# Colorful subhypergraphs in Kneser hypergraphs 

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#### Abstract

Using a $Z_{q}$-generalization of a theorem of Ky Fan, we extend to Kneser hypergraphs a theorem of Simonyi and Tardos that ensures the existence of multicolored complete bipartite graphs in any proper coloring of a Kneser graph. It allows to derive a lower bound for the local chromatic number of Kneser hypergraphs (using a natural definition of what can be the local chromatic number of a uniform hypergraph).


Keywords: colorful complete p-partite hypergraph; combinatorial topology; Kneser hypergraphs; local chromatic number.

## 1 Introduction

### 1.1 Motivations and results

A hypergraph is a pair $\mathcal{H}=(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ a family of subsets of $V(\mathcal{H})$. The set $V(\mathcal{H})$ is called the vertex set and the set $E(\mathcal{H})$ is called the edge set. A graph is a hypergraph each edge of which is of cardinality two. A $q$ uniform hypergraph is a hypergraph each edge of which is of cardinality $q$. The notions of graphs and 2-uniform hypergraphs therefore coincide. If a hypergraph has its vertex set partitioned into subsets $V_{1}, \ldots, V_{q}$ so that each edge intersects each $V_{i}$ at exactly one vertex, then it is called a $q$-uniform $q$-partite hypergraph. The sets $V_{1}, \ldots, V_{q}$ are called the parts of the hypergraph. When $q=2$, such a hypergraph is a graph and said to be bipartite. A $q$-uniform $q$-partite hypergraph is said to be complete if all possible edges exist.

A coloring of a hypergraph is a map $c: V(\mathcal{H}) \rightarrow[t]$ for some positive integer $t$. A coloring is said to be proper if there is no monochromatic edge, i.e. no edge $e$ with $|c(e)|=1$. The chromatic number of such a hypergraph, denoted $\chi(\mathcal{H})$, is the minimal value of $t$ for which a proper coloring exists. Given $X \subseteq V(\mathcal{H})$, the hypergraph with vertex set $X$ and with edge set $\{e \in E(\mathcal{H}): e \subseteq X\}$ is the subhypergraph of $\mathcal{H}$ induced by $X$ and is denoted $\mathcal{H}[X]$.

Given a hypergraph $\mathcal{H}=(V(\mathcal{H}), E(\mathcal{H}))$, we define the Kneser graph $\mathrm{KG}^{2}(\mathcal{H})$ by

$$
\begin{aligned}
V\left(\mathrm{KG}^{2}(\mathcal{H})\right) & =E(\mathcal{H}) \\
E\left(\mathrm{KG}^{2}(\mathcal{H})\right) & =\{\{e, f\}: e, f \in E(\mathcal{H}), e \cap f=\emptyset\} .
\end{aligned}
$$

The "usual" Kneser graphs, which have been extensively studied - see [20, 21] among many references, some of them being given elsewhere in the present paper - are the special cases $\mathcal{H}=\left([n],\binom{[n]}{k}\right)$ for some positive integers $n$ and $k$ with $n \geqslant 2 k$. We denote them $\mathrm{KG}^{2}(n, k)$. The main result for "usual" Kneser graphs is Lovász's theorem [11].

Theorem (Lovász theorem). Given $n$ and $k$ two positive integers with $n \geqslant 2 k$, we have $\chi\left(\mathrm{KG}^{2}(n, k)\right)=n-2 k+2$.

The 2-colorability defect $\operatorname{cd}^{2}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ has been introduced by Dol'nikov [3] in 1988 for a generalization of Lovász's theorem. It is defined as the minimum number of vertices that must be removed from $\mathcal{H}$ so that the hypergraph induced by the remaining vertices is of chromatic number at most 2 :

$$
\operatorname{cd}^{2}(\mathcal{H})=\min \{|Y|: Y \subseteq V(\mathcal{H}), \chi(\mathcal{H}[V(\mathcal{H}) \backslash Y]) \leqslant 2\} .
$$

Theorem (Dol'nikov theorem). Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then $\chi\left(\mathrm{KG}^{2}(\mathcal{H})\right) \geqslant \operatorname{cd}^{2}(\mathcal{H})$.

It is a generalization of Lovász theorem since $\operatorname{cd}^{2}\left([n],\binom{[n]}{k}\right)=n-2 k+2$ and since the inequality $\chi\left(\operatorname{KG}^{2}(n, k)\right) \leqslant n-2 k+2$ is the easy one.

The following theorem proposed by Simonyi and Tardos in 2007 [19] generalizes Dol'nikov's theorem. The special case for "usual" Kneser graphs is due to Ky Fan [7].

Theorem (Simonyi-Tardos theorem). Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let $r=\operatorname{cd}^{2}(\mathcal{H})$. Then any proper coloring of $\mathrm{KG}^{2}(\mathcal{H})$ with colors $1, \ldots, t$ (t arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil r / 2\rceil,\lfloor r / 2\rfloor}$ such that the $r$ different colors occur alternating on the two parts of the bipartite graph with respect to their natural order.

In 1976, Erdős [4] initiated the study of Kneser hypergraphs $\mathrm{KG}^{q}(\mathcal{H})$ defined for a hypergraph $\mathcal{H}=(V(\mathcal{H}), E(\mathcal{H}))$ and an integer $q \geqslant 2$ by

$$
\begin{aligned}
V\left(\mathrm{KG}^{q}(\mathcal{H})\right) & =E(\mathcal{H}) \\
E\left(\mathrm{KG}^{q}(\mathcal{H})\right) & =\left\{\left\{e_{1}, \ldots, e_{q}\right\}: e_{1}, \ldots, e_{q} \in E(\mathcal{H}), e_{i} \cap e_{j}=\emptyset \text { for all } i, j \text { with } i \neq j\right\} .
\end{aligned}
$$

A Kneser hypergraph is thus the generalization of Kneser graphs obtained when the 2uniformity is replaced by the $q$-uniformity for an integer $q \geqslant 2$. There are also "usual" Kneser hypergraphs, which are obtained with the same hypergraph $\mathcal{H}$ as for "usual" Kneser graphs, i.e. $\mathcal{H}=\left([n],\binom{[n]}{k}\right)$. They are denoted $\operatorname{KG}^{q}(n, k)$. The main result for them is the following generalization of Lovász's theorem conjectured by Erdős and proved by Alon, Frankl, and Lovász [2].

