Combinatorial Proofs of Various q-Pell Identities via Tilings

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Abstract

Recently, Benjamin, Plott, and Sellers proved a variety of identities involving sums of Pell numbers combinatorially by interpreting both sides of a given identity as enumerators of certain sets of tilings using white squares, black squares, and gray dominoes. In this article, we state and prove q–analogues of several Pell identities via weighted tilings.

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1 Introduction

In a recent work by Benjamin, Plott, and Sellers [2], a combinatorial interpretation of the Pell numbers was introduced. The nth Pell number, denoted by p_n, is defined recursively by $p_0 = 1$, $p_1 = 2$, and $p_n = 2p_{n-1} + p_{n-2}$ for all $n \geq 2$. As shown in [2], p_n can be interpreted as the number of Pell tilings of length n, that is to say, the number of tilings of a $1 \times n$ board using white squares, black squares, and gray dominoes. So, for example, $p_2 = 5$ since a board of length 2, or a 2–board, can be covered by two white squares or two black squares or one white square and one black square (in either order) or one gray domino. Using this interpretation, Benjamin, Plott and Sellers were able to prove a variety of identities involving sums of Pell numbers.

For the purposes of this paper, we will focus on the following Pell identities.

Theorem 1 *[2, Lemma 5] For all $n \geq 0$,*

$$p_{2n+1} = 2\sum_{k=0}^{n} p_{2k}.$$

Theorem 2 *For all $n \geq 0$,*

$$p_{2n} = p_n^2 + p_{n-1}^2.$$

Theorem 3 *For all $n \geq 0$,*

$$p_{2n+1} = 2\left(p_n^2 + 2\sum_{k=0}^{n-1} p_k^2\right).$$

Theorem 4 *[2, Lemma 7] For all $n \geq 0$,*

$$p_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} 2^{n-2k}.$$

Theorem 5 *[2, Theorem 8] For all $n \geq 2$,*

$$p_{n-2} + p_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k}.$$

A natural question to ask is whether q–analogues for such identities as those above exist. We have answered this question in the affirmative for each of these identities. Our primary goal in this work is to state and prove these q–analogues, mimicking the combinatorial techniques of Benjamin, Plott, and Sellers as much as possible by considering *weighted Pell tilings*.

2 The q-Pell Numbers

In a recent work by Santos and Sills [3], the q–Pell numbers $P_n(q)$ were defined by

$$P_{n+1}(q) = (1+q^{n+1})P_n(q) + q^n P_{n-1}(q), \qquad P_0(q) = 1, P_1(q) = 1+q. \tag{1}$$

Clearly, these q–Pell numbers specialize to the values p_n studied in [2] whenever $q = 1$. That is, $p_n = P_n(1)$ for each $n \geq 0$. Thus, the recursion in (1) suggests that we can generalize the combinatorial interpretation of the values p_n in [2] by q–counting the white squares, black squares, and gray dominoes. To begin, we define the weight of tile t as follows:

$$w(t) = \begin{cases} i & t \text{ is a gray domino at position } (i, i+1) \\ i & t \text{ is a black square at position } i \\ 0 & t \text{ is a white square at position } i \end{cases}$$

Let $\mathcal{T}_n$ be the set of all tilings of an n–board with white squares, black squares, and gray dominoes. Then, for any tiling $T \in \mathcal{T}_n$, we define the q-weight of T by

$$w_q(T) = \prod_{t \in T} q^{w(t)},$$

and define

$$\widetilde{P}_n(q) = \sum_{T \in \mathcal{T}_n} w_q(T).$$

From this definition, it clearly follows that

$$P_n(q) = \widetilde{P}_n(q).$$

To see this note that there is only one empty tiling of q–weight 1, so that $\widetilde{P}_0(q) = 1$. Likewise, there are clearly two tilings of length one with q–weights 1 (a white square) and q (a black square). As such, $\widetilde{P}_1(q) = 1 + q$. For $n \geq 1$, we note that each Pell tiling of an $(n+1)$–board can be classified by its last tile. In particular, the contribution to $\widetilde{P}_{n+1}(q)$ of all tilings whose last tile is a gray domino is $q^n \widetilde{P}_{n-1}(q)$, the contribution to $\widetilde{P}_{n+1}(q)$ of all tilings whose last tile is a black square is $q^{n+1}\widetilde{P}_n(q)$, and the contribution to $\widetilde{P}_{n+1}(q)$ of all tilings whose last tile is a white square is $\widetilde{P}_n(q)$. Thus,

