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COLORING GRAPHS WITH FIXED GENUS AND GIRTH

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Abstract. It is well known that the maximum chromatic number of a graph on the orientable surface S_g is $\theta(g^{1/2})$. We prove that there are positive constants c_1, c_2 such that every triangle-free graph on S_g has chromatic number less than $c_2(g/\log(g))^{1/3}$ and that some triangle-free graph on S_g has chromatic number at least $c_1 \frac{g^{1/3}}{\log(g)}$. We obtain similar results for graphs with restricted clique number or girth on S_g or N_k. As an application, we prove that an S_g-polytope has chromatic number at most $O(g^{3/7})$. For specific surfaces we prove that every graph on the double torus and of girth at least six is 3-colorable and we characterize completely those triangle-free projective graphs that are not 3-colorable.

1. Introduction

Grötzsch [14] proved that every planar graph with no triangles can be 3-colored. A short proof is given in [23]. Kronk and White [18] proved that every toroidal graph with no triangles can be 4-colored and that every toroidal graph with no cycles of length less than six can be 3-colored. Kronk [17] studied the chromatic number of triangle-free graphs on certain surfaces. Thomassen [23] showed that every graph on the torus with girth at least five is 3-colorable (as conjectured in [18]) and in the same work showed that a graph which embeds on the projective plane with no contractible 3-cycle nor 4-cycle is 3-colorable. Cook [5] showed that if G is a graph of genus g and the girth of G is at least $\mathrm{Max}\{9, 6 + 2\log_2(g)\}$ then G is 3-colorable.

In the following, we will be concerned with finite undirected connected graphs without loops or multiple edges. With $c, c_1, c_2, c_3, \ldots$ we shall indicate positive constants. By $v(G)$ we indicate the number of vertices in a graph G; $g(G)$ and $k(G)$ indicate the genus and crosscap number, respectively. Further, $e(G)$ is the number of edges; $q(G)$ the girth. By S_g we mean the orientable surface of genus g. Further, N_k denotes the sphere with k crosscaps added. For a graph G embedded on a surface, we let $r(G)$ be the number of regions. Sometimes we write v, e, etc. instead of $v(G)$, $e(G)$, etc. For undefined terms and concepts the reader is referred to [3] and [4].

In Section 2 of this paper we estimate the maximum chromatic number of a triangle-free graph on a fixed surface as described in the abstract. In Section 3 we extend the method to graphs of fixed genus and bounded clique number or girth. As

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an application, we show that S_g-polytopes have chromatic number at most $O(g^{3/7})$ in response to a question of Croft, et al. [6]. In that section we also determine the asymptotic behavior of the maximum cochromatic number of a graph on S_g, a question raised by Straight [21, 22]. In Section 4 we investigate the complexity of coloring graphs on a fixed surface. In particular, we produce a polynomially bounded algorithm for finding the chromatic number of such graphs when the girth is at least six.

In Section 5 we prove that a projective graph of girth at least four has chromatic number at most three if and only if it does not contain a nonbipartite quadrangulation, as conjectured by D. Youngs [25]. In particular, we obtain a polynomially bounded algorithm for finding the chromatic number of triangle-free projective graphs.

In Section 6 we identify the relevant unsolved problems for the torus and Klein bottle. Finally, we prove in Section 7 that every graph of girth at least six on the double torus is 3-colorable.

It is perhaps remarkable that the only embedding properties that are needed for the upper bounds on the chromatic number described above (except for [14, 23]) and also those in the present paper (except for projective graphs) are the upper bounds on the number of edges which follow from Euler's formula. In other words, the actual structure of the embedding plays no role. We shall make this precise as follows. If G is a graph on S_g or N_k, Euler's formula says:

$$v - e + r \geq 2 - 2g$$

or

$$v - e + r \geq 2 - k.$$

If, in addition, G has girth at least $q (q \geq 3)$ and at least one cycle, then

$$2e \geq qr.$$

Hence,

$$e(1 - 2/q) \leq v - 2 + 2g$$

or

$$e(1 - 2/q) \leq v - 2 + k.$$

Motivated by this we say that a graph G is (a, b)-*restricted* ($a \geq 0$ and $b \geq 3$) if G has girth at least b and for each induced subgraph H of G,

$$e(H)(1 - 2/b) \leq v(H) - 2 + a.$$

Now all known upper bounds for chromatic number of graphs of girth q on S_g, including those in the present paper but excluding those of [14, 23], hold for $(2g, q)$-restricted graphs. Hence they hold for nonorientable surfaces as well. This leads to the following general question.