Theorem (Alon-Frankl-Lovász theorem). Given $n, k$, and $q$ three positive integers with $n \geqslant q k$, we have $\chi\left(\operatorname{KG}^{q}(n, k)\right)=\left\lceil\frac{n-q(k-1)}{q-1}\right\rceil$.

There exists also a $q$-colorability defect $\operatorname{cd}^{q}(\mathcal{H})$, introduced by Kříž, defined as the minimum number of vertices that must be removed from $\mathcal{H}$ so that the hypergraph induced by the remaining vertices is of chromatic number at most $q$ :

$$
\operatorname{cd}^{q}(\mathcal{H})=\min \{|Y|: Y \subseteq V(\mathcal{H}), \chi(\mathcal{H}[V(\mathcal{H}) \backslash Y]) \leqslant q\} .
$$

The following theorem, due to Kříz [9, 10], generalizes Dol'nikov's theorem. It also generalizes the Alon-Frankl-Lovász theorem since $\operatorname{cd}^{q}\left([n],\binom{[n]}{k}\right)=n-q(k-1)$ and since again the inequality $\chi\left(\mathrm{KG}^{q}(n, k)\right) \leqslant\left\lceil\frac{n-q(k-1)}{q-1}\right\rceil$ is the easy one.
Theorem (Křiž theorem). Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then

$$
\chi\left(\mathrm{KG}^{q}(\mathcal{H})\right) \geqslant\left\lceil\frac{\operatorname{cd}^{q}(\mathcal{H})}{q-1}\right\rceil
$$

for any integer $q \geqslant 2$.
Our main result is the following extension of Simonyi-Tardos's theorem to Kneser hypergraphs.

Theorem 1. Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let $p$ be a prime number. Then any proper coloring $c$ of $\mathrm{KG}^{p}(\mathcal{H})$ with colors $1, \ldots, t$ (t arbitrary) must contain a complete p-uniform p-partite hypergraph with parts $U_{1}, \ldots, U_{p}$ satisfying the following properties.

- It has $\operatorname{cd}^{p}(\mathcal{H})$ vertices.
- The values of $\left|U_{j}\right|$ for $j=1, \ldots, p$ differ by at most one.
- For any $j$, the vertices of $U_{j}$ get distinct colors.

We get that each $U_{j}$ is of cardinality $\left\lfloor\operatorname{cd}^{p}(\mathcal{H}) / p\right\rfloor$ or $\left\lceil\operatorname{cd}^{p}(\mathcal{H}) / p\right\rceil$.
Note that Theorem 1 implies directly Kříž's theorem when $q$ is a prime number $p$ : each color may appear at most $p-1$ times within the vertices and there are $\operatorname{cd}^{p}(\mathcal{H})$ vertices. There is a standard derivation of Křiž's theorem for any $q$ from the prime case, see [22, 23]. Theorem 1 is a generalization of Simonyi-Tardos's theorem except for a slight loss: when $p=2$, we do not recover the alternation of the colors between the two parts.

Whether Theorem 1 is true for non-prime $p$ is an open question.

## 2 Local chromatic number and Kneser hypergraphs

In a graph $G=(V, E)$, the closed neighborhood of a vertex $u$, denoted $N[u]$, is the set $\{u\} \cup\{v: u v \in E\}$. The local chromatic number of a graph $G=(V, E)$, denoted $\chi_{\ell}(G)$, is the maximum number of colors appearing in the closed neighborhood of a vertex minimized over all proper colorings:

$$
\chi_{\ell}(G)=\min _{c} \max _{v \in V}|c(N[v])|
$$

where the minimum is taken over all proper colorings $c$ of $G$. This number has been defined in 1986 by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [5]. For Kneser graphs, we have the following theorem, which is a consequence of the Simonyi-Tardos theorem: any vertex of the part with $\lfloor r / 2\rfloor$ vertices in the completely multicolored complete bipartite subgraph has at least $\lceil r / 2\rceil+1$ colors in its closed neighborhhod (where $r=\operatorname{cd}^{2}(\mathcal{H})$ ).

Theorem (Simonyi-Tardos theorem for local chromatic number). Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. If $\operatorname{cd}^{2}(\mathcal{H}) \geqslant 2$, then

$$
\chi_{\ell}\left(\mathrm{KG}^{2}(\mathcal{H})\right) \geqslant\left\lceil\frac{\mathrm{cd}^{2}(\mathcal{H})}{2}\right\rceil+1 .
$$

Note that we can also see this theorem as a direct consequence of Theorem 1 in [18] (with the help of Theorem 1 in [13]).

We use the following natural definition for the local chromatic number $\chi_{\ell}(\mathcal{H})$ of a uniform hypergraph $\mathcal{H}=(V, E)$. For a subset $X$ of $V$, we denote by $\mathcal{N}(X)$ the set of vertices $v$ such that $v$ is the sole vertex outside $X$ for some edge in $E$ :

$$
\mathcal{N}(X)=\{v: \exists e \in E \text { s.t. } e \backslash X=\{v\}\}
$$

We define furthermore $\mathcal{N}[X]:=X \cup \mathcal{N}(X)$. Note that if the hypergraph is a graph, $\mathcal{N}[\{v\}]=N[v]$ for any vertex $v$. The definition of the local chromatic number of a hypergraph is then:

$$
\chi_{\ell}(\mathcal{H})=\min _{c} \max _{e \in E, v \in e}|c(\mathcal{N}[e \backslash\{v\}])|,
$$

where the minimum is taken over all proper colorings $c$ of $\mathcal{H}$. When the hypergraph $\mathcal{H}$ is a graph, we get the usual notion of local chromatic number for graphs.

