COLORED DECISION PROCESS PETRI NETS: MODELING, ANALYSIS AND STABILITY

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In this paper we introduce a new modeling paradigm for developing a decision process representation called the Colored Decision Process Petri Net (CDPPN). It extends the Colored Petri Net (CPN) theoretic approach including Markov decision processes. CPNs are used for process representation taking advantage of the formal semantic and the graphical display. A Markov decision process is utilized as a tool for trajectory planning via a utility function. The main point of the CDPPN is its ability to represent the mark-dynamic and trajectory-dynamic properties of a decision process. Within the mark-dynamic properties framework we show that CDPPN theoretic notions of equilibrium and stability are those of the CPN. In the trajectory-dynamic properties framework, we optimize the utility function used for trajectory planning in the CDPPN by a Lyapunov-like function, obtaining as a result new characterizations for final decision points (optimum point) and stability. Moreover, we show that CDPPN mark-dynamic and Lyapunov trajectory-dynamic properties of equilibrium, stability and final decision points converge under certain restrictions. We propose an algorithm for optimum trajectory planning that makes use of the graphical representation (CPN) and the utility function. Moreover, we consider some results and discuss possible directions for further research.

Keywords: decision process, colored Petri nets, colored decision process Petri nets, stability, Lyapunov methods, optimization, game theory

1. Introduction

Colored Petri nets (CPNs) provide a framework for the design, validation and verification of systems. CPNs combine the strength of place-transition Petri nets with the strength of programming languages. In this sense, place-transitions Petri nets provide primitives for describing the synchronization of concurrent processes, while a programming language provides primitives for defining data types (color sets) and manipulating data values. The formal definition of the syntax and semantics of CPNs can be found in (Jensen, 1981; 1986; 1994; 1997a; 1997b).

This paper introduces a modeling paradigm for developing decision process representation (Clempner, 2005) called the Colored Decision Process Petri Net (CDPPN). It extends the Colored Petri net theoretic approach including Markov decision processes, using a utility function as a tool for trajectory planning. On the one hand, Colored Petri nets are used for process representation, taking advantage of the well-known properties of this approach, namely, formal semantic and graphical display, giving a specific and unambiguous description of process behavior. On the other hand, Markov decision processes have become a standard model for decision theoretic planning problems, having as key drawbacks the exponential nature of dynamic policy construction algorithms. Although both perspectives are integrated in a CDPPN, they work on different execution levels. That is, the operation of a CPN is not modified and the utility function is used exclusively for establishing a trajectory tracking in a place-transition Petri net.

The main point of the CDPPN is its ability to represent mark-dynamic and trajectory-dynamic properties of a decision process application. We will identify the mark-dynamic properties of the CDPPN as properties related only with the Colored Petri net, and we will define the trajectory-dynamic properties of the CDPPN as properties related to the utility function at each place that depends on a probabilistic routing policy of the Colored Petri net.

Within the mark-dynamic properties framework we show that CDPPN theoretic notions of stability are those of the Colored Petri net. In this sense, we define the *equilibrium point* as a place in a CDPPN whose marking is bounded and does not change, and it is the last place in the net (a place without outgoing arcs).

In the trajectory-dynamic properties framework we define the utility function as a Lyapunov-like function (Massera, 1949). The core idea of our approach uses a utility function that is nonnegative and converges to an equilibrium point. For instance, in the arm race the level of defense of a nation is nonnegative. In economic models there are variables that correspond to, e.g., goods quantities that remain nonnegative. In a followers population model each variable remains nonnegative and corresponds to the population in a followers type.

By an appropriate selection of appropriate Lyapunov-like functions under a certain desired criterion it is possible to optimize the utility (Clempner *et al.*, 2005). *Optimizing* the utility amounts to the maximum or the minimum utility (depending on the concave or the convex shape of the application space definition). In addition to that, we used the notions of stability in the Lyapunov sense to characterize stability properties of the CDPPN. The core idea of our approach uses a non-negative utility function that converges in a decreasing form to a (set of) final decisions states. It is important to point out that the value of the utility function associated with the CDPPN implicitly determines a set of policies, not just a single policy, in the case of having several decision states that could be reached. We define the *optimum point* as the best choice selected from a number of possible final decisions states that may be reached (to select the optimum point, the decision process chooses the strategy that optimizes the utility).

As a result, we extend the mark-dynamic framework including trajectory-dynamic properties. We show that CDPPN mark-dynamic and trajectory-dynamic properties of the equilibrium, stability and optimum point conditions converge under certain restrictions: if the CDPPN is finite and nonblocking, then a final decision state is an equilibrium point iff it is an optimum point.

An algorithm for optimum trajectory planning used to find an optimum point is presented. It consists in finding a firing transition sequence such that an optimum decision state is reached in the CDPPN. For this propose the algorithm uses the graphical representation provided by the Colored Petri net and the utility function. It is important to note that algorithm complexity depends on the Lyapunov-like function chosen to represent the utility function.

The paper is structured in the following manner: The next section presents the necessary mathematical background and terminology needed to understand the rest of the paper. Section 3 discusses the main results of this paper, providing a definition of the CDPPN and giving a detailed analysis of the equilibrium, stability and optimum point conditions for the mark-dynamic and the trajectory-dynamic parts of the CDPPN. An algorithm for calculating the optimum trajectory used to find the optimum point is proposed. For illustration purposes we show how the standard notions of stability in CDPPN theory are applied to a practical example. Finally, some concluding remarks and suggestions for future work are provided in Section 4.

2. Preliminaries

In this section, we present some well-established definitions and properties (Lakshmikantham *et al.*, 1990, 1991), which will be used later. We set $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}_{n_0+} = \{n_0, n_0 + 1, \dots, n_0 + k, \dots\}$, $n_0 \geq 0$. Given $x, y \in \mathbb{R}^d$, we use the relation "$\leq$ " to denote componentwise inequalities with the same relation, i.e., $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$. A function $f(n, x)$, $f : \mathbb{N}_{n_0+} \times \mathbb{R}^d \to \mathbb{R}^d$ is called nondecreasing in x if, given $x, y \in \mathbb{R}^d$ such that $x \geq y$ and $n \in \mathbb{N}_{n_0+}$, we have $f(n, x) \geq f(n, y)$. A function $f(n, x)$, $f : \mathbb{N}_{n_0+} \times \mathbb{R}^d \to \mathbb{R}^d$ is called strictly increasing in x if, given $x, y \in \mathbb{R}^d$ such that $x > y$ and $n \in \mathbb{N}_{n_0+}$, we have $f(n, x) > f(n, y)$.

Consider the system of first-order difference equations given by

$$x(n + 1) = f\big[n, x(n)\big], \quad x(n_o) = x_0, \tag{1}$$

where $n \in \mathbb{N}_{n_0+}$, $x(n) \in \mathbb{R}^d$ and $f : \mathbb{N}_{n_0+} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous in $x(n)$.

Definition 1. The vector-valued function $\Phi(n, n_0, x_0)$ is said to be a solution of (1) if $\Phi(n_0, n_0, x_0) = x_0$ and $\Phi(n + 1, n_0, x_0) = f(n, \Phi(n, n_0, x_0))$ for all $n \in \mathbb{N}_{n_0+}$.

Definition 2. The system (1) is said to be (Lakshmikantham *et al.*, 1990):

(i) *practically stable*, if given (λ, A) with $0 < \lambda < A$, we have

$$|x_0| < \lambda \Rightarrow |x(n, n_0, x_0)| < A, \quad \forall n \in \mathbb{N}_{n_0+}, \quad n_0 \geq 0;$$

(ii) *uniformly practically stable*, if it is practically stable for every $n_0 \geq 0$.

The following class of functions is defined:

Definition 3. A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to a class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing.

2.1. Methods for Practical Stability

Consider (Lakshmikantham *et al.*, 1991) the vector function $v(n, x(n))$, $v : \mathbb{N}_{n_0+} \times \mathbb{R}^d \to \mathbb{R}^p_+$ and define the variation of v relative to (1) by

$$\Delta v = v\big(n+1, x(n+1)\big) - v\big(n, x(n)\big). \qquad (2)$$

Then the following result concerns the practical stability of (1):

Theorem 1. *Let* $v : \mathbb{N}_{n_0+} \times \mathbb{R}^d \to \mathbb{R}^p_+$ *be a continuous function in* x. *Define the function* $v_0(n, x(n)) = \sum_{i=1}^{p} v_i(n, x(n))$ *such that it satisfies the estimates*

$$b(|x|) \le v_0\left(n, x\left(n\right)\right) \le a(|x|) \ \text{for}\ a, b \in \mathcal{K},$$

$$\Delta v\big(n, x(n)\big) \le w\big(n, v\big(n, x(n)\big)\big)$$

for $n \in \mathbb{N}_{n_0+}$, $x(n) \in \mathbb{R}^d$, *where* $w : \mathbb{N}_{n_0+} \times \mathbb{R}^p_+ \to \mathbb{R}^p$ *is a continuous function in the second argument. Assume that* $g(n, e) \triangleq e + w(n, e)$ *is nondecreasing in* e, $0 < \lambda < A$ *are given and, finally, that* $a(\lambda) < b(A)$ *is satisfied. Then the practical stability properties of*

$$e(n+1) = g\big(n, e(n)\big), \quad e\big(n_0\big) = e_0 \ge 0, \qquad (3)$$

imply the corresponding practical stability properties of System (1).

Corollary 1.

1. *If* $w(n, e) \equiv 0$, *we get uniform practical stability of (1) which implies structural stability (Murata, 1989).*
2. *If* $w(n, e) = -c(e)$ *for* $c \in \mathcal{K}$, *we get uniform practical asymptotic stability of (1).*

2.2. Colored Petri Nets

In this section, we present the concepts of colored Petri nets (Jensen, 1981; 1986; 1994; 1997a; 1997b), multi-set, marking, step, firing rule and incidence matrix.

Definition 4. A multiset m over a nonempty set S is a function $m : S \to \mathbb{N}$ which we represent as a formal sum

$$\sum_{s \in S} m(s)s.$$

We denote by S_{ms} the set of all multisets over S. The nonnegative integers $\{m(s) : s \in S\}$ are the coefficients of the multiset, $s \in S$ iff $m(s) \neq 0$.

