1

COMBINATION OF ROUGH AND FUZZY SETS BASED ON α-LEVEL SETS

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ABSTRACT

A fuzzy set can be represented by a family of crisp sets using its α-level sets, whereas a rough set can be represented by three crisp sets. Based on such representations, this paper examines some fundamental issues involved in the combination of rough-set and fuzzy-set models. The rough-fuzzy-set and fuzzy-rough-set models are analyzed, with emphasis on their structures in terms of crisp sets. A rough fuzzy set is a pair of fuzzy sets resulting from the approximation of a fuzzy set in a crisp approximation space, and a fuzzy rough set is a pair of fuzzy sets resulting from the approximation of a crisp set in a fuzzy approximation space. The approximation of a fuzzy set in a fuzzy approximation space leads to a more general framework. The results may be interpreted in three different ways.

1 INTRODUCTION

Theories of rough sets and fuzzy sets are distinct and complementary generalizations of set theory [20, 30]. A fuzzy set allows a membership value other than 0 and 1. A rough set uses three membership functions, a reference set and its

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lower and upper approximations in an approximation space. There are extensive studies on the relationships between rough sets and fuzzy sets [4, 21, 25, 26]. Many proposals have been made for the combination of rough and fuzzy sets. The results of these studies lead to the introduction of the notions of rough fuzzy sets and fuzzy rough sets [2, 3, 5, 6, 10, 11, 15, 16].

In the theory of fuzzy sets, two equivalent representations of fuzzy sets have been suggested [9]. A fuzzy set can be represented either by a membership function, or by a family of crisp sets called the α-level sets of the fuzzy set. In many situations, it may be more convenient and simpler to use the set-method, i.e., the use of α-level sets of a fuzzy set [13, 19]. In contrast to the functional approach, i.e., the use of membership function of a fuzzy set, the set-method has many advantages in the definition, analysis, and operation with fuzzy concepts [22, 23]. Most of the studies on the combination of rough and fuzzy sets are based on the functional approach. Nakamura [14, 15] used the α-level sets of a fuzzy similarity relation in the study of fuzzy rough sets. The use of set-method in the combination of rough and fuzzy sets is briefly described by Klir and Yuan in a more general framework [9] recently. One may expect the same advantages of set-method in studying these extended notions.

The present study examines some of fundamental issues in the combination of rough and fuzzy sets from the perspective of α-level sets. One of the main objectives is to identify the relationships among rough fuzzy sets, fuzzy rough sets, and crisp sets. This will help us understand the inherent structures of these extended sets. In particular, a rough fuzzy set is defined as the approximation of a fuzzy set in a crisp approximation space, while a fuzzy rough set as the approximation of a crisp set in a fuzzy approximation space. Another objective is to study a more general framework in which a fuzzy set is approximated in a fuzzy approximation space. The results of the approximation can be interpreted in three different ways, a family of rough sets, a family of rough fuzzy sets, and a family of fuzzy rough sets.

2 FUZZY SETS

Let U be a set called universe. A fuzzy set $\mathcal{F}$ on U is defined by a membership function $\mu_{\mathcal{F}} : U \longrightarrow [0, 1]$. A crisp set can be regarded as a special case of fuzzy sets in which the membership function is restricted to the extreme points $\{0, 1\}$ of $[0, 1]$. The membership function of a crisp set is also referred to as a characteristic function. Given a number $\alpha \in [0, 1]$, an α-cut, or α-level set, of a fuzzy set is defined by:

$$\mathcal{F}_\alpha = \{x \in U \mid \mu_\mathcal{F}(x) \geq \alpha\}, \tag{1.1}$$

which is a subset of U. A strong α-cut is defined by:

$$\mathcal{F}_{\alpha^+} = \{x \in U \mid \mu_\mathcal{F}(x) > \alpha\}. \tag{1.2}$$

Through either α-cuts or strong α-cuts, a fuzzy set determines a family of nested subsets of U. Conversely, a fuzzy set $\mathcal{F}$ can be reconstructed from its α-level sets as follows:

$$\mu_\mathcal{F}(x) = \sup\{\alpha \mid x \in \mathcal{F}_\alpha\}. \tag{1.3}$$

The fuzzy-set equality and inclusion are expressed component-wise as:

$$\begin{aligned} \mathcal{A} = \mathcal{B} &\iff \mu_\mathcal{A}(x) = \mu_\mathcal{B}(x), \quad \text{for all} \quad x \in U, \\ \mathcal{A} \subseteq \mathcal{B} &\iff \mu_\mathcal{A}(x) \leq \mu_\mathcal{B}(x), \quad \text{for all} \quad x \in U. \end{aligned} \tag{1.4}$$

Using the notion of α-level sets, they can be equivalently defined by:

$$\begin{aligned} \mathcal{A} = \mathcal{B} &\iff \mathcal{A}_\alpha = \mathcal{B}_\alpha, \quad \text{for all} \quad a \in [0,1], \\ \mathcal{A} \subseteq \mathcal{B} &\iff \mathcal{A}_\alpha \subseteq \mathcal{B}_\alpha, \quad \text{for all} \quad a \in [0,1]. \end{aligned} \tag{1.5}$$

Therefore, we can use either definition of fuzzy sets. Each of these two representations has its advantages in the study of fuzzy sets. One of the main advantage of set based representation is that it explicitly establishes a connection between fuzzy sets and crisp sets. Such a linkage shows the inherent structure of a fuzzy set.

For an arbitrary family of subsets of U, $(\mathcal{A}_\alpha)_\alpha$, $\alpha \in [0,1]$, there is no guarantee that $\mathcal{A}_\alpha$ will be the α-level set of a fuzzy set. The necessary and sufficient conditions on $(\mathcal{A}_\alpha)_\alpha$ are given in the following representation theorems proved by Negoita and Ralescu [17, 18, 22].

Theorem 1 *Let $(\mathcal{A}_\alpha)_\alpha$, $\alpha \in [0,1]$, be a family of subsets of U. The necessary and sufficient conditions for the existence of a fuzzy set $\mathcal{F}$ such that $\mathcal{F}_\alpha = \mathcal{A}_\alpha$, $\alpha \in [0,1]$, are:*

$$\text{(i)} \quad \alpha_1 \leq \alpha_2 \Longrightarrow \mathcal{A}_{\alpha_1} \supseteq \mathcal{A}_{\alpha_2},$$

$$\text{(ii)} \quad \alpha_1 \leq \alpha_2 \leq \ldots, \text{ and } \alpha_n \longrightarrow \alpha \Longrightarrow \bigcap_{n=1}^{\infty} \mathcal{A}_{\alpha_n} = \mathcal{A}_\alpha.$$

Theorem 2 *Let $\psi : [0,1] \longrightarrow [0,1]$ be a given function, and $(\mathcal{A}_\alpha)_\alpha$, $\alpha \in [0,1]$, be a family of subsets of U. The necessary and sufficient conditions for the existence of a fuzzy set $\mathcal{F}$ such that $\mathcal{F}_{\psi(\alpha)} = \mathcal{A}_\alpha$, $\alpha \in [0,1]$, are:*

(i′) $\psi(\alpha_1) \leq \psi(\alpha_2) \Longrightarrow \mathcal{A}_{\alpha_1} \supseteq \mathcal{A}_{\alpha_2}$,

(ii′) $\psi(\alpha_1) \leq \psi(\alpha_2) \leq \ldots,$ and $\psi(\alpha_n) \longrightarrow \psi(\alpha) \Longrightarrow \bigcap_{n=1}^{\infty} \mathcal{A}_{\alpha_n} = \mathcal{A}_\alpha$.

An implication of Theorem 1 is that the family of α-level sets of a fuzzy set satisfies conditions (i) and (ii).

