Coloring with no 2-colored P_4's

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Abstract

A proper coloring of the vertices of a graph is called a *star coloring* if every two color classes induce a star forest. Star colorings are a strengthening of *acyclic colorings*, i.e., proper colorings in which every two color classes induce a forest.

We show that every acyclic k-coloring can be refined to a star coloring with at most $(2k^2 - k)$ colors. Similarly, we prove that planar graphs have star colorings with at most 20 colors and we exhibit a planar graph which requires 10 colors. We prove several other structural and topological results for star colorings, such as: cubic graphs are 7-colorable, and planar graphs of girth at least 7 are 9-colorable. We provide a short proof of the result of Fertin, Raspaud, and Reed that graphs with tree-width t can be star colored with $\binom{t+2}{2}$ colors, and we show that this is best possible.

1 Introduction

A *proper r-coloring* of a graph G is an assignment of labels from $\{1, 2, \ldots, r\}$ to the vertices of G so that adjacent vertices receive distinct labels. The minimum r so that G has a proper r-coloring is called the *chromatic number* of G, denoted by $\chi(G)$. The chromatic number is one of the most studied parameters in graph theory, and by convention, the term *coloring* of a graph is usually used instead of proper coloring. In 1973, Grünbaum [10] considered proper colorings with the additional constraint that the subgraph induced by every pair of color classes is *acyclic*, i.e., contains no cycles. He called such colorings *acyclic colorings*, and the minimum r such that G has an acyclic r-coloring is called the *acyclic chromatic number* of G, denoted by $a(G)$. In introducing the notion of an acyclic coloring, Grünbaum noted that the condition that the union of any two color classes induce a forest can be generalized to other bipartite graphs. Among other problems, he suggested requiring that the union of any two color classes induce a star forest, i.e., a proper coloring avoiding 2-colored paths with four vertices. We call such a coloring a *star coloring*. Star colorings have recently been investigated by Fertin, Raspaud and Reed [8], and Nešetřil and Ossona de Mendez [15].

In this paper we bound the minimum number of colors used in a star coloring when the graph is restricted to certain natural classes. In particular, we prove that planar graphs can be star colored with 20 colors, and we give analogous results for graphs embedded in arbitrary surfaces.

We begin by collecting some basic definitions and observations in Section 2. In Section 3 we define the central notion of an *in-coloring*. We use this concept, which is equivalent to a star coloring, in most of our proofs. For example, it leads to a simple proof of the fact that every graph of maximum degree Δ can be star colored with $\Delta(\Delta - 1) + 2$ colors. When $\Delta = 3$, we improve this to 7.

In Section 4, we investigate the connection between acyclic colorings and star colorings further. We define a refinement of acyclic colorings that allows us to improve the bound on the star chromatic number for planar graphs to 20. There are stronger results for planar graphs with large girth, and similar results for graphs embedded in an arbitrary surface in Section 5.

In Section 6, we bound the star chromatic number in terms of tree-width by showing that chordal graphs with clique number ω have star colorings using $\binom{\omega+1}{2}$ colors. This implies that outerplanar graphs have star colorings with at most 6 colors. We construct an example to show that these results are best possible and to obtain a planar graph with star chromatic number 10.

We conclude the paper by investigating the complexity of star coloring in Section 7. We show that even if G is planar and bipartite, the problem of deciding whether G has a star coloring with 3 colors is NP-complete. In Section 8, we collect some open questions for future investigation.

2 Definitions and preliminaries

Suppose $\mathcal{F}$ is a nonempty family of connected bipartite graphs, each with at least 3 vertices. An r-coloring of a graph G is said to be *$\mathcal{F}$-free* if G contains no 2-colored subgraph isomorphic to any graph F in $\mathcal{F}$. These $\mathcal{F}$-free colorings are a natural generalization of acyclic colorings: if $\mathcal{F}$ consists of all even cycles, then a coloring is $\mathcal{F}$-free if and only if it is acyclic. We denote the minimum number of colors in an $\mathcal{F}$-free coloring of G by $\chi_{\mathcal{F}}(G)$. If the family $\mathcal{F}$ consists of a single graph F, then we use $\chi_F(G)$. In this notation, if $\mathcal{F}$ is the family of all even cycles, then $\chi_{\mathcal{F}}(G) = a(G)$.