Definition 5. Addition, scalar multiplication, comparison and size of multisets are defined in the following way, for all m_1, m_2, $m_3 \in S_{ms}$ and all $n \in \mathbb{N}$:

(i) $m_1 + m_2 = \sum_{s \in S}(m_1(s) + m_2(s))s$ (addition),

(ii) $n * m = \sum_{s \in S}(n * m(s))s$ (scalar multiplication),

(iii) $m_1 \neq m_2 = \exists s \in S : m_1(s) \neq m_2(s)$ (comparison),

(iv) $m_1 \le m_2 = \forall s \in S : m_1(s) \le m_2(s)$ ($\ge$ and $=$ are defined analogously to $\le$),

(v) $|m| = \sum_{s \in S} m(s)$ ($|m| = 0$ iff $m = \emptyset$ the empty multiset) (size).

When $|m| = \infty$, we say that m is infinite. Otherwise, m is finite. When $m_1 \le m_2$, we also define the subtraction:

(vi) $m_2 - m_1 = \sum_{s \in S}(m_2(s) - m_1(s))s$ (subtraction).

Remark 1. The weighted sets over a set S, S_{ws}, are defined as multisets, but over Z, allowing negative coefficients. The operations for the weighted sets S_{ws} are the same for the operations with multisets but scalar multiplication is defined for negative integers and subtraction is defined also for all weighted sets.

Definition 6. A *colored Petri net* is the septuple CPN $= (\Sigma, P, Q, K, A^+, A^-, M_0)$, where

- Σ is a finite set of nonempty sets, called colors,
- P is the set of places,
- Q is the set of transitions,
- $P \cap Q = \varnothing$ and $P \cup Q \neq \varnothing$,
- $K : P \cup Q \longrightarrow \Sigma$ is the color function, where Σ is the set of finite nonempty sets,
- $A^+ : K(p) \times K(q) \to \mathbb{N}$ is the forward incidence matrix of $P \times Q$,
- $A^- : K(p) \times K(q) \to \mathbb{N}$ is the backward incidence matrix of $P \times Q$,
- M_0, the initial marking, is a vector indexed by the elements of P, where $M_0(p) : K(p) \to \mathbb{N}$.

Remark 2. A^+ and A^- are matrices of size $P \times Q$ with coefficients in $\mathbb{N}$ which define linear applications from $K(q)$ to $K(p)_{ms}$. The initial marking $M_0(p)$ takes its values in $K(p)_{ms}$.

Definition 7. A marking of CPN is a function M defined on P, such that $M(p) \in K(p)_{ms}$ for all $p \in P$.

Definition 8. A step of CPN is a function X defined on Q, such that $X(q) \in K(q)_{ms}$ for all $q \in Q$.

Definition 9. The transition firing rule is given by the following:

- A step X is enabled in a marking M iff the following property holds: $\forall p \in P$, $M(p) \geq \sum_{q\in Q} A^-(p,q)(X(q))$, which can also be written as $M \geq A^- * X$, where $*$ denotes generalized matrix multiplication. We then say that q is enabled or firable under the marking M.

- Firing a transition q leads to a new marking M_1 defined as follows: $\forall p \in P$,

$$M_1(p) = M(p) + \sum_{q\in Q} A^+(p,q)\big(X(q)\big) - \sum_{q\in Q} A^-(p,q)\big(X(q)\big)$$

or, in general,

$$M_1 = M + A^+ * X - A^- * X.$$

Remark 3. The condition $M(p) \geq \sum_{q\in Q} A^-(p,q)(X(q))$ tells us that the multiset of all the colors which are removed from p when q occurs (for all $q \in X$) is required to be less than or equal to the marking of p. It is important to mention that generalized matrix-multiplication, when is defined, behaves in relation to the size operation as follows:

$$|A_1 * A_2| = |A_1| * |A_2|.$$

Definition 10. The *incidence matrix* of a colored Petri net is defined by

$$A = A^+ - A^-, \quad A(p,q) \in K(q) \to K(p)_{ws}, \qquad (4)$$

where $A(p,q)$ is a linear mapping whose associated matrix $P \times Q$ takes values in Z.

Remark 4. If a transition q is fired with respect to a color $\kappa_q \in K(q)$, then for every color $\kappa_p \in K(p)$, $A(\kappa_p, \kappa_q)$ gives the number of colors κ_p to be added to (if the number is positive) or to be removed from (if the number is negative) place p. Notice that if M' can be reached from a marking M, i.e., there exists a sequence of enabled steps whose associated transitions have been fired, then we obtain

$$M' = M + A * X. \qquad (5)$$

Definition 11. Let a place $p \in P$, and a nonnegative $n \in \mathbb{N}$ be given. Then n is an *integer bound* for p iff for M' reachable from M we have $|M'(p)| \leq n$.

Let $(\mathbb{N}_{n_0+}, d)$ be a metric space where $d : \mathbb{N}_{n_0+} \times \mathbb{N}_{n_0+} \to \mathbb{R}_+$ is defined by

$$d(M_1, M_2) = \sum_{i=1}^{m} \zeta_i \big\| \big(M_1(p_i)(\kappa_p) - M_2(p_i)\big(\kappa_p\big)\big) \big\|, \quad (6)$$

where $\zeta_i > 0$, $\forall \kappa_p \in K(p_i)$, $i = 1, \dots, m$. Consider (5), which defines a continuous function in $(\mathbb{N}_{n_0+}, d)$. Now, we are ready to state and prove two main results of this subsection (Passino *et al.*, 1995).

Proposition 1. *Let CPN be a colored Petri net. The colored Petri net CPN is uniformly practically stable if there exists a strictly positive linear mapping* $\Phi : K(p)_{ws} \to \Upsilon_{ws}$ *(with* Υ *normally one of the color sets is already used in CPN) such that*

$$\Delta v = |\Phi * A * X| \leq 0. \qquad (7)$$

Proof. Let us choose $v(M(p)(c_p)) = |\Phi * M|$, $\forall c_p \in C(p)$, as a Lyapunov function candidate and let Φ be a strictly positive linear mapping. The Lyapunov function v satisfies the conditions of Theorem 1. Therefore, uniform practical stability is obtained if there exists a strictly positive linear mapping Φ such that (7) holds. ■

Remark 5. The condition given by (7) with a strictly equality sign is equivalent to the condition

$$\Phi * A = 0_f, \qquad (8)$$

where 0_f is the zero function.

The solution of this equation is not an easy task. However, various methods have been proposed (see Jensen, 1997b and the references given therein).

Proposition 2. *Let CPN be a colored Petri net. The CPN is stabilizable if there exists a step* X *such that*

$$\Delta v = |A * X| \leq 0. \qquad (9)$$

Proof. Let us choose

$$v\big(M(p)(c_p)\big) = \big[v_1\big(M(p)(c_p)\big), v_2\big(M(p)(c_p)\big), \dots, v_m\big(M(p)(c_p)(p)(c_p)\big)\big]^T, \ \forall c_p \in C(p),$$

as a vector Lyapunov function candidate, where $v_i(M(p)(c_p)) = |M(p_i)|, 1 \leq i \leq m$ (with m equal to the number of places in CPN). The Lyapunov function v satisfies the conditions of Theorem 1. Therefore, uniform practical stability is obtained if there exists a step X such that (9) holds. Therefore, we conclude that the CPN is stabilizable. ■

2.3. Decision Process

We assume that every discrete-event system with a finite set of states P to be controlled can be described as a fully observable, discrete-state Markov decision process (Bellman, 1957; Howard, 1960; Puterman, 1994). To control the Markov chain, there must exist a possibility of changing the probability of transitions through an external interference. We suppose that there exist a possibility of carrying out the Markov process by N different methods. In this sense, we suppose that the controlling of the discrete-event system has a finite set of actions Q which cause stochastic state transitions. We denote by $p_q(s,t)$ the probability that an action q generates a transition from a state s to a state t, where $s,t \in P$.

A stationary policy $\pi : P \to Q$ denotes a particular strategy or a course of action to be adopted by a discrete-event system, with $\pi(s,q)$ being the action to be executed whenever the discrete-event system is in a state $s \in P$. We refer the reader to (Bellman, 1957; Howard, 1960; Puterman, 1994) for a description of policy construction techniques.

Hereafter, we will consider the possibility of estimating every step of the process through a utility function that represents the utility generated by the transition from state s to state t in the case of using an action q. We assume an infinite time horizon, and that the discrete-event system accumulates the utility associated with the states it enters.

Let us define $U_\pi(s)$ as the maximum utility starting at the state s that guarantees choosing the optimal course of action $\pi(s,q)$. Let us suppose that at the state s we have an accumulated utility $B(s)$ and the previous transitions have been executed in an optimal form. In addition, suppose consider that the transition of going from the state s to the state t has a probability of $p_{\pi(s,q)}(s,t)$. Because the transition from the state s to the state t is stochastic, it is necessary to take into account the possibility of going through all the possible states from s to t. Then the utility of going from state s to state t is represented by

$$U_\pi(s) = B(s) + \beta \sum_{t\in P} p_{\pi(s,q)}(s,t) \cdot U_\pi(t), \qquad (10)$$

where $\beta \in [0,1)$ is the discount rate (Howard, 1960).

The value of π at any initial state s can be computed by solving this system of linear equations. A policy π is optimal if $U_\pi(t) \geq U_{\pi'}(t)$ for all $t \in P$ and policies π'. The function U establishes a preference relation.

3. Colored Decision Process Petri Net

We introduce the concept of Colored Decision Process Petri nets (CDPPN) by locally randomizing the possible choices, for each individual place of the Petri net (Clempner, 2005).

Definition 12. A *colored decision process Petri net* is the tentuple CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$, where

- Σ is a finite set of nonempty sets, called colors,
- $P = \{p_0, p_1, p_2, \dots, p_m\}$ is a finite set of places,
- $Q = \{q_1, q_2, \dots, q_n\}$ is a finite set of transitions,
- $F \subseteq I \cup O$ is a set of arcs where $I \subseteq (P \times Q)$ and $O \subseteq (Q \times P)$ such that $P \cap Q = \varnothing$ and $P \cup Q \neq \varnothing$,
- $K : P \cup Q \longrightarrow \Sigma$ is the color function, where Σ is the set of finite nonempty sets,
- $A^+ : K(p) \times K(q) \to \mathbb{N}$ is the forward incidence matrix of $P \times Q$,
- $A^- : K(p) \times K(q) \to \mathbb{N}$ is the backward incidence matrix of $P \times Q$,
- M_0, the initial marking, is a vector indexed by the elements of P, where $M_0(p) : K(p) \to \mathbb{N}$,
- $\pi(p,q) : K(p) \times K(q) \to \mathbb{R}_+$ is a routing policy representing the probability of choosing a particular transition (routing arc), such that for each $p \in P$,

$$\sum_{q_j:(p,q_j)\in I} \pi((\kappa_p, \kappa_{q_j})) = 1,$$

- $U(p) : K(p) \to \mathbb{R}_+$ is a utility function.

