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Colourful Simplicial Depth*

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Abstract. Inspired by Bárány's Colourful Carathéodory Theorem [4], we introduce a colourful generalization of Liu's simplicial depth [13]. We prove a parity property and conjecture that the minimum colourful simplicial depth of any core point in any d-dimensional configuration is d^2+1 and that the maximum is $d^{d+1}+1$. We exhibit configurations attaining each of these depths, and apply our results to the problem of bounding monochrome (non-colourful) simplicial depth.

1. Introduction

In statistics there are several measures of the depth of a point p in $\mathbb{R}^d$ relative to a fixed set S of sample points. Two recent surveys on data depth are [2] and [10], see references therein. The depth measure we are interested in is the *simplicial depth* of p, which is the number of simplices generated by points in S that contain p. A point of maximum simplicial depth can be viewed as a type of d-dimensional median. We would like to obtain a lower bound for the depth of simplicial medians.

To do this, we consider a generalized problem where the sample points are colourful. That is, in dimension d we consider sample points given in each of at least $(d+1)$ colours. Then we define the *colourful simplicial depth* of a point p relative to this sample to be the number of *colourful simplices* (i.e. simplices with one vertex of each colour) that contain p. We focus on the situation where the point p is in the intersection of the convex hulls of the individual colours, which is called the *core* of the configuration.

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If p is a core point we would typically expect the simplicial depth of p to be more than exponential in d. However, we exhibit configurations where p is a core point but is contained in only $d^2 + 1$ colourful simplices. We conjecture that any core point p of any d-dimensional colourful configuration is contained in at least $d^2 + 1$ colourful simplices. Along the way, we notice that both in the colourful and monochrome cases the simplicial depth of points in general position (relative to the sample set) sometimes has pleasant parity properties. We conclude by mentioning some other natural problems relating to the colourful and monochrome simplicial depth.

2. Definitions and Background

2.1. *Simplicial Depth*

The (closed) simplicial depth of a point p relative to a set S of $n = |S|$ points in $\mathbb{R}^d$ is the number of (closed) simplices generated by sets of $(d+1)$ points from S containing p in their convex hull. This was introduced by Liu [13] as a measure of how representative p is of the points in S. Denote the simplicial depth of p relative to S as $\text{depth}_S(p)$. The simplicial depth of p can be interpreted as the probability that p lies in a random simplex of S times a constant factor of n^{d+1} if we sample points from S uniformly with replacement, or times $\binom{n}{d+1}$ if we sample without replacement.

We are most interested in the case when $S \cup \{p\}$ is in *general position*, that is for all $k < d$ there are no k-dimensional affine subspaces that contain $k+2$ points from $S \cup \{p\}$. With this assumption, p will always be in the interior of any simplices that contain it, so the notions of closed and open simplicial depth coincide. Without this assumption the closed simplicial depth will be larger.

For a set of points S, define $f(S)$ to be the maximum simplicial depth of a point p relative to S, that is,

$$f(S) = \max_{p \in R^d} \text{depth}_S(p). \tag{1}$$

A point p maximizing $f(S)$ can be understood as a higher-dimensional median point. We call any such p a *simplicial median*. Indeed, for $d = 1$, this is the usual definition of a median $\mathbb{R}$. In higher dimensions this definition retains many desirable properties of the median, such as affine invariance and a high breakdown point (see, e.g. [2], [10], [11] and [13]). However, this maximum will not be attained at a point in general position.

We consider a similar quantity, the maximum simplicial depth of a point p that maintains $S \cup \{p\}$ in general position:

$$g(S) = \max_{S \cup \{p\} \text{ in general position}} \text{depth}_S(p). \tag{2}$$

Equivalently, g is the maximum open simplicial depth of a point $\mathbb{R}^d$. In this way the definition of g can be extended to the case when S is not in general position. While the maximum in (1) will be attained on a discrete set of points in $\mathbb{R}^d$, the maximum in (2) will be attained on an open set. For non-empty S, we will have $g(S) < f(S)$.

2.2. *Colourful Simplicial Depth*

Now consider a situation where points are given in each of $r \geq d+1$ colours. Then the sample consists of colourful sets $S_1, S_2, \ldots, S_r$ which define a colourful configuration $\mathbf{S}$. In the following we use a bold fount for colourful objects. A *colourful simplex* from these sets is any simplex whose vertices are chosen from distinct sets. We define $\mathbf{depth}_{\mathbf{S}}(p)$, the *colourful simplicial depth* of p relative to the configuration $\mathbf{S}$, as the number of colourful simplices containing p. As with monochrome simplicial depth, colourful depth can be interpreted probabilistically. In the case where $r = d+1$, colourful depth corresponds to specifying separate distributions for each vertex of the simplex. Dividing the depth by $|S_1| \cdot |S_2| \cdot \cdots \cdot |S_{d+1}|$ gives the probability that p lies in a random colourful simplex (sampled uniformly).

A choice of sets $S_1, \ldots, S_r$ specifies a *colourful configuration* $\mathbf{S}$ of points. We call the intersection of the convex hulls of the S_i's in a configuration the *core* of $\mathbf{S}$. Bárány proved that core points are contained in some colourful simplex; this is known at the Colourful Carathéodory Theorem [4]. In the remainder of the paper, except where noted, we assume that all configurations and p are in general position and have a non-empty (hence full-dimensional) core. We remark that our results hold under weaker conditions, such as p not lying on any hyperplanes generated by points from the configuration.

2.3. *Background*

Even before the notion of simplicial depth was studied in statistics, the question of computing bounds for $f(S)$ and $g(S)$ given n and d was studied in the combinatorics and computational geometry communities. The two-dimensional question dates back at least to Kárteszi [12] who showed that for n points in the plane, $g(S)$ is at most $\frac{1}{24}(n^3-n)$ for odd n and at most $\frac{1}{24}(n^3-4n)$ for even n, and showed that these bounds were attained when S is the set of vertices of a regular n-gon. In the early 1980s, Boros and Füredi [8] showed $g(S)$ is at least $n^3/27 + O(n^2)$, and gave configurations achieving this bound.

In a beautiful paper, Bárány [4] gave bounds for the monochrome simplicial depth in dimension d as an application of his Colourful Carathéodory Theorem. He obtained a lower bound by showing that after colouring the points, some point p must be contained in many colourful simplices. A key point of Bárány's proof is that a core point p of a colourful configuration must lie in at least *one* colourful simplex. Using this fact, for a set S of n points in general position in $\mathbb{R}^d$ Bárány obtains a lower bound of

$$g(S) \geq \frac{1}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d). \tag{3}$$

This result is asymptotically sharp up to a constant factor as a function of n (for fixed d). However, as Bárány remarks, the constant is probably quite far from the truth. Indeed, he gives a sharp upper bound of

$$g(S) \leq \frac{1}{2^d (d+1)!} n^{d+1} + O(n^d). \tag{4}$$

We speculate that the true lower bound is not much less than the upper bound.

One way to improve (3) would be to show that a core point p must lie in more than one colorful simplex. In Bárány's original paper, he notes that p must in fact lie in at least $(d + 1)$ colourful simplices, thereby improving (3) to

$$g(S) \geq \frac{1}{(d+1)^d}\binom{n}{d+1} + O(n^d). \tag{5}$$

More generally, if we could show that any core point p of a d-dimensional configuration is contained in at least $\mu(d)$ simplices, then we can improve the constant in (3) by a factor of $\mu(d)$.

3. Colourfully Covering the Core

This leads us to ask: What is the minimum number $\mu(d)$ of simplices that can contain a core point p in a colourful configuration? Given a colourful configuration $\mathbf{S}$ with colourful sets $S_1, \ldots, S_r$ we can define

$$\mathrm{m}(\mathbf{S}) = \min_{p \in \mathrm{core}(\mathbf{S})} \mathbf{depth}_{\mathbf{S}}(p). \tag{6}$$

We remark that if core($\mathbf{S}$) has a non-empty interior, the minimum in (6) will be attained on an open set of points that are in general position relative to $\mathbf{S}$.

