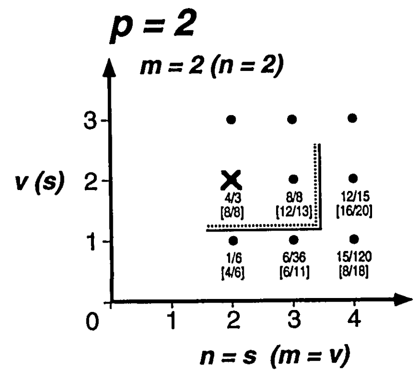


Fig. 4. Conditions for recovering spectral descriptions from chromatic change: dichromacy. Problems are shown where $p = m = 2$ or $p = n = 2$. The **X** marks the problem $(p\,m\,n\,v\,s) = (2\,2\,2\,2\,2)$ where recovery fails totally (see Subsection 4.C). See the caption for Fig. 3 and text for further discussion.

linear model with parameters $(p\,m\,n\,v\,s)$ provides a perfect color constancy algorithm, then there exists a bilinear model that provides a perfect algorithm with suitably altered parameters. We provide a complete list of entailments in the companion paper.[54]

The scope is broadened further by the following fact: a particular bilinear model that provides a perfect color constancy algorithm will continue to provide perfect recovery when its bilinear model matrices are subject to an invertible bilinear transformation $\mathbf{B}_j \leftrightarrow \mathbf{X}\mathbf{B}_j\mathbf{Y}$, for $j = 1, \ldots, n$, where **X** and **Y** are nonsingular linear transformations. In other words, equivalent models are generated from a given one by the application of both an invertible linear transformation to the photoreceptoral sensitivities and a possibly distinct invertible linear transformation to the basis functions for illumination. The same is true for the mapping $\mathbf{B}_i' \leftrightarrow \mathbf{X}\mathbf{B}_i'\mathbf{Y}$ (now **Y** transforms reflectances).

## 7. DISCUSSION

In the companion paper[54] we report the results of applying the model check algorithm to the color constancy problems listed in Figs. 3–5 of the current paper. Our aim there is to distinguish two-stage linear recovery procedures for color constancy that (1) recover perfectly reflectance and illuminant descriptors when provided adequate data from procedures that (2) work imperfectly, suffering partial failure, and to distinguish these from procedures that (3) fail totally.

### A. Distinguishing Problems and Models

The task has two parts. Problems identified by their parameters $(p\,m\,n\,v\,s)$ must be distinguished from one another. Furthermore, bilinear models with the parameters of a particular problem must be distinguished from one another. One could inadvertently choose a bad bilinear model with parameters that match those of a problem in which perfect recovery is possible. The simplest example of a poor choice is a bilinear model with matrices that are not invertible.[24] The model check algorithm distinguishes bilinear models that provide perfect recovery procedures from models that provide procedures that fail, either partially or totally. We shall use the model check algorithm to show that particular color constancy problems permit perfect recovery procedures by checking successfully the function of particular bilinear models.[54]

### B. Necessary and Sufficient Test for Problems with a Defect in Views?

The model check algorithm fails to handle a small number of problems, some of which are of great interest (see