The previous behavior of the CDPPN is described as follows: When a token reaches a place, it is reserved for the firing of a given transition according to the routing policy determined by U. A transition q must fire as soon as all the places $p_1 \in P$ contain enough tokens reserved for a transition q. Once the transition fires, it consumes the corresponding tokens and immediately produces an amount of tokens in each subsequent place $p_2 \in P$. When $\pi(p,q)(\kappa_p, \kappa_q) = 0$, this means that there are no arcs in the place-transition Petri net. In Figs. 1 and 2 we have represented partial routing policies π that generate a transition from a state p_1 to a state p_2 where $p_1, p_2 \in P$:

- **Case 1.** In Fig. 1 the probability that q_1 generates a transition from the state p_1 to p_2 is 1/3. But, because the transition q_1 to the state p_2 has two arcs, the probability of generating a transition from the state p_1 to p_2 is increased to 2/3.
- **Case 2.** In Fig. 2, by convention, the probability that q_1 generates a transition from the state p_1 to p_2 is 1/3 (1/6 plus 1/6). However, because the transition q_1 to the state p_2 has only one arc, the probability of generating a transition from the state p_1 to p_2 is decreased to 1/6.

- **Case 3.** Finally, we have the trivial case when there exists only one arc from p_1 to q_1 and from q_1 to p_2.

It is important to note that, by definition, the utility function U is employed only for establishing a trajectory tracking, working on a different execution level than that of the place-transition Petri net. The utility function U by no means changes the place-transition Petri net evolution or performance.

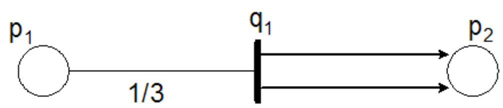

Fig. 1. Routing policy, Case 1.

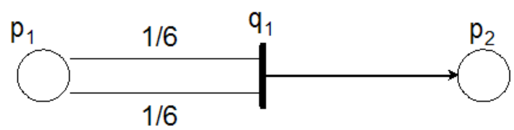

Fig. 2. Routing policy, Case 2.

Remark 6. In the previous definition we consider nets with a single initially marked place.

Remark 7. The previous definition by no means changes the behavior of the place-transition Petri net, and the *routing policy is used to calculate the utility value at each place of the net.*

Remark 8. It is important to note that the utility value can be renormalized after each transition or time k of the net.

$U_k(\cdot)$ denotes the utility at the place $p_i \in P$ at time k and let $U_k = [U_k(\cdot), \dots, U_k(\cdot)]^T$ denote the utility state of the CDPPN at the time k. $FN(p,q) : K(p) \times K(q) \to \mathbb{R}_+$ is the number of arcs from the place p to the transition q (the number of arcs from transition q to the place p). The rest of CDPPN functionality is as described in PN preliminaries.

Consider an arbitrary $p_i \in P$. For each fixed transition $q_j \in Q$ that forms an output arc $(q_j, p_i) \in O$, we look at all the previous places p_h of the place p_i denoted by the list (set) $p_{\eta_{ij}} = \{p_h : h \in \eta_{ij}\}$, where $\eta_{ij} = \{h : (p_h, q_j) \in I$ and $(q_j, p_i) \in O\}$, which form all the input arcs $(p_h, q_j) \in I$ and produce the sum

$$\sum_{h \in \eta_{ij}} \Psi(p_h, q_j, p_i) * U_k(p_h)(\kappa_{p_h}), \quad \forall \kappa_{p_h} \in K(p_h), \tag{11}$$

where

$$\Psi(p_h, q_j, p_i)(\kappa_{p_h}, \kappa_{q_j}, \kappa_{p_i}) = \pi(p_h, q_j)(\kappa_{p_h}, \kappa_{q_j}) * \frac{FN(q_j, p_i)(\kappa_{q_j}, \kappa_{p_i})}{FN(p_h, q_j)(\kappa_{p_h}, \kappa_{q_j})},$$

$$\forall \kappa_{p_h} \in K(p_h), \ \ \forall \kappa_{q_j} \in K(q_j), \ \ \forall \kappa_{p_i} \in K(p_i),$$

and the index sequence j is the set $\{j : q_j \in (p_h, q_j) \cap (q_j, p_i)$ and p_h running over the set $p_{\eta_{ij}}\}$.

Proceeding with all the q_js, we form the vector indexed by the sequence j identified by $(j_0, j_1, \dots, j_f)$ as follows:

$$\Bigg[\sum_{h \in \eta_{ij_0}} \Psi(p_h, q_{j_0}, p_i)(\kappa_{p_h}, \kappa_{q_{j_0}}, \kappa_{p_i}) * U_k(p_h)(\kappa_p), \sum_{h \in \eta_{ij_1}} \Psi(p_h, q_{j_1}, p_i)(\kappa_{p_h}, \kappa_{q_{j_1}}, \kappa_{p_i}) * U_k(p_h)(\kappa_p), \dots, \sum_{h \in \eta_{ij_f}} \Psi(p_h, q_{j_f}, p_i)(\kappa_{p_h}, \kappa_{q_{j_f}}, \kappa_{p_i}) * U_k(p_h)(\kappa_p) \Bigg]. \tag{12}$$

Intuitively, the vector (12) represents all the possible trajectories through the transitions q_js, where $(j_1, j_2, \dots, j_f)$, to a place p_i for a fixed i.

Continuing the construction of the utility function U, let us introduce the following definition:

Definition 13. Let $L : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous map. Then L is a *Lyapunov-like function* (Kalman and Bertram, 1960) iff it satisfies the following properties:

1. $\exists x^*, \ L(x^*) = 0$,
2. $L(x) > 0, \ \forall x \neq x^*$,
3. $L(x) \to \infty$ as $x \to \infty$,
4. $\Delta L = L(x_{i+1}) - L(x_i) < 0$ for all $x_i, x_{i+1} \neq x^*$.

Then, formally, we define the utility function U as follows:

Definition 14. The utility function U with respect to a colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is represented by the equation

$$U_k^{q_j}(p_i)(\kappa_{p_i}) = \begin{cases} U_k(p_0)(\kappa_{p_0}) & \text{if} \quad i = 0, \ k = 0, \\ L(\alpha) & \text{if} \quad i > 0, \ k = 0 \\ & \text{and} \quad i \geq 0, \ k > 0, \end{cases} \tag{13}$$

where

$$\alpha = \Bigg[\sum_{h\in\eta_{ij_0}} \Psi(p_h, q_{j_0}, p_i)(\kappa_{p_h}, \kappa_{q_{j_0}}, \kappa_{p_i}) * U_k^{q_{j_0}}(p_h)(\kappa_{p_h}), \sum_{h\in\eta_{ij_1}} \Psi(p_h, q_{j_1}, p_i)(\kappa_{p_h}, \kappa_{q_{j_1}}, \kappa_{p_i}) * U_k^{q_{j_1}}(p_h)(\kappa_{p_h}), \dots, \sum_{h\in\eta_{ij_f}} \Psi(p_h, q_{j_f}, p_i)(\kappa_{p_h}, \kappa_{q_{j_f}}, \kappa_{p_i}) * U_k^{q_{j_f}}(p_h)(\kappa_{p_h}) \Bigg]. \tag{14}$$

The function $L : D \subseteq \mathbb{R}_+^n \to \mathbb{R}_+$ is a Lyapunov-like function which optimizes the utility through all possible transitions (i.e., trough all possible trajectories defined by different q_js), D is the decision set formed by the js, $0 \le j \le f$ of all those possible transitions $(q_j \;\; p_i) \in O$,

$$\Psi(p_h, q_j, p_i)(\kappa_{p_h}, \kappa_{q_j}, \kappa_{p_i}) = \pi(p_h, q_j)(\kappa_{p_h}, \kappa_{q_j}) * \frac{FN(q_j, p_i)(\kappa_{q_j}, \kappa_{p_i})}{FN(p_h, q_j)(\kappa_{p_h}, \kappa_{q_j})},$$

$$\forall \kappa_{p_h} \in K(p_h), \;\; \forall \kappa_{q_j} \in K(q_j), \;\; \forall \kappa_{p_i} \in K(p_i),$$

η_{ij} is the index sequence of the list of places previous to p_i through the transition q_j, p_h $(h \in \eta_{ij})$ is a specific previous place of p_i through the transition q_j.

Remark 9.

- Note that the previous definition of the utility function U with respect to (10) considers the accumulated utility $B(\cdot) = 0$, and the Lyapunov-like function L guarantees that the optimal course of action is followed, taking into account all the the possible paths defined. In addition to that, the function L establishes a preference relation because, by definition, L is asymptotic. This condition gives the decision maker the opportunity to select a path that optimizes the utility.
- The iteration over k for U is as follows:
  1. For $i = 0$ and $k = 0$ the utility is $U_0(p_0)(\kappa_{p_0})$ in the place p_0 and for the rest of the places p_i the utility is 0.
  2. For $i \ge 0$ and $k > 0$ the utility is $U_k^{q_j}(p_i)(\kappa_{p_i})$ $\forall \kappa_{p_i} \in K(p_i)$ in each place p_i, computed by taking into account the utility value of the previous places p_h for k and $k-1$ (when needed).