In this notation our objective is to find the minimum value of m($\mathbf{S}$) over all configurations $\mathbf{S}$ with full-dimensional core in dimension d. For a fixed d, it is clear that some configuration with $(d+1)$ points in each of $(d+1)$ colours attains this minimum, which depends only on the dimension. Hence we can define

$$\mu(d) = \min_{d \text{ configurations } \mathbf{S},\ p \in \mathrm{core}(\mathbf{S})} \mathbf{depth}_{\mathbf{S}}(p). \tag{7}$$

One might suppose that m($\mathbf{S}$) is often large. As a thought experiment, consider choosing a configuration at random. If we take $(d + 1)$ points in $\mathbb{R}^d$ from a distribution that is "nice" and centrally symmetric about the origin $\mathbf{0}$, the probability that $\mathbf{0}$ is contained in their convex hull is $1/2^d$ (see, e.g. [17]). This suggests that for random $\mathbf{S}$, a typical value for $\mathbf{depth}_{\mathbf{S}}(\mathbf{0})$ would be $(1/2^d)(d + 1)^{d+1}$. For a set S of $(d + 1)^2$ points in the plane, plugging this value into Bárány's analysis gives us an estimate of $g(S)$ very close to Bárány's upper bound (4). However, it is not immediately clear if we should expect m($\mathbf{S}$) to be much smaller than $\mathbf{depth}_{\mathbf{S}}(\mathbf{0})$.

If we take a configuration $\mathbf{S}^{\triangle}$ with $S_1^{\triangle}$ given by $(d + 1)$ points in general position and $S_1^{\triangle} = S_2^{\triangle} = \cdots = S_{d+1}^{\triangle}$ we get $\mathrm{m}(\mathbf{S}^{\triangle}) = (d + 1)!$. In Section 3.4 we exhibit a configuration $\mathbf{S}^{-}$ with $\mathrm{m}(\mathbf{S}^{-}) \leq d^2 + 1$.

In the remainder of the paper, except where noted, we consider configurations with $(d + 1)$ points in each of $(d + 1)$ colours.

3.1. *Preliminaries*

In [7] Bárány and Onn consider the problem of *colourful linear programming*. This is the algorithmic version of the colourful Carathéodory problem: Given a core point p, how can we *find* a colourful simplex containing p? They begin with some preprocessing which is also helpful here.

Take a colourful configuration $\mathbf{S}$ of $(d+1)$ colourful sets in $\mathbb{R}^d$, $\mathbf{S} = \{S_1, \ldots, S_{d+1}\}$. Take $p \in \text{int}(\text{core}(\mathbf{S}))$. Without loss of generality we assume that the core point $p = \mathbf{0}$. Given any finite set of points $T \subseteq \mathbb{R}^d$, scaling the points of T does not affect whether $\mathbf{0}$ lies in the convex hull of T since the coefficients in a convex combination can themselves be rescaled. This allows us to normalize $\mathbf{S}$ by rescaling its points to unit vectors.

Let $\text{conv}(T)$ be the convex hull of the points in T and let $\text{cone}(T)$ be the set of non-negative linear combinations of points of T. A cone is *simplicial* if it can be generated by a set of d linearly independent points in $\mathbb{R}^d$. If $T \subseteq \mathbb{R}^d$ is a set of points, $\mathbf{0} \notin T$, but $\mathbf{0} \in \text{conv}(T)$, then $\text{cone}(T)$ must contain a non-trivial linear subspace of $\mathbb{R}^d$. A closed, convex cone is called *pointed* if it does not contain such a subspace, so we summarize this as:

Lemma 3.1. *Given any finite set of non-zero points $T \subseteq \mathbb{R}^d$, $\mathbf{0}$ is in $\text{conv}(T)$ if and only if $\text{cone}(T)$ is not pointed.*

When T is a finite set of points on the unit d-sphere $\mathbb{S}^d \subseteq \mathbb{R}^d$, Lemma 3.1 is equivalent to saying that $\mathbf{0} \in \text{conv}(T)$ if and only if T is not contained in any open hemisphere of $\mathbb{S}^d$. One direction is proved by building a hemisphere from a hyperplane through $\mathbf{0}$ whose normal lies in the interior of $\text{cone}(T)$ when this cone is pointed. The other direction is proved by observing that an open hemisphere never contains both a point p and its antipode $-p$.

We would like to put Lemma 3.1 in a form that is convenient for counting how many simplices generated from T contain $\mathbf{0}$. To do this, we find it helpful to think about the antipode of one of the points.

Lemma 3.2. *If $T = \{p_1, p_2, \ldots, p_{d+1}\}$ is a set of non-zero affinely independent points in $\mathbb{R}^d$, $\mathbf{0}$ is in $\text{conv}(T)$ if and only if the antipode $-p_{d+1}$ is in $\text{cone}(p_1, p_2, \ldots, p_d)$.*

Proof. Let $K = \text{cone}(p_1, p_2, \ldots, p_d)$. Since K is a cone generated by d linearly independent points in $\mathbb{R}^d$, K is simplicial and hence pointed. If $-p_{d+1} \in K$, then we can write it as a conic combination of the remaining p_i, that is, $-p_{d+1} = \sum_{i=1}^{d} a_i p_i$ for some $a_1, \ldots, a_d \geq 0$. Moving the p_{d+1} term to the right-hand side of the equation and dividing by $1 + \sum_{i=1}^{d} a_i$ gives $\mathbf{0}$ as a convex combination of the p_i's. If $-p_{d+1}$ is not in K, then we can strictly separate $-p_{d+1}$ from K with a hyperplane H through $\mathbf{0}$. Then both K and p_{d+1} lie strictly on the same side of H, and the cone generated by T must be pointed. □

3.2. *A Variational Approach*

Take a point p from a finite set $S \in \mathbb{R}^d$. Call a simplex generated by points in S a *p-simplex* if p is one of the points used to generate the simplex, and call a p-simplex *zero-containing* if it contains $\mathbf{0}$ in its interior. Define $z_S(p)$ to be the number of zero-containing p-simplices for a given S.

Lemma 3.2 tells us that $z_S(p)$ is the number of simplicial cones generated by $S\backslash\{p\}$ that contain $-p$. We find it useful to think about what happens to $z_S(p)$ if we move p while fixing the remaining points of S. This is particularly illustrative if we confine p to the surface of the unit sphere $\mathbb{S}^d$ centred at $\mathbf{0}$.

Let $U = S\backslash\{p\}$ with $|U| = u$. Initially $z_S(p)$ will be the number of simplicial cones generated by sets of d points from U that contain $-p$. Now consider what happens as p (and hence $-p$) move. The value of $z_S(p)$ will stay fixed until $-p$ crosses the boundary of some simplicial cone from U. These boundaries are defined by the hyperplanes generated by $\mathbf{0}$ and sets of $(d-1)$ points from U. Taking all $(d-1)$ sets from U, we can generate all such boundaries. They divide the surface of $\mathbb{S}^d$ into open *cells* that are $(d-1)$-dimensional open sets. We can define the *depth* of a cell of S to be the number of simplicial cones generated by S containing any given point in the interior of the cell.