$$\widetilde{P}_{n+1}(q) = q^n \widetilde{P}_{n-1}(q) + (1+q^{n+1})\widetilde{P}_n(q),$$

and hence the $\widetilde{P}_n(q)$ yields a combinatorial interpretation for $P_n(q)$. For example, all possible Pell tilings of a 3–board are illustrated in Figure 1. According to the q-weight of each such tiling, we find that

$$P_3(q) = 1 + 2q + 2q^2 + 3q^3 + 2q^4 + q^5 + q^6.$$

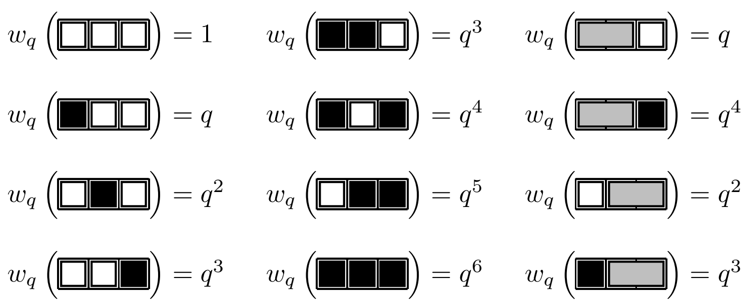

Figure 1: The q-enumeration of Pell tilings of a 3–board.

3 The q–Pell Identities

Given the definition of the q–Pell numbers $P_n(q)$ above, we now state and prove q–analogues of Theorems 1 through 5 via these weighted tilings. We begin with Theorem 1.

3.1 An Analogue of Theorem 1

For our first q-analogue, we focus on Pell tilings of an odd length. Clearly, any such tiling must contain at least one square. If the right–most square occurs in position $2i+1$ for $0 \le i \le n$, then the first $2i$ positions are covered by a Pell tiling of length $2i$ and the last $2n-2i$ positions are covered by $n-i$ gray dominoes. Therefore the q–weight of all such Pell tilings is given by

$$q^{n(n+1)-i(i+1)}(1+q^{2i+1})P_{2i}(q)$$

where the factor of $q^{n(n+1)-i(i+1)}$ accounts for the weight of the $n-i$ dominoes and the factor of $(1+q^{2i+1})$ accounts for the choice of white square or black square in position $2i+1$. Summing over all values of i produces our first result.

Theorem 6 *For all $n \ge 0$,*

$$P_{2n+1}(q) = \sum_{i=0}^{n} q^{n(n+1)-i(i+1)}(1+q^{2i+1})P_{2i}(q). \tag{2}$$

3.2 Analogues of Theorems 2 and 3

In order to state our q–analogues of Theorems 2 and 3, we must first introduce the notion of a shifted q–Pell number. If we think of $P_n(q)$ as the generating function for weighted Pell tilings covering positions 1 through n, then for any $m \ge 0$, let $P_n^{(m)}(q)$ denote the generating function for weighted Pell tilings covering only positions $m+1$ through $m+n$. We will refer to $P_n^{(m)}(q)$ as an *m–shifted q–Pell number.* Using the same reasoning as in Section 2, we see that $P_n^{(m)}(q)$ satisfies the recursion

$$P_{n+1}^{(m)}(q) = (1 + q^{m+n+1})P_n^{(m)}(q) + q^{m+n}P_{n-1}^{(m)}(q),$$

with initial conditions $P_0^{(m)}(q) = 1$ and $P_1^{(m)}(q) = 1 + q^{m+1}$. Furthermore, notice that $P_n^{(0)}(q) = P_n(q)$ and $P_n^{(m)}(1) = p_n$ for all $m, n \geq 0$.

We are now ready to state our q–analogues. We begin with an analogue of a generalization of Theorem 2.