There are a number of definitions for fuzzy-set complement, intersection, and union. We choose the standard max-min system proposed by Zadeh [30], in which fuzzy-set operations are defined component-wise as:

$$\begin{aligned} \mu_{\neg\mathcal{A}}(x) &= 1 - \mu_{\mathcal{A}}(x), \\ \mu_{\mathcal{A}\cap\mathcal{B}}(x) &= \min[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)], \\ \mu_{\mathcal{A}\cup\mathcal{B}}(x) &= \max[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)]. \end{aligned} \tag{1.6}$$

In terms of α-level sets, they can be expressed by:

$$\begin{aligned} (\neg\mathcal{A})_\alpha &= \neg\mathcal{A}_{(1-\alpha)^+}, \\ (\mathcal{A}\cap\mathcal{B})_\alpha &= \mathcal{A}_\alpha \cap \mathcal{B}_\alpha, \\ (\mathcal{A}\cup\mathcal{B})_\alpha &= \mathcal{A}_\alpha \cup \mathcal{B}_\alpha. \end{aligned} \tag{1.7}$$

An important feature of fuzzy-set operations is that they are truth-functional. One can obtain membership functions of the complement, intersection, and union of fuzzy sets based solely on the membership functions of the fuzzy sets involved.

3 ROUGH SETS

Let U denote a finite and non-empty set called the universe, and let $R \subseteq U \times U$ denote an equivalence relation on U, i.e., R is a reflexive, symmetric and transitive relation. If two elements x, y in U belong to the same equivalence class, i.e., xRy, we say that they are indistinguishable. The pair $apr_R = (U, R)$ is called an approximation space. The equivalence relation R partitions the set U into disjoint subsets. It defines the quotient set U/R consisting of equivalence classes of R. The equivalence class $[x]_R$ containing x plays dual roles. It is a subset of U if considered in relation to the universe, and an element of U/R if considered in relation to the quotient set. The empty set $\emptyset$ and equivalent classes are called the elementary sets. The union of one or more elementary sets is called a composed set. The family of all composed sets is denoted by Com(apr). It is a subalgebra of the Boolean algebra 2^U formed by the power set of U.

Given an arbitrary set $A \subseteq U$, it may not be possible to describe A precisely in the approximation space $apr_R = (U, R)$. Instead, one may only characterize A by a pair of lower and upper approximations. This leads to the concept of rough sets. In this study, a rough set is interpreted by three ordinary sets:

$$\begin{aligned}
\text{Reference set}: &\quad A \subseteq U, \\
\text{Lower approximation}: &\quad \underline{apr}_R(A) = \{x \in U \mid [x]_R \subseteq A\}, \\
\text{Upper approximation}: &\quad \overline{apr}_R(A) = \{x \in U \mid [x]_R \cap A \neq \emptyset\}. \qquad (1.8)
\end{aligned}$$

By definition, $\underline{apr}_R(A) \subseteq A \subseteq \overline{apr}_R(A)$ and $\overline{apr}_R(A) = \neg\underline{apr}_R(\neg A)$. The pair $(\underline{apr}_R(A), \overline{apr}_R(A))$ is called a rough set with a reference set A.

The characteristic functions of $\underline{apr}_R(A)$ and $\overline{apr}_R(A)$ are called strong and weak membership functions of a rough set [20]. Let μ_A and μ_R denote the membership functions of A and R, respectively. The physical meaning of lower and upper approximations may be understood better by the following two expressions:

$$\begin{aligned}
\mu_{\underline{apr}_R(A)}(x) &= \inf\{\mu_A(y) \mid y \in U, (x, y) \in R\}, \\
\mu_{\overline{apr}_R(A)}(x) &= \sup\{\mu_A(y) \mid y \in U, (x, y) \in R\}, \qquad (1.9)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\underline{apr}_R(A)}(x) &= \inf\{1 - \mu_R(x, y) \mid y \notin A\}, \\
\mu_{\overline{apr}_R(A)}(x) &= \sup\{\mu_R(x, y) \mid y \in A\}. \qquad (1.10)
\end{aligned}$$

For the two special sets $\emptyset$ and U, definition (1.10) is not defined. In this case, we simply define $\mu_{\underline{apr}_R(U)}(x) = 1$ and $\mu_{\overline{apr}_R(\emptyset)}(x) = 0$ for all $x \in U$. In subsequent discussion, we will not explicitly state these definitions for boundary cases. Based on the two equivalent definitions, lower and upper approximations may be interpreted as follows. An element x belongs to the lower approximation $\underline{apr}_R(A)$ if all elements equivalent to x belong to A. In other words, x belongs to the lower approximation of A if any element not in A is not equivalent to x, namely, $\mu_R(x, y) = 0$. An element x belongs to the upper approximation

approximation $\overline{apr}_R(A)$ if at least one element equivalent to x belongs to A. That is, x belongs to the upper approximation of A if any element in A is equivalent to x, namely, $\mu_R(x, y) = 1$. Therefore, the weak and strong membership functions of a rough set can be computed from the membership function of the reference set if the equivalence relation is used to select elements to be considered. Alternatively, they can also be computed from the membership function of the equivalent relation if the reference set is used to select elements to be considered. These two views are important in the combination of rough and fuzzy sets. For convenience, the strong and weak membership functions of a rough set can be expressed as:

$$\begin{aligned} \mu_{\underline{apr}_R(A)}(x) &= \inf\{\max[\mu_A(y), 1 - \mu_R(x, y)] \mid y \in U\}, \\ \mu_{\overline{apr}_R(A)}(x) &= \sup\{\min[\mu_A(y), \mu_R(x, y)] \mid y \in U\}. \end{aligned} \tag{1.11}$$

Although both membership functions of R and A are used, the inf and sup operations are in fact performed only on one membership function.

For two rough sets $(\underline{apr}_R(A), \overline{apr}_R(A))$ and $(\underline{apr}_R(B), \overline{apr}_R(B))$, their intersection and union are given by $(\underline{apr}_R(A \cap B), \overline{apr}_R(A \cap B))$ and $(\underline{apr}_R(A \cup B), \overline{apr}_R(A \cup B))$, with reference sets $A \cap B$ and $A \cup B$, respectively. The rough-set complement is defined by $(\underline{apr}_R(\neg A), \overline{apr}_R(\neg A))$, with a reference set $\neg A$. In contrast to fuzzy sets, rough-set intersection and union are not truth-functional as indicated by the properties:

$$\begin{array}{ll}
\text{(R0)} & \underline{apr}_R(\neg A) = \neg \overline{apr}_R(A), \\
 & \overline{apr}_R(\neg A) = \neg \underline{apr}_R(A), \\
\text{(R1)} & \underline{apr}_R(U) = U, \\
 & \overline{apr}_R(\emptyset) = \emptyset, \\
\text{(R2)} & \underline{apr}_R(A \cap B) = \underline{apr}_R(A) \cap \underline{apr}_R(B), \\
 & \overline{apr}_R(A \cup B) = \overline{apr}_R(A) \cup \overline{apr}_R(B), \\
 & \underline{apr}_R(A \cup B) \supseteq \underline{apr}_R(A) \cup \underline{apr}_R(B), \\
 & \overline{apr}_R(A \cap B) \subseteq \overline{apr}_R(A) \cap \overline{apr}_R(B), \\
\text{(R3)} & \underline{apr}_R(A) \subseteq A, \\
 & A \subseteq \overline{apr}_R(A), \\
\text{(R4)} & A \subseteq \underline{apr}_R(\overline{apr}_R(A)), \\
 & \overline{apr}_R(\underline{apr}_R(A)) \subseteq A, \\
\text{(R5)} & \underline{apr}_R(A) \subseteq \underline{apr}_R(\underline{apr}_R(A)), \\
 & \overline{apr}_R(\overline{apr}_R(A)) \subseteq \overline{apr}_R(A).
\end{array}$$

Property (R0) shows that lower and upper approximations are dual to each other. The above pairs of properties may be considered as dual properties. It is sufficient to only define one of the approximations and to define the other one using property (R0). The two conditions with equality sign in (R2) imply the other two conditions. They state that in general it is impossible to calculate the weak membership function of rough-set intersection and the strong membership function of rough-set union based only on the membership functions of two rough sets involved. One must also take into consideration the interaction between two reference sets, and their relationships to the equivalent classes of R. Properties (R0)-(R2) follow from the definition of lower and upper approximations. Property (R3) follows from the reflexivity of binary relations, property (R4) follows from the symmetry, and property (R5) follows from the transitivity. By removing the last two properties from (R2), properties (R1)-(R5) form an independent set. They are also sufficient in the sense that any other properties of rough sets can be derived from them [12].