In this paper, we concentrate on the case when $\mathcal{F} = \{P_4\}$, the path on 4 vertices. Recall that a *star* is a graph isomorphic to $K_{1,t}$ for some $t \geq 0$ and a graph all of whose components are stars is called a *star-forest*. In a proper coloring that avoids a 2-colored P_4, the union of any two color classes cannot induce a cycle since every even cycle contains P_4 as a subgraph. Hence the union induces a star-forest (every component must be a star, since otherwise it would contain a 2-colored P_4). We will use the following terminology.

Definition 2.1. An r-coloring of G is called a *star coloring* if there are no 2-colored paths on 4 vertices. The minimum r such that G has a star coloring using r colors is called the *star chromatic number* of G and is denoted by $\chi_{P_4}(G)$ or $\chi_s(G)$.

Observe that if H is a subgraph of F, then an H-free coloring of G is certainly an F-free coloring of G, i.e., $\chi_F(G) \leq \chi_H(G)$. Every member of the family of bipartite graphs $\mathcal{F}$ has a 3 vertex path as a subgraph, hence we can deduce the following proposition.

Proposition 2.2. $\chi_{\mathcal{F}}(G) \leq \chi_{P_3}(G) = \chi(G^2) \leq \min\{\Delta(G)^2 + 1, n\}$.

Proof. The second inequality follows from the observation that a coloring in which each bicolored path has at most two vertices can be obtained by coloring every pair of vertices that are at a distance two apart with distinct colors. The graph G^2 is obtained from G by inserting edges between any two vertices whose distance in G is two, and the bound follows since the chromatic number is always at most the maximum degree plus 1. □

The last inequality above can be exact (*e.g.*, C_5), but for families of graphs that have unbounded maximum degree (such as planar graphs), Proposition 2.2 provides no useful bound on the star chromatic number.

If the family $\mathcal{F}$ does not contain a star, then every graph in $\mathcal{F}$ has P_4 as a subgraph, so $\chi_{\mathcal{F}} \leq \chi_s$. Thus, for such a family, a bound on the star chromatic number also bounds $\chi_{\mathcal{F}}$. On the other hand, suppose that the family $\mathcal{F}$ contains $K_{1,t}$, and we consider $\mathcal{F}$-free colorings of planar graphs. Since a planar graph may contain an arbitrarily large star and every k-coloring of $K_{1,tk}$ contains a 2-colored $K_{1,t}$, we conclude that $\chi_{\mathcal{F}}$ cannot be bounded by an absolute constant. This suggests that the star chromatic number is the most interesting parameter to study, since it bounds $\chi_{\mathcal{F}}$ for all well-behaved choices of $\mathcal{F}$ for interesting families such as planar graphs. In Section 4, we will show that χ_s is bounded above by 20 for all planar graphs.

For some of our results, we use the more general language of *list-colorings*. A list-coloring of a graph G is a proper coloring where the colors come from lists assigned at each vertex. The *list-chromatic number* of G is the minimum size of lists that can be assigned to the vertices so that G can always be colored from them. Clearly, the list-chromatic number is always at least the chromatic number. We may also consider star colorings in which each vertex receives a color from its assigned list. The smallest list size that guarantees the existence of such a coloring of a graph is its *star list-chromatic number*.

3 Orientations and star colorings

It is convenient to define the following digraph coloring notion that is equivalent to star coloring.

Definition 3.1. A proper coloring of an orientation of a graph G is called an *in-coloring* if for every 2-colored P_3 in G, the edges are directed towards the middle vertex. We will call such a P_3 an *in-*P_3. A coloring of G is an in-coloring if it is an in-coloring of some orientation of G. A *list in-coloring* of G is an in-coloring of G where the colors are chosen from the lists assigned to each vertex.

Nešetřil and Ossona de Mendez [15] consider a very similar idea that they define in terms of a derived graph. We prove the following lemma, which corresponds to their Corollary 3.