The following theorem is a consequence of Theorem 1 and generalizes the SimonyiTardos theorem for local chromatic number to Kneser hypergraphs.

Theorem 2. Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then

$$
\chi_{\ell}\left(\mathrm{KG}^{p}(\mathcal{H})\right) \geqslant \min \left(\left\lceil\frac{\operatorname{cd}^{p}(\mathcal{H})}{p}\right\rceil+1,\left\lceil\frac{\mathrm{~cd}^{p}(\mathcal{H})}{p-1}\right\rceil\right)
$$

for any prime number $p$.

Proof. Denote $\operatorname{cd}^{p}(\mathcal{H})$ by $r$. Let $c$ be any proper coloring of $\mathrm{KG}^{p}(\mathcal{H})$. Consider the complete $p$-uniform $p$-partite hypergraph $\mathcal{G}$ in $\mathrm{KG}^{p}(\mathcal{H})$ whose existence is ensured by Theorem 1. Choose $U_{j}$ of cardinality $\lceil r / p\rceil$.

If $\lceil r /(p-1)\rceil>\lceil r / p\rceil$, then there is a vertex $v$ of $\mathcal{G}$ not in $U_{j}$ whose color is distinct of all colors used in $U_{j}$. Choose any edge $e$ of $\mathcal{G}$ containing $v$ and let $u$ be the unique vertex of $e \cap U_{j}$. We have then $|c(\mathcal{N}[e \backslash\{u\}])| \geqslant\left|U_{j}\right|+1=\lceil r / p\rceil+1$.

Otherwise, $\lceil r /(p-1)\rceil=\lceil r / p\rceil$, and for any edge $e$, we have $|c(\mathcal{N}[e \backslash\{u\}])| \geqslant\lceil r / p\rceil=$ $\lceil r /(p-1)\rceil$, with $u$ being again the unique vertex of $e \cap U_{j}$.

As for Theorem 1, we do not know whether this theorem remains true for non-prime p.

## 3 Combinatorial topology and proof of the main result

### 3.1 Tools of combinatorial topology

### 3.1.1 Basic definitions

We use the cyclic and muliplicative group $Z_{q}=\left\{\omega^{j}: j=1, \ldots, q\right\}$ of the $q$ th roots of unity. We emphasize that 0 is not considered as an element of $Z_{q}$. For a vector $X=\left(x_{1}, \ldots, x_{n}\right) \in\left(Z_{q} \cup\{0\}\right)^{n}$, we define $X^{j}$ to be the set $\left\{i \in[n]: x_{i}=\omega^{j}\right\}$ and $|X|$ to be the quantity $\left|\left\{i \in[n]: x_{i} \neq 0\right\}\right|$.

We assume basic knowledges in algebraic topology, see the book by Munkres for instance for an introduction to this topic [17]. A simplicial complex is said to be pure if all maximal simplices for inclusion have the same dimension. For K a simplicial complex, we denote by $\mathcal{C}(\mathrm{K})$ its chain complex. We always assume that the coefficients are taken in $\mathbb{Z}$.

### 3.1.2 Special simplicial complexes

For a simplicial complex K , its first barycentric subdivision is denoted by $\operatorname{sd}(\mathrm{K})$. It is the simplicial complex whose vertices are the nonempty simplices of $K$ and whose simplices are the collections of simplices of K that are pairwise comparable for $\subseteq$ (these collections are usually called chains in the poset terminology, with a different meaning as the one used above in "chain complexes").

As a simplicial complex, $Z_{q}$ is seen as being 0 -dimensional and with $q$ vertices. $Z_{q}^{* d}$ is the join of $d$ copies of $Z_{q}$. It is a pure simplicial complex of dimension $d-1$. A vertex $v$ taken in the $\mu$ th copy of $Z_{q}$ in $Z_{q}^{* d}$ is also written $(\epsilon, \mu)$ where $\epsilon \in Z_{q}$ and $\mu \in[d]$. Sometimes, $\epsilon$ is called the sign of the vertex, and $\mu$ its absolute value. This latter quantity is denoted $|v|$.

The simplicial complex $\operatorname{sd}\left(Z_{q}^{* d}\right)$ plays a special role. We have

$$
V\left(\operatorname{sd}\left(Z_{q}^{* d}\right)\right) \simeq\left(Z_{q} \cup\{0\}\right)^{d} \backslash\{(0, \ldots, 0)\}:
$$

a simplex $\sigma \in Z_{q}^{* d}$ corresponds to the vector $X=\left(x_{1}, \ldots, x_{d}\right) \in\left(Z_{q} \cup\{0\}\right)^{d}$ with $x_{\mu}=\epsilon$ for all $(\epsilon, \mu) \in \sigma$ and $x_{\mu}=0$ otherwise.

We denote by $\sigma_{q-2}^{q-1}$ the simplicial complex obtained from a $(q-1)$-dimensional simplex and its faces by deleting the maximal face. It is hence a ( $q-2$ )-dimensional pseudomanifold homeomorphic to the $(q-2)$-sphere. We also identify its vertices with $Z_{q}$. A vertex of the simplicial complex $\left(\sigma_{q-2}^{q-1}\right)^{* d}$ is again denoted by $(\epsilon, \mu)$ where $\epsilon \in Z_{q}$ and $\mu \in[d]$. For $\epsilon \in Z_{q}$ and a simplex $\tau$ of $\left(\sigma_{p-2}^{p-1}\right)^{* d}$, we denote by $\tau^{\epsilon}$ the set of all vertices of $\tau$ having $\epsilon$ as sign, i.e. $\tau^{\epsilon}:=\{(\omega, \mu) \in \tau: \omega=\epsilon\}$. Note that if $q$ is a prime number, $Z_{q}$ acts freely on $\sigma_{q-2}^{q-1}$.

