Ref.TH.2150-CERN

COLOURED SUPERSYMMETRY *)

R. Casalbuoni **), G. Domokos +) and S. Kövesi - Domokos +)
CERN - Geneva

ABSTRACT

We construct a supersymmetry algebra which satisfies the triality rules governing the multiplication of "coloured" quark variables. Quarks and leptons are treated on an equal footing. The affine structure imposed on the unified space of fermionic and Minkowski coordinates forces the introduction of separate leptonic and hadronic (quark) variables. Some basic properties of dynamical superstructures based on the coloured superspace are discussed. It is pointed out, in particular, that supersymmetric theories based on coloured superspace allow for an interpretation of superfield components in terms of lepton and quark fields; the theory permits separate conservation of lepton and baryon numbers.

---

*) Research supported in part by the U.S. Energy Research and Development Administration under Contract No. E(11-1)3285. Report No. COO 3285-28.

**) On leave from INFN, Sezione di Firenze, Florence, Italy.

+) On leave from Johns Hopkins University, Baltimore, MD 21218, U.S.A.

Ref.TH.2150-CERN
31 March 1976

1. INTRODUCTION.

Color symmetry is distinguished by the fact that all physically observable states are apparently color singlets. In this respect color is different from other known internal and space-time symmetries; its rôle resembles rather the one played by the symmetric group in physics (Pauli principle).

It is plausible to assume therefore that - similarly to the Pauli principle - the color properties of observed hadrons may be expressed through algebraic relations obeyed by the dynamical variables describing hadronic constituents. One may also expect that the apparent unobservability of hadronic constituents is a consequence of such algebraic relations rather than of some peculiar dynamical mechanism. This question, however, has not been answered in a satisfactory way until now[1].

Gürsey and his collaborators have pioneered such an algebraic approach to the description of color[2,3,4]. They observed that algebras belonging to a certain class (Cayley algebras and some of their generalizations) possess automorphism groups containing SU(3) (identifiable with the color group) as a subgroup. Moreover, the multiplication tables of those algebras exhibit triality properties which are isomorphic to those required by the multiplication rules of quarks[3]. An analysis of the automorphism groups of the relevant algebras can be found in Freudenthal's paper[5]; the possible physical significance of those group structures has been elucidated in ref.6.

Gürsey et al. explicitly assume that algebras of the Cayley type serve as a model for charge space properties of quarks and leptons. However, it would be desirable to achieve a unification of charge and space-time properties of quarks and leptons at a deeper level[7].

In this paper we report the first results of an investigation into such a possible unification. In particular, we propose that the unification can be achieved within the framework of supersymmetric theories[8]. In order to realise this idea, we construct a class of "supersymmetry" algebras which satisfy a number of physically reasonable criteria (discussed in Sec.3). The algebras so constructed admit a treatement of leptons and quarks on an equal footing, along the lines proposed by Gürsey[6]. However, we find that as soon as the quark-lepton algebra is unified with the Abelian algebra of space-time coordinates in a geometrical framework, the quark and lepton parts have to be dealt with separately.

This is a desirable feature, for it suggests that theories to be con-

structed along the lines proposed here have an inherent possibility of distinguishing between quarks and leptons on a geometrical basis.

The next Section serves primarily a paedagogical purpose. We introduce and analyse (at a very elementary level) the algebraic mechanism which endows the algebras investigated by Gürsey et al. with the desirable triality properties. Besides summarizing the necessary mathematical apparatus, we also gain some insight into the possibilities of generalizing the algebraic framework, so as to suit the physical requirements.This is carried out in Sec.3, while the basic geometrical aspects of the problem - in particular, the question of constructing superspaces - are dealt with in Sec.4.

The last Section contains comments and remarks concerning the possibility of developing the scheme described here into a dynamical theory.

The investigations described in this paper are confined to the algebraic and geometric aspects of the problem under consideration. The question of constructing dynamical models within this framework will be discussed in subsequent publications.

2. THE ALGEBRAIC ORIGIN OF TRIALITY: THE CAYLEY-DIXON CONSTRUCTION

The Cayley-Dixon (CD) process is devised to systematically enlarge algebras endowed with certain properties. The prescription itself is a simple one.

Assume that $\mathcal{A}$ is an algebra with involution[9] and let $a_i \in \mathcal{A}$ $(i=1,2,\ldots)$. We enlarge $\mathcal{A}$ by introducing a new abstract element, say v, such that $v^2 = \mu$ is a number (element of the ground field). We then consider elements of the form $(a_1 + v a_2)$ as elements of the "large" algebra, $\mathcal{B}$. Multiplication in $\mathcal{B}$ is defined by

$$(a_1+va_2)(a_3+va_4)=(a_1a_3+\mu a_4\bar{a}_2)+v(\bar{a}_1a_4+a_3a_2), \qquad (2.1)$$

where $\bar{a}_i$ is the conjugate of a_i in $\mathcal{A}$.