Lemma 3.2. *A coloring of a graph G is a star coloring if and only if it is an in-coloring of some orientation of G.*

Proof. Given a star coloring, we can form an orientation by directing the edges towards the center of the star in each star-forest corresponding to the union of two color classes.

Conversely consider an in-coloring of $\vec{G}$, an orientation of G. Let $uvwz$ be some P_4 in G. We may assume the edge vw is directed towards w in $\vec{G}$. For the given coloring to be an in-coloring at v, we must have three different colors on u, v, w. □

Thus $\chi_s(G)$ is the minimum number of colors used in an in-coloring of any orientation of G. If we restrict our attention to acyclic orientations, we can use Lemma 3.2 to improve the degree bound.

Theorem 3.3. *Let G be a graph with maximum degree Δ. If G has an acyclic orientation with maximum indegree k, then $\chi_s(G) \leq k\Delta + 1$.*

Proof. Let $\vec{G}$ denote the acyclic orientation of G, and let $v_1, v_2, \ldots, v_n$ be an acyclic ordering of the vertices obtained by iteratively deleting the vertices of indegree zero. Thus in $\vec{G}$ all edges are directed from the vertex of smaller index towards the vertex of larger index. Now greedily color $v_1, \ldots, v_n$ as follows: to color v_i select a color from its list that is not used on any vertex v_j where $j < i$ and the distance between v_i and v_j in G is at most two. This ensures that adjacent vertices receive different colors, and that there is no 2-colored P_3 in $\vec{G}$ in which the middle vertex has outdegree 1 or 2. Since each vertex has at most k colored neighbors when it is colored, and these in turn have at most $\Delta - 1$ other neighbors each, the greedy coloring can be completed from the assigned lists. □

The theorem gives a slightly better bound on χ_s in terms of the maximum degree.

Corollary 3.4. $\chi_s(G) \leq \Delta(\Delta-1)+2$. *Equality can occur only if some component of G is Δ-regular.*

Proof. We may assume that G is connected, and that T is a spanning tree in G. If G has a vertex v of degree less than Δ, then orient all edges in T towards v and extend this to an acyclic orientation in the natural way. The result now follows from Theorem 3.3.

If G is Δ-regular, then remove one vertex w, color the remaining graph using $\Delta(\Delta-1)+1$ colors and assign a new color to w. □

Although the bound in Corollary 3.4 is sharp (for example, for C_5), it is not asymptotically optimal: Fertin, Raspaud and Reed [8] claim to have a proof along the lines of Alon, McDiarmid and Reed [4] that $O(\Delta^{3/2})$ colors are sufficient and $\Omega(\Delta^{3/2}/\log\Delta)$ colors may be necessary in a star coloring of a graph of maximum degree Δ.

For cubic graphs Corollary 3.4 yields a bound of 8; however this can be improved to 7 by the following theorem. Note that the Möbius ladder M_8 obtained by adding edges between antipodal vertices of an 8-cycle has $\chi_s(M_8)=6$.

Theorem 3.5. *If G has maximum degree at most 3, then G can be star colored from lists of size 7.*

Proof. We will prove by induction on n that some orientation of G can always be in-colored from lists of size 7. If G is small we may color each vertex with its own color. We may assume that G is connected, else the components may be colored separately. If G is not cubic, then we remove a vertex, say x, of degree less than three and inductively color the smaller graph. Since x has at most six vertices in its first and second neighborhoods, we may color it with a different color and orient its incident edges towards x.

Thus we assume G is connected and cubic. Suppose that C is a minimal cycle in G, given by $C = < u_1, u_2, \ldots, u_t >$. Let $G' = G - C$. For $1 \leq j \leq t$, let v_j denote the neighbor of u_j in G'. Let c be an in-coloring of G'.

Orient all the edges between G' and C so that they are directed into C, and orient the edges on C so that they point from the smaller index to the larger index, except for edge u_4u_3, which is oriented in the opposite direction in the case when $t \geq 4$. Thus C has sinks at u_3 and u_t (if $t \geq 5$) and sources at u_1 and u_4 (if $t \geq 5$). We will now extend c to obtain an in-coloring of this orientation of G. This can be easily done when $t=3$ by coloring the vertices in decreasing order: u_3, u_2, u_1 (in fact, each vertex only has to avoid the colors of 5 vertices from its first and second neighborhoods).