### 3.1.3 Barycentric subdivision operator

Let K be a simplicial complex. There is a natural chain map $\mathrm{sd}_{\#}: \mathcal{C}(\mathrm{K}) \rightarrow \mathcal{C}(\operatorname{sd}(\mathrm{K}))$ which, when evaluated on a $d$-simplex $\sigma \in \mathrm{K}$, returns the sum of all $d$-simplices in $\operatorname{sd}(\mathrm{K})$ contained in $\sigma$, with the induced orientation. "Contained" is understood according to the geometric interpretation of the barycentric subdivision. If K is a free $Z_{q^{-}}$-simplicial complex, $\mathrm{sd}_{\#}$ is a $Z_{q}$-equivariant map.

### 3.1.4 The $Z_{q}$-Fan lemma

The following lemma plays a central role in the proof of Theorem 1. It is proved (implicitely and in a more general version) in [8, 14].

Lemma 3 ( $Z_{q}$-Fan lemma). Let $q \geqslant 2$ be a positive integer. Let $\lambda_{\#}: \mathcal{C}\left(\operatorname{sd}\left(Z_{q}^{* n}\right)\right) \rightarrow$ $\mathcal{C}\left(Z_{q}^{* m}\right)$ be a $Z_{q}$-equivariant chain map. Then there is an $(n-1)$-dimensional simplex $\rho$ in the support of $\lambda_{\#}\left(\rho^{\prime}\right)$, for some $\rho^{\prime} \in \operatorname{sd}\left(Z_{q}^{* n}\right)$, of the form $\left\{\left(\epsilon_{1}, \mu_{1}\right),\left(\epsilon_{2}, \mu_{2}\right), \ldots,\left(\epsilon_{n}, \mu_{n}\right)\right\}$, with $\mu_{i}<\mu_{i+1}$ and $\epsilon_{i} \neq \epsilon_{i+1}$ for $i=1, \ldots, n$.

This $\rho^{\prime}$ is an alternating simplex.
Proof. The proof is exactly the proof of Theorem 5.4 (p.415) of [8]. The complex $X$ in the statement of this Theorem 5.4 is our complex $\operatorname{sd}\left(Z_{q}^{* n}\right)$, the dimension $r$ is $n-1$, and the generalized $r$-sphere $\left(x_{i}\right)$ is any generalized $(n-1)$-sphere of $\operatorname{sd}\left(Z_{q}^{* n}\right)$ with $x_{0}$ reduced to a single point. The chain map $h_{\bullet}^{\ell}$ is induced by our chain map $\lambda_{\#}$, instead of being induced by the chain map $\ell_{\#}$ of [8] (itself induced by the labeling $\ell$ ). It does not change the proof since $h_{\bullet}^{\ell}$ only uses the fact that $\ell_{\#}$ is a $Z_{q}$-equivariant chain map. In the statement of Theorem 5.4 of [8], $\alpha_{i}$ is always a lower bound on the number of "alternating patterns" (i.e. simplices $\rho^{\prime}$ as in the statement of the lemma) in $\ell_{\#}\left(x_{i}\right)$, even for odd $i$ since the map $f_{i}$ in Theorem 5.4 of [8] is zero on non-alternating elements. Since $\alpha_{0}=1$, we get that $\alpha_{i} \neq 0$ for all $0 \leqslant i \leqslant n-1$.

In particular, for $q=2$, it gives the Ky Fan theorem [6] used for instance in $[7,15,18$ ] to derive properties of Kneser graphs.

### 3.2 Proof of the main result

Proof of Theorem 1. We first sketch some steps in the proof. We assume given a proper coloring $c$ of $\mathrm{KG}^{p}(\mathcal{H})$. With the help of the coloring $c$, we build a $Z_{p}$-equivariant chain $\operatorname{map} \psi_{\#}: \mathcal{C}\left(\operatorname{sd}\left(Z_{p}^{* n}\right)\right) \rightarrow \mathcal{C}\left(Z_{p}^{* m}\right)$, where $m=n-\operatorname{cd}^{p}(\mathcal{H})+t(p-1)$. We apply Lemma 3 to get the existence of some alternating simplex $\rho^{\prime}$ in $\operatorname{sd}\left(Z_{p}^{* n}\right)$. Using properties of $\psi_{\#}$ (especially the fact that it is a composition of maps in which simplicial maps are involved), we show that this alternating simplex provides a complete $p$-uniform $p$-partite hypergraph in $\mathcal{H}$ with the required properties.

Let $r=\operatorname{cd}^{p}(\mathcal{H})$. Following the ideas of $[12,22]$, we define

$$
f:\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\} \rightarrow Z_{p} \times[m]
$$

with $m=n-r+t(p-1)$. We choose a total ordering $\preceq$ on the subsets of $[n]$. This ordering is only used to get a clean definition of $f$.

If $X \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$ is such that $|X| \leqslant n-r$, then $f(X)$ is defined to be $(\epsilon,|X|)$ with $\epsilon$ being the first nonzero component in $X$.
If $X \in\left(Z_{p} \cup\{0\}\right)^{n} \backslash\{(0, \ldots, 0)\}$ is such that $|X| \geqslant n-r+1$, by definition of the colorability defect, at least one of the $X^{j}$ 's with $j \in[p]$ contains an edge of $\mathcal{H}$. Choose $j \in[p]$ such that there is $S \subseteq X^{j}$ with $S \in E(\mathcal{H})$. In case several $S$ are possible, choose the maximal one according to the total ordering $\preceq$. Its defines $F(X):=S$ and $f(X):=\left(\omega^{j}, n-r+c(F(X))\right)$.