Consider moving p along the surface of $\mathbb{S}^d$ to a new point p'. If $-p$ and $-p'$ are in the same cell, we will have $z_S(p) = z_S(p')$. Now suppose $-p$ is in a cell C adjacent to the cell C' containing $-p'$. Then as we move from $-p$ to $-p'$ we cross a single hyperplane H defined by a set U^0 of $(d-1)$ points from U belonging to H. Let us say that $-p$ is on the left of H and $-p'$ is on the right. For the moment we assume that only $(d-1)$ points of U lie on H. Let U^- be the set of k points from U on the left of H, and let U^+ be the $u-k-(d-1)$ points from U on the right. Since $-p$ is in a cell bordered by H, it lies in the cone defined by the points from U^0 and x for any point $x \in U^-$. On the other hand, $-p$ is separated by H from the cones formed by U^0 and y for any $y \in U^+$. Hence $-p$ is contained in exactly k simplicial cones from S generated by U^0 and a single other point. Similarly, $-p'$ is contained in exactly $u-k-(d-1)$ such cones. Simplicial cones that do not contain U^0 in their generating set will not have H as a facet, so they will contain $-p$ if and only if they contain $-p'$. Suppose $-p$ is in l such cones. Then $z_S(p) = l+k$, while $z_S(p') = l + u - k - (d-1)$.

We conclude that given the value of $z_S(p)$ at some point p, we can in principle compute $z_S(p')$ for any other point p' by tracing a path from $-p$ to $-p'$, and seeing how each hyperplane generated from points in $U = S\backslash\{p\}$ divides the points of U. To do this formally, we need a topological lemma that says we can always draw a path between two points on $\mathbb{S}^d$ that crosses only hyperplanes from U (as opposed to passing through cones generated by fewer than $(d-1)$ points). This reduces to the following fact which can be proved using algebraic topology, see, for example, [14]:

Lemma 3.3. *The sphere $\mathbb{S}^d$, a $(d-1)$-dimensional manifold, remains path connected after removing finitely many $(d-3)$-dimensional manifolds.*

3.3. *Parity*

The variational approach to computing $z_S(p)$ explains the following parity phenomenon:

Proposition 3.4. *For any colourful configuration* $\mathbf{S}$ *of $(d+1)$ points in each of $(d+1)$ colours in* odd *dimension d and any point p with* $\mathbf{S}$ *and p in general position, the colourful simplicial depth of p with respect to* $\mathbf{S}$ *is* even.

The authors were surprised by this fact while experimenting with configurations. However, it is easy to explain this via a colourful version of the method described in Section 3.2. Suppose we begin with a configuration $\mathbf{S}^0$ with $(d+1)$ points in each of $(d+1)$ colours clustered near the North Pole of $\mathbb{S}^d$. (We remarked in Section 3.1 that it is sufficient to consider configurations on the surface of $\mathbb{S}^d$.) Then we can move one point at a time from its initial position in $\mathbf{S}^0$ to its final position in $\mathbf{S}$ generating a sequence of configurations $\mathbf{S}^0, \mathbf{S}^1, \mathbf{S}^2, \ldots, \mathbf{S}^{(d+1)^2} = \mathbf{S}$. Clearly, $\mathbf{depth}_{\mathbf{S}^0}(\mathbf{0}) = 0$. As we move a given point p_i of colour j from its initial position in $\mathbf{S}^0$ (and $\mathbf{S}^i$) to its final position in $\mathbf{S}$ (and $\mathbf{S}^{i+1}$), we need only to know what happens when the antipode $-p_i$ crosses colourful hyperplanes defined by a set of $(d-1)$ points of $(d-1)$ colours, and not of colour j. Such a colourful hyperplane H will miss only one other colour, j'. There will be k points of colour j' on one side of H, and $(d+1-k)$ on the other side. Here we are assuming that the points from $\mathbf{S}$ are in general position, but we can argue by continuity that this assumption is not necessary. As $-p_i$ crosses H the number of simplicial cones containing $-p_i$ generated by points from H and a point of colour j' changes from k to $(d+1-k)$. As long as $(d+1)$ is even, the parity does not change.

Examining this proof, we can see that Proposition 3.4 can be generalized:

Theorem 3.5. *If* $\mathbf{S} = \{S_1, S_2, \ldots, S_r\}$ *is a* d*-dimensional colourful configuration of points and for each* $i = 1, 2, \ldots, r$ *we have* $|S_i|$ *even, and* p *is and any point* p *with* $\mathbf{S}$ *and* p *in general position, then the colourful simplicial depth of* p *with respect to* $\mathbf{S}$ *is also even.*

For monochrome depth, as we move point p around $\mathbb{S}^d$ we need to consider all possible hyperplanes formed from $S \backslash \{p\}$. Using the same reasoning as Theorem 3.5 we get:

Theorem 3.6. *If* S *is a set of* n *points in* $\mathbb{R}^d$, *and* $n-d$ *is even, and* p *is a point such that* $S \cup \{p\}$ *is in general position, then the simplicial depth of* p *with respect to* S *is even.*

Remark 3.7. The variational approach suggested in Section 3.2 has appeared in various guises in discussions of monochrome simplicial depth. In particular, it underlies many of the algorithms suggested for computing monochrome simplicial depth. Several such algorithms have been proposed recently, see, for example, the discussion in [2]. Many of these focus on the two-dimensional problem, but [11] and [9] use variational ideas in three- and four-dimensional algorithms.

For this reason, we believed that Theorem 3.6 existed as folklore for some time. Baker remarks on the two-dimensional version in a recreational mathematics note [3], but this fact, which impressed the authors with its simple elegance, seems curiously neglected in the literature. We speculate that one reason for this is that in statistics the focus has been on computing the depth of the sample points themselves, which are not in general position and do not retain nice parity conditions.

3.4. *Configurations with Small Minimal Colourful Depth*

We now describe how to build a colourful configuration $\mathbf{S}^-$ that contains $\mathbf{0}$ in its core, but where only $d^2 + 1$ colourful simplices contain $\mathbf{0}$. Our strategy is to fix the first d colourful sets $S_1^-, S_2^-, \ldots, S_d^-$ and then consider possible placements of $(d+1)$ points $p_1, p_2, \ldots, p_{d+1}$ to form S_{d+1}^-. We will place the points from $\mathrm{S}^- = S_1^- \cup S_2^- \cup \cdots \cup S_d^-$ on the sphere $\mathbb{S}^d$ in such a way that some regions of $\mathbb{S}^d$ are sparsely covered by simplices from S^-.

We begin by fixing $\varepsilon = 1/100d$. We will place the points from $\mathbf{S}^-$ in three locations on $\mathbb{S}^d$. The first on the *Tropic of Capricorn*, which we define to be the set of points on $\mathbb{S}^d$ whose dth coordinate is -2ε. The second is on the *Tropic of Cancer*, whose dth coordinate is ε. The two tropics are topologically copies of $\mathbb{S}^{d-1}$, but unlike their namesakes they are not equally spaced from the equator. The final region is the *polar region* of points in $\mathbb{S}^d$ which are within ε of the North Pole $p_{\text{north}} = (0, 0, \ldots, 0, 1)$ (see Fig. 1).

Now let us fix the positions of the points $\{x_1, x_2, \ldots, x_{d+1}\} \in S_1^-$. Take

$$x_1 = (\sqrt{1-4\varepsilon^2}, 0, 0, \ldots, 0, -2\varepsilon) \qquad \text{and} \qquad x_2 = (-\sqrt{1-\varepsilon^2}, 0, 0, \ldots, 0, \varepsilon).$$

Note that the line segment between x_1 and x_2 passes just below the origin in the sense that it contains a point whose first $(d-1)$ coordinates are 0, and whose dth coordinate is negative (and small). We now place the remaining points $x_3, \ldots, x_{d+1}$ in the polar region in such a way as to ensure that $\mathbf{0} \in \operatorname{int}(\operatorname{conv}(S_1^-))$. For $d = 2$ we can do this by placing x_3 at the North Pole. For $d \geq 3$ we can place the points on the section of the Arctic Circle (points with distance ε to the North Pole) with zero initial coordinate. Topologically the Arctic Circle is a copy of $\mathbb{S}^{d-2}$; we can take $x_3, \ldots, x_{d+1}$ to be the vertices of a regular simplex inscribed on this sphere.