For $t \geq 4$ we color the vertices in decreasing order, except that we color u_3 before u_4: $u_t, u_{t-1}, \ldots, u_5, u_3, u_4, u_2, u_1$. At each step, we claim that we can choose a color from the list of the vertex we are coloring to ensure that every 2-colored P_3 points towards the center vertex.

Each u_i loses potentially three colors from its list because of its neighbor v_i in G' and because of v_i's two other neighbors. So we may assume that the lists are of size 4 and we need to consider P_3's that are formed using at least two vertices on C.

To color u_t, we need only avoid the colors appearing on v_{t-1} and v_1 since $u_t u_{t-1} v_{t-1}$ and $u_t u_1 v_1$ are not in-P_3's. This leaves us a choice of 2 colors for u_t. To color u_{t-1}, we must avoid the color given to u_t in addition to the color of v_{t-2}, however we need not consider the color of v_t since $u_{t-1} u_t v_t$ is an in-P_3. When we need to choose a color for u_i (for i between $t-2$ and 5), we must avoid the color on u_{i+1}, u_{i+2} and v_{i-1}. Note that we need not consider the colors on u_{i-1} and v_{i+1} since u_{i-1} has not been colored yet, and $u_i u_{i+1} v_{i+1}$ is an in-P_3. Since the list has size 7, there is a color remaining.

When we choose a color for u_3, both its neighbors on C are uncolored. Our choices are thus constrained by the colors on v_4, v_2 and u_5. Again, there is a color remaining in the list. For u_4 now, the colors on u_5 and u_3 are both excluded, but $u_4 u_5 v_5$ and $u_4 u_3 v_3$ are in-P_3's, so the only other color that is potentially lost is the one of u_6.

In the penultimate step we choose a color on u_2 that is not the color on u_3 or u_t or v_1. Note that $u_2 u_3 u_4$ and $u_2 u_3 v_3$ are in-P_3's. Finally to color u_1, we observe that every relevant P_3 which ends in u_1, except for $u_1 u_2 u_3$, is an in-P_3. Hence at most 3 colors are excluded from the list: those of u_2, u_3 and u_t. Since there is a color remaining in the list, the coloring can be completed. □

4 Refining acyclic colorings

Recall that an acyclic coloring of a graph is a proper coloring with no 2-colored cycles. In his paper introducing acyclic colorings Grünbaum showed that planar graphs are acyclically 9-colorable [10]. There was a brief flurry of activity [14, 1, 12, 13, 2] culminating in Borodin's substantial accomplishment that planar graphs are acyclically 5-colorable [5]. Already in his paper, Grünbaum noted (without proof) that bounding the acyclic chromatic number bounds the star chromatic number. We state the result, a proof of which was given by Fertin, Raspaud and Reed [8].

Theorem 4.1. $a(G) \le \chi_s(G) \le a(G) 2^{a(G)-1}$.

For planar graphs, this gives a bound of 80 on the star chromatic number. Nešetřil and Ossona de Mendez [15] improved this to 30 by using an argument similar to our notion of in-coloring. To improve the bound further, we refine acyclic colorings to exploit the local structure.

Definition 4.2. Let F be a star forest in G with bipartition X, Y such that X consists of all centers in F. The *F-reduction* of G is obtained by considering the bipartite subgraph induced by the X, Y-cut in G, contracting all edges in F and removing any loops or multiple edges formed.

Note that the graph induced by the X, Y-cut contains F as a (usually proper) subgraph, so the F-reduction is well defined and X can be viewed as its vertex set. We illustrate this with an example in which vertices in X are denoted by $\otimes$ and those in Y by $\bullet$. Edges not in the cut are denoted by dotted lines and edges in F with double lines.

