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COMBINATORIAL AND ARITHMETICAL PROPERTIES OF INFINITE WORDS ASSOCIATED WITH NON-SIMPLE QUADRATIC PARRY NUMBERS

Lubomíra Balková, Edita Pelantová and Ondřej Turek [1]

Abstract. We study some arithmetical and combinatorial properties of β-integers for β being the larger root of the equation $x^2 = mx-n, m, n \in \mathbb{N}, m \geq n+2 \geq 3$. We determine with the accuracy of ± 1 the maximal number of β-fractional positions, which may arise as a result of addition of two β-integers. For the infinite word u_β coding distances between the consecutive β-integers, we determine precisely also the balance. The word u_β is the only fixed point of the morphism $A \to A^{m-1}B$ and $B \to A^{m-n-1}B$. In the case $n = 1$, the corresponding infinite word u_β is sturmian, and, therefore, 1-balanced. On the simplest non-sturmian example with $n \geq 2$, we illustrate how closely the balance and the arithmetical properties of β-integers are related.

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1. Introduction

In this paper, we focus on the study of some arithmetical and combinatorial properties of β-integers for β being a quadratic algebraic integer with a positive norm. β-integers are related to the so-called greedy algorithm searching for the expansion of a real number x in base $\beta > 1$; this algorithm has been introduced by Rényi [22]. A real number x is called a β-integer if its β-expansion has the form $\pm \sum_{k=0}^{n} x_k \beta^k$, i.e., if all of its coefficients at powers β^{-k} vanish for $k > 0$. The set of β-integers (denoted by $\mathbb{Z}_\beta$) equals in the case of $\beta \in \mathbb{N}$ to the set of integers $\mathbb{Z}$. If β is not an integer, the set $\mathbb{Z}_\beta$ has much more interesting properties:

1. (1) $\mathbb{Z}_\beta$ is not invariant under translation.
2. (2) $\mathbb{Z}_\beta$ has no accumulation points.
3. (3) $\mathbb{Z}_\beta$ is relatively dense (= distances between the successive elements of $\mathbb{Z}_\beta$ are bounded).
4. (4) $\mathbb{Z}_\beta$ is self-similar, i.e., $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$.

After the discovery of quasicrystals in 1982 [24], it has turned out that the set $\mathbb{Z}_\tau$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean, serves as a model describing the coordinates of atoms in these materials with long-range orientational order and sharp diffraction images of non-crystallographic 5-fold symmetry. Later on, quasicrystals with other

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[1] Doppler Institute for Mathematical Physics and Applied Mathematics and
Department of Mathematics, FNSPE, Czech Technical University,
Trojanova 13, 120 00 Praha 2, Czech Republic
e-mail: `l.balkova@centrum.cz & masakova@km1.fjfi.cvut.cz & oturek@centrum.cz`

non-crystallographic symmetries have been found. In order to serve as a convenient model for quasicrystals, the set $\mathbb{Z}_\beta$ must satisfy together with conditions $1.-4.$ also another natural property, the so-called finite local complexity. In one-dimensional case, it means that there exists only a finite number of distances between the successive elements of $\mathbb{Z}_\beta$. From results [21, 25], it follows that $\mathbb{Z}_\beta$ has this property if and only if the Rényi expansion of unity in base β is eventually periodic. Such numbers β are called Parry numbers. It can be easily shown that every Parry number β is an algebraic integer, i.e., it is a root of a monic polynomial having integer coefficients. The task to describe which algebraic integers are Parry numbers has not been solved yet. It is known that each Pisot number, i.e., an algebraic integer greater than 1 whose conjugates have modulus less than 1, is as well Parry. In the case of β being a Pisot number, β-integers form a Meyer set, i.e., it holds

$$\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$$

for a finite set $F \subset \mathbb{R}$. Thus, the notion of Meyer set generalizes the notion of lattice, which is crucial for the description of crystals. As we have already mentioned, in the case of β being a Parry number, the set $\mathbb{Z}_\beta$ has a finite number of distances between neighbors. If we associate to the different gaps different letters, it is possible to encode the set $\mathbb{Z}_\beta$ as an infinite word u_β over a finite alphabet. Combinatorial properties of words u_β have been studied in several papers: [11, 12] is devoted to the description of factor complexity of u_β, palindromes of u_β are described in paper [3]. So far the least studied problem is the balance of u_β, i.e., the maximal difference in numbers of different letters in factors of the same length. Balance is clearly known for $\mathbb{Z}_\beta$ which corresponds to sturmian words, i.e., for β being a quadratic unit. In paper [26], the balance property for u_β, where β is the larger root of the quadratic polynomial $x^2 - mx - n$, $m, n \in \mathbb{N}$, $m \geq n \geq 1$, has been studied. For other types of irrationalities, the balance property has not been described yet.

The sets of ordinary integers and β-integers are very different also from the arithmetical point of view. $\mathbb{Z}_\beta$ is not closed under addition and multiplication for any $\beta \notin \mathbb{N}$. Sum of two β-integers may even not have a finite β-expansion. So far unsolved and likely very difficult is the question of characterization of those β for which this pathological situation does not appear. Mathematically expressed it means to describe β for which the set $Fin(\beta)$, i.e., the set of numbers with a finite β-expansion, is a subring of $\mathbb{R}$. Frougny and Solomyak have shown in [13] that the necessary condition for this so-called finiteness property is that β is a Pisot number. Some sufficient conditions can be found in [2, 13, 15]. If a sum or a product of two β-integers has a finite β-expansion, there arises a question how long the β-fractional part of the sum or the product is. This problem has been investigated in [4, 6, 9, 14, 18].

Here, the main attention is devoted to the investigation of arithmetics of β-integers for a non-simple quadratic Parry number β, i.e., for β being the larger root of the quadratic polynomial $x^2 = mx - n, m, n \in \mathbb{N}, m \geq n + 2 \geq 3$. We determine with the accuracy of ± 1 the maximal number of β-fractional positions $L_\oplus(\beta)$, which may arise as a result of addition of two β-integers. In Theorem 6.2, it is proved that $\left\lfloor \frac{m-2}{m-n-1} \right\rfloor \leq L_\oplus(\beta) \leq \left\lceil \frac{m-1}{m-n-1} \right\rceil$. So we improve considerably the estimate from the paper [14]. In Theorem 5.9, we determine accurately also the balance of u_β; the difference in numbers of different letters in factors of the same length is at most $\left\lceil \frac{m-2}{m-n-1} \right\rceil$. On this easiest non-sturmian example, we illustrate how closely the arithmetical and combinatorial properties of $\mathbb{Z}_\beta$ are related. Particulary, we show the relation between $L_\oplus(\beta)$ and the balance property. Our method might be applied also for the determination of the balance property for words coding β-integers with irrationalities of a higher degree.

2. Preliminaries

An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A concatenation of letters is a *word*. The set $\mathcal{A}^*$ of all finite words (including the empty word ε) provided with the operation of concatenation is a free monoid. The length of a word $w = w_0w_1w_2\cdots w_{n-1}$ is denoted by $|w| = n$. We will deal also with infinite words $u = u_0u_1u_2\cdots$. A finite word w is called a *factor* of the word u (finite or infinite) if there exist a finite word $w^{(1)}$ and a word $w^{(2)}$ (finite or infinite) such that $u = w^{(1)}ww^{(2)}$. The word w is a *prefix* of u if $w^{(1)} = \varepsilon$. Analogously, w is a *suffix* of u if $w^{(2)} = \varepsilon$. A concatenation of k words w will be denoted by w^k, a concatenation of infinitely many finite words w by w^ω. An infinite word u is said to be *eventually periodic* if there exist words v, w such that $u = vw^\omega$. A word which is not eventually periodic is called *aperiodic*. We will denote by $\mathcal{L}(u)$ (language of u) the set of all factors of the word u. $\mathcal{L}_n(u)$ denotes the set of all factors of length n of the word u, clearly

$$\mathcal{L}(u) = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(u).$$

The measure of variability of local configurations in u is expressed by the *factor complexity function* $\mathcal{C}_u : \mathbb{N} \to \mathbb{N}$, which associates with $n \in \mathbb{N}$ the number $\mathcal{C}_u(n) := \#\mathcal{L}_n(u)$. Obviously, a word u is eventually periodic if and only if $\mathcal{C}_u(n)$ is bounded by a constant. On the other hand, one can show that a word u is aperiodic if and only if $\mathcal{C}_u(n) \geq n + 1$ for all $n \in \mathbb{N}$. Infinite aperiodic words with the minimal complexity $\mathcal{C}_u(n) = n + 1$ for all $n \in \mathbb{N}$ are called *sturmian words*. These words are studied intensively, several different definitions of sturmian words can be found in [7].