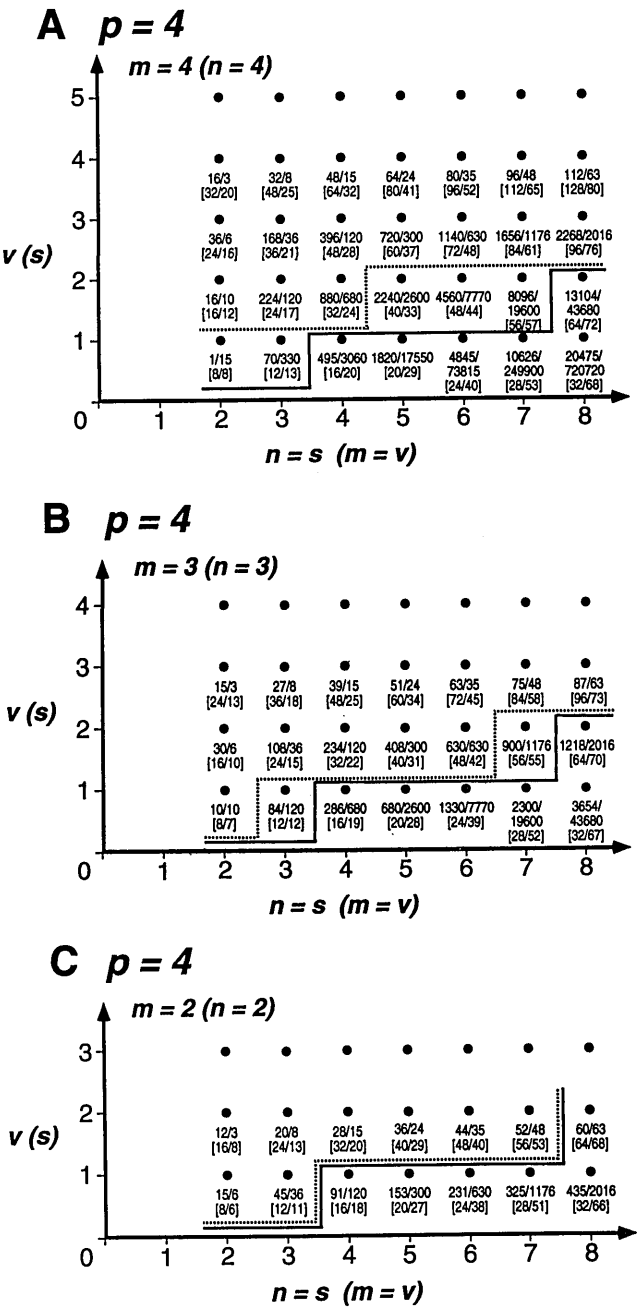


Fig. 5. Conditions for recovering spectral descriptions from chromatic change: tetrachromacy. A, Square problems where $p = m = 4$ or $p = n = 4$; B, rectangular problems where $p = 4$, $m = 3$ or $p = 4$, $n = 3$; C, rectangular problems where $p = 4$, $m = 2$ or $p = 4$, $n = 2$. See the caption for Fig. 3 and text for further discussion.

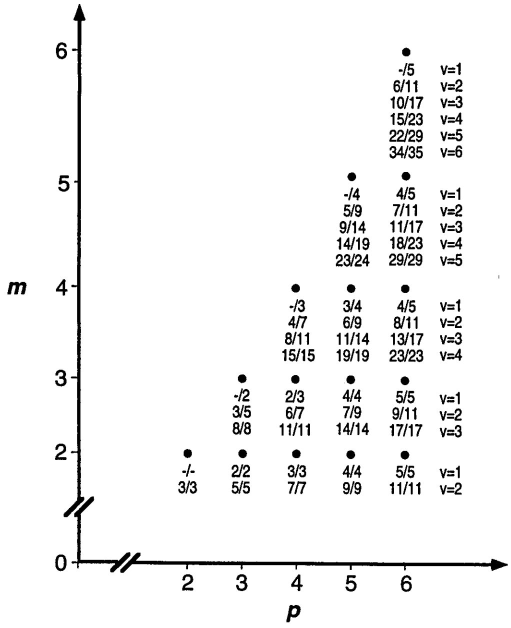


Fig. 6. Limits on the recovery of spectral descriptions from chromatic change. The horizontal axis marks the number $p$ of photoreceptoral types, and the vertical axis marks the dimension $m$ of the linear model for illumination. At each point is marked (1) the maximum number $N_s$ of surface reflectance descriptors that may be recovered by a bilinear model that can be tested with the model check algorithm and (2) the maximum number $N_n$ of reflectance descriptors that may be recovered by a bilinear model that satisfies the necessary condition $pv > n$. These two quantities are shown in the format $N_s/N_n$ for values of the number $v$ of views indicated in the legends at the right. See text for further discussion.

Section 6). In these problems the defect in views is nonzero ($m > v$), and one is led to ask more generally whether the sufficient test provided by the model check algorithm in cases where $m > v$ can be transformed into a necessary and sufficient test of a particular bilinear model that may be applied to all problems.

The set of homogeneous polynomial equations provided by the model check algorithm is of degree $d_v + 1$ in $n^2 - 1$ underlying variables, and the check linearizes this system by investigating solutions among the monomial unknowns rather than among the variables themselves. A necessary and sufficient test for all feasible problems with a defect in views involves determining solutions to the set of polynomial equations expressed directly in terms of the underlying variables. This is made possible by an elimination procedure that converts a set of homogeneous polynomial equations into a sequence of resultants, which provide criteria that let one determine in principle whether the bilinear model can be used to recover descriptors uniquely.[61–63] However, cursory examination shows that this algorithm is unlikely to provide a practical test of our problems: the algorithm is numerically intractable in all cases of interest to us. Our model check algorithm thus appears to be the best available practical method for checking whether a particular bilinear model can be used to recover spectral descriptions from chromatic change uniquely.