Remark. The CD process obviously generalizes the process of "complexification" of real numbers. Indeed, if a_i are real numbers ($\bar{a}_i = a_i$) and $v^2 = -1$, then the algebra $\mathcal{B}$ with multiplication (2.1) is the algebra of complex numbers.

In abstract algebra, the CD process is used to generate Hurwitz algebras which are interesting from an algebraic point of view; their properties are described in standard textbooks[10]. From our point of view, however, the CD process is interesting for a different reason. In order to understand this, we study the simplest model considered by Gürsey et al.[2],

based on the split octonion algebra.

We start from the quaternion algebra and we write its multiplication table in terms of standard basis elements e_i, (i=1,2,3):

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \,, \tag{2.2}$$

where ϵ_{ijk} is the Levi-Civitá tensor and the summation convention is used. The multiplication table is invariant under the involution $\bar{e}_i = -e_i$, $\overline{(e_i e_j)} = \bar{e}_j \bar{e}_i$. Moreover, the multiplication table (2.2) is invariant under a change of basis by means of a proper orthogonal transformation,

$$e'_i = O_{ij} e_j \,, \quad O_{ij} O_{kj} = \delta_{ik} \,, \quad \mathrm{Det}\, O_{ij} = 1 \,. \tag{2.3}$$

(Actually, the transformations (2.3) span the full automorphism group of the algebra (2.2).)

We now enlarge the quaternion algebra (2.2) by introducing the new "imaginary unit" v, $v^2 = 1$.

One verifies that the elements

$$\begin{aligned} u_i &= \tfrac{1}{2}(e_i + v e_i) \,, & u_i^* &= \tfrac{1}{2}(e_i - v e_i) \,, \\ u_o &= \tfrac{1}{2}(1 + v) \,, & u_o^* &= \tfrac{1}{2}(1 - v) \,, \end{aligned} \tag{2.4}$$

span a basis of the enlarged algebra[10]. The multiplication rules (2.1) give e.g.

$$\begin{aligned} u_i u_j &= \tfrac{1}{4}(e_i + v e_i)(e_j + v e_j) = \tfrac{1}{4}(e_i e_j + v^2 e_j \bar{e}_i) + \tfrac{1}{4} v (\bar{e}_i e_j + e_j e_i) \\ &= \tfrac{1}{4}(e_i e_j - e_j e_i) - \tfrac{1}{4} v (e_i e_j - e_j e_i) = \epsilon_{ijk} \tfrac{1}{2}(e_k - v e_k) = \epsilon_{ijk} u_k^* \,, \end{aligned}$$

where (2.2) has been used. One can now similarly deduce the entire multiplication table of the basis elements (2.4). The multiplication rules are identical to the ones established by Günaydin and Gürsey for elements of the split octonion algebra[11]. For the sake of completeness we exhibit the full multiplication table below.

$$\begin{aligned} u_i u_j &= \epsilon_{ijk} u_k^* & u_i^* u_j^* &= \epsilon_{ijk} u_k \\ u_i u_j^* &= -\delta_{ij} u_o & u_i^* u_j &= -\delta_{ij} u_o^* \end{aligned}$$

$$
\begin{aligned}
&u_o u_i = u_i && u_o^* u_i^* = u_i^* \\
&u_i u_o = 0 && u_i^* u_o^* = 0 \\
&u_o^* u_i = 0 && u_o u_i^* = 0 \\
&u_i u_o^* = u_i && u_i^* u_o = u_i \\
&u_o u_o = u_o && u_o^* u_o^* = u_o^* \\
&u_o u_o^* = u_o^* u_o = 0 .
\end{aligned}
\tag{2.5}
$$

The multiplication table (2.5) is invariant under the group SU(3) , which, in a sense, may be regarded as a " complexification " of the automorphism group, SO(3) , of quaternions. The group SU(3) is identifiable with the color group, since the rules (2.5) exhibit triality. Indeed, $u_i \sim (3)$, $u_i^* \sim (\bar{3})$, $u_o \sim (1)$ under SU(3). The multiplication rules (2.5) give $(3)\times(\bar{3}) \sim (1)$, $(3)\times(3) \sim (\bar{3})$, which is just what is needed for a "colored" quark algebra[12]. (We remark in parentheses that (2.5) is actually invariant under a larger group - a "split" form of G(2) - of which SU(3) is a subgroup, see ref.3. However, we are primarily interested in those automorphisms of (2.5) which leave the element ν invariant.)

We now observe the two features of the CD process which are important for what follows.

1.) Enlargement of the automorphism group. In infinitesimal form, the automorphism transformations (2.3) of the quaternion algebra read:

$$\delta e_i = a_{ij} e_j \quad , \quad (a_{ij} = - a_{ji} \quad , \quad a_{ij} \text{ real}) .$$

The enlargement of (2.2) to (2.5) leads to an enlargement of the automorphism group, since

$$\delta (e_i \pm \nu e_i) = a_{ij} e_j \pm \nu a_{ij} e_j$$

remains an automorphism. In addition to transformations with real antihermitean matrices a_{ij} , however, the "large" algebra (2.5) also remains invariant under transformations with "pure imaginary" antihermitean matrices of the form νS_{ij} , where S_{ij} is a symmetric matrix of real elements, viz.

$$
\begin{aligned}
\delta (e_i \pm \nu e_i) &= \nu S_{ij} (e_j \pm \nu e_j) \\
&= \pm (S_{ij} e_j \pm \nu S_{ij} e_j)
\end{aligned}
$$

see eq.(2.1). One easily verifies that the matrices a_{ij} and νS_{ij} - with their action on $(e_i \pm \nu e_i)$ just defined - span the Lie algebra of SU(3), which

is then identified with the color group.