Another way how to measure the degree of variability in an infinite word u is the balance property. Let us denote the number of letters $a \in \mathcal{A}$ in the word w by $|w|_a$. We say that an infinite word u is c-balanced, if for every $a \in \mathcal{A}$ and for every pair of factors w, $\hat{w}$ of u, with the same length $|w| = |\hat{w}|$, we have $||w|_a - |\hat{w}|_a| \leq c$. Note that in the case of binary alphabet $\mathcal{A} = \{\mathcal{A}, \mathcal{B}\}$, this condition may be written in a simpler way as $||w|_A - |\hat{w}|_A| \leq c$. Sturmian words are characterized by the property that they are 1-balanced (or simply balanced) [20]. To determine the minimal constant c for which an infinite word is c-balanced is a difficult task. Adamczewski [1] gives an upper bound on c for a certain class of infinite words. To describe his result, we must introduce the notion of morphism. A mapping φ on the free monoid $\mathcal{A}^*$ is called a *morphism* if $\varphi(vw) = \varphi(v)\varphi(w)$ for all $v, w \in \mathcal{A}^*$. Obviously, for determining the morphism it suffices to give $\varphi(a)$ for all $a \in \mathcal{A}$. The action of the morphism φ can be naturally extended to right-sided infinite words by the prescription

$$\varphi(u_0u_1u_2\cdots) := \varphi(u_0)\varphi(u_1)\varphi(u_2)\cdots$$

A non-erasing morphism φ, for which there exists a letter $a \in \mathcal{A}$ such that $\varphi(a) = aw$ for some non-empty word $w \in \mathcal{A}^*$, is called a *substitution*. An infinite word u such that $\varphi(u) = u$ is called a *fixed point* of the substitution φ. Obviously, every substitution has at least one fixed point, namely

$$\lim_{n\to\infty} \varphi^n(a).$$

To any substitution φ over a k-letter alphabet $\mathcal{A} = \{a_1, a_2, ..., a_k\}$, one can associate the so-called *incident matrix* M of size $k \times k$ defined by

$$M_{ij} := |\varphi(a_i)|_{a_j}.$$

The result of Adamczewski [1] concerns infinite words u being fixed points of primitive substitutions. Recall that a substitution φ is primitive if there exists a power k of φ such that each pair of letters $a, b \in \mathcal{A}$ satisfies $|\varphi^k(a)|_b \geq 1$. In accordance with the Perron-Frobenius theorem, the incident matrix of a primitive substitution has one real eigenvalue greater than one, which is moreover greater than the modulus of all the other eigenvalues. This eigenvalue, say Λ, is called the *Perron eigenvalue* of the substitution. In [1] it has been proved that if u is the fixed point of a primitive substitution with the incidence matrix M, then u is c-balanced for some constant c if and only if $|\lambda| < 1$ for all eigenvalues λ of M, $\lambda \neq \Lambda$.

2.1. Beta-expansions and beta-integers

Let $\beta > 1$ be a real number and let x be a non-negative real number. Any convergent series of the form

$$x = \sum_{i=-\infty}^{k} x_i \beta^i,$$

where $x_i \in \mathbb{N}$, is called a *β-representation* of x. As well as it is usual for the decimal system, we will denote the β-representation of x by

$$x_k x_{k-1} \cdots x_0 \bullet x_{-1} \cdots \qquad \text{if} \quad k \geq 0,$$

and

$$0 \bullet \underbrace{0 \cdots 0}_{(-1-k)-times} x_k x_{k-1} \cdots \qquad \text{otherwise.}$$

If a β-representation ends with infinitely many zeros, it is said to be *finite* and the ending zeros are omitted. If $\beta \notin \mathbb{N}$, for a given x there can exist several β-representations.

Any positive number x has at least one representation. This representation can be obtained by the following *greedy algorithm:*

(1) Find $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$ and put $x_k := \lfloor \frac{x}{\beta^k} \rfloor$ and $r_k := \{\frac{x}{\beta^k}\}$, where $\lfloor x \rfloor$ denotes the lower integer part and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.
(2) For $i < k$, put $x_i := \lfloor \beta r_{i+1} \rfloor$ and $r_i := \{\beta r_{i+1}\}$.

The representation obtained by the greedy algorithm is called *β-expansion* of x and the coefficients of a β-expansion clearly satisfy: $x_k \in \{1, \ldots, \lceil \beta \rceil - 1\}$ and $x_i \in \{0, \ldots, \lceil \beta \rceil - 1\}$ for all $i < k$, where $\lceil x \rceil$ denotes the upper integer part of x. We will use for the β-expansion of x the notation $\langle x \rangle_\beta$. If $x = \sum_{i=-\infty}^{k} x_i \beta^i$ is the β-expansion of a nonnegative number x, then $\sum_{i=-\infty}^{-1} x_i \beta^i$ is called the *β-fractional part* of x. Let us introduce some important notions connected with β-expansions:

- The set of nonnegative numbers x with a vanishing β-fractional part are called *nonnegative β-integers*, formally

$$\mathbb{Z}_\beta^+ := \{x \geq 0 \mid \langle x \rangle_\beta = x_k x_{k-1} \cdots x_0 \bullet\}.$$

- The set of *β-integers* is then defined by

$$\mathbb{Z}_\beta := \left(-\mathbb{Z}_\beta^+\right) \cup \mathbb{Z}_\beta^+.$$

- All the real numbers with a finite β-expansion of $|x|$ form the set $Fin(\beta)$, formally

$$Fin(\beta) := \bigcup_{n \in \mathbb{N}} \frac{1}{\beta^n} \mathbb{Z}_\beta.$$

For any $x \in Fin(\beta)$, we denote by $fp_\beta(x)$ the length of its fractional part, i.e.,

$$fp_\beta(x) = \min\{l \in \mathbb{N} \mid \beta^l x \in \mathbb{Z}_\beta\}.$$

The sets $\mathbb{Z}_\beta$ and $Fin(\beta)$ are generally not closed under addition and multiplication. The following notion is important for studying of the lengths of fractional parts which may appear as a result of addition and multiplication.

- $L_\oplus(\beta) := \min\{L \in \mathbb{N} \mid x, y \in \mathbb{Z}_\beta, x + y \in Fin(\beta) \Longrightarrow fp_\beta(x + y) \le L\}$.
- $L_\otimes(\beta) := \min\{L \in \mathbb{N} \mid x, y \in \mathbb{Z}_\beta, xy \in Fin(\beta) \Longrightarrow fp_\beta(xy) \le L\}$.

If such $L \in \mathbb{N}$ does not exist, we set $L_\oplus(\beta) := \infty$ or $L_\otimes(\beta) := \infty$.

The Rényi expansion of unity simplifies the description of elements of $\mathbb{Z}_\beta$ and $Fin(\beta)$. For its definition, we introduce the transformation $T_\beta(x) := \{\beta x\}$ for $x \in [0, 1]$. The *Rényi expansion of unity* in base β is defined as

$$d_\beta(1) = t_1 t_2 t_3 \cdots , \quad \text{where} \quad t_i := \lfloor \beta T_\beta^{i-1}(1) \rfloor.$$

Every number $\beta > 1$ is characterized by its Rényi expansion of unity. Note that $t_1 = \lfloor \beta \rfloor \ge 1$. Not every sequence of nonnegative integers is equal to $d_\beta(1)$ for some β. Parry studied this problem in his paper [21]: A sequence $(t_i)_{i\ge1}$, $t_i \in \mathbb{N}$, is the Rényi expansion of unity for some number β if and only if the sequence satisfies

$$t_j t_{j+1} t_{j+2} \cdots \prec t_1 t_2 t_3 \cdots \quad \text{for every } j > 1,$$

where $\prec$ denotes strictly lexicographically smaller.

The Rényi expansion of unity enables us to decide whether a given β-representation of x is the β-expansion or not. For this purpose, we define the *infinite Rényi expansion of unity*

$$d^*_\beta(1) = \begin{cases} d_\beta(1) & \text{if} \quad d_\beta(1) \quad \text{is infinite,} \\ (t_1 t_2 \cdots t_{m-1}(t_m - 1))^\omega & \text{if} \quad d_\beta(1) = t_1 \dots t_m \quad \text{with} \quad t_m \neq 0. \end{cases} \tag{1}$$

Parry has proved also the following proposition.