Problem 1. Do the following hold for all $k \geq 3$ and $g \geq 1$ and $q \geq 3$:

$$\text{Max}\{\chi(G)\colon g(G) \leq g,\ q(G) \geq q\} = \text{Max}\{\chi(G)\colon G \text{ is } (2g, q)\text{-restricted}\}$$

and

$$\text{Max}\{\chi(G)\colon k(G) \leq k,\ q(G) \geq q\} = \text{Max}\{\chi(G)\colon G \text{ is } (k, q)\text{-restricted}\}?$$

Given graphs G_1 and G_2 with edges u_1v_1 and u_2v_2 taken respectively from G_1 and G_2, let us form a new graph by identifying u_1 and u_2, adding the edge v_1v_2, and removing the edges u_1v_1 and u_2v_2. This operation, discussed in [15], is known as the Hajós construction.

We cannot have equality in Problem 1 for surfaces of positive Euler characteristic because of the Hajós construction. (If we apply the Hajós construction to two copies of K_5 we obtain a $(0,3)$-restricted graph of chromatic number five; whereas every graph on the sphere S_0 has chromatic number at most four by the 4-color theorem.) Also, we have to exclude the Klein bottle, since K_7 is a $(2,3)$-restricted graph of chromatic number seven whereas every graph on the Klein bottle N_2 has chromatic number at most six.

2. Triangle-free graphs on a fixed surface

Let C_g^m (respectively Q_g^m) be the maximum chromatic number of all graphs of genus g and clique number (girth) less than (greater than) m. Thus, $C_g^3 = Q_g^3$.

Theorem 2.1. *There exist c_1 and c_2 such that*

$$c_1 \frac{\sqrt[3]{g}}{\log g} \leq Q_g^3 \leq c_2 \sqrt[3]{\frac{g}{\log g}}.$$

Proof. Erdös [8] proved the existence of a triangle-free graph of order $\lfloor g^{2/3} \rfloor$ having at most g edges and an independence number less than $cg^{1/3}\log(g)$. Clearly, such a graph can be embedded on a surface of genus g. Dividing the order by the independence number establishes the lower bound. In [1, 2] the existence of a c_1 is established having the property that any triangle-free graph of order n must contain an independent set of order at least $c_1\sqrt{n\log(n)}$. Suppose G is a triangle-free graph of genus g. Let us set $s = \sqrt[3]{\frac{g}{\log g}}$. We wish to show the existence of a c_2 where $\chi(G) \leq c_2 s$. Now, successively remove from G vertices of degree less than s until all remaining vertices have degree at least s. Let H denote the graph that remains. Then $e(H) \geq \frac{sv(H)}{2}$. Note, $G - H$ can be colored with s colors. Assuming H is non-empty, we will color H with at most $c_3 s$ colors. By Euler's formula $e(H) \leq 3v(H) - 6 + 6g$. Hence, $v(H) < c_4 g^{2/3}(\log g)^{1/3}$. Let us set this last expression equal to w. Suppose some triangle-free graph has order t, where $\frac{w}{4^{m+1}} \leq t < \frac{w}{4^m}$. We note it must contain an independent set of order at least

$$c_1 \frac{\sqrt{w}}{2^{m+1}} \sqrt{\log(w) - (m+1)\log(4)}$$

by [1, 2]. After assigning this set a color, let us remove it and repeat the process until at most $\frac{w}{4^{m+1}}$ vertices remain. In doing so, we will have used at most

$$\frac{\sqrt{w}}{c_1 2^{m-1}\sqrt{\log(w) - (m+1)\log(4)}}$$

colors. Let us apply this process to H, increasing m until $\frac{w}{4^m} \leq s$. Such a process generates at most

$$\sum_{m=0}^{M} \frac{\sqrt{w}}{c_1 2^{m-1}\sqrt{\log(w) - (m+1)\log(4)}}$$

color classes, where $M = \lceil \log_4(w/s) \rceil + 1$. But this sum is bounded above by $\frac{8}{c_1}\sqrt{\frac{w}{\log w}}$. Further, this expression is less than some multiple of s. At this point, at most s vertices are uncolored. We may assign each a distinct color and in doing so, we color G with fewer than some multiple of s colors. □

Recently an improved bound on the Ramsey number $R(3, m)$ was found by R. H. Kim [16]. The proof allows $\log g$ in the lower bound of Theorem 2.1 to be replaced by $(\log g)^{2/3}$.