2.) The"triality rule" arises because the CD multiplication (2.2) selects the appropriate (symmetric or antisymmetric, respectively) parts of the quaternion product (2.2). Thus we can abstract the prescription by means of which algebras exhibiting triality can be constructed.

First, one starts with a "small" algebra which is a quaternion algebra; the coefficients of the quaternion units e_i , need not belong to a commutative field.

Next, one introduces a new element, υ , such that $\upsilon^2=1$ and constructs "complexified elements" in analogy with (2.4).

Finally, one defines a multiplication rule (the analogue of (2.1)), which selects the appropriately symmetrised or antisymmetrised quaternion products.

3. CONSTRUCTION OF A CHIRAL QUARK-LEPTON ALGEBRA.

We want to construct an algebra of quark and lepton "coordinates" (in the sense of supergauge theories) which exhibits correct triality properties for quarks. To this end, following the prescription given in the previous Section, we consider a general quaternion of the form:

$$q_\alpha = x_\alpha + e_a x^a_\alpha \tag{3.1}$$

where the e_a stands for quaternionic units with a multiplication rule given by (2.2). We want the quantities x^a_α to describe quark degrees of freedom, whereas the color singlet part, x_α is to be regarded as a leptonic coordinate[6,7]. (For the sake of brevity, we shall use the terms "quarks" and "leptons", respectively.)

It is evident therefore that x^a_α and x_α have to be elements of a Grassmann algebra,

$$\{ x^a_\alpha, x^b_\beta \} = \{ x_\alpha, x_\beta \} = 0. \tag{3.2}$$

However, there is no basic physical principle fixing the relative commutativity properties of x_α and x^a_α . The index set $\{\alpha\}$ consists of spinor and possibly of internal symmetry indices as well. For the time being, we leave the question of internal symmetries open; most of the subsequent considerations are carried out under the tacit assumption that α is just a (Weyl) spinor index.

In addition to q_α, we shall have to consider its quaternion conjugate, $\tilde{q}_\alpha = x_\alpha - e_a x^a_\alpha$, too. The usual spinor notation is used throughout[13]. The quantities $q_{r\alpha}$ have to span a representation of the Lorentz and internal symmetry groups. It follows that x_α and x^a_α have to transform according to the same representation of the Lorentz and internal symmetry groups; otherwise, there could not exist a Lorentz invariant (internal symmetry invariant, respectively) distinction between leptons and quarks. This is just the expression of the lepton-quark symmetry in this framework.

We now build up the algebra of the "complexified" quantities, $Q_\alpha = \frac{1}{2}(q_\alpha + v q_\alpha)$, with $v^2 = 1$.

We want i) to enforce the triality rule for the quark part,

ii) to exclude mixed (quark-lepton, diquark-lepton, etc) states, at least up to the three quark and quark-antiquark levels.

The CD prescription has to be modified for two reasons. First, the presence of conjugations in (2.1) does not allow one to define transformations of Q_α with complex matrices in any consistent way; in particular, Q_α cannot be considered a Weyl spinor. Second, the coefficients of the quaternion units are now elements of an anticommutative, rather than of a commutative algebra. For quaternions q_i (i=1,2,...) with anticommuting coefficients, and $v^2 = 1$ a suitable definition of the product is:

$$(q_1 + v q_2)(q_3 + v q_4) = (q_1 q_3 + q_4 q_2) + v(q_1 q_4 + q_3 q_2), \tag{3.3}$$

instead of (2.1).

Further, the relative commutativity properties of the coefficient algebra and of the quaternion units are to be left open.

We put:

$$\begin{aligned} x_\alpha e_\beta &= \gamma e_\beta x_\alpha, \\ x^a_\alpha e_\beta &= \delta e_\beta x^a_\alpha, \\ x_\alpha x^\beta_\gamma &= \epsilon x^\beta_\gamma x_\alpha, \end{aligned} \tag{3.4}$$

where γ, δ, ϵ are real, and $\gamma^2 = \delta^2 = \epsilon^2 = 1$. In addition, eqs.(2.2) and (3.2) are assumed to hold, of course.