The points of colours $2, 3, \ldots, d$ are chosen similarly. The first points from each of the d colours are arranged in a regular simplex on Capricorn. The remaining points in the same relative position to the first point, so that each S_i^- is a rotation of S_1^- around

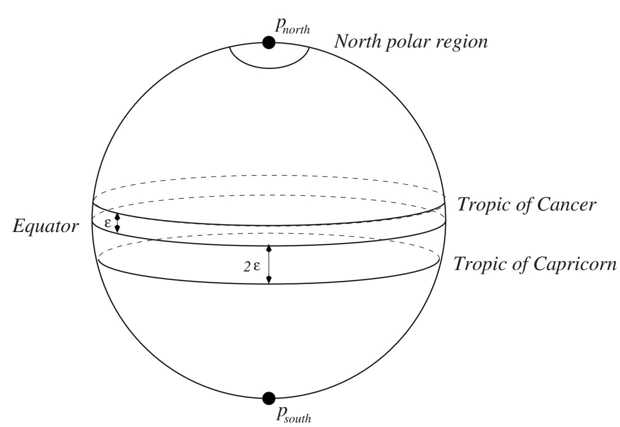

Fig. 1. Three-dimensional illustration of the regions used in constructing $\mathbf{S}^-$.

the dth coordinate axis. In particular, for each $i = 1, 2, \ldots, d$, the second point of S_i^- will lie on Cancer and the final $(d-1)$ points will lie in the polar region.

We finish our construction by considering possible placements of the points $p_1, \ldots, p_{d+1}$ of S_{d+1}^-. We want to place the p_i's in such a way that their antipodes (the $-p_i$'s) are contained in few colourful simplicial cones generated from $\mathbf{S}^-$.

Consider the cell C_{south} defined by colours $1, \ldots, d$ of $\mathbf{S}^-$ on $\mathbb{S}^d$ which contains the South Pole $p_{\text{south}} = (0, 0, \ldots, 0, -1)$. We claim this is exactly the intersection of $\mathbb{S}^d$ with the single colourful simplicial cone K_{Cap} defined by the d colourful points on Capricorn. This follows since any other colourful cone is generated by a set of d coloured points chosen from Capricorn, Cancer and the northern polar region. Fix such a cone and call these sets G_{Cap}, G_{Can} and G_{Pole} and let $K_G = \text{cone}(G_{\text{Cap}} \cup G_{\text{Can}} \cup G_{\text{Pole}})$. We assume that we have $|G_{\text{Cap}}| < d$. We need to show that $\text{int}(K_{\text{Cap}}) \cap \text{int}(K_G) = \emptyset$. To do this, we find a hyperplane separating K_{Cap} and K_G. If $G_{\text{Cap}} = \emptyset$ the hyperplane through the Equator will do. Otherwise, take the colours from G_{Cap} and consider any facet F of K_{Cap} containing generators of each of these colours. Then F separates K_{Cap} from all the polar points and all the Cancer points of colours from $\{1, 2, \ldots, n\} \backslash G_{\text{Cap}}$. (To be absolutely proper, in higher dimension we would have to move Capricorn up towards the equator to ensure the separation of the Cancer points, i.e. we would have to reduce the constant 2ε to $(1+\delta)\varepsilon$ for some $\delta > 0$.) This completes the proof. We conclude that the cell C_{south} is covered only by the colourful cone K_{Cap} and closely approximates the spherical cap bounded by Capricorn.

It is a good strategy to place the antipodes $-p_i$ in C_{south}. If we do this for all of S_{d+1}^-, however, the resulting configuration will not have $\mathbf{0} \in \text{conv}(S_{d+1}^-)$ (S_{d+1}^- would certainly be contained in an open hemisphere). So we must have at least one antipode, say $-p_1$ above Capricorn. Indeed, if we place the remaining $-p_i$ below Capricorn, we would need to have $-p_1$ above the ring of the antipodes of Capricorn. More precisely, this is the set of points on $\mathbb{S}^d$ with final coordinate value exactly 2ε. In particular, it is above Cancer.

Let $A = \{a_1, a_2, \ldots, a_d\}$ be the points from $S_1^-, S_2^-, \ldots, S_d^-$ on Capricorn. Similarly, let $B = \{b_1, b_2, \ldots, b_d\}$ be the points on Cancer. Let us count how many simplicial cones from $\mathbf{S}^-$ must contain $-p_1$ if we place $-p_1$ above Cancer. To do this, we start with $-p_1$ in C_{south} and then move it above Cancer noting which cell boundaries it crosses as suggested in Section 3.2. This structure of the cell boundaries is a topological question, so we find it convenient to remove the p_{south} and equate $\mathbb{S}^d$ with $\mathbb{R}^{d-1}$.

With the exception of the single colourful cone that contains C_{south}, the colourful simplicial cones generated by $\mathbf{S}^-$ correspond to colourful simplices in $\mathbb{R}^{d-1}$. The polar points on $\mathbb{S}^d$ will be clustered near the origin in $\mathbb{R}^{d-1}$. Let $A' = \{a'_1, \ldots, a'_d\}$ and $B' = \{b'_1, \ldots, b'_d\}$ be the projections of A and B respectively in $\mathbb{R}^{d-1}$. Then $\text{conv}(A')$ and $\text{conv}(B')$ form nested simplices which contain the projection of the polar region. The boundaries of the colourful simplicial cones on $\mathbb{S}^d$ map to facets of simplices in $\mathbb{R}^{d-1}$; both are defined by sets of $(d-1)$ colourful points. Moving $-p_1$ from below Capricorn to above Cancer corresponds to moving $-p'_1$ from outside $\text{conv}(A')$ to inside $\text{conv}(B')$.

Let us now see what simplicial facets $-p'_1$ must cross to do this. If we keep $-p'_1$ far away from the a'_i's and b'_i's themselves, we can avoid any facets involving the polar points: These facets involve at most $(d-2)$ generators from A' and B', and hence have ends that are at most $(d-3)$-dimensional manifolds in $\text{conv}(A') \setminus \text{int}(\text{conv}(B'))$. The ends can be avoided by Lemma 3.3.

This still leaves $d2^{d-1}$ colourful facets defined by choosing $(d-1)$ colourful points from A' and B'. We can enumerate them by first choosing an index (colour) to omit and then representing the choices of a_i''s and b_i''s by a 0-1 vector of length $(d-1)$. Letting 0 represent the choice of an a_i', $\text{conv}(A')$ is bounded by the facets defined by d index choices and a vector of 0's, while $\text{conv}(B')$ is bounded by the facets defined by d index choices and a vector of 1's. In fact there are 2^d colourful simplices generated by A' and B', and they are enumerated by 0-1 vectors of length d. Their facets are enumerated by choosing an index to drop from the enumerating sequence. Therefore the sums of the 0-1 vectors enumerating the facets of a given simplex can differ by at most 1.

Now start with $-p_1'$ outside $\text{conv}(A')$. To bring $-p_1'$ inside $\text{conv}(B')$, we must start by bringing it into $\text{conv}(A')$. This involves crossing some boundary face of $\text{conv}(A')$, say the one defined by $a_1, \ldots, a_{d-1}$. This is enumerated as $(d, 0, 0, \ldots, 0, 0)$. We can proceed through facets $(d-1, 0, 0, \ldots, 0, 1)$, $(d-2, 0, 0, \ldots, 0, 1, 1)$ until finally we cross $(1, 1, 1, \ldots, 1)$ into a cell of $\text{conv}(B')$. This involves d facet crossings, which is minimal since at each crossing we can only add a single 1 to the 0-1 part of the enumerating vector.

We claim that as $-p_1'$ crosses each facet, it makes a net gain of $d-1$ containing simplices. At the first facet, $(d, 0, 0, \ldots, 0, 0)$, $-p_1'$ leaves the single exterior simplex defined by the points A' projected from Capricorn and enters the d simplices defined by $a_1', \ldots, a_{d-1}'$ and the d points of colour d other than a_d'. At subsequent facet crossings, the same thing happens for the remaining colours: $-p_1'$ leaves the simplex defined by the crossing facet and a_i'. As $-p_1'$ leaves, it enters the simplices defined by this facet and the d remaining points of colour i. Hence the number of simplices containing $-p_1'$ immediately after crossing into $\text{conv}(B')$ is exactly $1 + d(d-1)$.