3. Clique number, girth, polyhedral surfaces, cochromatic number

The method of Theorem 2.1 extends to the following result.

Theorem 3.1. *For fixed s, there exist c_1 and c_2 where for sufficiently large g,*

$$c_1 \frac{g^{\frac{s-1}{2s}}}{(\log g)} \le C_g^s \le c_2 \left(\frac{g}{\log g}\right)^{\frac{s-2}{2s-3}}.$$

Theorem 3.2. *For fixed s, there exist c_1 and c_2 where for arbitrarily small $\varepsilon > 0$ and sufficiently large g,*

$$c_1 g^{\frac{1-\varepsilon}{2s+2}} \le Q_g^s \le c_2 g^{\frac{2}{s+3}}.$$

The proofs of these remarks are similar to the proof of Theorem 2.1. And so, we simply point out that the lower bound of Theorem 3.1 follows from a result of Bollobás [3]. (See the proof of Theorem 11, pages 287–289.) For the upper bound we use [3, Theorem 17, page 298] instead of [1, 2]. The lower bound of Theorem 3.2 follows from the proof of inequality (4) in [7] and the upper bound follows from inequality (5) in the same document.

An S_g-*polytope* is a topological subspace in the 3-space R^3 which is homeomorphic to S_g and which is the union of a finite collection of convex polygons. Its chromatic number is the smallest number of colors needed to color its dual graph. Croft, Falconer and Guy [6] suggest a comparison between the maximum chromatic number of an S_g-polytope with the Heawood bound $\theta(g^{1/2})$. In [24] it is pointed out that the dual graph of an S_g-polytope contains no K_5. Hence, Theorem 3.1 implies:

Theorem 3.3. *Every S_g-polytope has chromatic number $o(g^{3/7})$.*

No doubt, it is possible in the preceding theorem to decrease the constant 3/7. But perhaps it would be more interesting to look for some lower bound.

Problem 2 ([24])**.** Does there exist an S_g-polytope with chromatic number at least 100?

Given G, the *cochromatic number*, $z(G)$, of G is the minimum order of all partitions of $V(G)$ where each part induces a complete or empty graph. Straight asked in [21, 22] what is the largest possible z for graphs embeddable on a given surface. Using a proof-technique similar to that above, we can give an asymptotic solution for orientable surfaces. Let us denote by $z(S_g)$ the maximum cochromatic number of all graphs which embed on S_g.

Theorem 3.4. *With the preceding notation, $z(S_g) = \theta(\frac{\sqrt{g}}{\log g})$.*

Proof. As is shown in [13], there exists a graph of order $\lfloor \sqrt{g} \rfloor$ which has cochromatic number at least $c_1 \frac{\sqrt{g}}{\log g}$ and embeds on S_g. So, suppose G_g is a graph of genus g. Remove from G_g all vertices of degree less than $\frac{\sqrt{g}}{\log g}$. Repeat this process until all vertices that remain have degree at least $\frac{\sqrt{g}}{\log g}$. Let H_g be the graph that remains at the end of this process. As before, we can color $G_g - H_g$ using at most $\frac{\sqrt{g}}{\log g}$ colors. We note $e(H_g) \geq v(H_g)\frac{\sqrt{g}}{2\log g}$. By Euler's formula $e(H_g) < 7g$, for g sufficiently large. From [9] we know that a graph with this size can have cochromatic number at most $c_2\frac{\sqrt{g}}{\log g}$. □

4. Critical graphs and computational complexity

Let us say G is *k-critical* if for each edge $e, \chi(G-e) < \chi(G) = k$. In this section we consider the following general question. Let g, k, q be natural numbers where $q \geq 3$. Does there exist infinitely many k-critical graphs of girth at least q embedded on S_g?

Since the 3-critical graphs are the odd cycles, we shall assume that $k \geq 4$. For $g = 0$ and $q = 3$, the answer is affirmative as shown by repeatedly applying Hajós' construction to K_4. (Note, Hajós' construction applied to two k-critical graphs produces a new k-critical graph.) By Grötzsch's theorem (see [23]) there exists no 4-critical graph of girth at least four on the sphere. So for $g = 0$, the above question is settled completely.