Property 1. *The continuous function $U(\cdot)$ satisfies the following properties:*

1. *There is a $p^{\triangle} \in P$ such that*
   (a) if there exists an infinite sequence $\{p_i\}_{i=1}^{\infty} \in P$ with $p_n \underset{n\to\infty}{\to} p^{\triangle}$ such that $0 \le \dots < U(p_n)(\kappa_{p_n}) < U(p_{n-1})(\kappa_{p_{n-1}}) \dots < U(p_1)(\kappa_{p_1})$, then $U(p^{\triangle})(\kappa_{p^{\triangle}})$ is the infimum, i.e., $U(p^{\triangle})(\kappa_{p^{\triangle}}) = 0$,
   (b) if there exists a finite sequence $p_1, \dots, p_n \in P$ with $p_1, \dots, p_n \to p^{\triangle}$ such that $C = U(p_n)(\kappa_{p_n}) < U(p_{n-1})(\kappa_{p_{n-1}}) \dots < U(p_1)(\kappa_{p_1})$, then $U(p^{\triangle})(\kappa_{p^{\triangle}})$ is the minimum, i.e., $U(p^{\triangle})(\kappa_{p^{\triangle}}) = C$, where $C \in \mathbb{R}$, $p^{\triangle} = p_n$,
2. *$U(p)(\kappa_p) > 0$ or $U(p)(\kappa_p) > C$, where $C \in \mathbb{R}$, $\forall p \in P$ such that $p \ne p^{\triangle}$.*
3. *If $\forall p_i, p_{i-1} \in P$ such that $p_{i-1} \le_U p_i$, then $\Delta U = U(p_i)(\kappa_{p_i}) - U(p_{i-1})(\kappa_{p_{i-1}}) < 0$.*
4. *The routing policies decrease monotonically, i.e., $\pi_i \ge \pi_j$ (notice that the indices i and j are taken so that $j > i$ along a trajectory to the infimum or the minimum).*

Remark 10. In Property 1 we state that $\Delta U = U(p_i)(\kappa_{p_i}) - U(p_{i-1})(\kappa_{p_{i-1}}) < 0$ for determining the asymptotic condition of the Lyapunov-like function. However, it is easy to show that such a property is convenient for deterministic systems. In Markov decision process systems it is necessary to include probabilistic decreasing asymptotic conditions to guarantee the asymptotic condition of the Lyapunov-like function.

Property 2. *The utility function $U(p) : K(p) \to \mathbb{R}_+$ is a Lyapunov-like function.*

Remark 11. From Properties 1 and 2 we have the following:

- $U(p^{\triangle})(\kappa_{p^{\triangle}}) = 0$ or $U(p^{\triangle})(\kappa_{p^{\triangle}}) = C$ means that a final state is reached. Without loss of generality we can say that $U(p^{\triangle})(\kappa_{p^{\triangle}}) = 0$ by means of a translation to the origin.
- In Property 1 we conclude that the Lyapunov-like function $U(p)(\kappa_p)$ approaches an infimum/minimum when p is large thanks to Point 4 of Definition 13.
- Property 1, Point 3, is equivalent to the following statement: There is an $\varepsilon > 0$ such that $|U(p_i)(\kappa_{p_i}) - U(p_{i-1})(\kappa_{p_{i-1}})| > \varepsilon$, $\forall p_i, p_{i-1} \in P$ such that $p_{i-1} \le_U p_i$.

For instance, the utility function U in terms of the entropy is a specific Lyapunov-like function used in information theory as a measure of the information disorder. Another possible choice is the $\min$ function used in business process re-engineering to evaluate the job performance.

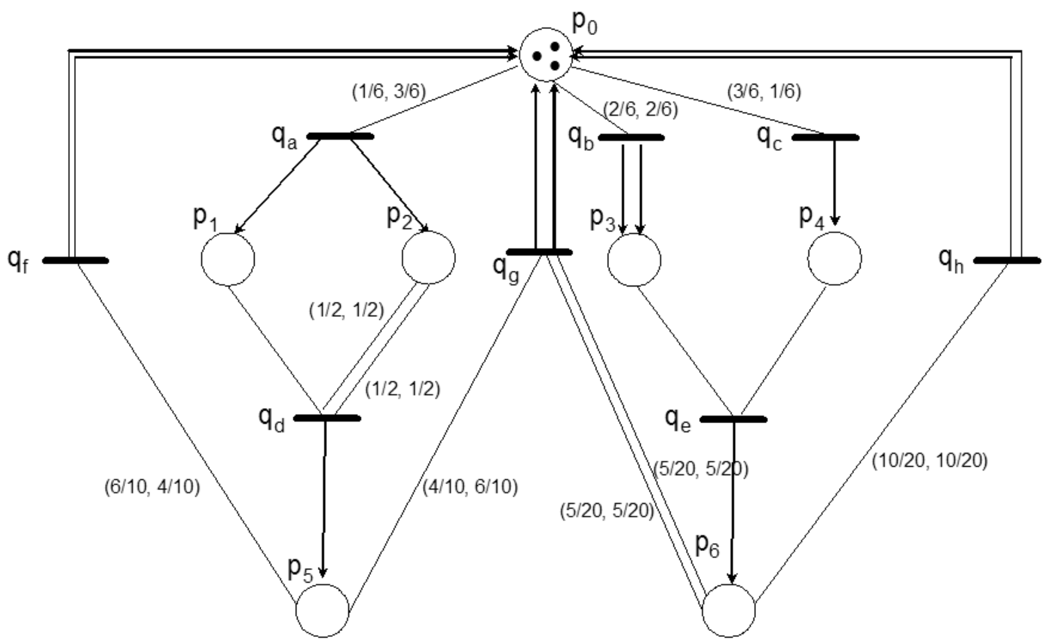

Fig. 3. Setting of Example 1.

Example 1. Define the Lyapunov-like function L in terms of the entropy $H(p_i) = -p_i \ln p_i$ as $L = \max_{i=1,\dots,|\alpha|}(-\alpha_i \ln \alpha_i)$. We will conceptualize H as the average amount of uncertainty created by moving one step ahead (the uncertainty is high when H is close to 0 and low when H is close to 1). In the CDPPN the token will have two colors identified as l and r. Every arc has associated a set of probabilities corresponding to the colors l and r, i.e., (probability_l, probability_r).

$$U_{k=0}(p_0)(l) = 1,$$

$$U_{k=0}(p_0)(r) = 0.8,$$

$$\begin{aligned} U^{q_a}_{k=0}(p_1)(l) &= L\big[\Psi(p_0, q_a, p_1)(l, q_a, l) * U^{q_a}_{k=0}(p_0)(l)\big] \\ &= L[1/6 * 1] = \max H[1/6 * 1] = 0.298, \end{aligned}$$

$$\begin{aligned} U^{q_a}_{k=0}(p_1)(r) &= L\big[\Psi(p_0, q_a, p_1)(r, q_a, r) * U^{q_a}_{k=0}(p_0)(r)\big] \\ &= L[3/6 * 0.8] = \max H[3/6 * 0.8] \\ &= 0.366, \end{aligned}$$

$$\begin{aligned} U^{q_a}_{k=0}(p_2)(l) &= L\big[\Psi(p_0, q_a, p_2)(l, q_a, l) * U^{q_a}_{k=0}(p_0)(l)\big] \\ &= L[1/6 * 1] = \max H[1/6 * 1] = 0.298, \end{aligned}$$

$$\begin{aligned} U^{q_a}_{k=0}(p_2)(r) &= L\big[\Psi(p_0, q_a, p_2)(r, q_a, r) * U^{q_a}_{k=0}(p_0)(r)\big] \\ &= L[3/6 * 0.8] = \max H[3/6 * 0.8] \\ &= 0.366, \end{aligned}$$

$$\begin{aligned} U^{q_b}_{k=0}(p_3)(l) &= L\big[\Psi(p_0, q_b, p_3)(l, q_b, l) * U^{q_b}_{k=0}(p_0)(l)\big] \\ &= L[(2/6 * 2) * 1] = \max H[4/6 * 1] \\ &= 0.270, \end{aligned}$$

$$\begin{aligned} U^{q_b}_{k=0}(p_3)(r) &= L\big[\Psi(p_0, q_b, p_3)(r, q_b, r) * U^{q_b}_{k=0}(p_0)(r)\big] \\ &= L[(2/6 * 2) * 0.8] = \max H[4/6 * 0.8] \\ &= 0.335, \end{aligned}$$

$$\begin{aligned} U^{q_c}_{k=0}(p_4)(l) &= L\big[\Psi(p_0, q_c, p_4)(l, q_c, l) * U^{q_c}_{k=0}(p_0)(l)\big] \\ &= L[3/6 * 1] = \max H[3/6 * 1] = 0.346, \end{aligned}$$

$$\begin{aligned} U^{q_c}_{k=0}(p_4)(r) &= L\big[\Psi(p_0, q_c, p_4)(r, q_c, r) * U^{q_c}_{k=0}(p_0)(r)\big] \\ &= L[1/6 * 0.8] = \max H[1/6 * 0.8] \\ &= 0.268, \end{aligned}$$

$$\begin{aligned} U^{q_d}_{k=0}(p_5)(l) &= L\big[\Psi(p_1, q_d, p_5)(l, q_d, l) U^{q_d}_{k=0}(p_1)(l) \\ &\quad + \Psi(p_2, q_d, p_5)(l, q_d, l) U^{q_d}_{k=0}(p_2)(l)\big] \\ &= L[1 * 0.298 + 1/2 * 0.298] \\ &= \max H[0.447] = 0.359, \end{aligned}$$

$$\begin{aligned} U^{q_d}_{k=0}(p_5)(r) &= L\big[\Psi(p_1, q_d, p_5)(r, q_d, r) U^{q_d}_{k=0}(p_1)(r) \\ &\quad + \Psi(p_2, q_d, p_5)(r, q_d, r) U^{q_d}_{k=0}(p_2)(r)\big] \\ &= L[1 * 0.366 + 1/2 * 0.366] \\ &= \max H[0.549] = 0.329, \end{aligned}$$

$$\begin{aligned} U^{q_e}_{k=0}(p_6)(l) &= L\big[\Psi(p_3, q_e, p_6)(l, q_e, l) U^{q_e}_{k=0}(p_3)(l) \\ &\quad + \Psi(p_4, q_e, p_6)(l, q_e, l) U^{q_e}_{k=0}(p_4)(l)\big] \end{aligned}$$

$$\begin{aligned}
&= L[1 * 0.270 + 1 * 0.346] \\
&= \max H[0.616] = 0.298,
\end{aligned}$$