Note that $f$ induces a $Z_{p}$-equivariant simplicial map $f: \operatorname{sd}\left(Z_{p}^{* n}\right) \rightarrow \mathrm{L} * \mathrm{M}$, where $\mathrm{L}:=Z_{p}^{*(n-r)}$ and $\mathrm{M}:=\left(\sigma_{p-2}^{p-1}\right)^{* t}$.

Let $W_{a}$ be the set of simplices $\tau \in \mathrm{M}$ such that $\left|\tau^{\epsilon}\right|=0$ or $\left|\tau^{\epsilon}\right|=a$ for all $\epsilon \in Z_{p}$. Let $W=\bigcup_{a=1}^{m} W_{a}$. Choose an arbitrary equivariant map $s: W \rightarrow Z_{p}$. Such a map can be easily built by choosing one representative in each orbit ( $Z_{p}$ acts freely on each $W_{a}$ ). We build also an equivariant map $s_{0}: \sigma_{p-2}^{p-1} \rightarrow Z_{p}$, again by choosing one representative in each orbit of the action of $Z_{p}$. We define now a simplicial map $\left.g: \operatorname{sd}(\mathrm{L} * \mathrm{M})\right) \rightarrow Z_{p}^{* m}$ as follows.

Take a vertex in $\operatorname{sd}(\mathrm{L} * \mathrm{M})$. It is of the form $\sigma \cup \tau \neq \emptyset$ where $\sigma \in \mathrm{L}$ and $\tau \in \mathrm{M}$.
If $\tau \neq \emptyset$. Let $\alpha:=\min _{\epsilon \in Z_{p}}\left|\tau^{\epsilon}\right|$.

- If $\alpha=0$, define $\bar{\tau}:=\left\{\epsilon \in Z_{p}: \tau^{\epsilon}=\emptyset\right\}$ and $g(\sigma \cup \tau)=\left(s_{0}(\bar{\tau}), n-r+|\tau|\right)$ (we have indeed $\left.\bar{\tau} \in \sigma_{p-2}^{p-1}\right)$.
- If $\alpha>0$, define $\bar{\tau}:=\bigcup_{\epsilon:\left|\tau^{\epsilon}\right|=\alpha} \tau^{\epsilon}$ and $g(\sigma \cup \tau):=(s(\bar{\tau}), n-r+|\tau|)$.


Figure 1: An example of a simplex $\tau \in \mathrm{M}$.

The definition of $\bar{\tau}$ is illustrated on Figures 1 and 2.
If $\tau=\emptyset$. Choose $(\epsilon, \mu)$ in $\sigma$ with maximal $\mu$. Define $g(\sigma \cup \tau):=(\epsilon, \mu)$. Note that $\mathbf{L}$ is such that there is only one $\epsilon$ for which the maximum is attained.

We check now that $g$ is a simplicial map. Assume for a contradiction that there are $\sigma \subseteq \sigma^{\prime}, \tau \subseteq \tau^{\prime}$ such that $g(\sigma \cup \tau)=(\epsilon, \mu)$ and $g\left(\sigma^{\prime} \cup \tau^{\prime}\right)=\left(\epsilon^{\prime}, \mu\right)$ with $\epsilon \neq \epsilon^{\prime}$. If $\tau=\emptyset$, then $\mu \leqslant n-r$ and $\tau^{\prime}=\emptyset$. We should then have $\epsilon=\epsilon^{\prime}$, which is impossible. If $\tau \neq \emptyset$, then $|\tau|=\left|\tau^{\prime}\right|$, and thus $\tau=\tau^{\prime}$. We should again have $\epsilon=\epsilon^{\prime}$ which is impossible as well.

Note that $g$ is increasing: for $\sigma \subseteq \sigma^{\prime}$ and $\tau \subseteq \tau^{\prime}$, we have $|g(\sigma \cup \tau)| \leqslant\left|g\left(\sigma^{\prime} \cup \tau^{\prime}\right)\right|$.
We get our map $\psi_{\#}$ by defining: $\psi_{\#}=g_{\#} \circ \operatorname{sd}_{\#} \circ f_{\#}$. It is a $Z_{p}$-equivariant chain map from $\mathcal{C}\left(\operatorname{sd}\left(Z_{p}^{* n}\right)\right)$ to $\mathcal{C}\left(Z_{p}^{* m}\right)$.

This chain map $\psi_{\#}$ satisfies the condition of Lemma 3. Hence, there exists $\rho \in Z_{p}^{* m}$ of the form $\rho=\left\{\left(\epsilon_{1}, \mu_{1}\right), \ldots,\left(\epsilon_{n}, \mu_{n}\right)\right\}$ with $\mu_{i}<\mu_{i+1}$ and $\epsilon_{i} \neq \epsilon_{i+1}$ for $i=1, \ldots, n-1$ such that $\rho$ is in the support of $\psi_{\#}\left(\rho^{\prime}\right)$ for some $\rho^{\prime} \in \operatorname{sd}\left(Z_{p}^{* n}\right)$.

We exhibit now some properties of $\rho$ and $\rho^{\prime}$.
Since $g$ is a simplicial map, we know that there is a permutation $\pi$ and a sequence $\sigma_{\pi(1)} \cup \tau_{\pi(1)} \subseteq \cdots \subseteq \sigma_{\pi(n)} \cup \tau_{\pi(n)}$ of simplices of $\mathrm{L} * \mathrm{M}$ such that $g\left(\sigma_{i} \cup \tau_{i}\right)=\left(\epsilon_{i}, \mu_{i}\right)$ with $\mu_{i}<\mu_{i+1}$ and $\epsilon_{i} \neq \epsilon_{i+1}$ for $i=1, \ldots, n-1$. To ease the following discussion, we define $\tau_{0}:=\emptyset$.


Figure 2: The simplex $\bar{\tau}$ which leads to the definition of $g$.