We now turn to the case g at least one. If G is k-critical, $G \neq K_k$ and $k \geq 4$, then

$$2e(G) \geq (k-1)v(G) + \frac{k-3}{k^2-3}v(G) + \frac{k-1}{k^2-3}.$$

This inequality was first established by Gallai [11] with a slight improvement by Dirac (see [19]). This bound leads to the following.

Theorem 4.1. *If G is triangle-free of genus g and k-critical, $k \geq 4$, with order v, then*

$$8g - 8 - \frac{k-1}{k^2-3} \geq \left(k - 5 + \frac{k-3}{k^2-3}\right)v.$$

Proof. If G embeds on S_g with r regions, then $4r \leq 2e$. Thus, from Euler's formula, $4v - 8 + 8g \geq 2e$. Using the preceding bound we achieve the desired bound. □

If g and k are fixed, and $k \geq 5$, then the preceding bounds v from above. Hence the following.

Corollary 4.2. *If $k \geq 5$, there are only a finite number of k-critical triangle-free graphs which embed on a given surface.*

We note that this corollary is the best possible in the sense that Mycielski-Grötzsch graphs [14, 20] are 4-critical and triangle-free, yet embed on the torus.

In general, computing a bound on the chromatic number is difficult. If $k \geq 3$ then by [12] determining if $\chi(G) \leq k$ is NP-complete. Even if G is a planar graph with $\Delta(G) = 4$, by [12], determining if $\chi(G) \leq 3$ is NP-complete. However, for triangle-free graphs of bounded genus we have:

Corollary 4.3. *If $k \geq 4$, and G is a triangle-free graph of bounded genus, then we can determine in polynomial time if $\chi(G) \leq k$.*

Proof. If G is triangle-free and of bounded genus and $\chi(G) > k$ then G contains a $(k+1)$-critical subgraph. But as there are only a finite number of $(k+1)$-critical graphs, we can check in polynomial time if G contains such a subgraph. □

Of course, we can tell in polynomial time if $\chi(G) = 2$. This leads us to an open question:

Problem 3. If G is triangle-free and of bounded genus or crosscap number, can we tell in polynomial time if $\chi(G) = 3$?

In Section 5 we answer this for graphs in the projective plane N_1.

If $\chi(G) \geq 5$, then in any embedding of G, there must be a non-contractible cycle. Therefore Corollary 4.2 includes the following result—discovered by Fisk and Mohar [10].

Corollary 4.4. *Given g, there exists a constant c_g so that if G is triangle-free and embeds on S_g with all non-bounding cycles having length at least c_g then $\chi(G) \leq 4$.*

By similar arguments we get

Theorem 4.5. *If $t \geq 4$, there are only a finite number of t-critical graphs of girth at least six which embed on a given surface.*

Corollary 4.6. *The chromatic number of a graph of girth at least six on a fixed surface can be found in polynomial time.*

We do not know if Corollary 4.6 remains true when "six" is replaced by "five" or "four".

Problem 4. Can the chromatic number of a graph of girth five (respectively four) on a fixed surface be found in polynomial time?

By Corollary 4.3 only the 3-color problem in Problem 4 remains open.

As we pointed out, k-critical graphs are useful for polynomial time algorithms. Thomassen [24] proved that there are only finitely many 6-critical graphs on a fixed surface. Examples due to Fisk (see [24]) show that infinitely many 5-critical graphs embed on the torus. Corollary 4.2 and Theorem 4.5 are in the same spirit. One case remains open:

Problem 5. Does there exist a surface S with an infinite family of 4-critical graphs of girth five?

5. The projective plane N_1

Albertson and Hutchinson [26] showed that each graph in N_1 is 5-colorable unless it contains K_6. It is easy to see that each graph of girth four in N_1 is 4-colorable. Thomassen [23] proved that every projective graph of girth at least five is 3-colorable. More precisely

Theorem 5.1 ([23]). *If G is a graph in N_1 with no contractible cycle of length three or four, then G is 3-colorable.*

Theorem 5.1 is the best possible in a strong sense by the following result of Youngs [25].