Rough sets are monotonic with respect to set inclusion:

$$\begin{array}{ll} \text{(RM1)} & A \subseteq B \Longrightarrow \underline{apr}_R(A) \subseteq \underline{apr}_R(B), \\ \text{(RM2)} & A \subseteq B \Longrightarrow \overline{apr}_R(A) \subseteq \overline{apr}_R(B). \end{array}$$

Let R^1 and R^2 be two equivalence relations on U. R^1 is a refinement of R^2, or R^2 is a coarsening of R^1, if $R^1 \subseteq R^2$. A refinment relation further divides the equivalence classes of a coarsening relation. That is, R^1 is a refinement of R^2 if and only if $[x]_{R^1} \subseteq [x]_{R^2}$ for all $x \in U$. The finest equivalence relation is the identity relation, whereas the coarsest relation is the Cartesian product $U \times U$. Rough sets are monotonic with respect to refinement of equivalence relations. If an equivalence relation R^1 is a refinement of another equivalence relation R^2, for any $A \subseteq U$ we have:

$$\begin{array}{ll} \text{(rm1)} & R^1 \subseteq R^2 \Longrightarrow \underline{apr}_{R^1}(A) \supseteq \underline{apr}_{R^2}(A), \\ \text{(rm2)} & R^1 \subseteq R^2 \Longrightarrow \overline{apr}_{R^1}(A) \subseteq \overline{apr}_{R^2}(A). \end{array}$$

Approximation of a set in a refined approximation space is more accurate in the sense that both lower and upper approximations are closer to the set. The two monotonicities of rough sets are useful in the combination of rough and fuzzy sets.

4 COMBINATION OF ROUGH AND FUZZY SETS

The combination of rough and fuzzy sets leads to the notions of rough fuzzy sets and fuzzy rough sets. Different proposals have been suggested for defining such notions. Before presenting rigorous analysis of these concepts based on α-level sets, we briefly review the main results of existing studies.

4.1 Overview

Given an equivalence relation R on a universe U, it defines a quotient set U/R of equivalent classes. For any subset A of the universe, Dubois and Prade [6, 7] defined a rough set as a pair of subsets of U/R:

$$\begin{aligned} \underline{qapr}_R(A) &= \{[x]_R \mid [x]_R \subseteq A\}, \\ \overline{qapr}_R(A) &= \{[x]_R \mid [x]_R \cap A \neq \emptyset\}. \end{aligned} \tag{1.12}$$

The first $[x]_R$ is used as an element of U/R, and the second $[x]_R$ is used as a subset of U. The pair $(\underline{qapr}_R(A), \overline{qapr}_R(A))$ is called a rough set on U/R with reference set A. Although this definition differs from the original proposal of Pawlak [20], they are consistent with each other. Pawlak's lower and upper approximations may be viewed as extensions of $\underline{qapr}_R$ and $\overline{qapr}_R$:

$$\begin{aligned} \underline{apr}_R(A) &= \bigcup_{[x]_R \in \underline{qapr}_R(A)} [x]_R, \\ \overline{apr}_R(A) &= \bigcup_{[x]_R \in \overline{qapr}_R(A)} [x]_R. \end{aligned} \tag{1.13}$$

The notion of rough fuzzy sets defined by Dubois and Prade deals with the approximation of fuzzy sets in an approximation space [6, 7]. Given a fuzzy set $\mathcal{F}$, the result of approximation is a pair of fuzzy sets on the quotient set U/R:

$$\begin{aligned} \mu_{\underline{qapr}_R(\mathcal{F})}([x]_R) &= \inf\{\mu_{\mathcal{F}}(y) \mid y \in [x]_R\}, \\ \mu_{\overline{qapr}_R(\mathcal{F})}([x]_R) &= \sup\{\mu_{\mathcal{F}}(y) \mid y \in [x]_R\}. \end{aligned} \tag{1.14}$$

By using the extension principle, the pair can be extended to a pair of rough sets on the universe U:

$$\begin{aligned} \mu_{\underline{apr}_R(\mathcal{F})}(x) &= \inf\{\mu_{\mathcal{F}}(y) \mid y \in [x]_R\}, \\ \mu_{\overline{apr}_R(\mathcal{F})}(x) &= \sup\{\mu_{\mathcal{F}}(y) \mid y \in [x]_R\}. \end{aligned} \tag{1.15}$$

Similar to equation (1.11), they can be expressed as:

$$\begin{aligned} \mu_{\underline{apr}_R(\mathcal{F})}(x) &= \inf\{\max[\mu_{\mathcal{F}}(y), \mu_R(x,y)] \mid y \in U\}, \\ \mu_{\overline{apr}_R(\mathcal{F})}(x) &= \sup\{\min[\mu_{\mathcal{F}}(y), 1 - \mu_R(x,y)] \mid y \in U\}. \end{aligned} \quad (1.16)$$

The pair $(\underline{qapr}_R(\mathcal{F}), \overline{qapr}_R(\mathcal{F}))$ is called a rough fuzzy set on U/R, and the pair $(\underline{apr}_R(\mathcal{F}), \overline{apr}_R(\mathcal{F}))$ is called a rough fuzzy set on U, with reference fuzzy set $\mathcal{F}$.

The notion of fuzzy rough sets defined by Dubois and Prade [6] is originated from Willaeys and Malvache [24] for defining a fuzzy set with respect to a family of fuzzy sets. It deals with the approximation of fuzzy sets in a fuzzy approximation space defined by a fuzzy similarity relation $\Re$ or defined by a fuzzy partition. We only review the results obtained from a fuzzy similarity relation. A fuzzy similarity relation $\Re$ is a fuzzy subset of $U \times U$ and has three properties:

$$\begin{aligned} &\text{reflexivity}: \quad \text{for all } x \in U, \mu_{\Re}(x,x) = 1, \\ &\text{symmetry}: \quad \text{for all } x, y \in U, \mu_{\Re}(x,y) = \mu_{\Re}(y,x), \\ &\text{transitivity}: \quad \text{for all } x, y, x \in U, \mu_{\Re}(x,z) \geq \min[\mu_{\Re}(x,y), \mu_{\Re}(y,z)]. \end{aligned}$$

Given a fuzzy similarity relation $\Re$, the pair $apr_{\Re} = (U, \Re)$ is called a fuzzy approximation space. A fuzzy similarity relation can be used to define a fuzzy partition of the universe. A fuzzy equivalence class $[x]_{\Re}$ of elements close to x is defined by:

$$\mu_{[x]_{\Re}}(y) = \mu_{\Re}(x,y). \quad (1.17)$$

The family of all fuzzy equivalence classes is denoted by $U/\Re$. For a fuzzy set $\mathcal{F}$, its approximation in $apr_{\Re}$ is called a fuzzy rough set, which is a pair of fuzzy sets on $U/\Re$:

$$\begin{aligned} \mu_{\underline{qapr}_{\Re}(\mathcal{F})}([x]_{\Re}) &= \inf\{\max[\mu_{\mathcal{F}}(y), 1 - \mu_{[x]_{\Re}}(y)] \mid y \in U\}, \\ \mu_{\overline{qapr}_{\Re}(\mathcal{F})}([x]_{\Re}) &= \sup\{\min[\mu_{\mathcal{F}}(y), \mu_{[x]_{\Re}}(y)] \mid y \in U\}. \end{aligned} \quad (1.18)$$

They can be extended to a pair of fuzzy sets on the universe:

$$\begin{aligned} \mu_{\underline{apr}_{\Re}(\mathcal{F})}(x) &= \inf\{\max[\mu_{\mathcal{F}}(y), 1 - \mu_{\Re}(x,y)] \mid y \in U\}, \\ \mu_{\overline{apr}_{\Re}(\mathcal{F})}(x) &= \sup\{\min[\mu_{\mathcal{F}}(y), \mu_{\Re}(x,y)] \mid y \in U\}. \end{aligned} \quad (1.19)$$

The approximation of a crisp set in a fuzzy approximation space may be considered as a special case. By comparing equations (1.16) and (1.19), one can conclude that rough fuzzy sets are special cases of fuzzy rough sets as defined by Dubois and Prade. Although the names of rough fuzzy sets and fuzzy rough sets are symmetric, the role played by them are not symmetric.