Proposition 2.1. *[Parry condition] Let $d^*_\beta(1)$ be an infinite Rényi expansion of unity. Let $\sum_{i=-\infty}^k x_i \beta^i$ be a β-representation of a positive number x. Then $\sum_{i=-\infty}^k x_i \beta^i$ is a β-expansion of x if and only if*

$$x_i x_{i-1} \cdots \prec d^*_\beta(1) \quad \textit{for all } i \le k. \tag{2}$$

2.2. Infinite words associated with β-integers

If β is an integer, then, clearly, $\mathbb{Z}_\beta = \mathbb{Z}$ and the distance between the neighboring elements of $\mathbb{Z}_\beta$ for a fixed β is always 1. The situation changes dramatically if $\beta \notin \mathbb{N}$. In this case, the number of different distances between the neighboring elements of $\mathbb{Z}_\beta$ is at least 2. In paper [25], it is shown that distances occurring between neighbors of $\mathbb{Z}_\beta$ form the set $\{\Delta_k \mid k \in \mathbb{N}\}$, where

$$\Delta_k := \sum_{i=1}^{\infty} \frac{t_{i+k}}{\beta^i} \quad \text{for } k \in \mathbb{N}\,. \tag{3}$$

It is evident that the set $\{\Delta_k \mid k \in \mathbb{N}\}$ is finite if and only if $d_\beta(1)$ is eventually periodic.

If $d_\beta(1)$ is eventually periodic, we will call β a *Parry number*. If $d_\beta(1)$ is finite, β is said to be a *simple Parry number*. Every Pisot number, i.e., a real algebraic integer greater than 1, all of whose conjugates are of modulus strictly less than 1, is a Parry number [8].

From now on, we will restrict our considerations to the quadratic Parry numbers which are necessarily Pisot numbers. The Rényi expansion of unity for a simple quadratic Parry number β is equal to $d_\beta(1) = pq$, where $p \geq q$. Hence, β is exactly the positive root of the polynomial $x^2 - px - q$. Whereas the Rényi expansion of unity for a non-simple quadratic Parry number β is equal to $d_\beta(1) = pq^\omega$, where $p > q \geq 1$. Consequently, β is the larger root of the polynomial $x^2-(p+1)x+p-q$. Drawn on the real line, there are only two distances between neighboring points of $\mathbb{Z}_\beta$. The longer distance is always $\Delta_0 = 1$, the smaller one is $\Delta_1 = \beta - p$. In the non-simple case, we will often use the expression of the smaller distance in the form $\Delta_1 = 1 - \frac{p-q}{\beta}$. Conversely, if there are exactly two types of distances between the neighboring points of $\mathbb{Z}_\beta$ for $\beta > 1$, then β is a quadratic Pisot number.

If we assign letters A, B to the two types of distances Δ_0 and Δ_1, respectively, and write down the order of distances in $\mathbb{Z}_\beta^+$ on the real line, we naturally obtain an infinite word; we will denote this word by u_β. Since $\beta\mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+$, it can be shown easily that the word u_β is a fixed point of a certain substitution φ (see [10]); in particular, for a simple quadratic Parry number β, the generating substitution is

$$\varphi(A) = A^pB, \quad \varphi(B) = A^q, \tag{4}$$

and the beginning of the only fixed point of the substitution is

$$u_\beta = \underbrace{A^pB \ \cdots \ A^pB}_{p \text{ times}} \ A^q \ \underbrace{A^pB \ \cdots \ A^pB}_{p \text{ times}} \ A^q \ \cdots\,, \tag{5}$$

for a non-simple quadratic Parry number β, the generating substitution is

$$\varphi(A) = A^pB, \quad \varphi(B) = A^qB, \tag{6}$$

and the beginning of the only fixed point of the substitution is

$$u_\beta = \underbrace{A^pB \ \cdots \ A^pB}_{p \text{ times}} \ A^qB \ \underbrace{A^pB \ \cdots \ A^pB}_{p \text{ times}} \ A^qB \ \cdots \tag{7}$$

Let us remark that the matrices of these substitutions are $\begin{pmatrix} p & 1 \\ q & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 1 \\ q & 1 \end{pmatrix}$, respectively, i.e., both substitutions are primitive. Therefore, it follows from result [1] that there exists a positive integer c such that u_β is c-balanced.

In the case of β being the root of $x^2 - px - q$, i.e., β is a quadratic simple Parry number, the smallest possible constant c has been found. In paper [26], it is shown that the infinite word generated by substitution (4) is $(1 + \lfloor (p-1)/(p+1-q) \rfloor)$-balanced. Also the values of $L_\oplus(\beta)$ have been quite precisely estimated in [14]:

$$L_\oplus(\beta) = 2p \quad \text{if} \quad q = p,$$

$$2\left\lfloor \frac{p+1}{p-q+1} \right\rfloor \leq L_\oplus(\beta) \leq 2\left\lceil \frac{p}{p-q+1} \right\rceil \quad \text{if} \quad q < p.$$

In this paper, we will consider the arithmetical properties of of the infinite word u_β associated with $\mathbb{Z}_\beta$ in the less studied case, where β is the larger root of the equation $x^2 - (p+1)x + p - q$.

3. Beta-arithmetics for non-simple quadratic Parry numbers

The aim of this section is to improve the upper bound on the number $L_\oplus(\beta)$ for β having the Rényi expansion of unity equal to $d_\beta(1) = pq^\omega$ for $q \leq p-1$. In the case of $q = p-1$, β is the larger root of the equation $x^2 - (p+1)x + 1 = 0$, thus β is a quadratic unit. For quadratic units in [9], it is shown that $L_\oplus(\beta) = L_\otimes(\beta) = 1$. Let us focus on the case of $q < p-1$. In [14], one can find the following estimates:

$$L_\oplus(\beta) \leq 3(p+1)\ln(p+1) \quad \text{and} \quad L_\otimes(\beta) \leq 4(p+1)\ln(p+1).$$

Here, the estimate on $L_\oplus(\beta)$ will be improved. In [4] and in [13], it is shown that if $d_\beta(1) = t_1t_2\cdots t_m(t_{m+1})^\omega$ and $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1}$, then $Fin(\beta)$ is closed under addition of positive elements. Every β-representation can be written as a sum of finitely many numbers whose β-representation contains only one non-zero coefficient and is equal to the β-expansion. This fact implies that if a number x has a certain finite β-representation, then x has as well a finite β-expansion. It follows from the definition of the greedy algorithm that if $x_kx_{k-1}\cdots x_0 \bullet x_{-1}x_{-2}\cdots$ is the β-expansion of $x > 0$ and $\tilde{x}_k\tilde{x}_{k-1}\cdots\tilde{x}_0 \bullet \tilde{x}_{-1}\tilde{x}_{-2}\cdots$ is a β-representation of x, then

$$\tilde{x}_k\tilde{x}_{k-1}\cdots\tilde{x}_0\tilde{x}_{-1}\tilde{x}_{-2}\cdots \preceq x_kx_{k-1}\cdots x_0x_{-1}x_{-2}\cdots$$

Thus, the β-expansion of x is the lexicographically greatest β-representation of x.

Let us limit our considerations to the special case of $d_\beta(1) = pq^\omega$. The shortest and lexicographically smallest words that do not fulfill the Parry condition (2) are the words

$$(p+1) \text{ and } pq^s(q+1), \text{ where } s \geq 0.$$

Using the equation $\beta^2 = (p+1)\beta - (p-q)$, one can easily obtain:

$$(p+1)\bullet = 10 \bullet (p-q) \tag{8}$$

$$pq^s(q+1)\bullet = 10^{s+2} \bullet (p-q) \tag{9}$$

Let us remark that on the right-hand side of the equations, there are already the β-expansions.

Applying the rules (8) and (9) to coefficients of a finite β-representation of a number x from left to right, one obtains a lexicographically greater β-representation of x, and, at the same time, the sum of digits in the new β-representation is reduced. It follows that repeating the rules (8) and (9) finitely many times, it is possible to transform any finite β-representation of x into the β-expansion of x.

Example 3.1. $(p+2)q(q+1)\bullet \;=\; (p+1)00\bullet \;+\; 1q(q+1)\bullet \;=\; 10(p-q)0\bullet \;+\; 1q(q+1)\bullet \;=\; 11p(q+1)\bullet \;=\; 1200\bullet(p-q)$

On the other hand, the rules (8) and (9) raise the sum of digits on the right-hand side of the fractional point $\bullet$. It means that the number of digits in the β-expansion of x on the right-hand side of $\bullet$ is greater than or equal to the number of digits in any β-representation of x.

In all what follows, the same method of a multiple application of the rules (8) and (9) as in Example 3.1 will be used.