Theorem 5.2 ([25]). *If G is a nonbipartite quadrangulation of N_1, then $\chi(G) = 4$.*

In other words, a quadrangulation of N_1 may have chromatic number two or four, but it cannot have chromatic number three. Youngs [25] also conjectured that if a triangle-free graph in N_1 has chromatic number four, then it contains a nonbipartite quadrangulation. We shall prove that conjecture. First we prove an extension of Grötzsch's theorem which is of interest in its own right. As the method is the same as [23], some details which can be found in [23] are omitted here.

Theorem 5.3. *Let G be a planar triangle-free graph with chordless outer cycle $C\colon x_1, x_2, x_3, x_4, x_5, x_6, x_1$. Let c be a coloring of C in colors* 1, 2, 3*. Then c can be extended to a* 3*-coloring of G if and only if G doesn't contain a* 2*-connected subgraph H with outer cycle C such that all other facial cycles are* 4*-cycles and such that opposite vertices of C have the same color.*

Proof. The "only if" part of Theorem 5.3 follows from Theorem 5.2. For if H exists and we identify any two opposite vertices of C, then H becomes a nonbipartite quadrangulation of N_1. Assume therefore that H does not exist. We prove, by induction on $V(G)$, that c can be extended to a 3-coloring.

Assume first that $G - C$ has a vertex x joined to at least two vertices of C. If x cannot be colored by 1, 2, or 3, then $G(V(C) \cup \{x\})$ can play a role of H. If x can be colored, then we color x and apply the induction hypothesis to the two or three faces of $G(V(C) \cup \{x\})$. (In Grötzsch's theorem a cycle of length 4 or 5 is allowed to be precolored, see [23].) So assume that no vertex of $G - C$ is joined to two vertices of C.

We may also assume that each vertex of $G - C$ has degree at least three since otherwise we delete it and use induction.

If G has a separating cycle S of length at most five, then we delete the interior and apply the induction hypothesis to the remaining graph. Then we apply Grötzsch's Theorem to the interior of S. So assume that G has no separating cycle of length four or five.

If G has girth at least five, then c can be extended to a 3-coloring by [23, Theorem 3.1]. So assume that $C_1 : y_1, y_2, y_3, y_4, y_1$ is a 4-cycle in G. As G has no separating 4-cycle, C_1 is facial. Since $G - C$ has no vertex joined to two vertices of C, it is possible to identify either y_1, y_3 or y_2, y_4 (say the former) into a vertex y_0 such that C is still chordless. Also, we do not create a triangle, since G has no separating 5-cycle. By the induction hypothesis, we may assume that the new graph (containing y_0) contains a quadrangulation, say H', and that $x_1, x_2, x_3, x_4, x_5, x_6$ are colored $1, 2, 3, 1, 2, 3$, respectively. If $y_0 \notin V(H')$, then we are finished. So assume that $y_0 \in V(H')$. Let H'' be obtained from H' by splitting y_0 into y_1 and y_2 and adding the edges and vertices of C_1. If $y_2, y_4 \in V(H')$, then H'' can play the role of H. So assume that $y_2 \notin V(H')$. Using the induction hypothesis, we may assume that c can be extended to a 3-coloring of G minus the interior of the unique 6-cycle C_2 which contains two edges incident with y_2 and four edges of H'. Now the resulting coloring can be extended to a 3-coloring of G unless the interior of C_2 contains a subgraph H_1 with outer cycle C_2 such that all other facial cycles are 4-cycles. If $y_4 \notin V(H')$ then we find similarly a subgraph H_2 inside a 6-cycle containing y_1, y_4, y_3 and four edges of H' such that all bounded faces of H_2 are bounded by 4-cycles. Now $H' \cup H_1 \cup H_2$ can play the role of H. □

Theorem 5.4. *Let G be a graph in the projective plane N_1 such that all contractible cycles have length at least four. Then G is* 3*-colorable if and only if G does not contain a nonbipartite quadrangulation.*

Proof. The "only if" part follows from Theorem 5.2. We prove the "if" part by induction.

If G has a 3-cycle C, then C is noncontractible. We color C by 1, 2, 3 and we cut C into a 6-cycle and apply Theorem 5.3. So assume that G has no 3-cycle.

We may also assume that G has no vertex of degree less than three nor a separating contractible 4-cycle or 5-cycle since otherwise we use the method of Theorem 5.3.