Theorem 7 *For all $n \geq 2$ and $1 \leq i \leq n-1$,*

$$P_n(q) = P_i(q)P_{n-i}^{(i)}(q) + q^i P_{i-1}(q)P_{n-i-1}^{(i+1)}(q).$$

Proof: Clearly we can partition the collection of all Pell tilings of an n-board based on whether or not a tiling has a gray domino at position $(i, i+1)$. If there is a gray domino at this position, then the first $i-1$ positions are covered by a tiling of length $i-1$ and the last $n-i-1$ positions are covered by an $(i+1)$–shifted tiling of length $n-i-1$. All tilings of this form are counted by

$$q^i P_{i-1}(q)P_{n-i-1}^{(i+1)}(q)$$

where the factor of q^i accounts for the weight of the domino at position $(i, i+1)$.

On the other hand, if there is no domino at position $(i, i+1)$, the first i positions are covered by a tiling of length i and the last $n-i$ positions are covered by an i–shifted tiling of length $n-i$. All tilings of this form are counted by

$$P_i(q)P_{n-i}^{(i)}(q),$$

as required. □

In particular, applying Theorem 7 to Pell tilings of length $2n$ with $i = n$ produces

$$P_{2n}(q) = P_n(q)P_n^{(n)}(q) + q^n P_{n-1}(q)P_{n-1}^{(n+1)}(q)$$

which is a natural q–analogue of Theorem 2.

We finish this section with another q-analogue for Pell tilings of odd length.

Theorem 8 *For all $n \geq 0$,*

$$P_{2n+1}(q) = (1 + q^{n+1})\left(P_n(q)P_n^{(n+1)}(q) + \sum_{i=1}^{n} q^{in}(1 + q^i)P_{n-i}(q)P_{n-i}^{(n+i+1)}(q)\right).$$

Proof: We begin our proof by again pointing out that any Pell tiling of odd length must contain at least one square. But instead of grouping these tilings according to the right–most square (as in the proof of Theorem 6), we will now group tilings according to the square closest to position $n+1$. This square must be unique since there cannot be squares in positions $n+1-i$ and $n+1+i$ that are both considered closest to the center. If there were squares in these two positions, the region of the board from position $n+2-i$ to position $n+i$ is a board of length $2i+1$ and must contain a square that is even closer to the center.

If a tiling has a square in position $n+1$, then the first n positions are covered by a tiling of length n and the last n positions are covered by an $(n+1)$–shifted tiling of length n. Thus, all tilings that have a square in position $n+1$ are counted by

$$(1+q^{n+1})P_n(q)P_n^{(n+1)}(q), \tag{3}$$

where the factor of $(1+q^{n+1})$ represents the choice of color for the square in position $n+1$.

Now suppose that the square closest to position $n+1$ is in position $n+1-i$ for $1 \leq i \leq n$. Note that this means positions $n+2-i$ through $n+1+i$ must be covered with i gray dominoes. The total weight of these gray dominoes is given by

$$(n+2-i)+(n+4-i)+\cdots+(n+2i-i) = in+i.$$

Furthermore, all tilings of this form have the first $n-i$ positions covered by a tiling of length $n-i$ and the last $n-i$ tilings covered by an $(n+1+i)$–shifted tiling of length $n-i$. Therefore, all tilings of this form are counted by

$$q^{in+i}(1+q^{n+1-i})P_{n-i}(q)P_{n-i}^{(n+1+i)}(q)$$

where the factor of $(1+q^{n+1-i})$ represents the choice of color for the square in position $n+1-i$.

The only possibility left is that the square closest to position $n+1$ is in position $n+1+i$ for $1 \leq i \leq n$. This means positions $n+1-i$ through $n+i$ must be covered with i gray dominoes. The total weight of these gray dominoes is given by

$$(n+1-i)+(n+3-i)+\cdots+(n+2i-1-i) = in.$$

Similarly, all tilings of this form are counted by

$$q^{in}(1+q^{n+1+i})P_{n-i}(q)P_{n-i}^{(n+1+i)}(q)$$

where the factor of $(1+q^{n+1+i})$ represents the choice of color for the square in position $n+1+i$.