Since $g$ is increasing, we get that $\pi(i)=i$ for all $i$. Using the fact that $f$ is simplicial, we get that $\left|\sigma_{n} \cup \tau_{n}\right|=n$, and then that $\left|\sigma_{i} \cup \tau_{i}\right|=i$. Since $\left|\sigma_{n}\right| \leqslant n-r$, we have $\tau_{n} \neq \emptyset$. Note that $\tau_{i}=\tau_{i+1}$ implies that $\tau_{i}=\emptyset$ (otherwise $\mu_{i}$ would be equal to $\mu_{i+1}$ ). Therefore, defining $z$ to be the largest index such that $\tau_{z}$ is empty, we have $z<n$ and a sequence $\tau_{z+1} \subsetneq \tau_{z+2} \subsetneq \cdots \subsetneq \tau_{n}$. Finally, noting that $\sigma_{i+1} \cup \tau_{i+1}$ has only one more element than $\sigma_{i} \cup \tau_{i}$ for $i=1, \ldots, n-1$, we get that $\left|\tau_{z+\ell}\right|=\ell$ for $\ell=0, \ldots, n-z$.

Consider now the sequence $\left(\omega_{1}, \nu_{1}\right), \ldots,\left(\omega_{n-z}, \nu_{n-z}\right)$, where $\left(\omega_{\ell}, \nu_{\ell}\right)$ is the unique vertex of $\tau_{z+\ell} \backslash \tau_{z+\ell-1}$ for $\ell=1, \ldots, n-z$. The sign $\omega_{\ell+1}$ is necessarily such that $\tau_{z+\ell}^{\omega_{\ell+1}}$ has minimum cardinality among the $\tau_{z+\ell}^{\epsilon}$, otherwise the set of $\epsilon$ for which $\left|\tau_{z+\ell+1}^{\epsilon}\right|$ is minimum would be the same as for $\left|\tau_{z+\ell}^{\epsilon}\right|$, and, according to the definition of the maps $s$ and $s_{0}$, we would have $\epsilon_{\ell+1}=\epsilon_{\ell}$.

We clearly have $\left|\left|\tau_{z+1}^{\epsilon}\right|-\left|\tau_{z+1}^{\epsilon^{\prime}}\right|\right| \leqslant 1$ for all $\epsilon, \epsilon^{\prime}$ since $\left|\tau_{z+1}\right|=1$. Now assume that for $k \geqslant z+1$ we have $\left|\left|\tau_{k}^{\epsilon}\right|-\left|\tau_{k}^{\epsilon^{\prime}}\right|\right| \leqslant 1$ for all $\epsilon, \epsilon^{\prime}$. Since the element added to $\tau_{k}$ to get $\tau_{k+1}$ is added to a $\tau_{k}^{\epsilon}$ with minimum cardinality, we have $\left|\left|\tau_{k+1}^{\epsilon}\right|-\left|\tau_{k+1}^{\epsilon^{\prime}}\right|\right| \leqslant 1$ for all $\epsilon, \epsilon^{\prime}$. By induction we have in particular

$$
\begin{equation*}
\left|\left|\tau_{n}^{\epsilon}\right|-\left|\tau_{n}^{\epsilon^{\prime}}\right|\right| \leqslant 1 \quad \text { for all } \epsilon, \epsilon^{\prime} \tag{1}
\end{equation*}
$$

We can now conclude. Using the fact that $f$ is simplicial, we get that $\rho^{\prime}=\left\{X_{1}, \ldots, X_{n}\right\}$ where the $X_{i}$ are signed vectors with $\left|X_{i}\right|=i$ and $X_{1} \subseteq \cdots \subseteq X_{n}$. Moreover, we have
$f\left(\left\{X_{z+1}, \ldots, X_{n}\right\}\right)=\tau_{n}$. Each $X_{i}$ provides a vertex $F\left(X_{i}\right)$ of $\mathrm{KG}^{p}(\mathcal{H})$ for $i=z+1, \ldots, n$. For each $j$, define $U_{j}$ to be the set of such vertices $F\left(X_{i}\right)$ such that the sign of $f\left(X_{i}\right)$ is $\omega^{j}$. The $U_{j}$ are subsets of vertices of $\mathrm{KG}^{p}(\mathcal{H})$. For two distinct $j$ and $j^{\prime}$, if $F\left(X_{i}\right) \in U_{j}$ and $F\left(X_{i^{\prime}}\right) \in U_{j}^{\prime}$, we have $F\left(X_{i}\right) \cap F\left(X_{i^{\prime}}\right)=\emptyset$. Thus, the $U_{j}$ induce in $\mathrm{KG}^{p}(\mathcal{H})$ a complete $p$-partite $p$-uniform hypergraph with $n-z$ vertices. Equation (1) indicates that the cardinalities of the $U_{j}$ differ by at most one. Since the $f\left(X_{i}\right)$ are all distinct, each $U_{j}$ has all its vertices of distinct colors.

It remains to prove that $z=n-r$ (actually, $z \leqslant n-r$ would be enough). First, we have $\mu_{i} \geqslant i$ for all $i=1, \ldots, n$ and $\mu_{z+1}=n-r+1$, thus $z \leqslant n-r$. Second, $\left|f\left(X_{z+1}\right)\right| \geqslant n-r+1$, which implies $\left|X_{z+1}\right| \geqslant n-r+1$, i.e. $z \geqslant n-r$. We get $z=n-r$, as required.

## 4 Alternation number

### 4.1 Definition

Alishahi and Hajiabolhassan [1], going on with ideas introduced in [16], defined the $q$ alternation number $\operatorname{alt}^{q}(\mathcal{H})$ of a hypergraph $\mathcal{H}$. Using this parameter, we can improve upon some theorems involving the $q$-colorability defect. The $q$-alternation number is defined as follows.