Observation 3.2. *If $x, y \geq 0$ and $x, y \in Fin(\beta)$, then $fp_\beta(x+y) \geq fp_\beta(x)$.*

The following lemma is the most important tool to estimate $L_\oplus(\beta)$.

Lemma 3.3. *Let $x_kx_{k-1}\cdots x_0\bullet$ be the β-expansion of a positive β-integer x and let $l \in \mathbb{N}$. Then either $x + \beta^l \in \mathbb{Z}_\beta$ or there exists $s \geq l$ such that*

(1) *for* $l = 0$,

$$\langle x + \beta^l \rangle_\beta = x_k \cdots (x_{s+1} + 1)0^{s+1} \bullet (p - q),$$

(2) *for* $l \geq 1$, *the coefficients satisfy* $x_i \geq q$ *for* $1 \leq i \leq l - 1$, $x_0 \geq q + 1$, *and*

$$\langle x + \beta^l \rangle_\beta = x_k \cdots (x_{s+1} + 1)0^{s-l+1}(x_{l-1} - q) \cdots (x_1 - q)(x_0 - q - 1) \bullet (p - q).$$

Proof. (1) For $l = 0$. Let us suppose that $x + \beta^0 = x + 1 \notin \mathbb{Z}_\beta$. Then $x_k x_{k-1} \cdots (x_0 + 1)\bullet$ is not a β-expansion of $x + 1$. Therefore it has a suffix of the form $(p + 1)$ or $pq^{s-1}(q + 1)$, where $s \geq 1$. Applying the rule (8), resp. (9), the β-representation of $x + 1$ can be rewritten as

$$x_k x_{k-1} \cdots x_1 (p + 1)\bullet \ = \ x_k \cdots x_2 (x_1 + 1)0 \bullet (p - q)$$

or

$$x_k x_{k-1} \cdots x_{s+1} p q^{s-1}(q + 1)\bullet \ = \ x_k \cdots (x_{s+1} + 1)0^{s+1} \bullet (p - q).$$

Now, it suffices to show that the expressions on the right-hand side are already β-expansions, or, equivalently, they fulfill the Parry condition (2). It follows immediately from the fact that if $x_k \cdots x_1 p$ and $x_k \cdots x_{s+1} p q^s$ fulfill the Parry condition, then $x_k \cdots (x_1 + 1)0$ and $x_k \cdots (x_{s+1} + 1)0^{s+1}$ fulfill this condition, too.

(2) For $l \geq 1$. Let us suppose that $x + \beta^l \notin \mathbb{Z}_\beta$. Clearly $l \leq k$. Then

$$x_k \cdots x_{l+1}(x_l + 1)x_{l-1} \cdots x_0 \tag{10}$$

does not fulfill the Parry condition (2). This implies three possibilities for the values of x_l.

(a) $x_l = q - 1$,
(b) $x_l = p$,
(c) $x_l = q$.

(a) Let $x_l = q - 1$. Denote $s = \min\{i > l \mid x_i = p\}$. Obviously, $x_i = q$ for all i, $s > i > l$. Necessarily, $x_{s+1} < p$. Suppose that for all $i < l$ it holds $x_i \geq q$ and $x_0 \geq q + 1$, then we can apply the rule (9) for rearranging the β-representation of $x + \beta^l$ in the following way:

$$x_k \cdots x_{s+1} p q^{s-l} x_{l-1} \cdots x_0 \bullet \ =$$

$$(x_{l-1} - q) \cdots (x_1 - q)(x_0 - q - 1) \bullet \ + x_k \cdots x_{s+1} p q^{s-1}(q + 1)\bullet \ =$$

$$(x_{l-1} - q) \cdots (x_1 - q)(x_0 - q - 1) \bullet \ + x_k \cdots (x_{s+1} + 1)0^{s+1} \bullet (p - q) \ =$$

$$x_k \cdots (x_{s+1} + 1)0^{s-l+1}(x_{l-1} - q) \cdots (x_1 - q)(x_0 - q - 1) \bullet (p - q)$$

Since the last expression fulfills the Parry condition (2), we have obtained the β-expansion of $x + \beta^l$. Let us show now that the conditions $x_0 \geq q + 1$ and $x_i \geq q$ for all $i < l$ are satisfied. Firstly, we prove that $x_i \geq q$ for all $i < l$. Let us prove it by contradiction. Let us denote by i_0 the maximal index $< l$ such that $x_{i_0} \leq q - 1$. Then, let us denote by j_0 the minimal index $> i_0$ such that $x_{j_0} \geq q + 1$. Such an index exists because (10) does not fulfill the Parry condition (2). Hence, the chain (10) has the following form:

$$x_k \cdots x_{s+1} p q^{s-l} x_{l-1} \cdots x_{j_0+1} x_{j_0} q^{j_0 - i_0 - 1} x_{i_0} x_{i_0 - 1} \cdots x_0,$$

where $x_{l-1},\ldots,x_{j_0} \geq q$. Using the rule (9), we get the β-representation of $x+\beta^l$ in the form:
if $j_0 > i_0+1$,

$$x_k \cdots (x_{s+1}+1)0^{s-l+1}(x_{l-1}-q)\cdots(x_{j_0+1}-q)(x_{j_0}-q-1)pq^{j_0-i_0-2}x_{i_0}x_{i_0-1}\cdots x_0\bullet$$

if $j_0 = i_0+1$,

$$x_k \cdots (x_{s+1}+1)0^{s-l+1}(x_{l-1}-q)\cdots(x_{j_0+1}-q)(x_{j_0}-q-1)(x_{i_0}+p-q)x_{i_0-1}\cdots x_0\bullet$$

In both cases, these β-representations are already the β-expansions, thus we get a contradiction with the fact that $x+\beta^l \notin \mathbb{Z}_\beta$.
Secondly, we show that $x_0 \geq q+1$. Let us prove it again by contradiction. Let us suppose that $x_0 = q$, then there exists $t \geq 1$ such that q^t is the suffix of the chain $x_k \cdots x_0$. Let us consider the maximal such t. Then the β-representation of $x+\beta^l$ has the following form:

$$x_k \cdots x_{s+1}pq^{s-l}x_{l-1}\cdots x_{t+1}x_tq^t\bullet$$

where $x_i \geq q$ for all $i \in \{t+1,\ldots,l-1\}$ and $x_t \geq q+1$. Applying the rule (9), we can rewrite the β-representation as

$$x_k \cdots (x_{s+1}+1)0^{s-l+1}(x_{l-1}-q)\cdots(x_{t+1}-q)(x_t-q-1)pq^{t-1}\bullet$$

which is a contradiction with $x+\beta^l \notin \mathbb{Z}_\beta$.

(b) Let $x_l = p$. Then $x_{l+1} < p$ and $x_{l-1} \leq q$. Using the rule (8), we obtain

$$x_k \cdots x_{l+1}(p+1)x_{l-1}\cdots x_0\bullet \;=\; x_k \cdots (x_{l+1}+1)0(x_{l-1}+p-q)x_{l-2}\cdots x_0\bullet \quad (11)$$

Since $x_lx_{l-1}\cdots x_0 = px_{l-1}\cdots x_0 \prec pq^\omega$, we have $x_{l-1}\cdots x_0 \prec q^\omega$, and, consequently, $(x_{l-1}+p-q)x_{l-2}\cdots x_0 \prec pq^\omega$. Thus, the expression on the right-hand side of (11) is already the β-expansion of $x+\beta^l$, which is a contradiction with $x+\beta^l \notin \mathbb{Z}_\beta$.

(c) Let $x_l = q$. Since addition of 1 to the l^{th} digit x_l breaks the Parry condition, there exists $t \geq l$ such that $x_k\cdots x_0 = x_k\cdots x_{t+1}pq^{t-l}x_{l-1}\cdots x_0$. The β-representation of $x+\beta^l$, equal to $x_k\cdots x_{t+1}pq^{t-l-1}(q+1)x_{l-1}\cdots x_0\bullet$ can be rewritten, using the rule (9), as

$$x_k \cdots (x_{t+1}+1)0^{t-l+1}(x_{l-1}+p-q)x_{l-2}\cdots x_0\bullet$$

which is already the β-expansion of $x+\beta^l$. Thus, we arrive again at a contradiction with $x+\beta^l \notin \mathbb{Z}_\beta$.