We take now the product,

$$Q_\alpha Q_\beta = \tfrac{1}{4}\{q_\alpha, q_\beta\} + \tfrac{1}{4}\nu\{q_\alpha, q_\beta\}.$$

Using (3.4) and (3.2) together with (3.1) we find:

$$\{q_\alpha, q_\beta\} = (\gamma+\epsilon)\, e_\tau (x_\alpha x^\tau_\beta + x_\beta x^\tau_\alpha) + 2\epsilon\, \epsilon_{abc}\, e_\tau\, x^a_\alpha x^b_\beta .$$

The absence of quark-lepton compounds requires $\gamma+\epsilon=0$. (Dileptons are absent automatically, since $\{x_\alpha, x_\beta\}=0$.)

For the triple product we find:

$$(Q_\alpha Q_\beta) Q_\gamma = \tfrac{\epsilon}{2}\epsilon_{abc}\left[\tfrac{1}{2}(e_\tau e_c + \delta e_c e_\tau) x^a_\alpha x^b_\beta x^c_\gamma + (1+\gamma) x^a_\alpha x^b_\beta x_\gamma\right] + \nu[\ldots].$$

The three-quark system is a color singlet if $\delta=1$; the absence of a diquark-lepton system requires $\gamma=-1$. Hence, all coefficients in (3.4) are uniquely determined:

$$\gamma=-1\,, \quad \delta=1\,, \quad \epsilon=1. \tag{3.5}$$

We note in passing that the product (3.3) is associative up to the three-quark level (but not beyond that): $(Q_\alpha Q_\beta)Q_\gamma = Q_\alpha (Q_\beta Q_\gamma)$.

We now consider the left-handed part of the quark-antiquark system. We define

$$Q_{\dot\alpha} = \tfrac{1}{2}(q_{\dot\alpha} - \nu q_{\dot\alpha}), \tag{3.6}$$

where $q_{\dot\alpha} = x_{\dot\alpha} - e_a x^a_{\dot\alpha}$, $x_{\dot\alpha}$ and $x^a_{\dot\alpha}$ being spinor conjugates of x_α and x^a_α, respectively. (This means, in particular that Q_α and $Q_{\dot\alpha}$ are <u>conjugates</u> of each other. The conjugation so defined is an automorphism of the algebra generated by the Q_α and $Q_{\dot\alpha}$.) Evidently the conjugate spinors have to have the same commutation properties among themselves and with the quaternion units as the undotted spinors do. The products $Q_\alpha Q_{\dot\alpha}$ and $Q_{\dot\alpha} Q_\alpha$ are given by:

$$Q_\alpha Q_{\dot\alpha} = \tfrac{1}{4}[q_\alpha, q_{\dot\alpha}] - \tfrac{1}{4} v [q_\alpha, q_{\dot\alpha}] ,$$

$$Q_{\dot\alpha} Q_\alpha = \tfrac{1}{4}[q_{\dot\alpha}, q_\alpha] + \tfrac{1}{4} v [q_{\dot\alpha}, q_\alpha] .$$

On working out the product as illustrated on the example of the two- and three-quark products, one finds that lepton-quark cross terms are absent and the quark-antiquark system is a color singlet if and only if

$$\{X_\alpha, X_{\dot\beta}\} = \{X_\alpha, X^b_{\dot\beta}\} = \{X_{\dot\alpha}, X^b_\beta\} = 0 . \tag{3.7}$$

It is worth remarking that the very existence of an algebra which satisfies all the physical criteria listed above has not been evident <u>a priori</u>. Our construction shows not only that such an algebra <u>exists</u>, but <u>it is uniquely determined</u>.

The left-handed algebra spanned by the Q_α and $Q_{\dot\alpha}$ possesses no non-trivial bilinear invariant[14],

$$\epsilon^{\alpha\beta} Q_\alpha Q_\beta = \epsilon^{\dot\alpha\dot\beta} Q_{\dot\alpha} Q_{\dot\beta} = 0$$

in virtue of (3.3). However, there exists a <u>real</u> color singlet four-vector[15], <u>viz.</u>

$$[Q_\alpha, Q_{\dot\alpha}] = (X_\alpha X_{\dot\alpha} + X^a_\alpha X^a_{\dot\alpha}) .$$

In order to construct a non- trivial invariant (and also, in order to span a representation of the extended Lorentz group), a right-handed spinor has to be introduced, <u>viz.</u>

$$R^{\dot\alpha} = \tfrac{1}{2}(\tau^{\dot\alpha} + v\tau^{\dot\alpha}) , \quad R^\alpha = \tfrac{1}{2}(\tau^\alpha - v\tau^\alpha), \tag{3.8}$$

with $\tau^\alpha = y^\alpha - e_a y^{a\alpha}$. An invariant is then given by

$$[Q_\alpha, R^\alpha] = X_\alpha y^\alpha + X^a_\alpha y^{a\alpha} .$$

Finally, we point out that elements of the form (a+va) and (b-vb), (where a and b may carry any number of spinor indices) form subalgebras of the algebra generated by Q_α and $Q_{\dot\alpha}$.

4. COLORED SUPERSPACE.

We recall that the concept of a <u>superspace</u>, forming the basis of supergauge invariant theories, can be approached from a purely geometrical point of view[16].