Nakamura [14, 15] defined a fuzzy rough set by using a family of equivalence relations induced by different level sets of a fuzzy similarity relation $\Re$. For a $\beta \in [0,1]$, the level set $\Re_\beta$ is an equivalence relation. It defines an approximation space $apr_{\Re_\beta} = (U, \Re_\beta)$. The approximation of a fuzzy set $\mathcal{F}$ in $apr_{\Re_\beta}$ turns out to be a rough fuzzy set $(\underline{apr}_{\Re_\beta}(\mathcal{F}), \overline{apr}_{\Re_\beta}(\mathcal{F}))$. The family of rough fuzzy sets, $(\underline{apr}_{\Re_\beta}(\mathcal{F}), \overline{apr}_{\Re_\beta}(\mathcal{F}))$, $\beta \in [0,1]$, is related to a fuzzy rough set of Dubois and Prade [6, 7]. Lin [11] studied the concept of fuzzy rough sets from the view point of the topology of function spaces. However, it needs to be clarified that fuzzy rough sets called by Lin are in fact rough fuzzy sets called by Dubois and Prade.

All above proposals agree with Pawlak's formulation regarding the interpretation of rough-set intersection and union. These operations are defined based on the approximations of the intersection and union of reference sets or fuzzy sets. Some other studies on the combination of rough and fuzzy sets do not have this feature. Iwinski [8] suggested an alternative definition of rough sets, which is related to but quite different from the one proposed by Pawlak [20]. An Iwinski rough set is defined as a pair of subsets taking from a sub-Boolean algebra of 2^U, without reference to a subset of the universe. For simplicity, we consider the sub-Boolean algebra formed by the set of all composed sets $\mathrm{Com}(apr)$. An Iwinski rough set is defined as pair of sets (A_L, A_U) with $A_L \subseteq A_U$ from $\mathrm{Com}(apr)$. We may refer to A_L and A_U as lower and upper bounds, respectively. The intersection and union are defined component-wise as:

$$\begin{aligned}(A_L, A_U) \cap (B_L, B_U) &= (A_L \cap B_L, A_U \cap B_U),\\ (A_L, A_U) \cup (B_L, B_U) &= (A_L \cup B_L, A_U \cup B_U).\end{aligned} \tag{1.20}$$

One of the difficulties with such a definition is that the physical meaning of (A_L, A_U) is not entirely clear. This notion may perhaps be interpreted in relation to the concept of interval sets [27]. Biswas [3] adopted the same definition of rough fuzzy sets from Dubois and Prade. For rough-fuzzy-set intersection and union, a definition similar to that of Iwinski is used. Their use of two different models may lead to inconsistency in the interpretation of the concepts involved.

Nanda and Majumdar [16] suggested a different proposal for the definition of fuzzy rough sets by extending the work of Iwinski. Their definition is based on a fuzzification of the lower and upper bounds of Iwinski rough sets. It may be related to the concept of interval-valued fuzzy sets, also known as Φ-fuzzy sets [5, 31]. The same definition was also used by Biswas [2].

Kuncheva [10] defined the notion of fuzzy rough sets which models the approximation of a fuzzy set based on a weak fuzzy partition. It uses measures of fuzzy-set inclusion. A number of of different definitions may indeed be obtained with various measures of fuzzy-set inclusion. The intersection and union operations were not explicitly discussed by Kuncheva. This model is different from the above mentioned works. It is related to the probabilistic rough set model [29] and the variable precision rough set model [32].

The review of existing results shows that the same notions of rough fuzzy set and fuzzy rough sets are used with different meanings by different authors. The functional approaches clearly define various notions mathematically. However, the physical meanings of these notions are not clearly interpreted. In the rest of this section, we attempt to address these issues. The approximation of a fuzzy set in a crisp approximation space is called a rough fuzzy set, to be consistent with the naming of rough set as the approximation of a crisp set in a crisp approximation space. The approximation of a crisp set in a fuzzy approximation space is called a fuzzy rough set. Such a naming scheme has been used by Klir and Yuan [9], and Yao [28]. Under this scheme, these two models are complementary to each other, in a similar way that rough sets and fuzzy sets complementary to each other. In contrast to the proposal of Dubois and Prade [7], rough fuzzy sets are not considered as special cases of fuzzy rough sets. As a result, the framework of the approximation of a fuzzy set in a fuzzy approximation space is considered to be a more general model which unifies rough fuzzy sets and fuzzy rough sets. All these notions are interpreted based on the concept of α-level sets, which may be useful for their successful applications.

4.2 Approximation of fuzzy sets in crisp approximation spaces: rough fuzzy sets

Consider the approximation of a fuzzy set $\mathcal{F} = (\mathcal{F}_\alpha)_\alpha$, $\alpha \in [0, 1]$, in an approximation space $apr_R = (U, R)$, where R is an equivalence relation. For each α-level set $\mathcal{F}_\alpha$, we have a rough set:

$$
\begin{aligned}
\text{Reference set}: \quad & \mathcal{F}_\alpha, \\
\text{Lower approximation}: \quad & \underline{apr}_R(\mathcal{F}_\alpha) = \{x \in U \mid [x]_R \subseteq \mathcal{F}_\alpha\}, \\
\text{Upper approximation}: \quad & \overline{apr}_R(\mathcal{F}_\alpha) = \{x \in U \mid [x]_R \cap \mathcal{F}_\alpha \neq \emptyset\}. \quad (1.21)
\end{aligned}
$$

That is, $(\underline{apr}_R(\mathcal{F}_\alpha), \overline{apr}_R(\mathcal{F}_\alpha))$ is a rough set with reference set $\mathcal{F}_\alpha$. For the family of α-level sets, we have a family of lower and upper approximations, $(\underline{apr}_R(\mathcal{F}_\alpha))_\alpha$ and $(\overline{apr}_R(\mathcal{F}_\alpha))_\alpha$, $\alpha \in [0,1]$. A crucial question is whether they are the families of α-level sets of two fuzzy sets. Since the family $\mathcal{F}_\alpha$, $\alpha \in [0,1]$, is constructed from a fuzzy set $\mathcal{F}$, we have $\alpha_1 \leq \alpha_2 \Longrightarrow \mathcal{F}_{\alpha_1} \supseteq \mathcal{F}_{\alpha_2}$. By the monotonicity of lower and upper approximations with respect to set inclusion, i.e., properties (RM1) and (RM2), one can conclude that both $(\underline{apr}_R(\mathcal{F}_\alpha))_\alpha$ and $(\overline{apr}_R(\mathcal{F}_\alpha))_\alpha$ satisfy condition (i). The α-level sets of the fuzzy set $\mathcal{F}$ satisfy condition (ii). This implies that both families, $(\underline{apr}_R(\mathcal{F}_\alpha))_\alpha$ and $(\overline{apr}_R(\mathcal{F}_\alpha))_\alpha$, $\alpha \in [0,1]$, satisfy condition (ii). By Theorem 1, they define a pair of fuzzy sets $\underline{apr}_R(\mathcal{F})$ and $\overline{apr}_R(\mathcal{F})$ such that,

$$\begin{aligned}
(\underline{apr}_R(\mathcal{F}))_\alpha &= \underline{apr}_R(\mathcal{F}_\alpha), \\
(\overline{apr}_R(\mathcal{F}))_\alpha &= \overline{apr}_R(\mathcal{F}_\alpha).
\end{aligned} \tag{1.22}$$