□

Proposition 3.4. *Let $x, y \in \mathbb{Z}_\beta$, $x \geq y \geq 0$, and let all digits in the β-expansion of y be $\leq q$. Then the β-fractional part of $x+y$ is either 0 or $\frac{p-q}{\beta}$.*

Proof. We will proceed by induction on the positive elements of $\mathbb{Z}_\beta$. Let $x_k \ldots x_0\bullet$ be the β-expansion of x. For $y \in \{1,\ldots,q\}$, it follows from Lemma 3.3 that either $x+q \in \mathbb{Z}_\beta$ or there exists $i \in \{1,\ldots,q-1\}$ such that $\langle x+i\rangle_\beta = x_k\cdots(x_{s+1}+1)0^{s+1}\bullet(p-q)$. In the latter case, it is clear that also $x+j$, where $j \in \{i+1,\ldots,q\}$, has the fractional part $\frac{p-q}{\beta}$. Let $y \geq q+1$, $\langle y\rangle_\beta = y_ly_{l-1}\cdots y_0\bullet$, where $y_l \geq 1$ and $y_i \leq q$ for all $i \in \{0,\ldots,l\}$. If $x+\beta^l \in \mathbb{Z}_\beta$, then $x+y = \tilde{x}+\tilde{y}$, where $\tilde{x} = x+\beta^l$ and $\tilde{y} = y-\beta^l$, and the statement follows by applying the induction assumption on

$\tilde{y} = y - \beta^l < y$. If $x + \beta^l \notin \mathbb{Z}_\beta$, then using Lemma 3.3, we get

$$x+y = x+\beta^l+(y-\beta^l) = x_k \cdots (x_{s+1}+1)0^{s-l}(y_l-1)(x_{l-1}+y_{l-1}-q)\cdots(x_0+y_0-q-1)\bullet(p-q) \tag{12}$$

Moreover, $y_l - 1 \le q - 1$ and $(x_{l-1} + y_{l-1} - q)\cdots(x_0 + y_0 - q - 1) \preceq x_{l-1}\cdots x_0$. Consequently, the right-hand side of (12) is already the β-expansion of $x + y$. $\square$

It is known that if $d_\beta(1)$ is eventually periodic, then the set $Fin(\beta)$ is not closed under subtraction of positive elements. In our case, we have for instance: $\beta - 1 = (p-1)\bullet q^\omega$.

Observation 3.5. *Let $x \ge y \ge 0$, $x, y \in \mathbb{Z}_\beta$, then $x - y \in \mathbb{Z}_\beta$ or $x - y \notin Fin(\beta)$.*

To prove this statement by contradiction one assumes that $x - y$ has a finite β-expansion, but $x - y$ is not a β-integer, i.e., $fp_\beta(x - y) \ge 1$. Observation 3.2 implies that $fp_\beta(x) = fp_\beta(x - y + y) \ge fp_\beta(x - y) \ge 1$ and it is a contradiction with $x \in \mathbb{Z}_\beta$.

Theorem 3.6. *Let $d_\beta(1) = pq^\omega$. Then $L_\oplus(\beta) \le \lceil \frac{p}{q} \rceil$.*

Proof. Let $x, y \in \mathbb{Z}_\beta$ and $x, y \ge 0$. If $x-y \in Fin(\beta)$, then necessarily $fp_\beta(x-y) = 0$, as we have mentioned in Observation 3.5. Consequently, it suffices to consider the addition $x + y$. Without loss of generality, we can limit to the case $x \ge y$. Clearly, y can be written as:

$$y = y^{(1)} + y^{(2)} + \cdots + y^{(s)},$$

where $s \le \lceil \frac{p}{q} \rceil$ and the digits of $y^{(i)}$ are $\le q$ for all $i = 1, \ldots, s$. According to Proposition 3.4, if we add to a number of $Fin(\beta)$ a β-integer with small digits, the length of fractional part increases at most by 1. This proves the statement. $\square$

As an immediate consequence of the previous proof, we have the following corollary.

Corollary 3.7. *Let $x, y \in \mathbb{Z}_\beta$ and $x, y \ge 0$. Then there exists $\varepsilon \in \{0, 1, \ldots, \lceil \frac{p}{q} \rceil\}$ such that*

$$x + y \in \mathbb{Z}_\beta + \varepsilon\, \frac{p-q}{\beta}\ .$$

4. An upper bound on the constant c in the balance property of u_β

Let us show how we can derive an upper bound on the constant c in the balance property applying results on arithmetics, in particular applying Corollary 3.7. Let us remind that we obtain the infinite word u_β if we associate with the longer gap between the neighboring β-integers the letter A and with the shorter one the letter B. The length of the longer gap is $\Delta_A = 1$ and that of the shorter one $\Delta_B = 1 - \frac{p-q}{\beta}$.

Proposition 4.1. *u_β is $\lceil \frac{p}{q} \rceil$-balanced. Moreover, any prefix of u_β contains at least the same number of letters A as any other factor of u_β of the same length.*

Proof. Let w be a factor of u_β of length n and $\hat{w}$ be the prefix of u_β of the same length. Find β-integers x and y, $x < y$, such that the sequence of distances between neighboring β-integers in the segment of $\mathbb{Z}_\beta$ from x to y corresponds to the factor w. Clearly,

$$y = x + |w|_A \Delta_A + |w|_B \Delta_B. \tag{13}$$

The prefix $\hat{w}$ corresponds to the β-integer

$$z = |\hat{w}|_A \Delta_A + |\hat{w}|_B \Delta_B. \tag{14}$$

Corollary 3.7 implies that there exists $\hat{z} \in \mathbb{Z}_\beta$ such that

$$x + z = \hat{z} + \varepsilon(\Delta_A - \Delta_B), \text{ for some } \varepsilon \in \{0, 1, \dots, \left\lceil \frac{p}{q} \right\rceil\}. \tag{15}$$

Since $y, \hat{z} \in \mathbb{Z}_\beta$, it is possible to express the distance between y and $\hat{z}$ as a combination of the lengths of gaps Δ_A and Δ_B, i.e., there exist $L, M \in \mathbb{N}$ such that

$$\hat{z} - y = \pm(L\Delta_A + M\Delta_B). \tag{16}$$

Using (13), (14), and (15), we get

$$\hat{z} - y = x + z - \varepsilon(\Delta_A - \Delta_B) - x - |w|_A\Delta_A - |w|_B\Delta_B =$$

$$\begin{aligned} &(|\hat{w}|_A - |w|_A - \varepsilon)\Delta_A + (|\hat{w}|_B - |w|_B + \varepsilon)\Delta_B = \\ &(|\hat{w}|_A - |w|_A - \varepsilon)\Delta_A - (|\hat{w}|_A - |w|_A - \varepsilon)\Delta_B \end{aligned} \tag{17}$$

In the last equation, we have used the fact that the factors w and $\hat{w}$ have the same lengths, and, consequently, $|\hat{w}|_A - |w|_A = |w|_B - |\hat{w}|_B$. As $\Delta_A = 1$ and $\Delta_B = 1 - \frac{p-q}{\beta}$ are linearly independent over $\mathbb{Q}$, the expression of $\hat{z} - y$ in (17) as an integer combination of the lengths of gaps is unique. Since L, M are nonnegative, from (16) and (17) it follows that $|\hat{w}|_A - |w|_A - \varepsilon = 0$, i.e.,

$$|\hat{w}|_A = |w|_A + \varepsilon,$$

where $\varepsilon \in \{0, 1, \dots, \lceil \frac{p}{q} \rceil\}$, which is exactly the statement of the proposition. □

5. Balance property of u_β

In the previous section, we have proved, using some arithmetical properties of β-integers, that the infinite word u_β is $\lceil \frac{p}{q} \rceil$-balanced. In this section, we will even show that u_β is $\lceil \frac{p-1}{q} \rceil$-balanced, which is a better estimate in the case when q divides $p - 1$. We will as well prove that this estimate cannot be improved.

At first, let us state without any proof some trivial properties of the fixed point u_β of the substitution φ given by

$$\varphi(A) = A^pB, \qquad \varphi(B) = A^qB, \qquad \text{for } \ p > q > 1\,.$$

Observation 5.1. *Let BA^kB be a factor of u_β. Then $k = p$ or $k = q$. In particular, if A^k is a factor of u_β, then $k \le p$.*

Observation 5.2. *If v is a finite factor of u_β, then $B\varphi(v)$ is also a factor of u_β.*

Observation 5.3. *Let BvB be a factor of u_β. Then there exists a unique factor w of u_β such that $vB = \varphi(w)$.*

Now, we will describe two sequences of factors of u_β denoted by $\left(w_\beta^{(n)}\right)_{n=1}^\infty$ and $\left(u_\beta^{(n)}\right)_{n=1}^\infty$, whose behaviour fully determines the balance property of u_β.