Let $q$ and $n$ be positive integers. An alterning sequence is a sequence $s_{1}, s_{2}, \ldots, s_{n}$ of elements of $Z_{q}$ such that $s_{i} \neq s_{i+1}$ for all $i=1, \ldots, n-1$. For a vector $X=\left(x_{1}, \ldots, x_{n}\right) \in$ $\left(Z_{q} \cup\{0\}\right)^{n}$ and a permutation $\pi \in \mathcal{S}_{n}$, we denote $\operatorname{alt}_{\pi}(X)$ the maximum length of an alternating subsequence of the sequence $x_{\pi(1)}, \ldots, x_{\pi(n)}$. Note that by definition this subsequence has no zero element.
Example. Let $n=9, q=3$, and $X=\left(\omega^{2}, \omega^{2}, 0,0, \omega^{1}, \omega^{3}, 0, \omega^{3}, \omega^{2}\right)$, we have $\operatorname{alt}_{\mathrm{id}}(X)=4$. If $\pi$ is a permutation acting only on the first four positions, then $\operatorname{alt}_{\text {id }}(X)=\operatorname{alt}_{\pi}(X)$. If $\pi$ exchanges the last two elements of $X$, we have $\operatorname{alt}_{\pi}(X)=5$.

Let $\mathcal{H}=(V, E)$ be a hypergraph with $n$ vertices. We identify $V$ and $[n]$. The $q$ alternation number alt $^{q}(\mathcal{H})$ of a hypergraph $\mathcal{H}$ with $n$ vertices is defined as:

$$
\begin{equation*}
\operatorname{alt}^{q}(\mathcal{H})=\min _{\pi \in \mathcal{S}_{n}} \max \left\{\operatorname{alt}_{\pi}(X): X \in\left(Z_{q} \cup\{0\}\right)^{n} \text { with } E\left(\mathcal{H}\left[X^{j}\right]\right)=\emptyset \text { for } j=1, \ldots, q\right\} . \tag{2}
\end{equation*}
$$

Note that this number does not depend on the way $V$ and $[n]$ have been identified.

### 4.2 Improving the results with the alternation number

Alishahi and Hajiabolhassan improved the Křiž theorem by the following theorem.
Theorem (Alishahi-Hajiabolhassan theorem). Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then

$$
\chi\left(\mathrm{KG}^{q}(\mathcal{H})\right) \geqslant\left\lceil\frac{|V(\mathcal{H})|-\mathrm{alt}^{q}(\mathcal{H})}{q-1}\right\rceil
$$

for any integer $q \geqslant 2$.
It is an improvement since we have

$$
|V(\mathcal{H})|-\operatorname{alt}^{q}(\mathcal{H}) \geqslant \operatorname{cd}^{q}(\mathcal{H})
$$

as it can be easily checked. This inequality is often strict, see [1].
Theorem 1 and Theorem 2 can be similarly improved with the alternation number. Let $\pi$ be the permutation on which the minimum is attained in Equation (2). We replace $r=\operatorname{cd}^{p}(\mathcal{H})$ by $r=|V(\mathcal{H})|-\operatorname{alt}^{p}(\mathcal{H})$ in both proofs of Theorem 1 and Theorem 2, and we replace $|X|$ in the definition of $f$ by $\operatorname{alt}_{\pi}(X)$ in the proof of Theorem 1. There are no other changes and we get the following theorems.

Theorem 4. Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let $p$ be a prime number. Then any proper coloring $c$ of $\mathrm{KG}^{p}(\mathcal{H})$ with colors $1, \ldots, t$ ( $t$ arbitrary) must contain a complete p-uniform p-partite hypergraph with parts $U_{1}, \ldots, U_{p}$ satisfying the following properties.

- It has $|V(\mathcal{H})|-\operatorname{alt}^{p}(\mathcal{H})$ vertices.
- The values of $\left|U_{j}\right|$ for $j=1, \ldots, p$ differ by at most one.
- For any $j$, the vertices of $U_{j}$ get distinct colors.

Theorem 5. Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then

$$
\chi_{\ell}\left(\mathrm{KG}^{p}(\mathcal{H})\right) \geqslant \min \left(\left\lceil\frac{|V(\mathcal{H})|-\operatorname{alt}^{p}(\mathcal{H})}{p}\right\rceil+1,\left\lceil\frac{|V(\mathcal{H})|-\operatorname{alt}^{p}(\mathcal{H})}{p-1}\right\rceil\right)
$$

for any prime number $p$.
The special case of Theorem 4 when $p=2$ is proved in [1] in a slightly more general form.

### 4.3 Complexity

It remains unclear whether the alternation number, or a good upper bound of it, can be computed efficiently. However, we can note that given a hypergraph $\mathcal{H}$, computing the alternation number for a fixed permutation is an NP-hard problem.

Proposition 6. Given a hypergraph $\mathcal{H}$, a permutation $\pi$, and a number $q$, computing

$$
\max \left\{\operatorname{alt}_{\pi}(X): X \in\left(Z_{q} \cup\{0\}\right)^{n} \text { with } E\left(\mathcal{H}\left[X^{j}\right]\right)=\emptyset \text { for } j=1, \ldots, q\right\}
$$

is NP-hard.

Proof. The proof consists in proving that the problem of finding a maximum independent set in a graph can be polynomially reduced to our problem for $q=2, \pi=\mathrm{id}$, and $\mathcal{H}$ being some special graph.

Let $G$ be a graph. Define $G^{\prime}$ to be a copy of $G$ and consider the join $\mathcal{H}$ of $G$ and $G^{\prime}$. The join of two graphs is the disjoint union of the two graphs plus all edges $v w^{\prime}$ with $v$ a vertex of $G$ and $w^{\prime}$ a vertex of $G^{\prime}$. We number the vertices of $G$ arbitrarily with a bijection $\rho: V \rightarrow[|V|]$. It gives the following numbering for the vertices of $\mathcal{H}$. In $\mathcal{H}$, a vertex $v$ receives number $2 \rho(v)-1$ and its copy $v^{\prime}$ receives the number $2 \rho(v)$. Let $n=2|V|$. As usual, we denote the maximum cardinality of an independent set of $G$ by $\alpha(G)$.