They are defined by the following membership functions:

$$\begin{aligned}
\mu_{\underline{apr}_R(\mathcal{F})}(x) &= \sup\{\alpha \mid x \in (\underline{apr}_R(\mathcal{F}))_\alpha\} \\
&= \sup\{\alpha \mid \underline{apr}_R(\mathcal{F}_\alpha)\} \\
&= \sup\{\alpha \mid [x]_R \subseteq \mathcal{F}_\alpha\}, \\
\mu_{\overline{apr}_R(\mathcal{F})}(x) &= \sup\{\alpha \mid x \in (\overline{apr}_R(\mathcal{F}))_\alpha\} \\
&= \sup\{\alpha \mid x \in \overline{apr}_R(\mathcal{F}_\alpha)\} \\
&= \sup\{\alpha \mid [x]_R \cap \mathcal{F}_\alpha \neq \emptyset\}.
\end{aligned} \tag{1.23}$$

For an equivalence class $[x]_R$, $[x]_R \subseteq \mathcal{F}_\alpha$ if and only if $\mu_\mathcal{F}(y) \geq \alpha$ for all $y \in [x]_R$, and $[x]_R \cap \mathcal{F}_\alpha \neq \emptyset$ if and only if there exists a $y \in [x]_R$ such that $\mu_\mathcal{F}(y) \geq \alpha$. Therefore, the membership value of x belonging to $\underline{apr}_R(\mathcal{F})$ is the minimum of membership values of elements in the equivalent class containing x, and the membership value of x belonging to $\overline{apr}_R(\mathcal{F})$ is the maximum. They can be equivalently defined by:

$$\begin{aligned}
\mu_{\underline{apr}_R(\mathcal{F})}(x) &= \sup\{\alpha \mid [x]_R \subseteq \mathcal{F}_\alpha\} \\
&= \sup\{\alpha \mid \text{for all } y, y \in [x]_R \Longrightarrow \mu_\mathcal{F}(y) \geq \alpha\} \\
&= \inf\{\mu_\mathcal{F}(y) \mid y \in [x]_R\} \\
&= \inf\{\mu_\mathcal{F}(y) \mid (x,y) \in R\} \\
&= \inf\{\max[\mu_\mathcal{F}(y), 1 - \mu_R(x,y)] \mid y \in U\},
\end{aligned}$$

$$\begin{aligned}
\mu_{\overline{apr}_R(\mathcal{F})}(x) &= \sup\{\alpha \mid [x]_R \cap \mathcal{F}_\alpha \neq \emptyset\} \\
&= \sup\{\alpha \mid \text{there exists a } y \text{ such that } y \in [x]_R \text{ and } \mu_{\mathcal{F}}(y) \geq \alpha\} \\
&= \sup\{\mu_{\mathcal{F}}(y) \mid y \in [x]_R\} \\
&= \sup\{\mu_{\mathcal{F}}(y) \mid (x,y) \in R\} \\
&= \sup\{\min(\mu_{\mathcal{F}}(y), \mu_R(x,y)) \mid y \in U\}. \qquad (1.24)
\end{aligned}$$

They may be considered as a generalization of a rough set based on the interpretation of rough sets given by equation (1.9). Moreover, these membership functions can be expressed conveniently by the same formula (1.16).

The use α-level sets provides a clear interpretation of rough fuzzy sets. A fuzzy set $\mathcal{F}$ is described by a pair of fuzzy sets in an approximation space. It lies between the lower and upper approximations $\underline{apr}_R(\mathcal{F})$ and $\overline{apr}_R(\mathcal{F})$. We call the pair $(\underline{apr}_R(\mathcal{F}), \overline{apr}_R(\mathcal{F}))$ a rough fuzzy set with a reference fuzzy set $\mathcal{F}$. In other words, a rough fuzzy set is characterized by three fuzzy sets:

$$\begin{aligned}
\text{Reference fuzzy set}: &\quad \mu_{\mathcal{F}}, \\
\text{Lower approximation}: &\quad \mu_{\underline{apr}_R(\mathcal{F})}(x) = \inf\{\mu_{\mathcal{F}}(y) \mid y \in U, (x,y) \in R\}, \\
\text{Upper approximation}: &\quad \mu_{\overline{apr}_R(\mathcal{F})}(x) = \sup\{\mu_{\mathcal{F}}(y) \mid y \in U, (x,y) R\}. \qquad (1.25)
\end{aligned}$$

An α-level set of a rough fuzzy set is defined by in terms of the α-level sets of a fuzzy set $\mathcal{F}$:

$$\begin{aligned}
(\underline{apr}_R(\mathcal{F}), \overline{apr}_R(\mathcal{F}))_\alpha &= (\underline{apr}_R(\mathcal{F}_\alpha), \overline{apr}_R(\mathcal{F}_\alpha)) \\
&= ((\underline{apr}_R(\mathcal{F}))_\alpha, (\overline{apr}_R(\mathcal{F}))_\alpha), \qquad (1.26)
\end{aligned}$$

which is a rough set. By combining the results given in equations (1.5), (1.7), and the properties (R0)-(R5) of rough sets, rough fuzzy sets have properties: for two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$,

(RF0) $\underline{apr}_R(\neg\mathcal{A}) = \neg\overline{apr}_R(\mathcal{A})$,
$\overline{apr}_R(\neg\mathcal{A}) = \neg\underline{apr}_R(\mathcal{A})$,

(RF1) $\underline{apr}_R(U) = U$,
$\overline{apr}_R(\emptyset) = \emptyset$,

(RF2) $\underline{apr}_R(\mathcal{A} \cap \mathcal{B}) = \underline{apr}_R(\mathcal{A}) \cap \underline{apr}_R(\mathcal{B})$,
$\overline{apr}_R(\mathcal{A} \cup \mathcal{B}) = \overline{apr}_R(\mathcal{A}) \cup \overline{apr}_R(\mathcal{B})$,
$\underline{apr}_R(\mathcal{A} \cup \mathcal{B}) \supseteq \underline{apr}_R(\mathcal{A}) \cup \underline{apr}_R(\mathcal{B})$,
$\overline{apr}_R(\mathcal{A} \cap \mathcal{B}) \subseteq \overline{apr}_R(\mathcal{A}) \cap \overline{apr}_R(\mathcal{B})$,

(RF3) $\underline{apr}_R(\mathcal{A}) \subseteq \mathcal{A}$,
$\mathcal{A} \subseteq \overline{apr}_R(\mathcal{A})$,

$$
\begin{array}{ll}
\text{(RF4)} & \mathcal{A} \subseteq \underline{apr}_R(\overline{apr}_R(\mathcal{A})), \\
 & \overline{apr}_R(\underline{apr}_R(\mathcal{A})) \subseteq \mathcal{A}, \\
\text{(RF5)} & \underline{apr}_R(\mathcal{A}) \subseteq \underline{apr}_R(\underline{apr}_R(\mathcal{A})), \\
 & \overline{apr}_R(\overline{apr}_R(\mathcal{A})) \subseteq \overline{apr}_R(\mathcal{A}).
\end{array}
$$