By Theorem 5.1, we may then assume that G has a facial 4-cycle y_1, y_2, y_3, y_4, y_1. We now identify y_1 and y_3. As G has no 3-cycle we create no loops. As G has no vertex of degree two and no separating contractible cycle of length at most five, we create no contractible 3-cycle. Now we complete the proof by induction as in Theorem 5.3. If $\chi(G) > 3$, then we obtain a subgraph H of G which is a quadrangulation of N_1. It only remains to prove that $\chi(H) > 2$. But, otherwise any 2-coloring of H can be extended to a 3-coloring of G by Grötzsch's theorem. □

Corollary 5.5. *There exists a polynomially bounded algorithm for finding the chromatic number of a triangle-free projective graph G.*

Proof. It is easy to decide if $\chi(G) = 1$ or $\chi(G) = 2$. So assume that $\chi(G) \geq 3$. By the first remark of Section 5, $\chi(G) \leq 4$. To decide whether $\chi(G) < 4$ we use Theorem 5.4. We delete successively the interior of every contractible 4-cycle until we have a graph H in N_1 with no contractible 4-cycle. Then $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$ by Grötzsch's theorem. By Theorem 5.4, $\chi(H) \leq 3$ if and only if H is not a quadrangulation or H is a bipartite quadrangulation (which is easy to test). □

6. The torus S_1 and the Klein bottle N_2

It is well known that each graph in S_1 or N_2 is 6-colorable unless it contains K_7. (Note that K_7 can be embedded in S_1 but not in N_2.) Kronk and White [18] proved that every graph of girth at least four in S_1 is 4-colorable. The proof also works for $(2, 4)$-restricted graphs and hence also for graphs on the Klein bottle. The result is the best possible for S_1 because of the Mycielski-Grötzsch graphs [14, 20] and for N_2 because of the graphs in Theorem 5.2. Going to girth five, Thomassen [23] proved

Theorem 6.1 ([23]). *Every graph on the torus with no contractible cycle of length less than five is* 3*-colorable.*

As K_4 can be drawn on N_2 such that all cycles are noncontractible, Theorem 6.1 does not extend to N_2. Instead we have the following problem.

Problem 6. Is every graph of girth at least five on N_2 3-colorable?

We may also seek an analogue of Theorem 5.4.

Problem 7. Characterize the graphs of girth four on S_1 which are not 3-colorable.

Or even more generally:

Problem 8. Characterize the $(2, 4)$-restricted graphs which are not 3-colorable.

Recall that $(2, 4)$-restricted guarantees that no subgraph has more than twice as many edges as vertices. A precise solution to Problem 8 would also solve Problems 6 and 7 and prove Theorem 6.1.

7. The double torus S_2

Problem 9. Is every graph on S_2 of girth four 4-colorable?

Problem 10. Is every graph on S_2 of girth five 3-colorable?

By Corollary 4.2, Problem 9 is decidable whereas this is not known for Problem 10 (compare with Problem 5). So we focus on Problem 10.

An affirmative answer to Problem 10 would imply the corollary of Theorem 6.1 that every toroidal graph of girth at least five is 3-colorable.

Our final result is weaker than Problem 10 but stronger than the result of Cook [5] that a graph of girth at least six which embeds on the double torus has chromatic number at most four.

Theorem 7.1. *Every graph of girth at least six and genus two is* 3*-colorable.*

This theorem can be expanded to the more general family of $(4,6)$-restricted graphs. As the problem is finite, we merely present an outline of the proof. Suppose the statement is false. Let G be a 4-critical $(4,6)$-restricted graph. We wish to show that G is 3-colorable. The *minor* of G is the graph induced by all vertices of degree three in G. The *major* of G is the graph induced by all vertices of G not in the minor. From Gallai [11] we know that each block of the minor is a clique or odd cycle. Given the restrictions on the girth of G, each block of the minor is an edge or an odd cycle of length at least seven. We note that $\frac{2}{3}e \leq v+2$. As $2e$ equals the sum of degrees in G, there are at most six vertices in the major. We can color G by first 3-coloring the major and then extending this to each component of the minor. We note that if two vertices in the minor are adjacent in some cycle in the minor, and these two vertices are adjacent to vertices assigned different colors in the major, then the 3-coloring can be extended to the entire component. Likewise, if a vertex v has degree zero or one in the minor, and this vertex is adjacent to two vertices in the major which are given the same color, then the 3-coloring of the major can be extended to the component containing v. There are a number of cases to consider, but in each case one of these two coloring extensions can be applied to 3-color all of G.

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