### C. Scope

The restrictions $p \geq m \geq v$ and $s \geq n$ on problem parameters rule out problems that have structures incompatible with two-stage linear recovery procedures, such as the procedures of Maloney and Wandell[25] and D'Zmura.[52] Through the transposition entailment [relation (78)], one sees that two-stage linear recovery is possible only when the number of photoreceptoral types equals or exceeds one or both of the dimensions for reflectance and illumination.

Yet we shall find that almost all color constancy problems that are relevant to biological, human, and machine vision can be treated by the methods described here. The reason is that relatively many views of a set of surfaces are needed for recovery of $n$ descriptors for each surface and $m$ descriptors for each light in situations where $n, m > p$. Almost all problems that involve either unchanging illumination (one view) or a single change in illumination (two views) are captured by the present analysis.

Finally, the restriction $s \geq n$ is lifted, partially, through use of entailments. In general, however, problems that possess both a defect in views and a defect in surfaces escape the net.

## APPENDIX A: MODEL CHECK ALGORITHM FOR RECTANGULAR BILINEAR MODEL MATRICES

The equations that express the equivalence of quantum catch data in the case where $p \geq m \geq v$ and $n = s$ are

$$\mathbf{B}_i\left[\mathbf{z}_1\ldots\mathbf{z}_v\ \ \underbrace{\mathbf{0}\ldots\mathbf{0}}_{d_v}\right] = \left(\sum_{j=1}^{n} e_{ij}\mathbf{B}_j\right)\left[\mathbf{a}_1\ldots\mathbf{a}_v\ \ \underbrace{\mathbf{0}\ldots\mathbf{0}}_{d_v}\right] \quad \text{for } i = 1,\ldots,n. \tag{A1}$$

The bilinear model matrices are $p \times m$ in these $n$ equations, while the matrices $\mathbf{Z}$ and $\mathbf{A}$ that compose the illuminant descriptors and the $d_v$ columns of zeros are of size $m \times m$. Assuming that the columns of the bilinear model matrices are linearly independent, we can find $d_m = p - m$ further columns for each matrix that span the unique subspace that is orthogonal to the existing columns. The difference $d_m$ is the defect in illumination that specifies the number of columns $\mathbf{b}^*$ that must be adjoined to each bilinear model matrix to transform it from a $p \times m$ matrix $\mathbf{B}$ to a square, invertible $p \times p$ matrix $\mathbf{B}^*$ (see Subsection 3.C):

$$\mathbf{B}_j^* = [\mathbf{B}_j\, \mathbf{b}_{j,1}^* \ldots \mathbf{b}_{j,d_m}^*]. \tag{A2}$$

We also expand the matrices $\mathbf{Z}$ and $\mathbf{A}$ with zero entries to make them $p \times p$:

$$\mathbf{Z}^* = \begin{bmatrix} \mathbf{Z} & \overbrace{\mathbf{0}\ldots\mathbf{0}}^{d_m} \\ & \vdots\ \ddots\ \vdots \\ \left.\begin{matrix}\mathbf{0}\ldots\mathbf{0} & \ldots & \mathbf{0} \\ \vdots\ \ddots\ \vdots & \ddots & \vdots \\ \mathbf{0}\ldots\mathbf{0} & \ldots & \mathbf{0}\end{matrix}\right\}d_m & \end{bmatrix},$$

where

$$\mathbf{Z} = \left[\mathbf{z}_1\ldots\mathbf{z}_v\ \ \underbrace{\mathbf{0}\ldots\mathbf{0}}_{d_v}\right], \tag{A3}$$

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{A} & \overbrace{\mathbf{0}\ldots\mathbf{0}}^{d_m} \\ & \vdots \ddots \vdots \\ \mathbf{0}\ldots\mathbf{0} & \ldots\mathbf{0} \\ \vdots \ddots \vdots & \ddots \vdots \\ \mathbf{0}\ldots\mathbf{0} & \ldots\mathbf{0} \end{bmatrix}\Big\} d_m,$$

where

$$\mathbf{A} = \left[\mathbf{a}_1\ldots\mathbf{a}_v \quad \underbrace{\mathbf{0}\ldots\mathbf{0}}_{d_v}\right]. \tag{A4}$$