$$\begin{aligned}
U_{k=0}^{q_e}(p_6)(r) &= L\big[\Psi(p_3, q_e, p_6)(r, q_e, r)U_{k=0}^{q_e}(p_3)(r) \\
&\quad + \Psi(p_4, q_e, p_6)(r, q_e, r)U_{k=0}^{q_e}(p_4)(r)\big] \\
&= L[1 * 0.335 + 1 * 0.268] \\
&= \max H[0.603] = 0.305,
\end{aligned}$$

$$\begin{aligned}
U_{k=1}^{q_{(f,g,h)}}(p_0)(l) &= L\big[\Psi(p_5, q_f, p_0)(l, q_f, l)U_{k=1}^{q_f}(p_5)(l), \\
&\quad \Psi(p_5, q_g, p_0)(l, q_g, l)U_{k=1}^{q_g}(p_5)(l) \\
&\quad +\Psi(p_6, q_g, p_0)(l, q_g, l)U_{k=1}^{q_g}(p_6)(l), \\
&\quad \Psi(p_6, q_h, p_0)(l, q_h, l)U_{k=1}^{q_h}(p_6)(l)\big] \\
&= L[6/10 * 2 * 0.359, \\
&\quad (4/10 * 0.359 + 5/20 * 0.298) \\
&\quad * 2, 10/20 * 2 * 0.298] \\
&= \max H[0.430, 0.436, 0.298] \\
&= \max[0.362, 0.361, 0.360] = 0.362
\end{aligned}$$

$$\begin{aligned}
U_{k=1}^{q_{(f,g,h)}}(p_0)(r) &= L\big[\Psi(p_5, q_f, p_0)(r, q_f, r)U_{k=1}^{q_f}(p_5)(r), \\
&\quad \Psi(p_5, q_g, p_0)(r, q_g, r)U_{k=1}^{q_g}(p_5)(r) \\
&\quad + \Psi(p_6, q_g, p_0)(r, q_g, r)U_{k=1}^{q_g}(p_6)(r), \\
&\quad \Psi(p_6, q_h, p_0)(r, q_h, r)U_{k=1}^{q_h}(p_6)(r)\big] \\
&= L[4/10 * 2 * 0.329, (6/10 * 0.329 \\
&\quad + 5/20 * 0.305) * 2, 10/20 * 2 * 0.305] \\
&= \max H[0.263, 0.547, 0.305] \\
&= \max[0.351, 0.329, 0.362] = 0.362.
\end{aligned}$$

For $U_{k=1}^{q_{(f,g,h)}}(p_0)(l)$ we have

$$j = (f, g, h), \quad q_j = (q_f, q_g, q_h),$$

$$\eta_{0f} = \{5\}, \quad \eta_{0g} = \{5, 6\}, \quad \eta_{0h} = \{6\},$$

$$p_{\eta_{0f}} = \{p_5\}, \quad p_{\eta_{0g}} = \{p_5, p_6\}, \quad p_{\eta_{0h}} = \{p_6\}.$$

The case of $U_{k=1}^{q_{(f,g,h)}}(p_0)(r)$ is similar to $U_{k=1}^{q_{(f,g,h)}}(p_0)(l)$. However, the utilities for $U_{k=1}^{q_{(f,g,h)}}(p_0)(l)$ and $U_{k=1}^{q_{(f,g,h)}}(p_0)(r)$ are different. ♦

3.1. CDPPN Mark-Dynamic Properties

We will identify mark-dynamic properties of the CDPPN as properties related to the PN.

Definition 15. An *equilibrium point* with respect to CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is a place $p^* \in P$ such that $M_l(p^*) = S < \infty$, $\forall l \geq k$ and p^* is the last place of the net.

Theorem 2. *The colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is uniformly practically stable iff there exists a strictly positive linear mapping $\Phi : K(p)_{WS} \to \Upsilon_{MS}$ (with Υ being normally one of the color sets already used in the CDPPN) such that $\Delta v = |\Phi * A * X| \leq 0$.*

Proof. (Necessity) It follows directly from Proposition 1.

(Sufficiency) Let us suppose by contradiction that $|\Phi * A * X| > 0$ with fixed Φ. From $M' = M + A * X$ we have that $\Phi M' = \Phi M + \Phi * A * X$. Then, it is possible to construct an increasing sequence $\Phi M < \Phi M' < \cdots < \Phi M^n < \cdots$ which grows up without bounds. Therefore, the CDPPN is not uniformly practically stable. ■

Remark 12. It is important to stress that the only places where the CDPPN will be allowed to get blocked are those which correspond to equilibrium points.

3.2. CDPPN Trajectory-Dynamic Properties

We will identify trajectory-dynamic properties of the CDPPN as those properties related to the utility at each place of the PN. In this sense, we will relate an optimum point to the best possible performance choice. Formally, we will introduce the following definition:

Definition 16. A final decision point $p_f \in P$ with respect to the colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is a place $p \in P$ where the infimum is asymptotically approached (or the minimum is attained), i.e., $U(p)(\kappa_p) = 0$ or $U(p)(\kappa_p) = C$.

Definition 17. An optimum point $p^{\triangle} \in P$ with respect to the colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is a final decision point $p_f \in P$ where the best choice is selected 'according to some criteria'.

Property 3. *Every colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ has a final decision point.*

Remark 13. In the case where $\exists p_1, \ldots, p_n \in P$, such that $U(p_1) = \cdots = U(p_n) = 0$, we have that $p_1, \ldots, p_n$ are optimum points.

Proposition 3. *Let CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ be a colored decision process Petri net and let $p^{\triangle} \in P$ an optimum point. Then $U(p^{\triangle})(\kappa_{p^{\triangle}}) \leq U(p)(\kappa_p)$, $\forall p \in P$ such that $p \leq_U p^{\triangle}$.*

Proof. $U(p^{\triangle})(\kappa_{p^{\triangle}})$ is equal to the minimum or the infimum. Therefore, $U(p^{\triangle})(\kappa_{p^{\triangle}}) \leq U(p)(\kappa_p)$, $\forall p \in P$ such that $p \leq_U p^{\triangle}$. ■

Theorem 3. *The colored decision process Petri net CDPPN* $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ *is uniformly practically stable iff* $U(p_{i+1})(\kappa_{p_{i+1}}) - U(p_i)(\kappa_{p_i}) \leq 0$.

Proof. (Necessity) Let us choose $v = Id(U(p_i))$. Then $\Delta v = U(p_{i+1})(\kappa_{p_{i+1}}) - U(p_i)(\kappa_{p_i}) \leq 0$, and by the autonomous version of Theorem 1 and Corollary 1, the CDPPN is stable.

(Sufficiency) We want to show that the CDPPN is practically stable, i.e., given $0 < \lambda < A$, we must show that $|U(p_i)(\kappa_{p_i})| < A$. We know that $U(p_0)(\kappa_{p_0}) < \lambda$ and, since U is non-decreasing, we have that $|U(p_i)(\kappa_{p_i})| < |U(p_0)(\kappa_{p_0})| < \lambda < A$. ■

Definition 18. A strategy with respect to a colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is identified by σ and consists of the routing policy transition sequence represented in the CDPPN graph model such that some point $p \in P$ is reached.

Definition 19. An optimum strategy with respect to a colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is identified by $\sigma^{\triangle}$ and consists of the routing policy transition sequence represented in the CDPPN graph model such that an optimum point $p^{\triangle} \in P$ is reached.

Equivalently, we can represent (13) and (14) as follows:

$$U_k^{\sigma_{hj}}(p_i)(\kappa_{p_i}) = \begin{cases} U_k(p_0)(\kappa_{p_0}) & \text{if} \quad i = 0,\ k = 0, \\ L(\alpha) & \text{if} \quad i > 0,\ k = 0 \\ & \text{and} \quad i \geq 0,\ k > 0, \end{cases} \tag{15}$$

$$\alpha = \Big[\sum_{h \in \eta_{ij_0}} \sigma_{hj_0}(p_i)(\kappa_{p_i}) * U_k^{\sigma_{hj_0}}(p_h)(\kappa_{p_h}), \sum_{h \in \eta_{ij_1}} \sigma_{hj_1}(p_i)(\kappa_{p_i}) * U_k^{\sigma_{hj_1}}(p_h)(\kappa_{p_h}), \dots, \sum_{h \in \eta_{ij_f}} \sigma_{hj_f}(p_i)(\kappa_{p_i}) * U_k^{\sigma_{hj_f}}(p_h)(\kappa_{p_h}) \Big], \tag{16}$$

where $\sigma_{hj}(p_i)(\kappa_{p_i}) = \Psi(p_h, q_j, p_i)(\kappa_{p_h}, \kappa_{q_{j_0}}, \kappa_{p_i})$. The rest is as previously defined.

Notation 1. *With the intention to facilitate even more the notation, we will represent the utility function* U *as follows:*

1. $U_k(p_i)(\kappa_{p_i}) \triangleq U_k^{q_j}(p_i)(\kappa_{p_i}) \triangleq U_k^{\sigma_{hj}}(p_i)(\kappa_{p_i})$ *for any transition and any strategy,*
2. $U_k^{\triangle}(p_i)(\kappa_{p_i}) \triangleq U_k^{q_j^{\triangle}}(p_i)(\kappa_{p_i}) \triangleq U_k^{\sigma_{hj}^{\triangle}}(p_i)(\kappa_{p_i})$ *for an optimum transition and an optimum strategy.*

The reader will easily identify which notation is used depending on the context.