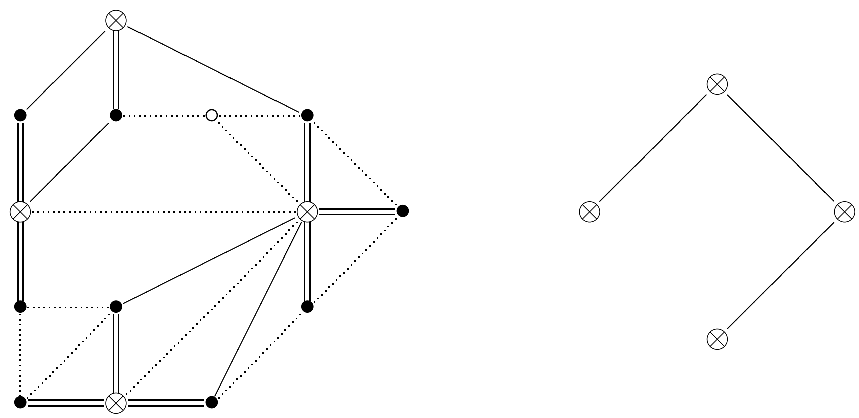

Figure 1: Graph and F-reduction

Theorem 4.3. *If every F-reduction of G is k-colorable, then every acyclic r-coloring of G can be refined to a star coloring of G with at most rk colors.*

Proof. Consider an acyclic coloring of G with r colors, i.e., every pair of color classes induces a forest. We will orient G according to the coloring: in each component of the forest induced by two color classes, pick a root and orient the edges towards this root. Observe that in every 2-colored P_3 of this coloring at least one edge is directed towards the middle vertex. We will now refine this coloring to obtain an in-coloring. Consider the i-th color class X_i in the acyclic coloring. Let F_i be the subgraph of G that consists of all edges that point into X_i. By the observation above F_i is a star forest. By hypothesis, the F_i-reduction of G is k-colorable, and we refine the colors on the vertices in X_i accordingly. This results in a coloring of G with rk colors. This coloring must be an in-coloring, since two vertices in X_i are connected by a directed P_3 precisely if they are adjacent in the F_i-reduction of G. □

Theorem 4.3 allows us to improve the current bound of 30 given by Nešetřil and Ossona de Mendez [15] for the star chromatic number for planar graphs.

Corollary 4.4. *If G is planar, then $\chi_s(G) \leq 20$.*

Proof. Planar graphs are 4-colorable, acyclically 5-colorable, and closed under taking minors (and thus, F-reductions). It follows that $\chi_s(G) \leq a(G)k \leq 5 \cdot 4 = 20$. □

Using other results from acyclic coloring, we also improve bounds on the star chromatic number for planar graphs of girth at least 5 and 7 mentioned in [6].

Corollary 4.5. *If G is a planar graph of girth at least 7, then $\chi_s(G) \leq 9$. If G is a planar graph of girth at least 5, then $\chi_s(G) \leq 16$.*

Proof. Borodin, Kostochka and Woodall [6] have shown that if the girth of a planar graph is at least 7, then $a(G) \leq 3$. Furthermore every F-reduction of a graph of girth g has girth at least $g/2$. Thus every F-reduction of G is planar and triangle-free and consequently 3-colorable by Grötzsch's theorem [9]. If the girth of G is at least 5, then $a(G) \leq 4$ (again, see [6]), so the second bound follows from the Four Color Theorem. □

Closer examination of the coloring in Theorem 4.3 also leads to an improvement of the bound in Theorem 4.1.

Corollary 4.6. *For any graph G, $\chi_s(G) \leq a(G)(2a(G) - 1)$.*

Proof. In the orientation of G produced in the proof of Theorem 4.3, a vertex in X_i has outdegree at most $a(G) - 1$, hence the F_i-reduction of G has maximum degree $2a(G) - 2$ and is thus $(2a(G) - 1)$-colorable. □

We show in Section 6 that this bound is optimal up to a factor of about 4.

Remark 4.7. *Theorem 4.3 can be strengthened by replacing acyclic coloring by the slightly weaker notion of a weakly acyclic coloring defined in [11]. A* weakly acyclic coloring *is a proper coloring such that every connected 2-colored set of vertices contains at most one cycle (as opposed to none). In other words it is an $\mathcal{F}$-free coloring, where $\mathcal{F}$ is the family of all connected bipartite graphs with more edges than vertices. The proof is identical, except that in a unicyclic 2-colored component the cycle is oriented cyclically and all other edges are oriented towards the cycle.*

5 Graphs on higher surfaces

How low can we push χ_s if we allow for a sufficiently high girth? Since there are graphs of arbitrarily high girth and high chromatic number we obviously need additional constraints, such as an embedding on a surface. The following lemma is part of the folklore.