The generalization of Eq. (67) to the case of rectangular bilinear-model matrices thus stands:

$$\mathbf{Z}^* = \left(\sum_{j=1}^{n} e_{1j}\mathbf{\Gamma}_{1j}{}^*\right)\mathbf{A}^* = \ldots = \left(\sum_{j=1}^{n} e_{nj}\mathbf{\Gamma}_{nj}{}^*\right)\mathbf{A}^*, \tag{A5}$$

where

$$\mathbf{\Gamma}_{ij}{}^* = \mathbf{B}_i{}^{*-1}\mathbf{B}_j{}^*. \tag{A6}$$

Two sorts of equations of degree $d_v + 1 = m - v + 1$ in the $e_{ij}$ arise from Eq. (A5). The first sort is found, again by forming $n - 1$ difference matrices,

$$\mathbf{W}_{1-j}{}^* = \left(\sum_{q=1}^{n} e_{1q}\mathbf{\Gamma}_{1q}{}^* - \sum_{q=1}^{n} e_{jq}\mathbf{\Gamma}_{jq}{}^*\right) \qquad \text{for } j = 2,\ldots,n, \tag{A7}$$

whence

$$\mathbf{W}_{1-j}{}^*\mathbf{A}^* = \mathbf{0}. \tag{A8}$$

The first $m$ entries of the first $m$ rows of each of these $n - 1$ differences provide $m$ $m$-dimensional vectors that all lie in the $d_v$-dimensional subspace orthogonal to that spanned by the $v$ linearly independent vectors of illuminant descriptors. There are a total of $(n - 1) \times m$ of these $m$-dimensional rows. The second sort is found by considering the equations

$$\mathbf{Z}^* = \left(\sum_{j=1}^{n} e_{ij}\mathbf{\Gamma}_{ij}{}^*\right)\mathbf{A}^* \qquad \text{for } i = 1,\ldots,n. \tag{A9}$$

Noting the zero entries in the last $p - m$ rows of $\mathbf{Z}^*$, one finds that the first $m$ entries of the last $p - m$ rows of each sum of starred gamma matrices provide vectors perpendicular to the $v$ column vectors $\mathbf{a}_1\ldots\mathbf{a}_v$ of $\mathbf{A}^*$. There are $n \times (p - m)$ of these $m$-dimensional rows. The total number of $m$-dimensional rows from the difference system [Eq. (A8)] and the sum system [Eq. (A9)] is $(n - 1)m + n(p - m)$, namely, $np - m$.

Each row lies in the same $d_v$-dimensional subspace orthogonal to the illuminants $\mathbf{a}_1\ldots\mathbf{a}_v$, so that all possible determinants of order $d_v + 1$ of the $np - m$ matrix of rows are zero. Each such determinant provides a single equation in monomial unknowns of degree $d_v + 1$ in the underlying variables $e_{ij}$. The total number $\mathsf{E}$ of equations is the number of distinct determinants of order $d_v + 1$ that may be formed within the $(np - m) \times m$ matrix of rows and subsumes the result for the square case [Eq. (71)]:

$$\underset{(p\geq m=v+d_v,\, n=s)}{\text{Number of Equations}} = \mathsf{E} = \binom{np - m}{d_v + 1}\binom{m}{d_v + 1} = \binom{np - m}{m - v + 1}\binom{m}{m - v + 1}. \tag{A10}$$

For a particular bilinear model to provide a perfect color constancy algorithm, it is necessary and sufficient for these $\mathsf{E}$ equations in the variables $(e_{11} - e_{jj})$ for $j = 2,\ldots,n$, and $e_{ij}$ for $i \neq j$, with coefficients determined by the model matrix entries, to possess only the scaling solution.

We again linearize the model check procedure by investigating solutions in the space of the monomial unknowns. We find the number $\mathsf{U}$ of monomial unknowns by allocating $d_v + 1$ items among a list of $n^2 - 1$ items $(e_{11} - e_{jj})$ for $j = 2,\ldots,n$, and $e_{ij}$ for $i \neq j$, and it is given by the expression for $\mathsf{U}$ in Eq. (72). As in Subsection 5.B.1, we form a homogeneous set of linear equations in the unknowns of degree $d_v + 1$ in the variables $(e_{11} - e_{jj})$ for $j = 2,\ldots,n$, and $e_{ij}$ for $i \neq j$. If the model check matrix representing these equations for a particular bilinear model has a rank equal to the number $\mathsf{U}$ of monomial unknowns, then its kernel has dimension zero, which implies that the only solution to Eq. (A5) is the scaling solution.

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