Example 2. For Example 1 we have

$$U_{k=0}(p_0)(l) = 1,$$

$$U_{k=0}(p_0)(r) = 0.8,$$

$$U_{k=0}^{\sigma_{hj}}(p_1)(l) = L\big[\sigma_{0a}(p_1)(l) * U_{k=0}^{\sigma_{0a}}(p_0)(l)\big], \quad \text{where } \{\sigma_{ha}\} = \{\sigma_{0a}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_1)(r) = L\big[\sigma_{0a}(p_1)(r) * U_{k=0}^{\sigma_{0a}}(p_0)(r)\big], \quad \text{where } \{\sigma_{ha}\} = \{\sigma_{0a}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_2)(l) = L[\sigma_{0a}(p_2)(l) * U_{k=0}^{\sigma_{0a}}(p_0)(l)], \quad \text{where } \{\sigma_{ha}\} = \{\sigma_{0a}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_2)(r) = L\big[\sigma_{0a}(p_2)(r) * U_{k=0}^{\sigma_{0a}}(p_0)(r)\big], \quad \text{where } \{\sigma_{ha}\} = \{\sigma_{0a}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_3)(l) = L\big[\sigma_{0b}(p_3)(l) * U_{k=0}^{\sigma_{0b}}(p_0)(l)\big], \quad \text{where } \{\sigma_{hb}\} = \{\sigma_{0b}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_3)(r) = L\big[\sigma_{0b}(p_3)(r) * U_{k=0}^{\sigma_{0b}}(p_0)(r)\big], \quad \text{where } \{\sigma_{hb}\} = \{\sigma_{0b}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_4)(l) = L\big[\sigma_{0c}(p_4)(l) * U_{k=0}^{\sigma_{0c}}(p_0)(l)\big], \quad \text{where } \{\sigma_{hc}\} = \{\sigma_{0c}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_4)(r) = L\big[\sigma_{0c}(p_4)(r) * U_{k=0}^{\sigma_{0c}}(p_0)(r)\big], \quad \text{where } \{\sigma_{hc}\} = \{\sigma_{0c}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_5)(l) = L\big[\sigma_{1d}(p_5)(l) * U_{k=0}^{\sigma_{1d}}(p_1)(l) + \sigma_{2d}(p_5)(l) * U_{k=0}^{\sigma_{2d}}(p_2)(l)\big], \quad \text{where } \{\sigma_{hd}\} = \{\sigma_{1d}, \sigma_{2d}\},$$

$$U_{k=0}^{\sigma_{hj}}(p_5)(r) = L\big[\sigma_{1d}(p_5)(r) * U_{k=0}^{\sigma_{1d}}(p_1)(r) + \sigma_{2d}(p_5)(r) * U_{k=0}^{\sigma_{2d}}(p_2)(r)\big], \quad \text{where } \{\sigma_{hd}\} = \{\sigma_{1d}, \sigma_{2d}\}$$

$$U^{\sigma_{hj}}_{k=0}(p_6)(l) = L\big[\sigma_{3e}(p_6)(l) * U^{\sigma_{3e}}_{k=0}(p_3)(l) + \sigma_{4e}(p_4)(l) * U^{\sigma_{4e}}_{k=0}(p_4)(l)\big], \text{ where } \{\sigma_{hj}\} = \{\sigma_{3e}, \sigma_{4e}\},$$

$$U^{\sigma_{hj}}_{k=0}(p_6)(r) = L\big[\sigma_{3e}(p_6)(r) * U^{\sigma_{3e}}_{k=0}(p_3)(r) + \sigma_{4e}(p_4)(r) * U^{\sigma_{4e}}_{k=0}(p_4)(r)\big], \text{ where } \{\sigma_{hj}\} = \{\sigma_{3e}, \sigma_{4e}\},$$

$$U^{\sigma_{hj}}_{k=1}(p_0)(l) = L\big[\sigma_{5f}(p_0)(l) * U^{\sigma_{5f}}_{k=0}(p_5)(l), \sigma_{5g}(p_0)(l) * U^{\sigma_{5g}}_{k=0}(p_5)(l) + \sigma_{6g}(p_0)(l) * U^{\sigma_{6g}}_{k=0}(p_6)(l), \sigma_{6h}(p_0)(l) * U^{\sigma_{6h}}_{k=0}(p_6)(l)\big], \text{ where } \{\sigma_{hf}\} = \{\sigma_{5f}\}, \{\sigma_{hg}\} = \{\sigma_{5g}, \sigma_{6g}\} \text{ and } \{\sigma_{hh}\} = \{\sigma_{6h}\},$$

$$U^{\sigma_{hj}}_{k=1}(p_0)(r) = L\big[\sigma_{5f}(p_0)(r) * U^{\sigma_{5f}}_{k=0}(p_5)(r), \sigma_{5g}(p_0)(r) * U^{\sigma_{5g}}_{k=0}(p_5)(r) + \sigma_{6g}(p_0)(r) * U^{\sigma_{6g}}_{k=0}(p_6)(r), \sigma_{6h}(p_0)(r) * U^{\sigma_{6h}}_{k=0}(p_6)(r)\big], \text{ where } \{\sigma_{hf}\} = \{\sigma_{5f}\}, \{\sigma_{hg}\} = \{\sigma_{5g}, \sigma_{6g}\} \text{ and } \{\sigma_{hh}\} = \{\sigma_{6h}\}.$$

Some possible routing policy transition sequence are

1. A strategy σ for a time $k = 1$ is as follows:

$$U_{k=0}(p_0)(l) = 1,$$

$$U^{\sigma_{hj}}_{k=0}(p_1)(l) = L\big[\sigma_{0a}(p_1)(l) * U^{\sigma_{0a}}_{k=0}(p_0)(l)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_2)(l) = L\big[\sigma_{0a}(p_2)(l) * U^{\sigma_{0a}}_{k=0}(p_0)(l)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_5)(l) = L\big[\sigma_{1c}(p_5)(l) * U^{\sigma_{1c}}_{k=0}(p_1)(l) + \sigma_{2d}(p_5)(l) * U^{\sigma_{2d}}_{k=0}(p_2)(l)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_3)(l) = L\big[\sigma_{0b}(p_3)(l) * U^{\sigma_{0b}}_{k=0}(p_0)(l)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_4)(l) = L\big[\sigma_{0c}(p_4)(l) * U^{\sigma_{0c}}_{k=0}(p_0)(l)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_6)(l) = L\big[\sigma_{3e}(p_6)(l) * U^{\sigma_{3e}}_{k=0}(p_3)(l) + \sigma_{4e}(p_4)(l) * U^{\sigma_{4e}}_{k=0}(p_4)(l)\big],$$

$$U_{k=0}(p_0)(r) = 1,$$

$$U^{\sigma_{hj}}_{k=0}(p_1)(r) = L\big[\sigma_{0a}(p_1)(r) * U^{\sigma_{0a}}_{k=0}(p_0)(r)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_2)(r) = L\big[\sigma_{0a}(p_2)(r) * U^{\sigma_{0a}}_{k=0}(p_0)(r)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_5)(r) = L\big[\sigma_{1c}(p_5)(r) * U^{\sigma_{1c}}_{k=0}(p_1)(r) + \sigma_{2d}(p_5)(r) * U^{\sigma_{2d}}_{k=0}(p_2)(r)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_3)(r) = L\big[\sigma_{0b}(p_3)(r) * U^{\sigma_{0b}}_{k=0}(p_0)(r)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_4)(r) = L\big[\sigma_{0c}(p_4)(r) * U^{\sigma_{0c}}_{k=0}(p_0)(r)\big],$$

$$U^{\sigma_{hj}}_{k=0}(p_6)(r) = L\big[\sigma_{3e}(p_6)(r) * U^{\sigma_{3e}}_{k=0}(p_3)(r) + \sigma_{4e}(p_4)(r) * U^{\sigma_{4e}}_{k=0}(p_4)(r)\big],$$

$$U^{\sigma_{5f}}_{k=1}(p_0)(l) = L\big[\sigma_{5f}(p_0)(l) * U^{\sigma_{5f}}_{k=0}(p_5)(l)\big], \quad (*)$$

$$U^{\sigma_{5f}}_{k=1}(p_0)(r) = L\big[\sigma_{5f}(p_0)(r) * U^{\sigma_{5f}}_{k=0}(p_5)(r)\big]. \quad (**)$$

An alternative strategy σ' for a time $k = 1$ is the same as in (1), just change (*) and (**) by

$$U^{\{\sigma_{5g},\sigma_{6g}\}}_{k=1}(p_0)(l) = L\big[\sigma_{5g}(p_0)(l) * U^{\sigma_{5g}}_{k=0}(p_5)(l) + \sigma_{6g}(p_0)(l) * U^{\sigma_{6g}}_{k=0}(p_6)(l)\big],$$

$$U^{\{\sigma_{5g},\sigma_{6g}\}}_{k=1}(p_0)(r) = L\big[\sigma_{5g}(p_0)(r) * U^{\sigma_{5g}}_{k=0}(p_5)(r) + \sigma_{6g}(p_0)(r) * U^{\sigma_{6g}}_{k=0}(p_6)(r)\big],$$

respectively. An alternative strategy σ'' for a time $k = 1$ the same as in (1), just change (*) and (**) by

$$U^{\sigma_{6h}}_{k=1}(p_0)(l) = L\big[\sigma_{6h}(p_0)(l) * U^{\sigma_{6h}}_{k=0}(p_6)(l)\big],$$

$$U^{\sigma_{6h}}_{k=1}(p_0)(r) = L\big[\sigma_{6h}(p_0)(r) * U^{\sigma_{6h}}_{k=0}(p_6)(r)\big],$$

respectively. ♦

3.3. Convergence of CDPPN Mark-Dynamic and Trajectory-Dynamic Properties

Theorem 4. *Let CDPPN* $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ *be a colored decision process Petri net. If* $p^* \in P$ *is an equilibrium point, then it is a final decision point.*

Proof. Let us suppose that p^* is an equilibrium point. We want to show that its utility has asymptotically approached an infimum (or reached a minimum). Since p^* is an equilibrium point, by definition, it is the last place of the net and its marking cannot be modified. But this implies that the routing policy attached to the transition(s) that follows p^* is 0, (in this case there is such a transition(s), i.e., the worst case). Therefore, its utility cannot be modified and since the utility is a decreasing function of p_i, an infimum or a minimum is attained. Then, p^* is a final decision point. ■

Theorem 5. *Let CDPPN* $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ *be a finite and non-blocking colored decision* *process Petri net (unless $p \in P$ is an equilibrium point). If $p_f \in P$ is a final decision point, then it is an equilibrium point.*

Proof. If p_f is a final decision point, since the CDPPN is finite, there exists a k such that $U_k(p_f)(\kappa_{p_f}) = C$. Let us suppose that p_f is not an equilibrium point.

Case 1. It is not bounded. So, it is possible to increment the marks of p_f in the net. Therefore, it is possible to modify its utility. As a result, it is possible to obtain a lower utility than C.