Rough fuzzy sets are monotonic with respect to fuzzy set inclusion: namely, for two fuzzy sets $\mathcal{A}, \mathcal{B}$,

$$
\begin{array}{ll}
\text{(RFM1)} & \mathcal{A} \subseteq \mathcal{B} \Longrightarrow \underline{apr}_R(\mathcal{A}) \subseteq \underline{apr}_R(\mathcal{B}), \\
\text{(RFM2)} & \mathcal{A} \subseteq \mathcal{B} \Longrightarrow \overline{apr}_R(\mathcal{A}) \subseteq \overline{apr}_R(\mathcal{B}).
\end{array}
$$

They are also monotonic with respect to refinement of equivalence relations. For two equivalence relations R^1 and R^2 and a fuzzy set $\mathcal{F}$, we have:

$$
\begin{array}{ll}
\text{(rfm1)} & R^1 \subseteq R^2 \Longrightarrow \underline{apr}_{R^1}(\mathcal{F}) \supseteq \underline{apr}_{R^2}(\mathcal{F}), \\
\text{(rfm2)} & R^1 \subseteq R^2 \Longrightarrow \overline{apr}_{R^1}(\mathcal{F}) \subseteq \overline{apr}_{R^2}(\mathcal{F}).
\end{array}
$$

4.3 Approximation of crisp sets in fuzzy approximation spaces: fuzzy rough sets

The concept of approximation spaces can be generalized by using fuzzy relations [1, 6]. Consider a fuzzy approximation space $apr_{\Re} = (U, \Re)$, where $\Re$ is a fuzzy similarity relation. Each of $\Re$'s β-level sets is an equivalence relation [15]. One can represent $\Re$ by a family of equivalence relations:

$$\Re = (\Re_\beta)_\beta, \qquad \beta \in [0, 1]. \tag{1.27}$$

This family defines a family of approximation spaces:

$$apr_{\Re} = (apr_{\Re_\beta} = (U, \Re_\beta))_\beta, \qquad \beta \in [0, 1]. \tag{1.28}$$

Given a subset A of U, consider its approximation in each of the approximation spaces. For a $\beta \in [0, 1]$, we have a rough set:

$$
\begin{array}{rl}
\text{Reference set :} & A \subseteq U, \\
\text{Lower approximation :} & \underline{apr}_{\Re_\beta}(A) = \{x \in U \mid [x]_{\Re_\beta} \subseteq A\}, \\
\text{Upper approximation :} & \overline{apr}_{\Re_\beta}(A) = \{x \in U \mid [x]_{\Re_\beta} \cap A \neq \emptyset\}.
\end{array} \tag{1.29}
$$

With respect to a fuzzy approximation space, we obtain a family of rough sets:

$$(\underline{apr}_{\Re_\beta}(A), \overline{apr}_{\Re_\beta}(A))_\beta, \qquad \beta \in [0, 1]. \tag{1.30}$$

Consider the family of lower approximations $(\underline{apr}_{\Re_\beta}(A))_\beta$, $\beta \in [0,1]$. Recall that $\Re_\beta$'s are derived from a fuzzy similarity relation $\Re$. If $\beta_2 \leq \beta_1$, then $\Re_{\beta_1} \subseteq \Re_{\beta_2}$, i.e., $\Re_{\beta_1}$ is a refinement of $\Re_{\beta_2}$. By property (rm1), it follows that $\underline{apr}_{\Re_{\beta_1}}(A) \supseteq \underline{apr}_{\Re_{\beta_2}}(A)$. Let $\psi(\beta) = 1 - \beta$. We have $\psi(\beta_1) \leq \psi(\beta_2) \Longrightarrow \underline{apr}_{\Re_{\beta_1}}(A) \supseteq \underline{apr}_{\Re_{\beta_2}}(A)$. Therefore, property (i′) holds. Since $\Re_\beta$'s are derived from a fuzzy similarity relation $\Re$, they satisfy property (ii) in Theorem 1. Combining this result with the definition of lower approximation and property (rm1), one can conclude:

$$\psi(\beta_1) \leq \psi(\beta_2) \leq \ldots \quad \text{and} \quad \psi(\beta_n) \longrightarrow \psi(\beta) \Longrightarrow \bigcap_{n=1}^{\infty} \underline{apr}_{\Re_{\beta_n}}(A) = \underline{apr}_{\Re_\beta}(A). \tag{1.31}$$

Hence, property (ii′) holds. By Theorem 2, there exists a fuzzy set $\underline{apr}_{\Re}(A)$ such that $(\underline{apr}_{\Re}(A))_{\psi(\beta)} = \underline{apr}_{\Re_\beta}(A)$. Similarly, one can use Theorem 1 to show the existence of a fuzzy set $\overline{apr}_{\Re}(A)$ for the family of upper approximations $(\overline{apr}_{\Re_\beta}(A))_\beta$ such that $(\overline{apr}_{\Re}(A))_\beta = \overline{apr}_{\Re_\beta}(A)$. In this case, property (rm2) is used.

The membership functions of the derived two fuzzy sets are given by:

$$\begin{aligned}
\mu_{\underline{apr}_{\Re}(A)}(x) &= \sup\{\psi(\beta) \mid x \in (\underline{apr}_{\Re}(A))_{\psi(\beta)}\} \\
&= \sup\{1-\beta \mid x \in \underline{apr}_{\Re_\beta}(A)\} \\
&= \sup\{1-\beta \mid [x]_{\Re_\beta} \subseteq A\} \\
&= \sup\{1-\beta \mid \text{for all } y, \mu_{\Re}(x,y) \geq \beta \Longrightarrow y \in A\} \\
&= \sup\{1-\beta \mid \text{for all } y, y \notin A \Longrightarrow \mu_{\Re}(x,y) < \beta\} \\
&= \inf\{1-\mu_{\Re}(x,y) \mid y \notin A\} \\
&= \inf\{\max[\mu_A(y), 1-\mu_{\Re}(x,y)] \mid y \in U\}, \\
\mu_{\overline{apr}_{\Re}(A)}(x) &= \sup\{\beta \mid x \in (\overline{apr}_{\Re}(A))_\beta\} \\
&= \sup\{\beta \mid x \in \overline{apr}_{\Re_\beta}(A)\} \\
&= \sup\{\beta \mid [x]_{\Re_\beta} \cap A \neq \emptyset\} \\
&= \sup\{\beta \mid \text{these exists a } y \text{ such that } \mu_{\Re}(x,y) \geq \beta \text{ and } y \in A\} \\
&= \sup\{\mu_{\Re}(x,y) \mid y \in A\} \\
&= \sup\{\min[\mu_A(y), \mu_{\Re}(x,y)] \mid y \in U\}.
\end{aligned} \tag{1.32}$$

They may be regarded as a generalization of rough set according to the interpretation given by equation (1.10). They also conform to the general formula (1.16).

We call the pair of fuzzy sets $(\underline{apr}_{\Re}(A), \overline{apr}_{\Re}(A))$ a fuzzy rough sets with reference set A. A fuzzy rough set is characterized by a crisp set and two fuzzy sets:

$$\begin{aligned} \text{Reference set}: &\quad A \subseteq U, \\ \text{Lower approximation}: &\quad \mu_{\underline{apr}_{\Re}(A)}(x) = \inf\{1 - \mu_{\Re}(x,y) \mid y \notin A\}, \\ \text{Upper approximation}: &\quad \mu_{\overline{apr}_{\Re}(A)}(x) = \sup\{\mu_{\Re}(x,y) \mid y \in A\}. \end{aligned} \tag{1.33}$$

An β-level set of a fuzzy rough sets is in terms of the β-level sets of the fuzzy similarity relation as:

$$\begin{aligned} (\underline{apr}_{\Re}(A), \overline{apr}_{\Re}(A))_{\beta} &= (\underline{apr}_{\Re_{\beta}}(A), \overline{apr}_{\Re_{\beta}}(A)) \\ &= ((\underline{apr}_{\Re}(A))_{(1-\beta)}, (\overline{apr}_{\Re}(A))_{\beta}), \end{aligned} \tag{1.34}$$

which is a rough set with reference set A in the approximation space $apr_{\Re_{\beta}} = (U, \Re_{\beta})$.