For the sake of simplicity, we work in a flat space and with chiral spinors. Minkowski space possesses an affine structure. In order to achieve a unification of the spinorial and Minkowski spaces, an affine structure has to be defined on the unified space which is compatible with the affine connections defined on the subspaces.

There exists a simple and straightforward way of implementing an affine structure on a given space. It consists of defining an infinitesimal displacement ("flow") vector which remains invariant under "parallell transfer". In "conventional" supergauge theories, the infinitesimal displacement vector may be taken to be:

$$d\omega_{\alpha\dot\alpha} = dx_{\alpha\dot\alpha} + \frac{i}{2}\left([\varphi_{\dot\alpha}, d\varphi_\alpha] - [d\varphi_{\dot\alpha}, \varphi_\alpha]\right)$$
$$d\omega_\alpha = d\varphi_\alpha \ , \quad d\omega_{\dot\alpha} = d\varphi_{\dot\alpha} \tag{4.1}$$

where $dx_{\alpha\dot\alpha}$ is an infinitesimal vector in Minkowski space and φ_α , $\varphi_{\dot\alpha}$ are Grassmann coordinates. Under the infinitesimal displacement, $\delta\varphi_\alpha = \epsilon_\alpha$, $\delta d\varphi_\alpha = 0$, the requirement that $d\omega$ remain invariant, $\delta d\omega_{\alpha\dot\alpha} = \delta d\omega_\alpha = 0$, determines the transformation property of $dx_{\alpha\dot\alpha}$.

This approach to the construction of superspaces can be readily generalized to the " colored" algebra developed in the last Section. Indeed, if we formally define the infinitesimal spinor,

$$dQ_\alpha = \tfrac{1}{2}(dx_\alpha + e_a dx^a_\alpha) + \tfrac{1}{2}\vartheta\,(dx_\alpha + e_a dx^a_\alpha)$$

the vector

$$\frac{i}{2}\,[Q_{\dot\alpha}, dQ_\alpha] = i\,(x_{\dot\alpha}\, dx_\alpha + x^a_{\dot\alpha}\, dx^a_\alpha)$$

is real and it is a color singlet. Hence, the vector $d\Omega$ with components

$$d\Omega_{\alpha\dot\alpha} = dx_{\alpha\dot\alpha} + \frac{i}{2}\ \left([Q_{\dot\alpha}, dQ_\alpha] - [dQ_{\dot\alpha}, Q_\alpha]\right),$$
$$d\Omega_\alpha = dQ_\alpha,$$
$$d\Omega_{\dot\alpha} = dQ_{\dot\alpha} \tag{4.2}$$

is a good candidate for an infinitesimal displacement vector in superspace. As a natural generalization of infinitesimal supergauge transformations we take

$$\delta Q_\alpha = \tfrac{1}{2}(a_\alpha + \vartheta\, a_\alpha)$$
$$\delta Q_{\dot\alpha} = \tfrac{1}{2}(a_{\dot\alpha} - \vartheta\, a_{\dot\alpha})$$
$$a_\alpha \equiv \epsilon_\alpha + e_a \epsilon^a_\alpha . \tag{4.3}$$

In eq. (4.3) ϵ_α and ϵ^a_α are infinitesimal. In other words, we induce infinitesimal transformations on the Grassmann components of Q_α. The superspace spanned by the Q_α and $x_{\alpha\dot\alpha}$ is to be flat, hence $\delta\, d Q_\alpha = 0$.

Commutativity and conjugation properties of the infinitesimal parameters ϵ_α and ϵ^a_α have to be identical with those of the Q_α. This follows from the fact that we want (4.3) to be an infinitesimal translation: $Q_\alpha + \delta Q_\alpha$ has to enjoy the same properties as Q_α does.

The transformation properties of $dx_{\alpha\dot\alpha}$ under (4.3) can be read off from the requirement that $d\Omega$ remain invariant. We find:

$$\begin{aligned}\delta\, dx_{\alpha\dot\alpha} &= -\tfrac{i}{2}[\delta Q_{\dot\alpha}, dQ_\alpha] + \tfrac{i}{2}[dQ_{\dot\alpha}, \delta Q_\alpha] \\ &= -i\left(\epsilon_{\dot\alpha}\, dx_\alpha + \epsilon^a_{\dot\alpha}\, dx^a_\alpha - dx_{\dot\alpha}\,\epsilon_\alpha - dx^a_{\dot\alpha}\,\epsilon^a_\alpha\right) \qquad (4.4)\end{aligned}$$

Similarly to ordinary supergauge theories, one readily verifies that the commutator of two supergauge transformations is a translation with nilpotent parameter, cf. ref.8.

Having defined supergauge transformations on Q_α, the generators of the algebra, we proceed to investigate whether the transformations (4.3) can be consistently implemented on the full algebra.