Case 2. It is not the last place in the net. Therefore, it is possible to fire an output transition to p_f in such a way that its marking is modified. Therefore, it is possible to modify the utility over p_f. As a result, it is possible to obtain a lower utility than C. ■

Corollary 2. *Let CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ be a finite and non-blocking colored decision process Petri net (unless $p \in P$ is an equilibrium point). Then an optimum point $p^\triangle \in P$ is an equilibrium point.*

Proof. From the previous theorem we know that a final decision point is an equilibrium point and, since in particular $p^\triangle$ is a final decision point, then it is an equilibrium point. ■

Remark 14. The finite and non-blocking (unless $p \in P$ is an equilibrium point) condition over the CDPPN cannot be relaxed:

1. Let us suppose that the CDPPN is not finite, i.e., p is in a cycle. Then the Lyapunov-like function converges to zero as $k \to \infty$, i.e., $L(p) = 0$, but the CDPPN has no final place. Therefore, it is not an equilibrium point.
2. Let us suppose that the CDPPN blocks at some place (not an equilibrium point) $p_b \in P$. Then the Lyapunov-like function has a minimum at the place p_b, $L(p_b) = C$, say, but p_b is not an equilibrium point because it is not necessarily the last place of the net.

Definition 20. Let CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ be a colored decision process Petri net. A *trajectory* ω is a (finite or infinite) ordered subsequence of places $p_{\varsigma(1)} \leq_{U_k} p_{\varsigma(2)} \leq_{U_k} \cdots \leq_{U_k} p_{\varsigma(n)} \leq_{U_k} \cdots$ such that a given strategy σ holds.

Definition 21. Let CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ be a colored decision process Petri net. An *optimum trajectory* ω is a (finite or infinite) ordered subsequence of places $p_{\varsigma(1)} \leq_{U_k^\triangle} p_{\varsigma(2)} \leq_{U_k^\triangle} \cdots \leq_{U_k^\triangle} p_{\varsigma(n)} \leq_{U_k^\triangle} \cdots$ such that the optimum strategy $\sigma^\triangle$ holds.

Theorem 6. *Let CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ be a non-blocking colored decision process Petri net (unless $p \in P$ is an equilibrium point). Then we have*

$$U_k^\triangle(p^\triangle)(\kappa_{p^\triangle}) \leq U_k(p)(\kappa_p), \quad \forall \sigma, \sigma^\triangle.$$

Proof. We have

$$U_k^{\sigma_{hj}}(p_i)(\kappa_{p_i}) = \begin{cases} U_k(p_0)(\kappa_{p_0}) & \text{if} \quad i = 0, \ k = 0, \\ L(\alpha) & \text{if} \quad i > 0, \ k = 0 \\ & \text{and} \quad i \geq 0, \ k > 0, \end{cases}$$

$$\alpha = \Bigg[\sum_{h \in \eta_{ij_0}} \sigma_{hj_0}(p_i)(\kappa_{p_i}) * U_k^{\sigma_{hj_0}}(p_h)(\kappa_{p_h}), \sum_{h \in \eta_{ij_1}} \sigma_{hj_1}(p_i)(\kappa_{p_i}) * U_k^{\sigma_{hj_1}}(p_h)(\kappa_{p_h}), \dots, \sum_{h \in \eta_{ij_f}} \sigma_{hj_f}(p_i)(\kappa_{p_i}) * U_k^{\sigma_{hj_f}}(p_h)(\kappa_{p_h}) \Bigg].$$

Then, starting from p_0 and proceeding with the iteration, eventually the trajectory ω given by $p_0 = p_{\varsigma(1)} \leq_{U_k} p_{\varsigma(2)} \leq_{U_k} \cdots \leq_{U_k} p_{\varsigma(n)} \leq_{U_k} \cdots$ which converges to $p^\triangle$ is produced, i.e., the optimum trajectory is obtained. Since at the optimum trajectory the optimum strategy $\sigma^\triangle$ holds, we have $U_k^\triangle(p^\triangle)(\kappa_{p^\triangle}) \leq U_k(p)(\kappa_p)$, $\forall \sigma, \sigma^\triangle$. ■

Remark 15. The inequality $U_k^\triangle(p^\triangle)(\kappa_{p^\triangle}) \leq U_k(p)(\kappa_p)$ means that the utility is optimum when the optimum strategy is applied.

Corollary 3. *Let CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ be a nonblocking colored decision process Petri net (unless $p \in P$ is an equilibrium point) and let $\sigma^\triangle$ be an optimum strategy. Set $L = \min_{i=1,\dots,|\alpha|}\{\alpha_i\}$. Then $U_k^\triangle(p)(\kappa_p)$ is equal to (17), where p is a vector whose elements are those places which belong to the optimum trajectory ω given by $p_0 \leq p_{\varsigma(1)} \leq_{U_k} p_{\varsigma(2)} \leq_{U_k} \cdots \leq_{U_k} p_{\varsigma(n)} \leq_{U_k} \cdots$ which converges to $p^\triangle$.*

Proof. Since at each step of the iteration $U_k^\triangle(p_i)(\kappa_{p_i})$ is equal to one of the elements of the vector α, the representation that describes the dynamical utility behavior of tracking the optimum strategy $\sigma^\triangle$ is given by (17), where $j_m, j_n, \dots, j_v, \dots$ represent the indices of the optimal routing policy, defined by q_js. ■

Plane symmetry involves moving all points around the plane so that their positions relative to one another remain the same, although their absolute positions may change. By analogy, let us introduce the following definition:

$$
\underbrace{\begin{array}{|c|c|c|c|}
\hline
\sigma^{\triangle}_{0j_m}(p_{\varsigma(0)})(\kappa_{p_{\varsigma(0)}}) & \sigma^{\triangle}_{1j_m}(p_{\varsigma(0)})(\kappa_{p_{\varsigma(0)}}) & \dots & \sigma^{\triangle}_{nj_m}(p_{\varsigma(0)})(\kappa_{p_{\varsigma(0)}}) \\
\hline
\sigma^{\triangle}_{0j_n}(p_{\varsigma(1)})(\kappa_{p_{\varsigma(1)}}) & \sigma^{\triangle}_{1j_n}(p_{\varsigma(1)})(\kappa_{p_{\varsigma(1)}}) & \dots & \sigma^{\triangle}_{nj_n}(p_{\varsigma(1)})(\kappa_{p_{\varsigma(1)}}) \\
\hline
\dots & \dots & \dots & \dots \\
\hline
\sigma^{\triangle}_{0j_v}(p_{\varsigma(i)})(\kappa_{p_{\varsigma(i)}}) & \sigma^{\triangle}_{1j_v}(p_{\varsigma(i)})(\kappa_{p_{\varsigma(i)}}) & \dots & \sigma^{\triangle}_{nj_v}(p_{\varsigma(i)})(\kappa_{p_{\varsigma(i)}}) \\
\hline
\dots & \dots & \dots & \dots \\
\hline
\end{array}}_{\sigma^{\triangle}}
\underbrace{\begin{array}{|c|}
\hline
U_k(p_0)(\kappa_{p_0}) \\
\hline
U_k(p_1)(\kappa_{p_1}) \\
\hline
\dots \\
\hline
U_k(p_i)(\kappa_{p_i}) \\
\hline
\dots \\
\hline
\end{array}}_{U}
\tag{17}
$$

Definition 22. A colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ is said to be *symmetric* if it is possible to decompose it into some finite number (greater than 1) of sub-Petri nets in such a way that there exists a bijection ψ between all the sub-Petri nets such that

$$
\begin{aligned}
(p,q) \in I \Leftrightarrow \big(\psi(p), \psi(q)\big) \in I \\
\text{and } (q,p) \in O \Leftrightarrow \big(\psi(q), \psi(p)\big) \in O
\end{aligned}
$$

for all of the sub-Petri nets.

Corollary 4. *Let CDPPN* $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$ *be a nonblocking (unless* p *is an equilibrium point) symmetric colored decision process Petri net and let* $\sigma^{\triangle}$ *be an optimum strategy. Set* $L = \min_{i=1,\dots,|\alpha|}\{\alpha_i\}$. *Then*

$$
\sigma^{\triangle} U \leq \sigma U, \quad \forall \sigma, \sigma^{\triangle},
$$

where σ *and* $\sigma^{\triangle}$ *are represented by a matrix and* U *is represented by a vector.*

Proof. From the previous corollary, thanks to the symmetric property, we obtain that for all $\forall \sigma, \sigma^{\triangle}$ the vector inequality $\sigma^{\triangle} U \leq \sigma U$ holds. ■

Example 3. Consider the CDPPN of Example 1. The strategies for the color l can be recursively modeled as follows (the representation for the color r is similar—the reader has to change l by r to obtain the result):

I. A strategy σ for a time $k = 0, i \geq 0$ is given by (18). A strategy σ for a time $k = 1, i \geq 0$ is shown in (19).
II. An alternative strategy σ' for a time $k = 1, i \geq 0$ is given by (20).
III. An alternative strategy σ'' for a time $k = 1, i \geq 0$ is shown in (21).

The optimality of the three strategies $\sigma \cup \sigma'$, $\sigma \cup \sigma''$ and $\sigma \cup \sigma'''$ will depend on the Lyapunov-like function L we choose.

3.4. Optimum Trajectory Planning

Given a nonblocking (unless $p \in P$ is an equilibrium point) colored decision process Petri net CDPPN $= \{\Sigma, P, Q, F, K, A^+, A^-, M_0, \pi, U\}$, the optimum trajectory planning consists in finding the firing transition sequence u such that the optimum target state M_t with the optimum point is achieved. The target state M_t belongs to the reachability set $R(M_0)$, and satisfies the condition that it is the last and final task processed by the CDPPN with some fixed starting state M_0 with a utility U_0.

Theorem 7. *The optimum trajectory planning problem is solvable.*

Proof. From what was shown in Theorem 6, for each step we find $U_k^{\triangle}(p_{\varsigma(1)})(\kappa_{p_{\varsigma(1)}}), \dots, U_k^{\triangle}(p_{\varsigma(i)})(\kappa_{p_{\varsigma(i)}}), \dots, U_k^{\triangle}(p^{\triangle})(\kappa_{p^{\triangle}})$. Define a mapping (see Notation 1):

$$
u_r\big(U_k^{q_j^{\triangle}}(p_{\varsigma(i)})(\kappa_{p_{\varsigma(i)}})\big) = [0, \dots, 0, 1(\kappa_{p_{\varsigma(i)}}), 0, \dots, 0] \tag{22}
$$

with 1 in the position j and zero everywhere else, and set $u = \sum_r u_r((U_k^{q_j^{\triangle}}(p_{\varsigma(i)})(\kappa_{p_{\varsigma(i)}}))$, where the index r runs over all the transitions associated with the subsequence $\varsigma(i)$ such that $p_{\varsigma(i)}$ converges to $p^{\triangle}$. Then, by construction, the optimum point is attained. ■

Remark 16. The order in which the transitions are fired is given by the order of the transitions inherited from the order of the subsequence $p_{\varsigma(i)}$.