Lemma 5.1. *For every surface S there is a girth γ such that the vertex set of every graph of girth at least γ embedded in S can be partitioned into a forest and an independent set I such that the distance (in G) between any two vertices in I is at least 3.*

Corollary 5.2. *For every surface S there is a constant γ such that every graph G of girth at least γ embedded in S has $\chi_s(G) \leq 4$.*

Proof. Star color the forest with 3 colors (see, *e.g.*, Theorem 6.1) and use the fourth color on the independent set. □

The following example shows that this result is best possible.

Example 5.3. Consider the planar graph G obtained by adding a pendant vertex to every vertex of a cycle on n vertices, C_n, where n is not divisible by 3. We show that $\chi_s(G) = 4$. To obtain such a 4-coloring, orient the cycle cyclically and in-color it with 4 colors. Then orient the remaining edges towards the cycle and color the pendant vertices with the color of the predecessor of their neighbor on the cycle.

Now assume that there was a 3-in-coloring of G. Since n is not divisible by 3 the cycle cannot be cyclically oriented, since this would force it to be colored cyclically $(1, 2, 3, 1, 2, \ldots)$. Thus some vertex v on the cycle has outdegree 2. We may assume that v has color 1 and its neighbors on the cycle have colors 2 and 3. But then no matter what the color of the vertex pendant to v is we get a 2-colored P_3 with center vertex v of outdegree at least 1.

Our next result bounds χ_s for embedded graphs. For ease of exposition we state and prove the theorem for orientable surfaces.

Theorem 5.4. *If G is embedded on a surface of genus g, then $\chi_s(G) \leq 20 + 5g$.*

Proof. The proof uses induction on g; the base case is given by Corollary 4.4. For the inductive step consider a graph embedded on a surface of genus $g+1$. Let C be a shortest non-contractible cycle in G. Now $G - C$ consists of one graph (or perhaps two graphs) which can be embedded in a surface of genus g (or perhaps two such surfaces). By the inductive hypothesis $G - C$ can be star colored with $20 + 5g$ colors. Next color the square of C, C^2, using at most 5 new colors (Proposition 2.2). We claim that these colorings combine to form a star coloring of G. A potential 2-colored P_4 must contain two vertices from C, say u and w with the same color. Now the vertex v between u, w on P_4 is not in C, but since u, w are at distance at least 3 on C the path uvw together with one of the u, w-segments of C yields a shorter non-contractible cycle. □

We suspect that the bound in the preceding theorem is far from tight. It is, however, superior to the bound we get from Theorem 4.3. Suppose G is embedded on a surface of genus g. Alon, Mohar, and Sanders [3] have shown that $a(G) = O(g^{4/7})$ and this is nearly best possible since there are graphs with $a(G) = \Omega(g^{4/7}/\log^{1/7} g)$. Together with Heawood's bound $\chi(G) = O(g^{1/2})$ this only yields a bound of $O(g^{15/14})$ for χ_s.

6 Tree-Width and a construction

In this section, we use the tree-width of a graph to bound χ_s. The tree-width of a graph is a measure of how tree-like the graph is. Tree-width was introduced by Robertson and Seymour and is a fundamental parameter both for the study of minors and the development of algorithms. For an introduction to this topic see Diestel [7].

Fertin, Raspaud, and Reed [8] proved the following result for graphs with bounded tree-width.

Theorem 6.1. *If G has tree-width t, then G has a star coloring from lists of size $\binom{t+2}{2}$.*

Their proof uses the structure of k-trees. We give a slightly simpler proof below, using the notion of chordal graphs.

Definition 6.2. A graph without chordless (i.e., induced) cycles of length at least 4 is called *chordal*. The *clique number* of a graph G, denoted by $\omega(G)$, is the order of a largest complete subgraph of G.