Let us define a sequence $\left(w_\beta^{(n)}\right)_{n=1}^\infty$ recursively by

$$\begin{aligned} w_\beta^{(1)} &= B \\ w_\beta^{(n)} &= B\varphi(w_\beta^{(n-1)}) \quad \text{ for } n \in \mathbb{N},\ n \ge 2. \end{aligned} \tag{18}$$

According to Observation 5.2 the words $w_\beta^{(n)}$ are factors of u_β. Note that the sequence $\left(|w_\beta^{(n)}|\right)_{n=1}^{\infty}$ is strictly increasing.

Furthermore, we define sequence $\left(u_\beta^{(n)}\right)_{n=1}^{\infty}$ by

$$u_\beta^{(n)} = \text{prefix of } \ u_\beta \text{ of the length } |w_\beta^{(n)}|\,.$$

Observation 5.4. *For all $n \in \mathbb{N}, n \geq 1$,*

$$w_\beta^{(n+1)} = w_\beta^{(n)} \hat{u}^{(n)} B,$$

where $\hat{u}^{(n)}$ is a prefix of u_β.

Proof. Let us proceed by induction on n. For $n = 1$, we have $w_\beta^{(2)} = B\varphi(w_\beta^{(1)}) = B\varphi(B) = BA^qB = w_\beta^{(1)}A^qB$; $\hat{u}^{(1)} = A^q$. Suppose for some $n \geq 2$ that $w_\beta^{(n)} = w_\beta^{(n-1)}\hat{u}^{(n-1)}B$ and $\hat{u}^{(n-1)}$ is a prefix of u_β. Then

$$w_\beta^{(n+1)} = B\varphi(w_\beta^{(n)}) = B\varphi(w_\beta^{(n-1)}\hat{u}^{(n-1)}B) = B\varphi(w_\beta^{(n-1)})\varphi(\hat{u}^{(n-1)})A^qB = w_\beta^{(n)}\hat{u}^{(n)}B\,,$$

where $\hat{u}^{(n)} = \varphi(\hat{u}^{(n-1)})A^q$ is a prefix of u_β according to Observation 5.1. □

Observation 5.4 allows us to define an infinite word w_β over $\mathcal{A}$ as

$$w_\beta = \lim_{n\to\infty} w_\beta^{(n)}.$$

It follows from the definition of $w_\beta^{(n)}$ that this infinite word fulfils

$$w_\beta = B\varphi(w_\beta). \tag{19}$$

Consequently, using Observation 5.3, we get the following observation.

Observation 5.5. *Let $w'B$ be a prefix of w_β. Then the unique factor w'' of u_β satisfying $w'B = B\varphi(w'')$ is a prefix of w_β.*

We know already from Proposition 4.1 that prefixes of u_β are the factors with the largest number of letters A. The infinite word w_β plays the same role for letters B.

Proposition 5.6. *Any prefix of w_β contains at least the same number of letters B as any other factor of the same length.*

Proof. We will prove the statement by contradiction. Let us assume that there exist a $k \in \mathbb{N}$ and a factor $v = v_0v_1v_2\cdots v_{k-1}$ of u_β such that $|w|_B < |v|_B$, where $w = w_0w_1w_2\cdots w_{k-1}$ is a prefix of w_β. We choose the minimal k with this property. Then

$$|v|_B = |w|_B + 1. \tag{20}$$

The minimality of k implies that $v_0 = B$, $v_{k-1} = B$, and $w_{k-1} = A$. The fact that w is a prefix of w_β which satisfies (19), implies $w_0 = B$. Thus $v_{k-1-q}v_{k-q}\cdots v_{k-3}v_{k-2} = A^q$ according to Observation 5.1, hence $w_{k-1-q}w_{k-q}\cdots w_{k-3}w_{k-2} = A^q$ by virtue of minimality of k. Observation 5.1 together with the fact $w_{k-1} = A$ implies that there is a uniquely determined integer j satisfying $0 \leq j \leq p-q-1$ such that wA^jB is a factor of u_β. Since $v_0 = B$, $w_0 = B$ and $v_{k-1} = B$, we may use Observation 5.3 to deduce that there are unique factors v' and w' of u_β such that

$\varphi(v') = v_1 v_2 \cdots v_{k-1}$ and $\varphi(w') = w_1 w_2 \cdots w_{k-1} A^j B, k \geq 1$. Since $\varphi(v')$ and $\varphi(w')$ contain the same number of letters B, clearly $|v'| = |w'| < k$. Moreover, it follows from Observation 5.5 that the factor w' is a prefix of w_β. As $\varphi(v')$ is shorter than $\varphi(w')$, the word v' contains more letters B than w', which is a prefix of w_β. It is a contradiction with the minimality of k. □

Lemma 5.7. *Let v, v' be factors of u_β of the same length k, let n be such a positive integer that $|w_\beta^{(n)}| \leq k < |w_\beta^{(n+1)}|$. Then*

$$||v|_B - |v'|_B| \leq |w_\beta^{(n)}|_B - |u_\beta^{(n)}|_B \,.$$

Proof. Propositions 4.1 and 5.6 imply

$$||v|_B - |v'|_B| \leq |w'|_B - |u'|_B \,,$$

where u' and w' are prefixes of u_β and w_β, respectively, of length k. Observation 5.4 together with the assumption $k < |w_\beta^{(n+1)}|$ implies that $w' = w_\beta^{(n)} \hat{u}$ for some prefix $\hat{u}$ of u_β. Let us write the factor u' in the form $u' = u_\beta^{(n)} \hat{v}$. Using Proposition 4.1, we get

$$|w'|_B - |u'|_B = |w_\beta^{(n)}|_B - |u_\beta^{(n)}|_B + |\hat{u}|_B - |\hat{v}|_B \leq |w_\beta^{(n)}|_B - |u_\beta^{(n)}|_B \,,$$

which concludes the proof of the statement. □

Lemma 5.7 will be very useful in the investigation of the balance property of u_β, since it enables us to find out the optimal balance bound of u_β by investigation of the sequence $(D_n)_{n=1}^{\infty}$, where

$$D_n := |w_\beta^{(n)}|_B - |u_\beta^{(n)}|_B.$$

The optimal balance bound c is then equal

$$c = \max\{D_n \mid n \in \mathbb{N}\}. \tag{21}$$

In the sequel, we will show that the sequence (D_n) has the form depicted in Figure 1 below, which shows that u_β is $\lceil \frac{p-1}{q} \rceil$-balanced and that this bound cannot be diminished.

To determine the value of D_{n+1} using the value of $D_n = |w_\beta^{(n)}|_B - |u_\beta^{(n)}|_B$, it is important to take in account the following facts.

(1) Since the number of letters A in the word $u_\beta^{(n)}$ is by D_n greater than in $w_\beta^{(n)}$, the length of $\varphi(u_\beta^{(n)})$ is by $(p-q)D_n$ letters longer than the length of $\varphi(w_\beta^{(n)})$.
(2) $w_\beta^{(n+1)} = B\varphi(w_\beta^{(n)})$.
(3) $u_\beta^{(n+1)}$ is a prefix of u_β chosen so that $|u_\beta^{(n+1)}| = |w_\beta^{(n+1)}|$.
(4) Since u_β is the fixed point of the substitution, $\varphi(u_\beta^{(n)})$ is a prefix of u_β as well.
(5) $u_\beta^{(n+1)}$ can be obtained from $\varphi(u_\beta^{(n)})$ by erasing its suffix of length $(p-q)D_n - 1$.
(6) As the lengths of $w_\beta^{(n)}$ and $u_\beta^{(n)}$ are the same, $\varphi(w_\beta^{(n)})$ and $\varphi(u_\beta^{(n)})$ contain the same number of letters B.

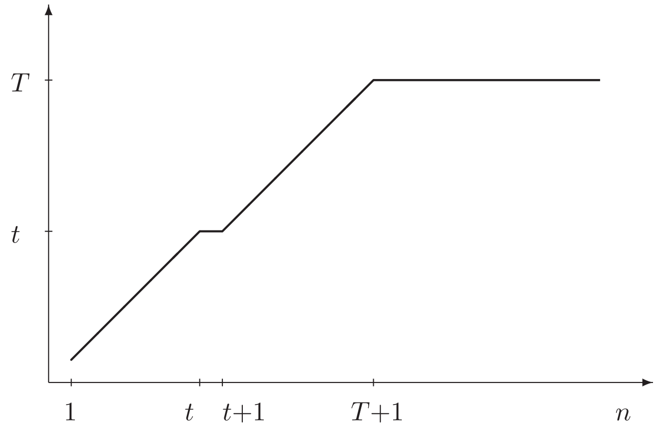

FIGURE 1. Illustration of the sequence (D_n), where $D_n = |w_\beta^{(n)}|_B - |u_\beta^{(n)}|_B$. The consecutive values are connected by a line and $t := \lfloor \frac{p+q}{q+1} \rfloor$ and $T = \lceil \frac{p-1}{q} \rceil$.