Taking again ordinary supergauge theories as a guide, we enlarge the algebra of Grassmann coefficients, x_α and x^a_α by the algebra of its (outer) derivations[17], p^α and $p^{a\alpha}$, respectively. The generators $p^\alpha, p^{a\alpha}, \ldots$ satisfy the relations,

$$\begin{aligned}&\{p^\alpha, p^\beta\} = 0, \quad \{p^{a\alpha}, p^{b\beta}\} = 0, \quad [p^\alpha, p^{b\beta}] = 0, \\ &\{p^{\dot\alpha}, p^{b\beta}\} = 0, \ldots \qquad (4.5)\end{aligned}$$

in complete analogy with eqs. (3.2), (3.4), (3.5) and (3.7). Further,

$$\begin{aligned}&\{p^\alpha, x_\beta\} = \delta^\alpha_\beta, \quad \{p^{a\alpha}, x^b_\beta\} = \delta^{ab}\delta^\alpha_\beta, \\ &[p^\alpha, x^b_\beta] = 0, \quad \{p^{\dot\alpha}, x^b_\beta\} = 0, \ldots \qquad (4.6)\end{aligned}$$

The commutation(anticommutation) rules are dictated by the requirements discussed in Sec.3 and by the requirement that the p^α and $p^{a\alpha}$ act as generators of infinitesimal translations on the coefficients of Q_α.

Next, we proceed to complexify the algebra given by (4.5) and (4.6). In particular, we conjecture that the infinitesimal map,

$$\begin{aligned}M(\epsilon_\alpha, \epsilon^a_\alpha) &= \tfrac{1}{4}(a_\alpha + v a_\alpha)(h^\alpha - v h^\alpha), \\ h^\alpha &\equiv p^\alpha + e_a p^{a\alpha} \qquad (4.7)\end{aligned}$$

induces infinitesimal translations on the subalgebra generated by the Q_α.

On working out the product we find:

$$M(\epsilon_\alpha, \epsilon^a_\alpha) = \tfrac{1}{2}(\epsilon \cdot p - v\,\epsilon \cdot p) + \tfrac{1}{2} e_r(\epsilon^r_\alpha p^\alpha + \epsilon_\alpha p^{r\alpha}) - \tfrac{1}{2} v\, e_r(\epsilon^r_\alpha p^\alpha + \epsilon_\alpha p^{r\alpha}),$$

where $\epsilon \cdot p \equiv \epsilon_\alpha p^\alpha - \epsilon^r_\alpha p_r^{\ \alpha}$. Apart from an irrelevant numerical factor, $M(\epsilon_\alpha, \epsilon^a_\alpha)$ indeed acts as an infinitesimal <u>left</u> translation on Q_α, <u>viz.</u>

$$M(\epsilon_\alpha, \epsilon^a_\alpha) Q_\beta \approx a_\alpha + v\, a_\alpha .$$

However, M <u>does not act as a derivation</u>[18] on the full algebra; in particular,

$$M(\epsilon_\alpha, \epsilon^a_\alpha)(Q_\beta Q_\gamma) \neq (M(\epsilon_\alpha, \epsilon^a_\alpha) Q_\beta) Q_\gamma + Q_\beta (M(\epsilon_\alpha, \epsilon^a_\alpha) Q_\gamma).$$

<u>The transformations which act as derivations are of the type $M(0, \epsilon^a_{\dot\alpha})$ and $M(\epsilon_\alpha, 0)$ only</u>. In other words, the supergauge parameters have to be either of a purely leptonic or of a purely hadronic (quark) type. An entirely similar result holds for the subalgebra generated by the $Q^\cdot_{\dot\alpha}$ too.

The explicit expressions of the "good" transformations are the following.

$$L(\epsilon_\alpha) = \tfrac{1}{4}(\epsilon_\alpha + v\epsilon_\alpha)(p^\alpha - v\,p^\alpha) = \tfrac{1}{2}(\epsilon_\alpha p^\alpha - v\,\epsilon_\alpha p^\alpha),$$

$$\overline{L}(\epsilon^\cdot_{\dot\alpha}) = \tfrac{1}{4}(\epsilon^\cdot_{\dot\alpha} - v\epsilon^\cdot_{\dot\alpha})(p^{\dot\alpha} + v\,p^{\dot\alpha}) = \tfrac{1}{2}(\epsilon^\cdot_{\dot\alpha} p^{\dot\alpha} + v\,\epsilon^\cdot_{\dot\alpha} p^{\dot\alpha}),$$

$$H(\epsilon^a_\alpha) = \tfrac{1}{4}(\epsilon^a_\alpha e_a + v\epsilon^a_\alpha e_a)(p^{b\alpha} e_b - v\,p^{b\alpha} e_b) = -\tfrac{1}{2}(\epsilon^a_\alpha p^{a\alpha} - v\,\epsilon^a_\alpha p^{a\alpha}),$$

$$\overline{H}(\epsilon^a_{\dot\alpha}) = \tfrac{1}{4}(\epsilon^a_{\dot\alpha} e_a - v\epsilon^a_{\dot\alpha} e_a)(p^{b\dot\alpha} e_b + v\,p^{b\dot\alpha} e_b) = -\tfrac{1}{2}(\epsilon^a_{\dot\alpha} p^{a\dot\alpha} + v\,\epsilon^a_{\dot\alpha} p^{a\dot\alpha}) \qquad (4.8)$$

The transformations listed above act as <u>left</u> translations on the spinorial coordinates. One realises, however, that L and H act as translations on the subalgebra composed of elements of the form $(a + v\,a)$, whereas $\overline{L}$ and $\overline{H}$ act as translations on the subalgebra composed of elements of the form $(b - v\,b)$ only. There exist no left translations with a domain extendable to the full algebra. Furthermore, the algebra (or any of its subalgebras) does not admit right translations.