Combining the last two cases and summing over all values of i produces

$$\sum_{i=1}^{n} q^{in}(1+q^{n+1})(1+q^{i})P_{n-i}(q)P_{n-i}^{(n+1+i)}(q). \tag{4}$$

Finally, adding together (3) and (4) completes the proof. $\square$

3.3 An Analogue of Theorem 4

In preparation for stating and proving our q–analogue of Theorem 4, we provide an important lemma which involves the *q-multinomial coefficients* which are defined by

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix}_q = \frac{[n]_q!}{[n_1]_q![n_2]_q!\cdots[n_r]_q!}$$

where $n_1+n_2+\cdots+n_r = n$. Here $[n]_q = \frac{1-q^n}{1-q}$ and $[n]_q! = \prod_{i=1}^n [i]_q$ respectively denote the usual q-analogues of n and $n!$.

Lemma 9 *The generating function for tilings with exactly j black squares, k gray dominoes and l white squares is given by*

$$q^{k^2+\binom{j+1}{2}+kj}\begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q.$$

Proof: Let $\mathcal{T}_{j,k,l}$ represent the collection of tilings of an n-board using exactly j black squares, k gray dominoes and l white squares where $n = j+2k+l$. To each tiling $T \in \mathcal{T}_{j,k,l}$, we will associate a sequence, σ_T, by replacing each black square with a zero, each gray domino with a one and each white square with a two. Thus σ_T is really just an element of $\mathcal{R}(0^j1^k2^l)$, the collection of all rearrangements of j zeros, k ones and l twos. For any $\sigma \in \mathcal{R}(0^j1^k2^l)$, let T_σ represent the corresponding tiling.

For a given $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{R}(0^j1^k2^l)$, we say that (σ_i, σ_j) is an inversion of σ if $i < j$ and $\sigma_i > \sigma_j$. Let $inv(\sigma)$ represent the total number of inversions of σ. In other words,

$$inv(\sigma) = |\{i < j \mid \sigma_i > \sigma_j\}|.$$

With these definitions in mind, we begin the process of calculating the weight of a generic tiling $T \in \mathcal{T}_{j,k,l}$. First, note that the tiling of minimum weight, T_{min}, corresponds to the rearrangement σ_{min}, where

$$\sigma_{min} = \underbrace{000...00}_{j}\underbrace{111...11}_{k}\underbrace{222...22}_{l}.$$

This is a simple consequence of the following facts:

1. the weight of a black square followed by a gray domino is less than that of a gray domino followed by a black square,

2. the weight of a black square followed by a white square is less than that of a white square followed by a black square, and

3. the weight of a gray domino followed by a white square is less than that of a white square followed by a gray domino.

Furthermore, the weight of T_{min} is given by

$$\sum_{r=1}^{j} r + \sum_{s=1}^{k}(j+2s-1) = \binom{j+1}{2} + kj + k^2.$$

Next, we consider how much the weight of T differs from the weight of T_{min}. First, pick $1 \leq r \leq j$ and consider the change in weight of the rth black square, from left to right. The weight of this tile in T_{min} is r. In T, the weight of this tile is increased by one for each white square to its left and by two for each gray domino to its left. In other words, the change in position of this tile is the number of inversions in σ_T of the form $(2, 0_r)$ plus twice the number of inversions of the form $(1, 0_r)$, where 0_r represents the rth zero. Summing over all r, the total increase in weight of the black squares is

$$|\{i < j \mid \sigma_i = 2,\ \sigma_j = 0\}| + 2|\{i < j \mid \sigma_i = 1,\ \sigma_j = 0\}|.$$

Now select $1 \leq s \leq k$ and consider the change in weight of the sth gray domino, from left to right. The weight of this tile in T_{min} is $j + 2s - 1$. In T, the weight of this tile is increased by one for each white square to its left and decreased by one for each black square to its right. In other words, the change in position of this tile is the number of inversions in σ_T of the form $(2, 1_s)$ minus the number of inversions of the form $(1_s, 0)$, where 1_s represents the sth one. Summing over all s, the total increase in weight of the gray dominoes is

$$|\{i < j \mid \sigma_i = 2,\ \sigma_j = 1\}| - |\{i < j \mid \sigma_i = 1,\ \sigma_j = 0\}|.$$