We now return our attention to $\mathbb{S}^d$. Denote by C_p the cell containing $-p_1$ whose projection lies inside $\text{conv}(B')$. From our construction, C_p is a cell above Cancer. We want to claim that in fact it contains some point above the set of antipodes of Capricorn,

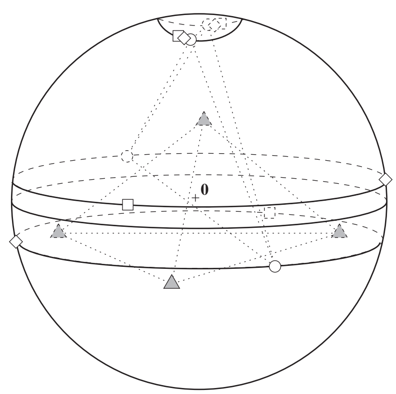

Fig. 2. Placement of points of colours 1, 2, 3 and antipodes of colour 4 in the three-dimensional $\mathbf{S}^-$.

that is, a point whose antipode is in C_{south}. This is a complicated geometric calculation. However, we observe that nothing in our topological argument above changes if we change the constant 2ε in our definition of Capricorn to $c\varepsilon$ for any $c \geq 0$. In particular, the cell C_p does not degenerate if we move the antipodes of Capricorn towards Cancer by decreasing c to 1. Therefore for some $c > 1$ (this condition maintains $\mathbf{0} \in \text{int}(\text{conv}(S_i^-))$ for $i = 1, \dots, d$), C_p includes some point above the antipodes of Capricorn. Any such c and point in C_p would be sufficient for our construction. We have used $c = 2$ for concreteness and take it as an article of faith that this is a small enough for our choice of ε.

The construction can now be completed. Take $-p_2$ to be the midpoint of the shortest spherical segment between Capricorn and p_1 (which lies below Capricorn). Let $z < -2\varepsilon$ be the final coordinate of $-p_2$ and arrange the remaining points so that $-p_2, -p_3, \dots, -p_{d+1}$ form a regular simplex on $\mathbb{S}^d \cap \{x \in \mathbb{R}^d | x_d = z\}$. Then $\mathbf{0}$ is in the convex hull of the $-p_i$ (and hence S_{d+1}^-). Finally we can calculate $\mathbf{depth}_{\mathbf{S}^-}(\mathbf{0})$ from the location of the $-p_i$: $\mathbf{0}$ lies in $1 + d(d-1)$ colourful simplices generated with p_1 and one simplex each including $p_2, p_3, \dots, p_{d+1}$. Hence,

$$\mathbf{depth}_{\mathbf{S}^-}(\mathbf{0}) = 1 + d(d-1) + d = d^2 + 1.$$

Remark 3.8. There are other nice configurations with $\mathbf{depth}_{\mathbf{S}^-}(\mathbf{0}) = d^2 + 1$. Consider a configuration $\mathbf{S}'$ similar to $\mathbf{S}^-$ but with the tropics pushed to the north, taking Cancer's final coordinate to 3ε and Capricorn's to $-\varepsilon$. We can then move each of $-p_1, \dots, -p_d$ across Capricorn and the equator through a single boundary facet. Finally, place $-p_{d+1}$ at the South Pole. Using the same analysis as above, we have $p_1, \dots, p_d$ points forming $1 + (d-1)$ simplices containing $\mathbf{0}$, and p_{d+1} forming one such simplex for a total of $d^2 + 1$.

Both $\mathbf{S}^-$ and $\mathbf{S}'$ have symmetry for the first d colours, but not the last one. We can also propose a configuration $\mathbf{S}''$ with symmetry between all the colours. Follow the recipe for $\mathbf{S}^-$ but place one point of each colour on Cancer and Capricorn and place the remaining points in the polar region. This brings a number of technical difficulties, however. The points will not be in general position, since the tropical hyperplanes include $(d+1)$ points. It is also a bit less natural to evenly space $(d+1)$ points on copies of $\mathbb{S}^{d-1}$, indeed, for $d = 2$ this construction does not make sense. When there is a nice way to do this for $d \geq 3$ (e.g. four points on $\mathbb{S}^2$) we may end up with some points being antipodes. This would cause $\mathbf{0}$ to be on the faces of some simplices and increase its colourful simplicial depth. Most of these problems can be fixed by perturbing $\mathbf{S}''$, but even so $\mathbf{S}''$ is not well-suited to our proof technique. One might also consider configurations that are not confined to the sphere.

3.5. *Evaluating $\mu(d)$*

The configuration $\mathbf{S}^-$ of Section 3.4 satisfies $\mathrm{m}(\mathbf{S}^-) \leq d^2+1$ where $\mathrm{m}(\mathbf{S})$ is the minimum colourful simplicial depth of the core point defined in (6). We would like to prove that $\mathrm{m}(\mathbf{S}^-) = d^2 + 1$ and in fact that for any colourful configuration $\mathbf{S}$ we will have $\mathrm{m}(\mathbf{S}) \geq d^2 + 1$, or equivalently $\mu(d) \geq d^2 + 1$. The second half of this proposition clearly implies the first. We suggest it is also more approachable since we can move the core point of minimum depth to $\mathbf{0}$ during preprocessing, whereas a direct attack on $\mathrm{m}(\mathbf{S}^-)$ requires understanding the shape of the core of $\mathbf{S}^-$.

Bárány's original Colourful Carathéodory theorem is exactly that $\mu(d) \geq 1$. He further shows that for any $\mathbf{S}$ any colourful point from $\mathbf{S}$ is part of some generating set for a colourful simplex containing $\mathbf{0}$. This immediately yields $\mu(d) \geq d+1$. In $\mathbf{S}^-$ we see that $p_2, p_3, \ldots, p_{d+1}$ all generate a *unique* colourful simplex containing $\mathbf{0}$. Thus the minimum number of colourful simplices containing $\mathbf{0}$ generated by an arbitrary point in a configuration is 1. To get a stronger lower bound than $\mu(d) \geq d+1$ we need to understand some global information about configurations.

Lemma 3.9. *Fix the sets $S_1, \ldots, S_d$ from a colourful configuration $\mathbf{S}$ with $\mathbf{0}$ in its core, and consider the cells created on $\mathbb{S}^d$ by the colourful simplicial cones from these sets. Then every cell has depth at least* 1, *and if there is a cell of depth* 1 *it is unique and all other cells have depth at least d.*

Proof. The fact that every cell has depth at least 1 is equivalent to the fact that every colourful point generates some colourful simplex that contains $\mathbf{0}$, proved in [4]. Suppose now that there is a cell C of depth 1. Any point exiting C through a bounding hyperplane will be exiting some colourful simplex. Since the depth of C is 1, this will always be the same simplex. Thus the extreme points of C must be a colourful set $A = \{a_1, \ldots, a_d\}$ with $a_i \in S_i$ generating this simplex. We can puncture $\mathbb{S}^d$ at $p \in C$ and project $\mathbb{S}^d \backslash \{p\}$ into $\mathbb{R}^{d-1}$. The a_i's project to a set $A' = \{a'_1, \ldots, a'_d\}$ that forms a $(d-1)$-simplex in $\mathbb{R}^{d-1}$. The remaining colourful points project to points in $\mathrm{conv}(A')$.