The infinitesimal transformations (4.8) satisfy the following "orthogonality" relations:

$$\bar{L}H = H\bar{L} = L\bar{L} = \bar{L}L = H\bar{H} = \bar{H}H = L\bar{H} = \bar{H}L = 0$$
$$[L, H] = [\bar{L}, \bar{H}] = 0 \qquad (4.9)$$

Finite transformations can be generated on the respective subalgebras by means of a formal exponentiation. We have, for instance,

$$\begin{aligned}(\exp L)Q_\alpha &\overset{\text{def.}}{=} Q_\alpha + LQ_\alpha + \tfrac{1}{2!}L(LQ_\alpha) + \cdots \\ &= Q_\alpha + \tfrac{1}{2}(\epsilon_\rho p^\rho - v\epsilon_\rho p^\rho)(q_\alpha + v q_\alpha) + \cdots \\ &= Q_\alpha + \tfrac{1}{2}\left([\epsilon_\rho p^\rho, q_\alpha] + v[\epsilon_\rho p^\rho, q_\alpha]\right) \\ &= Q_\alpha + \tfrac{1}{2}(\epsilon_\alpha + v\epsilon_\alpha), \end{aligned} \qquad (4.10)$$

and similarly for the other transformations listed in (4.8). Due to the fact that $L, \ldots, \bar{H}$ act as derivations, the action of any finite transformation is well-defined (and possesses the requisite group properties) on the subalgebra which forms its domain of definition.

We conclude that it is possible to impose an affine structure on a "colored" superspace defined by the infinitesimal vector (4.2) in a limited sense. Supergauge transformations defined by (4.3) and (4.4) possess linear realizations if and only if

a) the superspace is split into its leptonic and hadronic parts,

b) the domain of definition of the supergauge operators is restricted to appropriate subalgebras.

We note that a) corresponds to splitting off the trace of the quaternion (3.1), i.e. one replaces q_α by $\frac{1}{2}(q_\alpha \pm \tilde{q}_\alpha)$ everywhere.

5. SUMMARY AND COMMENTS.

Regarding the triality rule for quarks as a generalization of the Pauli principle (as explained in Sec.1), we have shown that a superspace can be constructed with the following properties.

i) The triality rule is exactly implemented as a consequence of algebraic relations.

ii) Quarks and leptons are treated symmetrically.

iii) The "fusion" of the quark and lepton coordinates with Minkowski space into an enlarged affine space forces a splitting of the fermionic coordinates into their leptonic and hadronic parts. Thus, the existence of separately conserved lepton and baryon numbers is not only made possible,

but it is strongly suggested by the geometrical structure. Whether or not these two quantum numbers are separately conserved, depends, of course, on the dynamical superstructure built upon such a geometry.

While we have not succeeded so far in constructing a consistent dynamical model based on this superspace, a few comments about the dynamical superstructure can be made already at this stage.

a) As pointed out before, the algebra constructed in Sec.3 is non-associative. As a consequence, the quantities $Q_\alpha, Q_{\dot\alpha}, \cdots$, cannot be regarded as dynamical variables in the usual sense. In particular, there exists no realization of their algebra in terms of linear operators acting on a vector space. Therefore, we conjecture that theories based on superspaces constitute the most natural framework in which colored quarks can be handled on an equal footing with leptons.

b) One can envisage the implementation of the supersymmetry devised here in terms of superfields.

Consider, for the sake of simplicity, scalar superfields only. These are basically of two types.

Type 1 : fields based on a subalgebra. These have the general form:

$$\Phi = \Phi(x, l_\alpha, h_\alpha),$$

where we used the notation:

$$l_\alpha = \tfrac{1}{2}(x_\alpha + v x_\alpha)$$
$$h_\alpha = \tfrac{1}{2}(e_a x^a_\alpha + v e_a x^a_\alpha).$$

Upon expanding Φ in powers of l_α and h_α, we get:

$$\Phi = L^\alpha(x) l_\alpha + H^\alpha(x) h_\alpha + H^{\alpha\beta}(x)(h_\alpha h_\beta) + B^{\alpha\beta\gamma}(x)(h_\alpha h_\beta h_\gamma) + \cdots$$

The ordinary fields L^α and $B^{\alpha\beta\gamma}$ can be interpreted as lepton and baryon fields, respectively. The baryon field is symmetric in the indices α, β, γ, due to the fact that $(h_\alpha h_\beta h_\gamma) = \tfrac{1}{2}\epsilon_{abc}(\xi^a_\alpha \xi^b_\beta \xi^c_\gamma + v\,\xi^a_\alpha \xi^b_\beta \xi^c_\gamma)$. There appear no "dilepton fields" (since $l_\alpha l_\beta = 0$) or lepton-quark, lepton-diquark fields, cf.Sec.3. The coordinate space may be enlarged by the inclusion of right-handed spinors, cf. eq.(3.8). Supersymmetry transformations can be realised on Φ in terms of the operators L and H. Together with a conjugate field, $\bar\Phi(x, l_{\dot\alpha}, h_{\dot\alpha})$, one may construct chiral currents.