Therefore the net change in weight from T_{min} to T, is precisely the number of inversions in σ_T. In other words, the q-weight of T_σ is given by

$$\begin{aligned} w_q(T_\sigma) &= q^{inv(\sigma)} w_q(T_{min}) \\ &= q^{k^2+\binom{j+1}{2}+kj+inv(\sigma)} \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{T \in \mathcal{T}_{j,k,l}} w_q(T) &= \sum_{\sigma \in \mathcal{R}(0^j 1^k 2^l)} w_q(T_\sigma) \\ &= \sum_{\sigma \in \mathcal{R}(0^j 1^k 2^l)} q^{k^2+\binom{j+1}{2}+kj+inv(\sigma)} \\ &= q^{k^2+\binom{j+1}{2}+kj} \sum_{\sigma \in \mathcal{R}(0^j 1^k 2^l)} q^{inv(\sigma)} \\ &= q^{k^2+\binom{j+1}{2}+kj} \begin{bmatrix} j+k+l \\ j,k,l \end{bmatrix}_q \end{aligned}$$

The last line follows from the classic result of MacMahon. (See [1, page 41] for more on this result.) □

We can now prove the following q–analogue of Theorem 4.

Theorem 10 *For all $n \geq 0$,*

$$P_n(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \prod_{j=1}^{n-2k} (1 + q^{j+k})$$

Proof: Clearly, the left–hand side q-counts the set of all Pell tilings of an n–board. Note that any Pell tiling of length n that has k gray dominoes, for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, must have $n - 2k$ squares, for a total of $n - k$ tiles. Applying Lemma 9, we have that the sum of all q-weighted tilings with k gray dominoes, j black squares for $0 \leq j \leq n - 2k$, and $n - j - 2k$ white squares is given by

$$q^{k^2 + \binom{j+1}{2} + kj} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q .$$

Summing over all possible j and k yields

$$\begin{aligned} P_n(q) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2k} q^{k^2 + \binom{j+1}{2} + kj} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \sum_{j=0}^{n-2k} q^{\binom{j+1}{2} + kj} \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \prod_{j=1}^{n-2k} (1 + q^{j+k}) \end{aligned}$$

where the last step follows from the q–analogue of the binomial theorem (see [1]). □

3.4 An Analogue of Theorem 5

In order to present an analogue of Theorem 5, it will be necessary to first consider the following m–shifted version of Lemma 9.

Lemma 11 *The generating function for m–shifted tilings with exactly j black squares, k gray dominoes and l white squares is given by*

$$q^{m(k+j) + k^2 + \binom{j+1}{2} + kj} \begin{bmatrix} j+k+l \\ j, k, l \end{bmatrix}_q .$$

Proof: To construct any m–shifted tiling of length n, simply take a tiling of length n and move each tile m positions to the right. Thus, the weight of each black square and gray domino increases by m. In our case, there are a total of $j + k$ black squares and gray dominoes, so this process of constructing an m–shifted Pell tiling increases the q–weight by a factor of $q^{m(k+j)}$. Applying Lemma 9 completes the proof. □

We are now in position to prove the following q–analogue of Theorem 5.

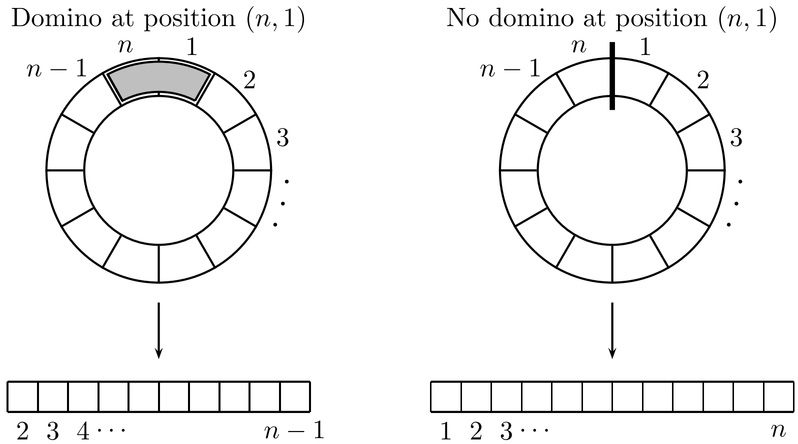

Figure 2: Two types of bracelets.