Take a point q inside $\mathrm{conv}(A')$. We want to show that q is contained in at least d colourful simplices in addition to $\mathrm{conv}(A')$ after projection. To do this, it is sufficient to show that if we take any colourful set $B' = \{b'_1, \ldots, b'_d\}$ of projected points with b'_i of colour i and $A' \cap B' = \emptyset$, then q is in some colourful simplex generated from points of $A' \cup B'$ with some generators from B'. Equivalently, we want to show that $\mathrm{conv}(A')$ is covered by colourful simplices generated from $A' \cup B'$ (excluding $\mathrm{conv}(A')$ itself from the covering). Then by partitioning the projections of the colourful points into $(d+1)$ colourful sets $A', B'_1, B'_2, \ldots, B'_d$ we get Lemma 3.9.

Consider the collection X of colourful $(d-1)$-simplices generated by A' and B' in $\mathbb{R}^{d-1}$ and let $\tilde{X}$ be the set of points contained in the colourful simplices of X other than $\mathrm{conv}(A')$. The elements of X are all the simplices formed by taking for each colour $i = 1, 2, \ldots, d$ either a'_i or b'_i as a generating vertex. This construction resembles the d-dimensional cross-polytope β_d (the dual of the d-cube), a regular polytope in $\mathbb{R}^d$ with $2d$ vertices and 2^d facets. The cross-polytope β_d is generated by taking as vertices the standard unit vectors $E^+ = \{e_1, \ldots, e_d\}$ and their negatives $E^- = \{-e_1, \ldots, -e_d\}$. The facets of β_d are the convex hulls generated by choosing for each $i = 1, \ldots, d$ either e_i or $-e_i$.

We can see that X is obtained from β_d as follows: We have $A' \cup B' \subset \mathbb{R}^{d-1}$. Embed $\mathbb{R}^{d-1}$ as an affine subspace $\mathrm{Aff}(A')$ in $\mathbb{R}^d$. Take H to be an affine hyperplane in $\mathbb{R}^d$ parallel to $\mathrm{Aff}(A')$. For $i = 1, \ldots, d$ let p_i be the intersection point of H with the line through b'_i perpendicular to $\mathbb{R}^{d-1}$. Let $P = \{p_1, \ldots p_d\}$ and generate a set Q of $(d-1)$-simplices by taking for each $i = 1, \ldots, d$ either a'_i or p_i. By construction X is the projection of

Q into $\text{Aff}(A')$. Now we claim that Q is a continuous image of the facets of β_d. We can exhibit such a map by first finding an affine transformation T_1 with $T_1(e_i) = a_i'$ for $i = 1, \ldots, d$ and $T_1(\text{Aff}(E^-)) = \text{H}$. Note that $T_1(\beta_d)$ is a polytope. Then applying a further affine transformation t_2 to H with $t_2(-e_i) = p_i$ for $i = 1, \ldots, d$ and extending this to T_2 on $\mathbb{R}^d$ so that T_2 fixes $\text{Aff}(A')$, we see that the composition $T_2 \circ T_1$ is the required map.

We proceed by contradiction. Assume that $\tilde{X}$ does not cover $\text{conv}(A')$. Then we can find a retraction of $\tilde{X}$ to its boundary $\partial(\text{conv}(A'))$. By composing T_2, the projection taking Q onto X and the retraction of $\tilde{X}$, we get a retraction of $T_1(\beta_d) \setminus \text{conv}(A')$ onto $\partial(\text{conv}(A'))$. However, $T_1(\partial(\beta_d))$ is a d-dimensional polytope topologically equivalent to $\mathbb{S}^d$ and hence $T_1(\partial(\beta_d)) \setminus \text{conv}(A')$ is topologically equivalent to a $(d-1)$-dimensional disk $\mathbb{B}^{d-1}$. Nevertheless, $\partial(\text{conv}(A'))$ is topologically equivalent to $\mathbb{S}^{d-1}$ and a well-known theorem of algebraic topology says that there does not exist a retraction of $\mathbb{B}^{d-1}$ to $\mathbb{S}^{d-1}$ (see, for example, Section 21 of [14]). This is the required contradiction, hence the colourful simplices of $X \setminus \text{conv}(A')$ cover $\text{conv}(A')$. □

Corollary 3.10. *The minimum colourful simplicial depth of any core point in any colourful configuration is at least* $2d$. *That is, we have* $\mu(d) \geq 2d$.

Proof. It suffices to prove this for a configuration **S** with $(d+1)$ in $(d+1)$ colours. Observe that if we have no cell of depth 1 then each of the $(d+1)$ points of S_{d+1} will generate at least two colourful simplices containing $\mathbf{0}$, and if we do have such a cell C, we must place at least one point, say $p_1 \in S_{d+1}$, outside of C to get $\mathbf{0} \in \text{conv}(S_{d+1})$. Then p_1 generates at least d simplices containing $\mathbf{0}$ in addition to the d required of the remaining points in S_{d+1}. □

3.6. *The Two-Dimensional Case*

We briefly illustrate our methods by describing how core points can be contained in configurations in $\mathbb{R}^2$. Consider such a configuration $\mathbf{S} = \{X, Y, Z\}$ with core point p. We assume general position, and, as discussed in Section 3.1, we may without loss of generality take the core point $p = \mathbf{0}$ and place the points of **S** on the unit circle $\mathbb{S}^2$.

Then the points of X and Y divide $\mathbb{S}^2$ into six segments. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$. These points generate nine simplicial cones and divide $\mathbb{S}^2$ into six segments. The boundaries between cones are simply the rays through the x_i's and y_i's. Because no three points of X or Y lie in the same half-circle, each hyperplane through $\mathbf{0}$ and x_i divides the y_i's two to one and vice versa. Then as the antipode of a point from Z crosses x_i or y_i the number of containing simplicial cones changes by exactly one.

To get a configuration $\mathbf{S}^-$ where only five simplices contain 0, we take $x_1 = (-\sqrt{1-4\varepsilon^2}, -2\varepsilon)$, $x_2 = (\sqrt{1-\varepsilon^2}, \varepsilon)$, $x_3 = (-\varepsilon, \sqrt{1-\varepsilon^2})$, $y_1 = (\sqrt{1-4\varepsilon^2}, -2\varepsilon)$, $y_2 = (-\sqrt{1-\varepsilon^2}, \varepsilon)$ and $y_3 = (\varepsilon, \sqrt{1-\varepsilon^2})$. Observe there is a large cell of depth 1 between x_1 and y_1. The reader can verify that the sequence of colourful cell depths is 1,2,3,4,3,2 (see Fig. 3).

Let $Z = \{z_1, z_2, z_3\}$. Place $z_2 = (-\sqrt{1-9\varepsilon^2}, 3\varepsilon)$ and $z_3 = (\sqrt{1-9\varepsilon^2}, 3\varepsilon)$ so that their antipodes lie between x_1 and y_1. They each generate one simplex containing $\mathbf{0}$.

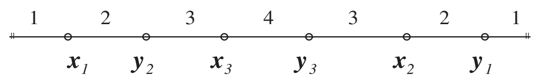

Fig. 3. Covering depths for a circle with a depth 1 cell.

Finally, to ensure that $\mathbf{0} \in \operatorname{conv}(Z)$, we see that $-z_1$ must lie above y_2 and x_2. Take $z_1 = (-\sqrt{1-16\varepsilon^2}, -4\varepsilon)$. Then $-z_1$ is contained in three colourful simplicial cones generated by X and Y. This configuration has $\mathbf{0}$ in the interior of its core and $\mathbf{0}$ lies in $1+1+3=5$ colourful simplices (see Fig. 4).

Using the analysis in Section 3.5 we see that the cells generated by X and Y have colourful covering depth at least 1. If no cell attains this, then our configuration must yield at least six colourful simplices containing $\mathbf{0}$. If some cell has depth 1, we can place at most two of the $-z_i$'s in this cell. The remaining z_i must then have depth at least 2, for a minimum of 4. In fact, we can strengthen this to show that our configuration is minimal by observing that we cannot place all of Z in two adjacent cells. We conclude that $\mu(2) = 5$. A similar observation in three dimensions shows that $\mu(3) \geq 8$. Given the construction of Section 3.4 and Proposition 3.4 we know that $\mu(3)$ is either 8 or 10. Bárány and Matoušek [5] have shown that $\mu(3) = 10$.