Based on the properties of rough sets, one can see that fuzzy rough sets satisfy the properties: for $A, B \subseteq U$,

$$\begin{array}{ll} \text{(FR0)} & \underline{apr}_{\Re}(\neg A) = \neg \overline{apr}_{\Re}(A), \\ & \overline{apr}_{\Re}(\neg A) = \neg \underline{apr}_{\Re}(A), \\ \text{(FR1)} & \underline{apr}_{\Re}(U) = U, \\ & \overline{apr}_{\Re}(\emptyset) = \emptyset, \\ \text{(FR2)} & \underline{apr}_{\Re}(A \cap B) = \underline{apr}_{\Re}(A) \cap \underline{apr}_{\Re}(B), \\ & \overline{apr}_{\Re}(A \cup B) = \overline{apr}_{\Re}(A) \cup \overline{apr}_{\Re}(B), \\ & \underline{apr}_{\Re}(A \cup B) \supseteq \underline{apr}_{\Re}(A) \cup \underline{apr}_{\Re}(B), \\ & \overline{apr}_{\Re}(A \cap B) \subseteq \overline{apr}_{\Re}(A) \cap \overline{apr}_{\Re}(B), \\ \text{(FR3)} & \underline{apr}_{\Re}(A) \subseteq A, \\ & A \subseteq \overline{apr}_{\Re}(A). \end{array}$$

For fuzzy rough sets, we do not have properties similar to (R4) and (R5), or (RF4) and (RF5). This stems from the fact the result of approximating a crisp set is a pair of fuzzy sets. Further approximations of the resulting fuzzy sets are not defined in this framework.

Fuzzy rough sets are monotonic with respect to set inclusion:

$$\begin{array}{ll} \text{(FRM1)} & A \subseteq B \Longrightarrow \underline{apr}_{\Re}(A) \subseteq \underline{apr}_{\Re}(B), \\ \text{(FRM2)} & A \subseteq B \Longrightarrow \overline{apr}_{\Re}(A) \subseteq \overline{apr}_{\Re}(B). \end{array}$$

They are monotonic with respect to the refinement of fuzzy similarity relations. A fuzzy similarity relation $\Re^1$ is a refinement of another fuzzy similarity relation $\Re^2$ if $\Re^1 \subseteq \Re^2$, which is a straightforward generalization of the refinement of crisp relations. The monotonicity of fuzzy rough sets with respect to refinement of fuzzy similarity relation can be expressed as:

$$\text{(frm1)} \qquad \Re^1 \subseteq \Re^2 \Longrightarrow \underline{apr}_{\Re^1}(A) \supseteq \underline{apr}_{\Re^2}(A),$$

$$\text{(frm2)} \qquad \Re^1 \subseteq \Re^2 \Longrightarrow \overline{apr}_{\Re^1}(A) \subseteq \overline{apr}_{\Re^2}(A).$$

4.4 Approximation of fuzzy sets in fuzzy approximation spaces

This section examines the approximation of a fuzzy set in a fuzzy approximation space. In this framework, on the one hand, we have a family of α-level sets $(\mathcal{F}_\alpha)_\alpha$, $\alpha \in [0,1]$, representing a fuzzy set $\mathcal{F}$, on the other hand, we have a family of β-level sets $(\Re_\beta)_\beta$, $\beta \in [0,1]$, prepresenting a fuzzy similarity relation $\Re$. Each α-level set $\mathcal{F}_\alpha$ is a crisp set, and each β-leval relation $\Re_\beta$ is an equivalence relation. Rough sets, rough fuzzy sets, and fuzzy rough sets can therefore be viewed as special cases of the generalized model.

For a fixed pair of numbers $(\alpha, \beta) \in [0,1] \times [0,1]$, we obtain a submodel in which a crisp set $\mathcal{F}_\alpha$ is approximated in a crisp approximation space $apr_{\Re_\beta} = (U, \Re_\beta)$. The result is a rough set $(\underline{apr}_{\Re_\beta}(\mathcal{F}_\alpha), \overline{apr}_{\Re_\beta}(\mathcal{F}_\alpha))$ with the reference set $\mathcal{F}_\alpha$. For a fixed β, we obtain a submodel in which a fuzzy set $(\mathcal{F}_\alpha)_\alpha$, $\alpha \in [0,1]$, is approximated in a crisp approximation space $apr_{\Re_\beta} = (U, \Re_\beta)$. The result is a rough fuzzy set $(\underline{apr}_{\Re_\beta}(\mathcal{F}), \overline{apr}_{\Re_\beta}(\mathcal{F}))$ with the reference fuzzy set $\mathcal{F}$. On the other hand, for a fixed α, we obtain a submodel in which a crisp set $\mathcal{F}_\alpha$ is approximated in a fuzzy approximation space $(apr_{\Re_\beta} = (U, \Re_\beta))_\beta$, $\beta \in [0,1]$. The result is a fuzzy rough set $(\underline{apr}_{\Re}(\mathcal{F}_\alpha), \overline{apr}_{\Re}(\mathcal{F}_\alpha))$ with the reference set $\mathcal{F}_\alpha$. In the generalized model, both α and β are not fixed. The result may be interpreted in three different views.

A family of rough sets: The first interpretation is based on a family of rough sets:

$$(\underline{apr}_{\Re_\beta}(\mathcal{F}_\alpha), \overline{apr}_{\Re_\beta}(\mathcal{F}_\alpha)), \qquad \alpha \in [0,1], \beta \in [0,1], \tag{1.35}$$

which represents the rough set approximation of each α-level set of a fuzzy set $\mathcal{F}$ in an approximation space induced by an β-level relation of a fuzzy similarity relation $\Re$. Under this interpretation, the relationships between different α-

level sets of $\mathcal{F}$, and the relationships between different β-level relations of $\Re$, are not taken into consideration.

A family of rough fuzzy sets: In the second view, we consider the following family of rough fuzzy sets:

$$(\underline{apr}_{\Re_\beta}(\mathcal{F}), \overline{apr}_{\Re_\beta}(\mathcal{F}))_\beta, \qquad \beta \in [0, 1], \tag{1.36}$$

which takes into consideration the relationships between different α-level sets of a fuzzy set $\mathcal{F}$. The relationships between different β-level relations of a fuzzy similarity relation $\Re$ are not considered.

A family of fuzzy rough sets: By employing the relationship between different β-level relations of a fuzzy relation $\Re$, we obtain a family of fuzzy rough sets:

$$(\underline{apr}_{\Re}(\mathcal{F}_\alpha), \overline{apr}_{\Re}(\mathcal{F}_\alpha)), \qquad \alpha \in [0, 1]. \tag{1.37}$$

It does not take account the relationships between different α-level sets of a fuzzy set $\mathcal{F}$.