Let $I \subseteq V$ be a independent set of $G$. Define $Y=\left(y_{1}, \ldots, y_{n}\right) \in\left(Z_{2} \cup\{0\}\right)^{n}$ as follows:

$$
y_{2 \rho(v)-1}=+1 \text { and } y_{2 \rho(v)}=-1 \text { for all } v \in I, \text { and } y_{i}=0 \text { for the other indices } i .
$$

By definition of the numbering, we have $\operatorname{alt}_{\mathrm{id}}(Y)=2|I|$ and thus

$$
\max \left\{\operatorname{alt}_{\text {id }}(X): X \in\left(Z_{2} \cup\{0\}\right)^{n} \text { with } E\left(\mathcal{H}\left[X^{j}\right]\right)=\emptyset \text { for } j=1,2\right\} \geqslant 2 \alpha(G)
$$

Conversely, any $X=\left(x_{1}, \ldots, x_{n}\right) \in\left(Z_{2} \cup\{0\}\right)^{n}$ with $E\left(\mathcal{H}\left[X^{j}\right]\right)=\emptyset$ for $j=1,2$ gives an independent set $I$ in $G$ and another $I^{\prime}$ in $G^{\prime}$ : take a longest alternating subsequence in $X$ and define the set $I$ as the set of vertices $v$ such that $x_{2 \rho(v)-1} \neq 0$ and the set $I^{\prime}$ as the set of vertices $v$ such that $x_{2 \rho(v)} \neq 0$. We have alt $\operatorname{tid}_{\text {id }}(X)=|I|+\left|I^{\prime}\right|$ because two components of $X$ with distinct index parities cannot be of same sign: each vertex of $G$ is the neighbor of each vertex of $G^{\prime}$. Thus

$$
\max \left\{\operatorname{alt}_{\mathrm{id}}(X): X \in\left(Z_{2} \cup\{0\}\right)^{n} \text { with } E\left(\mathcal{H}\left[X^{j}\right]\right)=\emptyset \text { for } j=1,2\right\} \leqslant 2 \alpha(G)
$$

The same proof gives also that computing the two-colorability defect $\operatorname{cd}^{2}(\mathcal{H})$ of any hypergraph $\mathcal{H}$ is an NP-hard problem.

## References

[1] M. Alishahi and H. Hajiabolhassan, On chromatic number of Kneser hypergraphs, preprint.
[2] N. Alon, P. Frankl, and L. Lovász, The chromatic number of Kneser hypergraphs, Transactions Amer. Math. Soc. 298 (1986), 359-370.
[3] V. L. Dol'nikov, A certain combinatorial inequality, Siberian Math. J. 29 (1988), 375-397.
[4] P. Erdős, Problems and results in combinatorial analysis, Colloquio Internazionale sulle Teorie Combinatorie (Rome 1973), Vol. II, No. 17 in Atti dei Convegni Lincei, 1976, pp. 3-17.
[5] P. Erdős, Z. Füredi, A. Hajnal, P. Komjáth, V. Rödl, and Á. Seress, Coloring graphs with locally few colors, Discrete Mathematics 59 (1986), 21-34.
[6] K. Fan, A generalization of Tucker's combinatorial lemma with topological applications, Annals Math., II Ser. 56 (1952), 431-437.
[7] K. Fan, Evenly distributed subset of $S^{n}$ and a combinatorial application, Pacific J. Math. 98 (1982), 323-325.
[8] B. Hanke, R. Sanyal, C. Schultz, and G. Ziegler, Combinatorial Stokes formulas via minimal resolutions, Journal of Combinatorial Theory, Series A 116 (2009), 404-420.
[9] I. Kříž, Equivariant cohomology and lower bounds for chromatic numbers, Transactions Amer. Math. Soc. 33 (1992), 567-577.
[10] I. Křiž, A correction to "Equivariant cohomology and lower bounds for chromatic numbers", Transactions Amer. Math. Soc. 352 (2000), 1951-1952.
[11] L. Lovász, Kneser's conjecture, chromatic number and homotopy, Journal of Combinatorial Theory, Series A 25 (1978), 319-324.
[12] J. Matoušek, A combinatorial proof of Kneser's conjecture, Combinatorica 24 (2004), 163-170.
[13] J. Matoušek and G. Ziegler, Topological lower bounds for the chromatic number: A hierarchy, Jahresber. Deutsch. Math.-Verein. 106 (2004), 71-90.
[14] F. Meunier, $A \mathbb{Z}_{q}$-Fan theorem, Tech. report, Laboratoire Leibniz-IMAG, Grenoble, 2005.
[15] F. Meunier, A topological lower bound for the circular chromatic number of Schrijver graphs, Journal of graph theory 49 (2005), 257-261.
[16] F. Meunier, The chromatic number of almost-stable Kneser hypergraphs, Journal of Combinatorial Theory, Series A 118 (2011), 1820-1828.
[17] J. R. Munkres, Elements of algebraic topology, Perseus Book Publishing, 1984.
[18] G. Simonyi and G. Tardos, Local chromatic number, Ky Fan's theorem, and circular colorings, Combinatorica 26 (2006), 587-626.
[19] G. Simonyi and G. Tardos, Colorful subgraphs of Kneser-like graphs, European Journal of Combinatorics 28 (2007), 2188-2200.
[20] S. Stahl, n-tuple colorings and associated graphs, Journal of Combinatorial Theory, Series B 20 (1976), 185-203.
[21] M. Valencia-Pabon and J. Vrecia, On the diameter of Kneser graphs, Discrete Mathematics 305 (2005), 383-385.
[22] G. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, Invent. Math. 147 (2002), 671-691.
[23] G. Ziegler, Erratum: Generalized Kneser coloring theorems with combinatorial proofs, Invent. Math. 163 (2006), 227-228.