4. Conclusions

Let us return to our original goals. Using the bound $\mu(d) \geq 2d$ from Section 3.5, we see that we can improve Bárány's lower bound (5) for the depth of the monochrome simplicial median to

$$g(S) \geq \frac{2d}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d). \tag{8}$$

This is a modest improvement. Unfortunately, the construction in Section 3.4 shows that

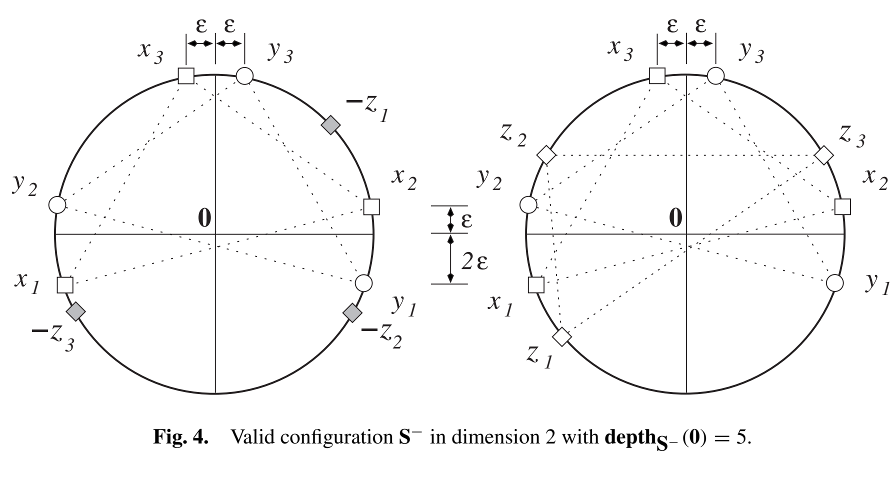

Fig. 4. Valid configuration $\mathbf{S}^-$ in dimension 2 with $\mathbf{depth}_{\mathbf{S}^-}(\mathbf{0}) = 5$.

simply bounding $\mu(d)$ cannot give a stronger bound than

$$g(S) \geq \frac{d^2+1}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d). \tag{9}$$

Quite recently, Wagner proved exactly the bound (9) in his thesis [16] as a special case of his First Selection Lemma. This is, to our knowledge, the first improvement of (5) since Bárány's original paper [4]. Wagner's result uses a continuous version of the Upper Bound Theorem for polytopes and other techniques from probability without any reference to colouring. We find the appearance of the constant $d^2 + 1$, which for us arrives from colourful combinatorics, quite remarkable.

4.1. *Bounds for Core Point Depth*

Recalling that $\mathrm{m}(\mathbf{S})$ is the minimum value of a core point in a configuration $\mathbf{S}$ and that $\mu(d)$ is the minimum value of $\mathrm{m}(\mathbf{S})$ over all d-dimensional colourful configurations $\mathbf{S}$, our main result is:

Theorem 4.1. *The minimal colourful simplicial depth of any interior core point in any colourful configuration is between* $2d$ *and* $d^2 + 1$. *That is*, *we have*: $2d \leq \mu(d) \leq d^2 + 1$.

Conjecture 4.2. *The minimum colourful simplicial depth of any interior core point in any colourful configuration is* $d^2 + 1$. *That is*, *we have* $\mu(d) = d^2 + 1$.

This conjecture implies that the configuration $\mathbf{S}^-$ minimizes $\mathrm{m}(\mathbf{S})$ for d-dimensional colourful configurations. It would also give an elementary proof of (9). It is easy to see that this holds for $d = 1$. As we noted in Section 3.6, Conjecture 4.2 holds for $d = 2$ and $d = 3$. The non-uniqueness of configurations attaining $\mathrm{m}(\mathbf{S}) = d^2 + 1$ suggests that any such proof cannot be completely trivial but it may be possible to do this through improved bookkeeping. The authors generated random low-dimensional configurations by computer and did not find any counterexamples to Conjecture 4.2.

Remark 4.3. The lower bound for $\mu(d)$ was improved very recently independently by Bárány and Matoušek [5] and Stephen and Thomas [15] to $\max(3d, \frac{1}{5}d(d+1))$ for $d > 2$ and $\lfloor (d+2)^2/4 \rfloor$ respectively. We know that $\mu(1) = 2$, $\mu(2) = 5$ and $\mu(3) = 10$. Combining the improved bounds with the parity conditions of Proposition 3.4 we have the following bounds on $\mu(d)$ for $d > 3$:

$$12 \leq \mu(4) \leq 17, \qquad 16 \leq \mu(5) \leq 26, \qquad 18 \leq \mu(6) \leq 37, \qquad 22 \leq \mu(7) \leq 50,$$

and for $d > 7$:

$$\left\lfloor \frac{(d+2)^2}{4} \right\rfloor \leq \mu(d) \leq d^2 + 1.$$

It is also natural to ask what type of colourful configuration has a core point of *maximum* colourful simplicial depth. For this question to be interesting, we must fix the number and size of the colourful sets. Hence we restrict our attention to d-configurations with $(d+1)$ points in each of $(d+1)$ colours. We also require p to lie in the interior of the core since moving to the boundary of a simplex increases the depth. We define

$$\nu(d) = \max_{d \text{ configurations } \mathbf{S},\ p \in \mathrm{int}(\mathrm{core}(\mathbf{S}))} \mathbf{depth}_{\mathbf{S}}(p). \tag{10}$$

Our method is well-suited to analyzing $\nu(d)$ simply by changing our objective to creating deep cells and placing antipodes in them. We remark that $\nu(1) = 2$. An analysis similar to that of Section 3.6 shows that $\nu(2) = 9$. The key observation is after placing two sets of three colourful points on the circle, the sequence of cell depths that we obtain is either 1,2,3,4,3,2 or 3,2,3,2,3,2. In the first case we also need to argue that the cells of depth at least 3 cover less than half the circle and that opposite every point of depth 4 is a point of depth 1.

The minimal core depth configuration $\mathbf{S}^-$ used to prove $\mu(2) = 5$ is topologically unique, so it is interesting to observe that, up to topology, there are two distinct configurations that contain $\mathbf{0}$ in nine colourful simplices. The first corresponds to the sequence of cell depths 1,2,3,4,3,2 and contains a point z_3 that generates a unique $\mathbf{0}$-containing colourful simplex. The second corresponds to the sequence 3,2,3,2,3,2 and is a combinatorially symmetric configuration where each colourful point is in exactly three $\mathbf{0}$-containing colourful simplices. The configurations are illustrated in Fig. 5.

We can build a configuration $\mathbf{S}^+$ with $\mathbf{depth}_{\mathbf{S}^+}(\mathbf{0}) = d^{d+1}+1$ by following the strategy for $\mathbf{S}^-$ but building a deep cell rather than a shallow one. To do this, we place the polar region points of colour i close to the geodesic between p_{north} and the point of colour i on Cancer. Then p_{north} is contained in every colourful cone generated by points from Cancer and the polar region (in fact these are all the colourful cones containing p_{north}). Hence the cell C_{north} containing p_{north} has depth d^d. By placing the points of S_{d+1}^+ so that d of their antipodes are in C_{north} and the final antipode is at p_{south}, we get $\mathbf{S}^+$ with $\mathbf{depth}_{\mathbf{S}^+}(\mathbf{0}) = d \cdot d^d + 1$. The two-dimensional $\mathbf{S}^+$ appears as the left element of Fig. 5.