It is well-known (see, *e.g.*, [7, Cor 12.3.9]) that the tree-width of a graph G can be expressed as

$$\min\{\omega(H) - 1 : E(G) \subset E(H); H \text{ chordal}\}.$$

We also use that a chordal graph has a *perfect elimination ordering* $v_1, \ldots, v_n$ of its vertices; for each vertex v_i, its neighbors with index larger than i form a complete graph.

Proof of Theorem 6.1. It suffices to prove that every chordal graph G with $\omega(G) = t$ has a star coloring from lists of size $\binom{t+1}{2}$. Let $v_1, \ldots, v_n$ be a perfect elimination ordering of G. Orient all edges to point from the earlier to the later vertex in this ordering. Now color the vertices, from last to first, by choosing a color for every vertex that appears neither in its first nor second out-neighborhood.

We first need to show that we have enough colors in every list. Let v be given. The first out-neighborhood $N = N^+(v)$ has size at most $t-1$, since $\{v\} \cup N$ forms a clique. The vertices in N are linearly ordered, so that the first vertex can have at most one out-neighbor outside of N, and so on. Altogether the first and second out-neighborhoods contain at most $(t-1) + 1 + 2 + \cdots + (t-1) = \binom{t+1}{2} - 1$ vertices.

The coloring obtained is clearly proper, but it remains to be seen that the coloring is an in-coloring. Let $v_i v_j v_k$ be any P_3 and assume $i < k$. If $i < j < k$, then v_i receives a color different from v_k. If $j < i < k$, then it follows from the elimination ordering that v_i and v_k are adjacent, and again receive different colors. Thus every 2-colored P_3 is an in-P_3. □

The next construction shows that Theorem 6.1 is best possible:

Theorem 6.3. *There is a sequence of chordal graphs $G_1, G_2, G_3, \ldots$ such that $\omega(G_t) = t$ and $\chi_s(G_t) = \binom{t+1}{2}$. Moreover, G_3 is outerplanar and G_4 is planar.*

Proof. We give a recursive construction with base cases $G_1 = K_1$ and $G_2 = P_4$. Let $t \geq 3$ and G_{t-1} be a chordal graph with $\omega(G_{t-1}) = t-1$ and $\chi_s(G_{t-1}) = \binom{t}{2}$. Let P be a path with vertices denoted $u_1, \ldots, u_n$, for $n = 2\binom{t}{2} + 2$. Make every u_i adjacent to every vertex of a clique with vertices $v_1, \ldots, v_{t-2}$. Take n copies of G_{t-1}, say $H_1, \ldots, H_n$ and for $1 \leq j \leq n$ add edges joining u_j with every vertex in H_j. Call the resulting graph G_t. It is easy to check that G_t is chordal and has clique number t, so that $\chi_s(G_t) \leq \binom{t+1}{2}$. Furthermore, G_3 is outerplanar and G_4 is planar.

Suppose that c is a star coloring of G_t with at most $\binom{t+1}{2} - 1$ colors and without loss of generality every v_i has color i. Call a vertex of P *redundant* if it has the same color as another vertex of P. At most $\binom{t+1}{2} - 1 - (t-2) = n/2$ colors can appear on P, so that there must be adjacent redundant vertices u_j and u_{j+1} on P. We may assume that $c(u_j) = t-1$ and $c(u_{j+1}) = t$. Since u_j and u_{j+1} are redundant, colors 1 through $t-2$ are not used on H_j and H_{j+1}. If some vertex in H_j were colored t and some vertex in H_{j+1} were colored $t-1$, then there would be a 2-colored P_4. Consequently we may assume that neither $t-1$ nor t appears as a color in H_j. Since $\chi_s(H) = \binom{t}{2}$ there must be at least $t + \binom{t}{2} = \binom{t+1}{2}$ colors used on G_t, a contradiction. □

Observe that since G_t is chordal we obtain $a(G_t) = \chi(G_t) = \omega(G_t) = t$ so that Corollary 4.6 is optimal within a factor of 4.

Theorem 6.1 and 6.3 also imply the following result obtained in [8], since outerplanar graphs have tree-width 2:

Corollary 6.4. *If G is outerplanar, then $\chi_s(G) \leq 6$ and this is best-possible.*

Observe that G_3 has 41 vertices, but we have a slightly more clever construction with 19 vertices. It is also worth pointing out that G_4 is the planar graph with the highest known star chromatic number.