Type 2 : Fields based on the full algebra. These are of the form:

$$\psi = \psi(x, l_\alpha, l_{\dot\alpha}, h_\alpha, h_{\dot\alpha})$$

On expanding ψ in powers of the fermionic coordinates, there appear

"meson fields" as coefficients of $l_\alpha l_{\dot\alpha}, h_\alpha h_{\dot\alpha}$, etc. However, supersymmetry transformations cannot be realised linearly on ψ , see Sec.4. Hence, fields of type 2 seem to be somewhat unattractive from a theoretical point of view.

ACKNOWLEDGEMENTS

We are grateful to D. Amati and S. Fubini for the hospitality extended to us at CERN, where this research has been carried out. Our thanks are due to B. Zumino for enlightening discussions on supersymmetric theories.

* * *

FOOTNOTES AND REFERENCES

1) See,however, the discussion of this question in refs. 4 and 7.

2) M.Günaydin and F.Gürsey, Lett. al Nuovo Cimento, 6, 401 (1973); Phys. Rev. D9, 3387 (1974).

3) M.Günaydin and F. Gürsey, J. Math. Phys.14 , 1651 (1973).

4) F. Gürsey, in Proc. JHU Workshop on Current Problems in High Energy Particle Theory (Ed. G. Domokos and S.K. Domokos), Johns Hopkins University, Baltimore, 1974.

5) H. Freudenthal, Adv. Math. 1, 145 (1964).

6) F. Gürsey, in Proc. Kyoto Conference (1975).

7) A preliminary investigation of this type has been carried out by the present authors, cf. R. Casalbuoni, G. Domokos and S. Kövesi-Domokos, Nuovo Cim. 31 A , 423 (1976). This paper is based on the assumption that color and space-time algebras can be realized in a direct product form.

8) J. Wess and B.Zumino, Nuc. Phys. B70, 39 (1974). There exist excellent reviews on the subject; cf. for instance, L. O'Raifertaigh, Comm. Dublin Inst. Adv. Studies, Ser. A. No. 22 (1975). B. Zumino, CERN preprint TH 2120 (1975). L. Corwin, Y. Neeman and S. Sternberg, Rev. Mod Phys. 47, 573 (1975).

9) An involution is an antiautomorphism defined on an algebra $\mathcal{A}$. If $a_i, a_j \ldots, \in \mathcal{A}$, their conjugates ($\bar a_i, \bar a_j \ldots$) obey the same multiplication rules as $a_i, a_j, \ldots$ do, provided the order of the factors is inverted.

10) See e.g. R.D.Schafer, An Introduction to Nonassociative Algebras. (Academic Press, New York, 1966).

11) Cf. Günaydin and Gürsey, ref. 3. It is to be observed, however, that Günaydin and Gürsey use an inequivalent realization of the split octonion algebra. Their construction is based on an enlargement of the ground field, whereas the construction given here realizes the split octonion algebra by means of a particular "complexification" of quaternions.

12) Cf. Casalbuoni et al., ref.7. A similar attempt at the unification of the color and space-time properties was also made by L.C.Biedenharn and H. van Dam, Duke University preprint, January 1976 (unpublished).

13) For a thorough analysis of supersymmetry algebras in a Weyl spinor basis see e.g. R.Haag, J.T.Lopuszanski and M.Sohnius, Nuc.Phys.B88, 257(1974).

14) This statement is true if either the internal symmetry group is trivial, or its metric is given by a symmetric matrix (e.g. orthogonal internal symmetry groups). However, a non-trivial bilinear invariant exists e.g. in the case of symplectic internal symmetry groups.

15) A real quantity is here and henceforth defined as one containing none of the "imaginary" units introduced before, $(\nu, e_a, \ldots)$.

16) The geometrical approach to supersymmetries has been developed by a number of authors. Cf. in particular, P.Nath and R.Arnowitt, Phys. Letters 56B, 177 (1975). R.Arnowitt, P.Nath and B.Zumino, Phys. Letters 56B, 81 (1975). L.N.Chang, K.Macrae and F.Mansouri, Phys. Letters 57B, 59 (1975). L.N.Chang, K.Macrae and F.Mansouri, Phys.Rev. D13, 235 (1976).

17) F.A.Berezin, The Method of Second Quantization (Academic Press, New York, 1966).

18) A derivation is a mapping of an algebra, A, into itself. If D is a derivation and a_1, $a_2 \in A$, then $D(a_1 a_2) = (Da_1)a_2 + a_1(Da_2)$; see ref. 10.