These six simple facts imply the following recurrence relation for the sequence (D_n):

$$D_{n+1} = 1 + |v|_B\,, \quad \text{where } v \text{ is a suffix of } \ \varphi(u_\beta^{(n)}) \ \text{ and } \ |v| = (p-q)D_n - 1. \quad (22)$$

Consequently, to determine the value of D_{n+1}, one needs to know the form of the suffix of $\varphi(u_\beta^{(n)})$, hence the form of the suffix of $u_\beta^{(n)}$.

Proposition 5.8. *Let $d_\beta(1) = pq^\omega$, where $p > q+1$ (we exclude the sturmian case) and let $t := \lfloor \frac{p+q}{q+1} \rfloor$ and $T = \lceil \frac{p-1}{q} \rceil$.*

1. *If $n \le t$, then $D_n = n$ and $u_\beta^{(n)}$ has the suffix A^n.*
2. *If $t+1 \le n \le T+1$, then $D_n = n-1$ and $u_\beta^{(n)}$ has the suffix $A^pBA^{(n-1)(q+1)-p}$.*
3. *If $T+1 \le n$, then $D_n = T$ and $u_\beta^{(n)}$ has the suffix A^{T-1}.*

Proof. Let us show the statement by induction on $n \in \mathbb{N}$.
Let $n = 1$, then $w_\beta^{(1)} = B, u_\beta^{(1)} = A$, hence we have $D_1 = |w_\beta^{(1)}|_B - |u_\beta^{(1)}|_B = 1$.
Let us suppose that for some n, $1 < n \le t-1$, it holds

$$D_n = n \text{ and } u_\beta^{(n)} \text{ has the suffix } A^n.$$

Let us use the rule (22) to calculate D_{n+1}. The word $\varphi(u_\beta^{(n)})$ has the suffix

$$\varphi(A^n) = \underbrace{(A^pB)\dots(A^pB)}_{n-\text{times}}. \quad (23)$$

We erase from this word of the length $(p+1)n$ the suffix v of length $(p-q)n-1$. Let us show that in this procedure, we have erased all n letters B, i.e., $|v|_B = n$, and, consequently, $D_{n+1} = 1+n$. To verify this statement, it suffices to prove inequality

$$(p+1)(n-1)+1 \leq (p-q)n-1, \tag{24}$$

which is equivalent to $n \leq \frac{p+q}{q+1} - 1$. Since n is an integer, the inequality means $n \leq \lfloor \frac{p+q}{q+1} \rfloor - 1 = t-1$, as we have supposed. Now we have to show, that at least $n+1$ letters remain in the word $\varphi(A^n)$ after removing the suffix v of the length $(p-q)n-1$, i.e., we have to verify $(p+1)n-(p-q)n+1 \geq n+1$. This inequality is easy to check.

Let us show how the statement 2. follows from 1. For $n = t$ the statement 1. implies that

$$D_t = t \ \text{ and } u_\beta^{(t)} \ \text{ has the suffix } A^t.$$

Clearly,

$$\varphi(A^t) = \underbrace{(A^pB)\dots(A^pB)}_{t-\text{times}}$$

is a suffix of $\varphi(u_\beta^{(t)})$.

In order to prove $D_{t+1} = t$, we have to show that the suffix of the length $(p-q)D_t - 1 = (p-q)t-1$ of the word $\varphi(A^t)$ contains exactly $t-1$ letters B. So we have to prove

$$(p+1)(t-2)+1 \leq (p-q)t-1 \leq (p+1)(t-1)$$

or equivalently $\frac{p}{q+1} \leq t \leq \frac{2p}{q+1}$, which is consequence of the definition of t.

By erasing the suffix v we have removed $t-1$ letters B from the word $\varphi(A^t)$ which has t letters B. The remaining part of this word (and therefore the suffix of $u_\beta^{(t+1)}$) is A^pBA^r, where $r = (p+1)(t-1) - |v| = (q+1)t-p$.

Now, suppose that for some $t+1 < n \leq T$, it holds

$$D_n = n-1 \ \text{ and } u_\beta^{(n)} \ \text{ has the suffix } A^pBA^{(n-1)(q+1)-p}.$$

Then $\varphi(u_\beta^{(n)})$ has the suffix

$$\varphi(A^pBA^{(n-1)(q+1)-p}) = \underbrace{(A^pB)\dots(A^pB)}_{p-\text{times}} A^qB \ \underbrace{(A^pB)\dots(A^pB)}_{(n-1)(q+1)-p-\text{times}} .$$

We want to prove that the suffix v of $\varphi(u_\beta^{(n)})$ of length $(p-q)(n-1)-1$ satisfies $|v|_B = n-1$ and that $u_\beta^{(n+1)}$ has the suffix $A^pBA^{n(q+1)-p}$. Before writing down the inequalities to be shown, notice the following two facts. If we erase v from the end of $\varphi(u_\beta^{(n)})$, we erase necessarily $A^qB \ \underbrace{(A^pB)\dots(A^pB)}_{(n-1)(q+1)-p-\text{times}}$. To see this, it suffices to prove the inequality (easily feasible using (27))

$$(q+1)+(p+1)\big((n-1)(q+1)-p\big) \leq (p-q)(n-1)-1. \tag{25}$$

At the same time, if we erase v from the end of $\underbrace{(A^pB)\dots(A^pB)}_{p-\text{times}} A^qB \ \underbrace{(A^pB)\dots(A^pB)}_{(n-1)(q+1)-p-\text{times}}$, it will still keep a prefix longer than $p+1$. This follows from the following inequality (easy to check)

$$(p-q)(n-1)-1 < (p+1)(p-1)+(q+1)+(p+1)\big((n-1)(q+1)-p\big). \tag{26}$$

Knowing the relations (25) and (26), what we have to show are the following two inequalities

$$(q+1)+(n-3)(p+1)+1 \leq (p-q)(n-1)-1 \leq (q+1)+(n-2)(p+1)-\big(n(q+1)-p\big).$$

The first one shows that $|v|_B \geq n-1$ while the second one shows that $|v|_B \leq n-1$ and that $u_\beta^{(n+1)}$ has the suffix $A^pBA^{n(q+1)-p}$. As it is easily verified for positive integers a, b

$$\left\lceil \frac{a}{b} \right\rceil \leq \frac{a}{b} + \frac{b-1}{b}\,,$$

we get

$$T = \left\lceil \frac{p-1}{q} \right\rceil \leq \frac{p-1}{q} + \frac{q-1}{q} = \frac{p+q-2}{q}\,. \tag{27}$$

The first inequality is equivalent to $n \leq \frac{2p}{q+1}$. Since $n \leq T \leq \frac{p+q-2}{q}$ it is enough to verify that $\frac{p+q-2}{q} \leq \frac{2p}{q+1}$, which is equivalent to $(q+1)(q-2) \leq p(q-1)$. This equation holds because in our substitution $p > q+1$. The second inequality is trivial.

Finally, let us show how the statement 3. follows from 2. For $n = T+1$ the statement 2. implies that

$$u_\beta^{(n)} \text{ has the suffix } A^{T-1} \text{ and } D_n = T. \tag{28}$$

Consequently, the word $\varphi(u_\beta^{(n)})$ has the suffix

$$\varphi(A^{T-1}) = \underbrace{(A^pB)(A^pB)\dots(A^pB)}_{(T-1)-\text{times}}.$$

We erase from this word the suffix v of length $(p-q)T-1$. Performing this procedure, we have erased all the letters B, i.e., $T-1$ letters B. To verify this statement, it suffices to prove the inequality

$$(p+1)(T-2)+1 \leq (p-q)T-1. \tag{29}$$

In order to prove that by erasing v, there are still at least $T-1$ letters left in the word $\varphi(A^{T-1})$, one has to show

$$T-1 \leq (p+1)(T-1)-(p-q)T+1. \tag{30}$$

Consequently, if we verify the equalities (29) and (30), it will be proved that $D_{n+1} = T$ and $u_\beta^{(n+1)}$ has the suffix A^{T-1}. It means by virtue of (28) for an index $n \geq T+1$, we have shown the virtue for the index $n+1$, thus, using induction, for all $n \geq T+1$.