Corollary 6.5. *There is a planar graph $G = G_4$ such that $\chi_s(G) = 10$.*

7 Complexity

Given a graph G, construct the *gadget* graph $G_{(t)}$ by replacing every edge uv of G with a separate copy of $K_{2,t}$ in such a fashion that u and v are identified with the vertices in the part of size 2. Note that $G_{(t)}$ is bipartite, and that every embedding of G in some surface extends to an embedding of $G_{(t)}$ in a natural fashion.

Lemma 7.1. *Let G be any graph, and $t \geq k \geq 3$. Then $\chi(G) \leq k$ if and only if $\chi_s(G_{(t)}) \leq k$. Thus when $k \geq 4$, equality holds in the first inequality if and only if it holds in the second.*

Proof. If there is a k-coloring of G, then it can be extended to a star coloring of $G_{(t)}$ on k colors by giving every vertex in the larger partite set of a $K_{2,t}$ a color different from that of its two (precolored) neighbors. Now suppose that $G_{(t)}$ has a star coloring with k colors. We claim that if u, v are adjacent vertices in G, then they must get different colors in $G_{(t)}$, so that the k-coloring of $G_{(t)}$ yields a k-coloring of G. Indeed, if the vertices in the partite set of size 2 in $K_{2,t}$ have the same color, then the only way to extend this to a star coloring is by giving all vertices in the set of size t a different color, resulting in at least $t + 1 > k$ different colors being used. □

The proof can be adjusted to show that G can be colored from lists of size k if and only if $G_{(t)}$ can be star colored from lists of size k. Thus we can easily build bipartite graphs in which the star chromatic and star list chromatic number can differ arbitrarily much.

A graph can be 2-colored such that no P_4 is 2-colored if and only if it is a star-forest. Lemma 7.1 shows that most other star coloring problems are NP-complete.

Theorem 7.2. *The problem of deciding if a planar bipartite graph has a star coloring with at most 3 colors is NP-complete.*

Proof. The problem is clearly in NP. To show that it is NP-complete it will suffice to reduce PLANAR-3-COLORABILITY to the problem. So let G be an instance of PLANAR-3-COLORABILITY. Observe that $G_{(3)}$ can be constructed from G in polynomial time, $G_{(3)}$ is planar and bipartite, and $\chi_s(G_{(3)}) \leq 3$ if and only if $\chi(G) \leq 3$. □

Corollary 7.3. *Given a graph G, it is NP-complete to decide if $\chi(G) = \chi_s(G)$ even if G is a planar graph with $\chi(G) = 3$.*

Proof. Add a disjoint triangle to $G_{(3)}$ in the transformation above. □

Theorem 7.4. *For $2 \leq t \leq k$ and $k > 2$, given a graph G with $\chi(G) = t$, it is NP-complete to decide if $\chi_s(G) \leq k$.*

Proof. We know that k-COLORABILITY is NP-complete. But by Lemma 7.1 we also see that k-COLORABILITY can be transformed into STAR k-COLORABILITY of bipartite graphs ($t = 2$). To obtain the case for $3 \leq t \leq k$ simply add in a disjoint K_t. □

8 Questions

We conclude with a number of questions.

1. What is the smallest integer r such that if G is planar, then $\chi_s(G) \leq r$? We know that $10 \leq r \leq 20$, but have no compelling reason to think that either of our bounds is correct. We also do not know the best bound if we restrict the class of planar graphs to those of girth, say 5.

2. Suppose G is planar. Nešetřil and Ossona de Mendez [15] have shown that if G is bipartite, then $\chi_s(G) \leq 18$. Recently we have been able to improve this bound to 14. Can this be improved further?

3. What is the smallest t such that if G is embedded on S_g, then $\chi_s(G) = O(g^t)$? We know that $4/7 \leq t \leq 1$ (see [3] and Theorem 5.4).

4. Suppose G is embedded on S_g. The *width* of G, denoted by $w(G)$, is the length of a shortest non-contractible cycle in G. If $w(G) \geq 100 \cdot 2^g$, is $\chi_s(G) \leq 25$?

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