Theorem 12 *For all $n \geq 2$,*

$$P_n(q) + q^n P^{(1)}_{n-2}(q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \prod_{j=1}^{n-2k} (1 + q^{j+k}) \tag{5}$$

Proof: As in the proof of Theorem 8 in [2], we will consider the sum of weighted Pell tilings of a *bracelet* of length n. In other words, we introduce a gray domino that simultaneously covers positions n and 1. Such a domino will have a q–weight of q^n.

We will now show that both sides of (5) q–count Pell tilings of a bracelet of length n. To begin, note that either a bracelet has a gray domino at position $(n, 1)$ or not, as illustrated in Figure 2. In the first case, the gray domino at position $(n, 1)$ contributes a factor of q^n to the q–weight of the tiling. Removing this domino leaves us with a 1–shifted Pell tiling of length $n-2$ (see Figure 2). In the second case, note that each tiling can be broken between positions n and 1 to produce a straight Pell tiling of length n with the same q-weight (again, see Figure 2). As such, we find that the sum of the q-weights of all Pell tilings of a bracelet of length n is given by $P_n(q) + q^n P^{(1)}_{n-2}(q)$.

Next, for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $0 \leq j \leq n-2k$, we determine the sum of the q–weights of the tilings of a bracelet of length n with exactly k gray dominoes and j black squares. By Lemma 9, it is clear that the sum of the q–weights of the bracelets that do not have a domino at position $(n, 1)$ is given by

$$q^{k^2 + \binom{j+1}{2} + kj} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q .$$

Similarly, applying Lemma 11 with $m=1$, we find that the sum of the q–weights of the bracelets with a domino at position $(n,1)$ is

$$\begin{aligned} & q^{n+(k-1+j)+(k-1)^2+\binom{j+1}{2}+(k-1)j} \begin{bmatrix} n-k-1 \\ j, k-1, n-j-2k \end{bmatrix}_q \\ = \quad & q^{k^2+\binom{j+1}{2}+kj+(n-k)} \begin{bmatrix} n-k-1 \\ j, k-1, n-j-2k \end{bmatrix}_q. \end{aligned}$$

Noting that

$$\begin{bmatrix} n-k-1 \\ j, k-1, n-j-2k \end{bmatrix}_q = \frac{[k]_q}{[n-k]_q} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q,$$

we find

$$\begin{aligned} & q^{k^2+\binom{j+1}{2}+kj} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q + q^{k^2+\binom{j+1}{2}+kj+(n-k)} \begin{bmatrix} n-k-1 \\ j, k-1, n-j-2k \end{bmatrix}_q \\ = \quad & q^{k^2+\binom{j+1}{2}+kj} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q \left(1 + \frac{q^{n-k}[k]_q}{[n-k]_q}\right) \\ = \quad & q^{k^2+\binom{j+1}{2}+kj} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q \left(\frac{[n-k]_q + q^{n-k}[k]_q}{[n-k]_q}\right) \\ = \quad & q^{k^2+\binom{j+1}{2}+kj} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q. \end{aligned}$$

Summing over all $0 \le k \le \lfloor \frac{n}{2} \rfloor$ and $0 \le j \le n-2k$ yields

$$\begin{aligned} P_n(q) + q^n P^{(1)}_{n-2}(q) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2k} q^{k^2+\binom{j+1}{2}+kj} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ j, k, n-j-2k \end{bmatrix}_q \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \sum_{j=0}^{n-2k} q^{\binom{j+1}{2}+kj} \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \prod_{j=1}^{n-2k} (1+q^{j+k}) \end{aligned}$$

where again, the last step follows from the q–analogue of the binomial theorem. □

4 Generalizations

In this final section, we ask a very natural question: Can any of the above q–analogues be generalized to arbitrary numbers of colors of squares and dominoes (as in the closing remarks in [2])? It is clear that the answer is yes, and we provide some of the more "obvious" generalizations below.