Property 4. *Let us denote the distance between the initial point* $p_0 \in P$ *and the optimum point* $p^{\triangle} \in P$ *by* $|p_0 - p^{\triangle}|$. *Then finding the firing vector* u *is bounded by the cost/benefit relation given by* $|p_0 - p^{\triangle}|/U_k(p^{\triangle})$.

The cost/benefit ratio provides information on the nature, magnitude and significance of the potential effects of a policy. It is applied when the policy analysis concerns the examination of the advantages and drawbacks of different proposed policies or of varying target levels of a policy. It is important to note that, intuitively, the distance $|p_0 - p^{\triangle}|$ represents the time taken to fire all the enabled transitions between p_0 and $p^{\triangle}$.

1	0	0	0	0	0	0	$U(p_0)(l)$
$\sigma_{0a}(p_1)(l)$	0	0	0	0	0	0	$U(p_1)(l)$
$\sigma_{0a}(p_2)(l)$	0	0	0	0	0	0	$U(p_2)(l)$
$\sigma_{0b}(p_3)(l)$	0	0	0	0	0	0	$U(p_3)(l)$
$\sigma_{0c}(p_4)(l)$	0	0	0	0	0	0	$U(p_4)(l)$
0	$\sigma_{1d}(p_5)(l)$	$\sigma_{2d}(p_5)(l)$	0	0	0	0	$U(p_5)(l)$
0	0	0	$\sigma_{3e}(p_6)(l)$	$\sigma_{4e}(p_6)(l)$	0	0	$U(p_6)(l)$

σ (18)

1	0	0	0	0	$\sigma_{5f}(p_0)(l)$	0	$U(p_0)(l)$
$\sigma_{0a}(p_1)(l)$	0	0	0	0	0	0	$U(p_1)(l)$
$\sigma_{0a}(p_2)(l)$	0	0	0	0	0	0	$U(p_2)(l)$
$\sigma_{0b}(p_3)(l)$	0	0	0	0	0	0	$U(p_3)(l)$
$\sigma_{0c}(p_4)(l)$	0	0	0	0	0	0	$U(p_4)(l)$
0	$\sigma_{1d}(p_5)(l)$	$\sigma_{2d}(p_5)(l)$	0	0	0	0	$U(p_5)(l)$
0	0	0	$\sigma_{3e}(p_6)(l)$	$\sigma_{4e}(p_6)(l)$	0	0	$U(p_6)(l)$

σ (19)

1	0	0	0	0	$\sigma_{5g}(p_0)(l)$	$\sigma_{6g}(p_0)(l)$	$U(p_0)(l)$
$\sigma_{0a}(p_1)(l)$	0	0	0	0	0	0	$U(p_1)(l)$
$\sigma_{0a}(p_2)(l)$	0	0	0	0	0	0	$U(p_2)(l)$
$\sigma_{0b}(p_3)(l)$	0	0	0	0	0	0	$U(p_3)(l)$
$\sigma_{0c}(p_4)(l)$	0	0	0	0	0	0	$U(p_4)(l)$
0	$\sigma_{1d}(p_5)(l)$	$\sigma_{2d}(p_5)(l)$	0	0	0	0	$U(p_5)(l)$
0	0	0	$\sigma_{3e}(p_6)(l)$	$\sigma_{4e}(p_6)(l)$	0	0	$U(p_6)(l)$

σ (20)

1	0	0	0	0	0	$\sigma_{6h}(p_0)(l)$	$U(p_0)(l)$
$\sigma_{0a}(p_1)(l)$	0	0	0	0	0	0	$U(p_1)(l)$
$\sigma_{0a}(p_2)(l)$	0	0	0	0	0	0	$U(p_2)(l)$
$\sigma_{0b}(p_3)(l)$	0	0	0	0	0	0	$U(p_3)(l)$
$\sigma_{0c}(p_4)(l)$	0	0	0	0	0	0	$U(p_4)(l)$
0	$\sigma_{1d}(p_5)(l)$	$\sigma_{2d}(p_5)(l)$	0	0	0	0	$U(p_5)(l)$
0	0	0	$\sigma_{3e}(p_6)(l)$	$\sigma_{4e}(p_6)(l)$	0	0	$U(p_6)(l)$

σ (21)

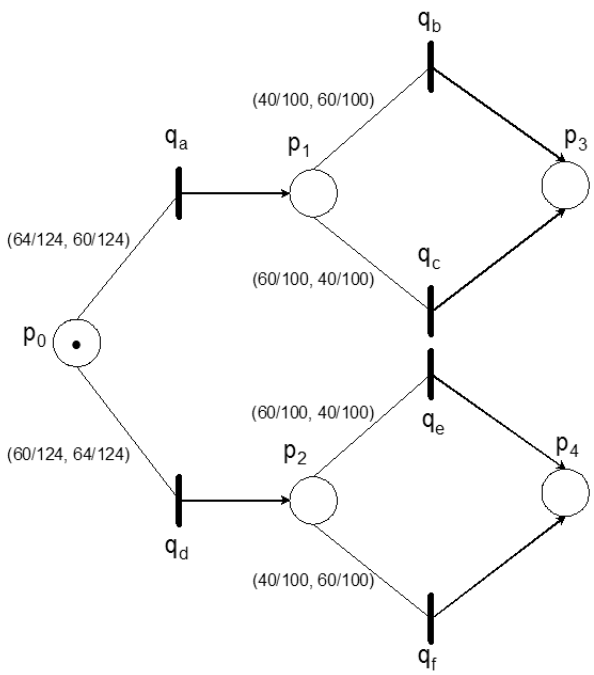

Fig. 4. Setting of Example 4.

Example 4. Let choose the Lyapunov-like function L in terms of the entropy $H(p_i) = -p_i \ln p_i$. We will conceptualize H as the average amount of uncertainty of moving one step ahead, where the uncertainty is high when H is close to 0 and low when H is close to 1. Because $L : D \subseteq \mathbb{R}^n \to \mathbb{R}_+$ we will use the function $\max_{i=1,\dots,|\alpha|}(-\alpha_i \ln \alpha_i)$ to select a proper element of the vector $\alpha \in D$. In the CDPPN the token will have two colors identified as l and r. Every arc has an associated tuple of probabilities corresponding to the colors l and r, i.e., (probability_l, probability_r).

(a) Then the optimum strategy $\sigma^{\triangle}$ is

$$U_{k=0}(p_0)(l) = 1,$$

$$\begin{aligned} U_{k=0}^{\sigma_{hj}}(p_2)(l) &= L\big[\sigma_{0d}(p_2)(l) * U_{k=0}^{\sigma_{0d}}(p_0)(l)\big] \\ &= \max H[60/124] = \max[0.351] = 0.351, \end{aligned}$$

$$\begin{aligned} U_{k=0}^{\sigma_{hj}}(p_4)(l) &= L\big[\sigma_{2e}(p_4)(l) \\ &\quad * U_{k=0}^{\sigma_{2e}}(p_2)(l), \sigma_{2f}(p_4)(l) * U_{k=0}^{\sigma_{2f}}(p_2)(l)\big] \\ &= \max H[(60/100)*0.351, (40/100)*0.351] \\ &= \max[0.328, 0.275] = 0.328. \end{aligned}$$

The firing transition vector is

$u =$

0	0	0	1	0	1
q_a	q_b	q_c	q_d	q_e	q_f

.

We do not compute $U_{k=0}^{\sigma_{hj}}(p_3)(l)$ because $U_{k=0}^{\sigma_{hj}}(p_2)(l)$ determines the optimum trajectory.

(b) An alternative strategy $\sigma \neq \sigma^{\triangle}$ is

$$U_{k=0}(p_0)(l) = 1,$$

$$\begin{aligned} U_{k=0}^{\sigma_{hj}}(p_1)(l) &= L\big[\sigma_{0a}(p_1)(l) * U_{k=0}^{\sigma_{0a}}(p_0)(l)\big] \\ &= \max H[64/124] = \max[0.341] = 0.341, \end{aligned}$$

$$\begin{aligned} U_{k=0}^{\sigma_{hj}}(p_3)(l) &= L\big[\sigma_{1b}(p_3)(l) * U_{k=0}^{\sigma_{1b}}(p_1)(l), \sigma_{1c}(p_3)(l) \\ &\quad * U_{k=0}^{\sigma_{1c}}(p_1)(l)\big] \\ &= \max H[(40/100)*0.341, (60/100)*0.341] \\ &= \max[0.271, 0.324] = 0.324. \end{aligned}$$

The firing transition vector is

$u' =$

1	1	0	0	0	0
q_a	q_b	q_c	q_d	q_e	q_f

.

As we can see, for σ we can obtain at most a value of 0.324 but for $\sigma^{\triangle}$ we obtain 0.328.

The computation for the color r is similar. However, the optimum strategy trajectory is opposite to the optimum strategy for the color l. ♦

4. Conclusions and Future Work

A formal framework for colored decision process Petri nets was presented. The expressive power and mathematical formality of the CDPPN contribute to bridging the gap between Petri nets and Markov decision processes. In this sense, there are a number of questions related to classical planning that may in the future be addressed satisfactorily within this framework. Traditional notions of stability in the Lyapunov sense were explored to characterize the stability properties of the CDPPN. We introduced the notion of uniformly practical stability and provided sufficient and necessary stability conditions for the CDPPN. In addition to that, we showed that the CDPPN mark-dynamic and trajectory-dynamic properties of equilibrium, stability and an optimum point converge under some mild restrictions. An algorithm for optimum trajectory planning used to identify the optimum point was described. Illustrative examples where equilibrium, stability and final decision point properties of the CDPPN were shown to hold were addressed. We are currently working on a generalization to game theory (Clempner *et al.*, 2005).

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