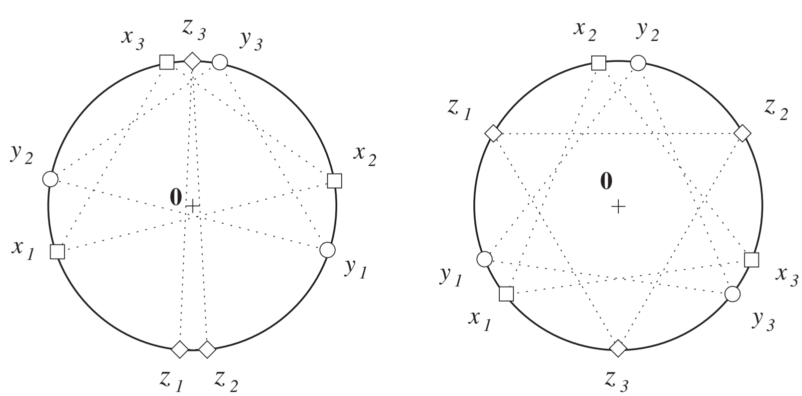

Fig. 5. The two configurations in dimension 2 with $\mathbf{depth}_{\mathbf{S}}(\mathbf{0}) = 9$.

A more symmetric (but similar) construction places one point of each colour at the vertices of a regular simplex, and the remaining points surround the antipode of the same colour.

It follows that $\nu(d) \geq d^{d+1} + 1$. We conjecture that this bound is tight. As with Conjecture 4.2 a computer search did not turn up any counterexamples.

Conjecture 4.4. *The maximum colourful simplicial depth of any point in the interior of the core of any colourful configuration of* $(d+1)$ *points in each of* $(d+1)$ *colours is* $d^{d+1} + 1$. *That is*, *we have* $\nu(d) = d^{d+1} + 1$.

Remark 4.5. For any d, there exists a colourful configuration $\mathbf{S}$ which contains $\mathbf{0}$ in at least 32% of its colourful simplices.

A configuration of $(d+1)$ points in each of $(d+1)$ colours generates $(d+1)^{d+1}$ colourful simplices, so Remark 4.5 follows immediately from the construction of $\mathbf{S}^+$. The minimum fraction of colourful simplices containing $\mathbf{0}$ from an $\mathbf{S}^+$ configuration is $82/256$ attained when $d = 3$.

5. Open Questions

We conclude by mentioning that there are many more natural questions relating to colourful and monochrome simplicial depth. The first is:

Question 5.1. What is a typical value of $\mathrm{m}(\mathbf{S})$ for a random configuration $\mathbf{S}$ of $(d+1)$ points in each of $(d+1)$ colours?

In Section 3 we remarked that such random configurations could be expected to have a simplicial depth on the order of $(1/2^d)(d+1)^{d+1}$ at the origin. We also gave a colourful configuration $\mathbf{S}^\triangle$ that has $\mathrm{m}(\mathbf{S}^\triangle) = (d+1)!$. However, $\mathbf{S}^\triangle$ is not in general position. Our construction $\mathbf{S}^-$ from Section 3.4 is in general position and has a low value of $\mathrm{m}(\mathbf{S}^-)$. It is not clear if this behaviour is typical, i.e. if most configurations have some point p near the edge of the core that drags down $\mathrm{m}(\mathbf{S})$, or if our configuration is statistically unlikely. Indeed, we can consider the possibility that *all* configurations in general position have such a point near the edge of the core.

Question 5.2. What is the maximum value of $\mathrm{m}(\mathbf{S})$ for a colourful configuration $\mathbf{S}$ of $(d+1)$ points in each of $(d+1)$ colours? What if $\mathbf{S}$ is not assumed to be in general position?

We observe that in fact our construction of a colourful configuration $\mathbf{S}^-$ with $\mathrm{m}(\mathbf{S}^-) = d^2 + 1$ contains points of high colourful simplicial depth, but away from $\mathbf{0}$. This leads us to consider the colourful analogues of the functions $f(S)$ and $g(S)$ of Section 2.1. For a colourful configuration $\mathbf{S}$, define

$$\mathbf{f}(\mathbf{S}) = \max_{p \in \mathbb{R}^d} \mathbf{depth}_{\mathbf{S}}(p) \qquad \text{and} \qquad \mathbf{g}(\mathbf{S}) = \max_{p \text{ in general position}} \mathbf{depth}_{\mathbf{S}}(p). \tag{11}$$

We focus on the case where we have $(d+1)$ colours. It is clear that given the sizes of the colourful sets $S_1, \ldots, S_{d+1}$ comprising $\mathbf{S}$ that the maximum of $\mathbf{f}(\mathbf{S})$ and $\mathbf{g}(\mathbf{S})$ is $|S_1| \cdot \cdots \cdot |S_{d+1}|$ and is attained by placing the points of each colour at (or near) the vertices of a simplex. If we restrict $\mathbf{S}$ to be a configuration of $(d+1)$ points in each of $(d+1)$ colours and take the maximum over the interior of the core, we get exactly the question of finding $\nu(d)$ (Conjecture 4.4). We are also interested in lower bounds for $\mathbf{f}(\mathbf{S})$ and $\mathbf{g}(\mathbf{S})$.

Question 5.3. For d-dimensional configurations consisting of n points in each of $(d+1)$ colours, find lower bounds for $\mathbf{f}(\mathbf{S})$ and $\mathbf{g}(\mathbf{S})$.

In a survey paper on the Colourful Carathéodory Theorem, Bárány and Onn [6] mention that the results of [1] can be applied to give a lower bound for $\mathbf{g}(\mathbf{S})$ when n is large of the form

$$\mathbf{g}(\mathbf{S}) \geq c_d \binom{n}{d+1}. \tag{12}$$

Unfortunately, the constant c_d is doubly exponential in d so the bound is only non-trivial if $n \gg e^{4d^2}$. In particular, it sheds no light on the $n = d+1$ case.

One can also get a lower bound for $\mathbf{g}(\mathbf{S})$ directly from the Colourful Tverberg Theorem [18], which is used to derive the results in [1]:

$$\mathbf{g}(\mathbf{S}) \geq \frac{1}{4}\left(\frac{n}{d+1} + 3\right). \tag{13}$$

This still does not help for $n = d+1$, but for small n the bound is stronger than (12) and comes with the additional guarantee that colourful simplices involved are disjoint! This suggests that there is much room for improvement.

5.1. *Monochrome Questions*

The authors would also like to mention that they do not know the answers to some fairly basic questions about monochrome simplicial depth. Recall the maximum closed and open depth functions $f(S)$ and $g(S)$ for a set of points S in $\mathbb{R}^d$ defined in Section 2.1.

Question 5.4. Are the points p attaining the maximum $f(S)$ in (1) always limit points of the set of maxima attaining $g(S)$ in (2)?

We feel that a positive answer to this question would provide a further natural justification for studying $g(S)$ in place of $f(S)$ when the former is more tractable. Similarly, it would be interesting to get conditions on S such that $f(S)$ is not much larger than $g(S)$.

We are also curious about the expected values of $f(S)$ and $g(S)$:

Question 5.5. Given n points in $\mathbb{R}^d$ distributed independently and symmetrically about $\mathbf{0}$, what is the expected deepest simplicial depth of the resulting configuration? That is, what is the expected depth of the simplicial median of the points?

Wagner and Welzl [17] give an expression for the expected depth of **0**, but **0** will not always be the deepest point. Indeed if $n = d + 1$ the expected simplicial depth of **0** will be $1/2^d$ while the simplicial median always has depth 1. For fixed d the expected depth of **0** is $(1/2^d)\binom{n}{d+1}$ which has the same asymptotic behaviour as Bárány's sharp upper bound (4) for $g(S)$. However, when n is not much larger than $(d + 1)$, the gap between the expected depth of **0** and Bárány's upper bound is substantial and it is not clear to us where the expected depth of the simplicial median lies.

Bárány's method of proving (3) combined with a solution to Question 5.1 might lend some insight into Question 5.5, but a direct approach would be better.

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