Suppose that there are $a \geq 1$ different colors of squares, $s_1, s_2, s_3, \ldots, s_a$ and $b \geq 1$ different colors of dominoes, $d_1, d_2, \ldots, d_b$. We define the q–weight of these colored tiles as follows:

$$w_q(t) = \begin{cases} q^{ij} & \text{if } t \text{ is a } d_j \text{ colored domino at position } (i, i+1) \\ q^{i(j-1)} & \text{if } t \text{ is an } s_j \text{ colored square at position } i. \end{cases}$$

Note that s_1 corresponds to a white square, s_2 corresponds to a black square, and d_1 corresponds to a gray domino of the previous sections. Letting $P_n(a, b; q)$ denote the corresponding generating function for Pell tilings of an n–board, we have

$$\begin{aligned} P_{n+1}(a, b; q) &= (1 + q^{n+1} + \cdots + q^{(a-1)(n+1)}) P_n(a, b; q) \\ &\quad + (q^n + q^{2n} + \cdots + q^{bn}) P_{n-1}(a, b; q) \\ &= \frac{1 - q^{a(n+1)}}{1 - q^{n+1}} P_n(a, b; q) + q^n \frac{1 - q^{bn}}{1 - q^n} P_{n-1}(a, b; q) \end{aligned}$$

with initial conditions $P_0(a, b; q) = 1$ and $P_1(a, b; q) = \frac{1-q^a}{1-q}$. Furthermore, letting $P_n^{(m)}(a, b; q)$ denote the generating function for m–shifted Pell tilings of an n–board, we also have

$$P_{n+1}^{(m)}(a, b; q) = \frac{1 - q^{a(m+n+1)}}{1 - q^{n+1}} P_n^{(m)}(a, b; q) + q^{m+n} \frac{1 - q^{bn}}{1 - q^n} P_{n-1}^{(m)}(a, b; q)$$

with initial conditions $P_0^{(m)}(a, b; q) = 1$ and $P_1^{(m)}(a, b; q) = \frac{1-q^{a(m+1)}}{1-q^{m+1}}$.

With these definitions in mind, we close with the following theorems, which can be easily proven using the same techniques described in Sections 3.1 and 3.2.

Theorem 13 *Generalization of Theorem 6: For all $n \geq 0$,*

$$P_{2n+1}(a, b; q) = \sum_{i=0}^{n} q^{n(n+1)-i(i+1)} \frac{1 - q^{a(2i+1)}}{1 - q^{2i+1}} \prod_{j=i+1}^{n} \frac{1 - q^{2bj}}{1 - q^{2j}} P_{2i}(q).$$

Theorem 14 *Generalization of Theorem 7: For all $n \geq 2$ and $1 \leq i \leq n-1$,*

$$P_n(a, b; q) = P_i(a, b; q) P_{n-i}^{(i)}(a, b, q) + q^i \frac{1 - q^{bi}}{1 - q^i} P_{i-1}(a, b; q) P_{n-i-1}^{(i+1)}(a, b; q).$$

Theorem 15 *Generalization of Theorem 8: For all $n \geq 0$,*

$$\begin{aligned} P_{2n+1}(a, b; q) &= \frac{1 - q^{a(n+1)}}{1 - q^{n+1}} P_n(a, b; q) P_n^{(n+1)}(a, b, q) \\ &\quad + \sum_{i=1}^{n} A_{n,i}(q) P_{n-i}(a, b; q) P_{n-i}^{(n+i+1)}(a, b; q) \end{aligned}$$

where

$$A_{n,i}(q) = q^{in+i}\frac{1-q^{a(n+1-i)}}{1-q^{n+1-i}}\prod_{j=1}^{i}\frac{1-q^{b(n-i+2j)}}{1-q^{n-i+2j}} + q^{in}\frac{1-q^{a(n+1+i)}}{1-q^{n+1+i}}\prod_{j=1}^{i}\frac{1-q^{b(n-i+2j-1)}}{1-q^{n-i+2j-1}}.$$

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