The above interpretations depend on the ways in which the family of rough sets $(\underline{apr}_{\Re_\beta}(\mathcal{F}_\alpha), \overline{apr}_{\Re_\beta}(\mathcal{F}_\alpha))$, $\alpha \in [0, 1]$, $\beta \in [0, 1]$, are grouped. An interesting problem is how to take into consideration both relationships between different α-level sets of fuzzy sets, and the relationships between different β-level relations of fuzzy similarity relations. By comparing equations (1.11), (1.24), and (1.32), one can conclude that the membership functions of rough sets, rough fuzzy sets, and fuzzy rough sets can be computed uniformly using the same scheme:

$$\begin{aligned} \mu_{\underline{apr}_\Gamma(\Delta)}(x) &= \inf\{\max[\mu_\Delta(y), 1 - \mu_\Gamma(x, y)] \mid y \in U\}, \\ \mu_{\overline{apr}_\Gamma(\Delta)}(x) &= \sup\{\min[\mu_\Delta(y), \mu_\Gamma(x, y)] \mid y \in U\}, \end{aligned} \tag{1.38}$$

where Γ is a variable that takes either an equivalence relation or a fuzzy similarity relation as its value, and Δ is a variable that takes either a crisp set or a fuzzy set as its value. The same scheme is used by Dubois and Prade [6] to define a pair of fuzzy sets as the result of approximating a fuzzy set in a fuzzy approximation space. This involves the combination of degrees of memberships of a fuzzy set and a fuzzy similarity relation. The physical meaning is not entirely clear. It is questionable that an element with α degree membership belonging to a fuzzy set would have the same physical interpretation as a pair with α degree membership belonging to a fuzzy relation, as the universes of the former and latter are quit different. For this reason, in this study we do not mix the membership functions of a fuzzy set and a fuzzy similarity relation. As seen from equations (1.11), (1.24), and (1.32), the inf and sup operations are indeed performed on one membership function. The use of other membership function is only for the seek of convenience.

5 CONCLUSION

A rough set is the approximation of a crisp set in a crisp approximation space. It is a pair of crisp set. A rough fuzzy set is derived from the approximation of a fuzzy set in a crisp approximation space. It is a pair of fuzzy sets in which all elements in the same equivalence class have the same membership. The membership of an element is determined by the original memberships of all those elements equivalent to that element. A fuzzy rough set is derived from the approximation of a crisp set in a fuzzy approximation space. It is a pair of fuzzy sets in which the membership of an element is determined by the degrees of similarity of all those elements in the set. By combining these submodels, we have proposed a more generalized model. In this model, we have studied the approximation of a fuzzy set in a fuzzy approximation space. The result of such an approximation is interpreted from three different point of views, a family of rough sets, a family of rough fuzzy sets, and a family of fuzzy rough sets.

By using a family of α-level sets for representing a fuzzy set, this study offered a different and complimentary perspective in understanding the combination of rough and fuzzy sets. More importantly, the investigation has clearly demonstrated the relationships among rough sets, rough fuzzy sets, fuzzy rough sets and ordinary sets. The inherent structures in each of these sets have also been exposed.

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REFERENCES

[1] Bezdek, J.C. and Harris, J.D., "Fuzzy partitions and relations: an axiomatic basis for clustering," Fuzzy Sets and Systems, **1**, pp. 111-127, 1978.

[2] Biswas, R., "On rough sets and fuzzy rough sets," Bulletin of the Polish Academy of Sciences, Mathematics, **42**, pp. 345-349, 1994.

[3] Biswas, R., "On rough fuzzy sets," Bulletin of the Polish Academy of Sciences, Mathematics, **42**, pp. 351-355, 1994.

[4] Chanas, S. and Kuchta, D., "Further remarks on the relation between rough and fuzzy sets," Fuzzy Sets and Systems, **47**, pp. 391-394, 1992.

[5] Dubois, D. and Prade, H., "Twofold fuzzy sets and rough sets – some issues in knowledge representation," Fuzzy Sets and Systems, **23**, pp. 3-18, 1987.

[6] Dubois, D. and Prade, H., "Rough fuzzy sets and fuzzy rough sets," International Journal of General Systems, **17**, pp. 191-209, 1990.

[7] Dubois, D. and Prade, H., "Putting rough sets and fuzzy sets together," in: Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory, Slowinski, R., Ed., Kluwer Academic Publishers, Boston, pp. 203-222, 1992.

[8] Iwinski, T.B. "Algebraic approach to rough sets," Bulletin of the Polish Academy of Sciences, Mathematics, **35**, 673-683, 1987.

[9] Klir, G.J. and Yuan, B., Fuzzy Sets and Fuzzy Logic, Theory and Applications, Prentice Hall, New Jersey, 1995.

[10] Kuncheva, L.I., "Fuzzy rough sets: application to feature selection," Fuzzy Sets and Systems, **51**, pp. 147-153, 1992.

[11] Lin, T.Y., "Topological and fuzzy rough sets," in: Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory, Slowinski, R., Ed., Kluwer Academic Publishers, Boston, pp. 287-304, 1992.

[12] Lin, T.Y. and Liu, Q., "Rough approximate operators: axiomatic rough set theory," in: Rough Sets, Fuzzy Sets and Knowledge Discovery, Ziarko, W.P., Ed., Springer-Verlag, London, pp. 256-260, 1994.

[13] Mizumoto, M. and Tanaka, K., "Some properties of fuzzy numbers," in: Advances in Fuzzy Set Theory and Applications, Gupta, M.M, Ragade, R.K., and Yager, R.R., Eds., North-Holland, New York, pp. 153-164, 1979.

[14] Nakamura, A., "Fuzzy rough sets," Note on Multiple-valued Logic in Japan, **9**, pp. 1-8, 1988.

[15] Nakamura, A. and Gao, J.M., "A logic for fuzzy data analysis," Fuzzy Sets and Systems, **39**, pp. 127-132, 1991.

[16] Nanda, S. and Majumdar, S., "Fuzzy rough sets," Fuzzy Sets and Systems, **45**, pp. 157-160, 1992.

[17] Negoita, C.V. and Ralescu, D.A., "Representation theorems for fuzzy concepts," Kybernetes, **4**, pp. 169-174, 1975.

[18] Negoita, C.V. and Ralescu, D.A., Applications of Fuzzy Sets to Systems Analysis, Wiley, New York, 1975.

[19] Nguyen, H.T., "A note on the extension principle for fuzzy sets," Journal of Mathematical Analysis and Applications, **64**, pp. 369-380, 1978.

[20] Pawlak, Z., "Rough sets," International Journal of Computer and Information Sciences, **11**, pp. 341-356, 1982.

[21] Pawlak, Z., "Rough sets and fuzzy sets," Fuzzy Sets and Systems, **17**, pp. 99-102, 1985.

[22] Ralescu, D.A., "A generalization of the representation theorem," Fuzzy Sets and Systems, **51**, pp. 309-311, 1992.

[23] Uehara, K. and Fujise, M., "Fuzzy inference based on families of α-level sets," IEEE Transactions on Fuzzy Systems, **1**, pp. 111-124, 1993.

[24] Willaeys, D. and Malvache, N., "The use of fuzzy sets for the treatment of fuzzy information by computer," Fuzzy Sets and Systems, **5**, pp. 323-328, 1981.

[25] Wong, S.K.M. and Ziarko, W., "Comparison of the probabilistic approximate classification and the fuzzy set model," Fuzzy Sets and Systems, **21**, pp. 357-362, 1987.

[26] Wygralak, M., "Rough sets and fuzzy sets – some remarks on interrelations," Fuzzy Sets and Systems, **29**, pp. 241-243, 1989.

[27] Yao, Y.Y. "Interval-set algebra for qualitative knowledge representation," Proceedings of the 5th International Conference on Computing and Information, pp. 370-375, 1993.

[28] Yao, Y.Y., "On combining rough and fuzzy sets," Proceedings of the CSC'95 Workshop on Rough Sets and Database Mining, Lin, T.Y. (Ed.), San Jose State University, 9 pages, 1995.

[29] Yao, Y.Y., and Wong, S.K.M., "A decision theoretic framework for approximating concepts," International Journal of Man-machine Studies, **37**, pp. 793-809, 1992.

[30] Zadeh, L.A., "Fuzzy sets," Information and Control, **8**, pp. 338-353, 1965.

[31] Zadeh, L.A., "The concepts of a linguistic variable and its application to approximate reasoning," Information Sciences, **8**, pp. 199-249, 1975.

[32] Ziarko, W., "Variable precision rough set model," Journal of Computer and System Sciences, **46**, pp. 39-59, 1993.