The equality (30) is equivalent to $T \geq \frac{p-1}{q}$, which is evidently satisfied as $T = \lceil \frac{p-1}{q} \rceil$. The equality (29) holds for being equivalent to $T \leq \frac{2p}{q+1}$.

□

As an immediate consequence of the just proved proposition and the relation (21), we have the following essential theorem.

Theorem 5.9. *Let $d_\beta(1) = pq^\omega$, where $p > q+1$, i.e., β is the larger root of the polynomial $x^2 - mx - n$, where $m = p+1$ and $n = p-q$. The infinite word u_β is c-balanced, where $c = \lceil \frac{p-1}{q} \rceil$. This value c is the smallest possible.*

6. A lower bound on $L_\oplus(\beta)$

In Section 4, we have derived an upper bound on the constant c in the balance property of u_β from the knowledge of an upper bound on $L_\oplus(\beta)$. Now, conversely, let us find a lower bound on $L_\oplus(\beta)$ using the optimal constant c in the balance property of u_β introduced in Theorem 5.9.

To derive a lower bound on $L_\oplus(\beta)$, we will use the fact that there exist a factor w and a prefix $\hat{w}$ of u_β of the same length such that $|\hat{w}|_A = |w|_A + \lceil \frac{p-1}{q} \rceil$. Let $x, y \in \mathbb{Z}_\beta$, $x < y$, such that the gaps in the segment of $\mathbb{Z}_\beta$ from x to y correspond to the word w. And, let $z \in \mathbb{Z}_\beta$ be the β-integer corresponding to the prefix $\hat{w}$. Then

$$x + z = y + \left\lceil \frac{p-1}{q} \right\rceil (\Delta_A - \Delta_B) = y + \left\lceil \frac{p-1}{q} \right\rceil \frac{p-q}{\beta}.$$

From Observation 3.2, it follows that

$$fp_\beta(x+z) = fp_\beta\left(y + \left\lceil \frac{p-1}{q} \right\rceil \frac{p-q}{\beta}\right) \geq fp_\beta\left(\left\lceil \frac{p-1}{q} \right\rceil \frac{p-q}{\beta}\right) \geq fp_\beta\left(\left\lfloor \frac{p-1}{q} \right\rfloor \frac{p-q}{\beta}\right).$$

Now, it suffices to show that $fp_\beta\big(\lfloor \frac{p-1}{q} \rfloor \frac{p-q}{\beta}\big) = \lfloor \frac{p-1}{q} \rfloor$.

Lemma 6.1. *For $j = 1, \ldots, \lfloor \frac{p-1}{q} \rfloor$, the β-expansion of the number $j\frac{p-q}{\beta}$ is*

$$\left\langle j \frac{p-q}{\beta} \right\rangle_\beta = (j-1) \bullet a_j \cdots a_1\,,$$

where $a_1 := p - q$ and $a_i := (p-1) - iq$ for $i = 2, \ldots, \lfloor \frac{p-1}{q} \rfloor$.

Proof. The numbers a_i are defined so that $a_i \geq 0$ and $(j-1)a_j a_{j-1} \cdots a_1 \prec pq^\omega$. Thus, the expression $(j-1) \bullet a_j \cdots a_1$ is the β-expansion of a positive number. Now, we have to show that

$$j\frac{p-q}{\beta} = j - 1 + \frac{a_j}{\beta} + \frac{a_{j-1}}{\beta^2} + \cdots + \frac{a_1}{\beta^j},$$

which can be done easily by mathematical induction on j. □

Lemma 6.1 shows that $fp_\beta\big(\lfloor \frac{p-1}{q} \rfloor \frac{p-q}{\beta}\big) = \lfloor \frac{p-1}{q} \rfloor$, in other words, it implies the announced lower bound on $L_\oplus(\beta)$. To sum up, we have derived the following theorem.

Theorem 6.2. *Let $d_\beta(1) = pq^\omega$, where $p > q+1$, i.e., β is the larger root of the polynomial $x^2 - mx - n$, where $m = p+1$ and $n = p - q$. Then*

$$\left\lfloor \frac{p-1}{q} \right\rfloor \leq L_\oplus(\beta) \leq \left\lceil \frac{p}{q} \right\rceil.$$

Let us mention that the difference between the upper bound $\lceil \frac{p}{q} \rceil$ and the lower bound $\lfloor \frac{p-1}{q} \rfloor$ is always 1. Our computer experiments support the conjecture $L_\oplus(\beta) = \lfloor \frac{p-1}{q} \rfloor$.

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References

[1] B. Adamczewski, *Balances for fixed points of primitive substitutions*, Theoret. Comput. Sci. **307** (2003), 47-75
[2] S. Akiyama, *Cubic Pisot units with finite beta expansions,* in Algebraic Number Theory and Diophantine Analysis, Graz 1998, Eds. F.Halter-Koch and R.F. Tichy, de Gruyter, Berlin, (2000), 11-26
[3] P. Ambrož, Ch. Frougny, Z. Masáková, E. Pelantová, *Palindromic complexity of infinite words associated with simple Parry numbers*, to appear in Annales de l'Institut Fourier (2005), pp. 25
[4] P. Ambrož, Ch. Frougny, Z. Masáková, E. Pelantová, *Arithmetics on number systems with irrational bases*, Bull. Belgian. Soc. Simon Stevin **10** (2003), 641-659
[5] P. Ambrož, Z. Masáková, E. Pelantová, *Addition and multiplication of beta-expansions in generalized Tribonacci base*, to appear in J. Autom. Lang. Comb. (2005)
[6] J. Bernat, *Computation of $L_\oplus$ for several cubic Pisot numbers*, to appear in J. Autom. Lang. Comb. (2005)
[7] J. Berstel, *Recent results on extension of sturmian words*, Int. J. Algebra Comput. **12** (2002), 371-385
[8] A. Bertrand, *Développements en base de Pisot et répartition modulo* 1, C. R. Acad. Sci. Paris **285**(1977), 419-421
[9] Č. Burdík, Ch. Frougny, J. P. Gazeau, R. Krejcar, *Beta-integers as natural counting systems for quasicrystals,* J. Phys. A **31** (1998), 6449-6472
[10] S. Fabre, *Substitutions et β-systèmes de numération,* Theoret. Comput. Sci. **137** (1995), 219-236
[11] Ch. Frougny, Z. Masáková, E. Pelantová, *Complexity of infinite words associated with beta-expansions*, RAIRO Theor. Inform. Appl. **38** (2004), 163-185; Corrigendum, RAIRO Theor. Inform. Appl. **38** (2004), 269-271
[12] Ch. Frougny, Z. Masáková, E. Pelantová, *Infinite special branches in words associated with beta-expansions*, to appear in J. Autom. Lang. Comb. (2005)
[13] Ch. Frougny, B. Solomyak, *Finite β-expansions,* Ergodic Theory Dynam. Systems 12 (1994), 713-723
[14] L. S. Guimond, Z. Masáková, E. Pelantová, *Arithmetics of β-expansions,* Acta Arithmetica 112.1 (2004), 23-40
[15] M. Hollander, *Linear numeration systems, finite beta-expansions, and discrete spectrum of substitution dynamical systems*, PhD Thesis, Washington University (1996)
[16] J. Justin, G. Pirillo, *On a combinatorial property of sturmian words*, Theor. Comput. Sci. **154(2)** (1996), 387-394
[17] J. Lagarias, *Geometric models for quasicrystals I. Delone sets of finite type*, Discrete Comput. Geom. **21** (1999), 161-191
[18] A. Messaoudi, *Généralisation de la multiplication de Fibonacci*, Math. Slovaca **50** (2000), 135-148
[19] Y. Meyer, *Quasicrystals, Diophantine approximation, and algebraic numbers*, in: F.Axel, D.Gratias (Eds.), Beyond Quasicrystals, Les éditions de physique, Springer, Berlin (1995), 3-16
[20] M. Morse, G.A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math. **62** (1940), 1-42
[21] W. Parry, *On the beta-expansions of real numbers*, Acta Math. Acad. Sci. Hungar.**11** (1960), 401-416
[22] A. Rényi, *Representations for real numbers and their ergodic properties,* Acta Math. Acad. Sci. Hungar. **8** (1957), 477-493
[23] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc. **12**(1980), 269-278
[24] D. Shechtman, I. Blech, D. Gratias, and J. Cahn, *Metallic phase with long-range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), 1951-1954
[25] W. P. Thurston, *Groups, tilings, and finite state automata*, Geometry supercomputer project research report GCG1, University of Minnesota (1989)
[26] O. Turek, *Balance properties of the fixed point of the substitution associated to quadratic simple Pisot numbers*, to appear in RAIRO Theor. Inform. Appl.

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