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Combinatorial *d*-Tori with a Large Symmetry Group

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Abstract. We construct a sequence of combinatorial triangulations of the d-dimensional torus with $2^{d+1}-1$ vertices and with a vertex transitive group action. This generalizes well-known constructions in the cases $d=2$ (7-vertex torus) and $d=3$.

1. Introduction and Result

Polyhedral tilings of euclidean d-space have been studied from several points of view. In particular, the class of *parallelohedra* provides tilings which are invariant under the action of a crystallographic group containing d linearly independent translations. It is clear that in this case the tiling may be regarded to live on a *d-torus* T^d which is defined to be the quotient of d-space by a pure translation group, and some of the euclidean symmetries of the tiling may carry over to the torus. Strictly speaking, euclidean symmetries of E^d will induce combinatorial automorphisms of the abstract complex T^d. However, we also call these *symmetries* of T^d because they are actually metric symmetries of the flat (= locally euclidean) geometry living on T^d. Of course, there might be more automorphisms than symmetries.

For $d=2$ it is easy to get triangulated tori with a flag transitive symmetry group (see [4]). However, the minimal 7-vertex triangulation has a symmetry group (and automorphism group) which is only half-flag transitive. Things are more complicated if we go to higher dimensions and look for simplicial decompositions with a small number of vertices. First, there are no combinatorial d-tori ($d \geq 3$) with a flag-transitive automorphism group because in the universal covering this would induce a regular tiling of E^d by simplices which do not exist. This indicates that the best one can hope for is a partial transitivity of the symmetry group. Secondly, the minimal number of vertices of a combinatorial d-torus ($d \geq 3$) is not known. For the cases $d = 1$ and $d = 2$ the minimal number of vertices of a d-torus are $n = 3$ and $n = 7$, respectively. For $d = 3$ there is a 15-vertex triangulation (see [9]) and nothing smaller has been observed. In this paper we continue this sequence and describe explicitly a quite symmetric simplicial decomposition of the d-dimensional torus by using $n = 2^{d+1} - 1$ vertices. This simplicial complex is a *combinatorial manifold* in the usual sense.

Definition. A simplicial complex is called a combinatorial d-manifold if its underlying set is homeomorphic to a topological d-manifold and if each vertex link is a combinatorial $(d-1)$-sphere.

This definition may be understood as recursive because it refers to itself for the case of a smaller dimension. For the dimension $d \leq 3$, it is known that any simplicial decomposition of a d-manifold is combinatorial (cf. [14]). This is not true in higher dimensions. A "simple" counterexample is the famous Edwards double suspension of a certain homology sphere (see [5]). It is also known that a combinatorial manifold admits a PL structure (cf. [15]). It is quite clear that the number of vertices must increase if the topology of the manifold becomes more complicated. However, the only precise statement of that kind which has been made has been for $d = 2$. In this case the inequality $n \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)})$, where n is the number of vertices, is easy to prove. Almost nothing is known for $d = 3$. The minimal number n of vertices has been shown to be $n = 9$ for the "3-dimensional Klein bottle," $n = 10$ for the sphere product $S^1 \times S^2$, and $n = 11$ for the real projective 3-space (see [1] and [16]).

The possible automorphism groups in these three cases are quite small. The 15-vertex 3-torus described in [9] has more symmetry but it is not known if this number, $n = 15$, is minimal.

Definition. A combinatorial manifold with n vertices is called k-neighborly ($1 \leq k \leq n-1$) if any k vertices determine a $(k-1)$-dimensional simplex which actually belongs to the triangulation.

This definition is motivated by the cyclic polytopes and questions about the upper bound conjecture (cf. [6] and [16]). It is quite clear that a k-neighborly manifold must be $(k-2)$-connected in the sense of homotopy theory. This means in particular that we cannot expect to have k-neighborly combinatorial tori for $k \geq 3$.

Theorem. *For any natural number d there exists a 2-neighborly combinatorial d-torus with $n = 2^{d+1} - 1$ vertices and $n \cdot d!$ d-dimensional simplices. It is a quotient of a simplicial tiling of euclidean d-space where each vertex star is a subdivided dual of the expanded simplex. Its symmetry group of order $2(d+1) \cdot n$ acts transitively on the set of vertices.*

Remark. Compare [9] for the particular case $d=3$.

2. The Geometry of the Expanded Simplex

We regard the euclidean n-space as the hyperplane $x_0+\cdots+x_d=0$ of $(d+1)$-space with coordinates $x_0,\ldots,x_d$. The projection of the $d+1$ vectors

$$E_i := (\underbrace{0,\ldots,0}_{i}, 1, \underbrace{0,\ldots,0}_{d-i})$$

onto this n-dimensional hyperplane is the key to the following observations.

Following Coxeter [3] the *expanded simplex* $e\alpha_d$ is defined to be the convex hull of the $d(d+1)$ points whose coordinates are permutations of

$$(1,-1,\underbrace{0,\ldots,0}_{d-1}).$$

Its dual is known to be a *parallelohedron*, meaning that there is a tiling of d-space by translated copies of it. It is also a *zonotope* with $d+1$ components or families of parallel edges (cf. [13]). These edges are parallel to the direction of the $d+1$ projected standard basis vectors of $(d+1)$-space. A description in terms of the configuration of $d+1$ hyperplanes in general position in real projective $(d-1)$-space can be found in [2].

For our purpose we have to analyze the structure of this dual of the expanded simplex in more detail. Up to a homothety the projections of $\pm E_k$ $(k=0,\ldots,d)$ are just

$$\pm A_k = \pm(\underbrace{-1,\ldots,-1}_{k}, d, \underbrace{-1,\ldots,-1}_{d-k}).$$

Lemma 1.

(i) *The vertices of $e\alpha_d$ (and consequently the facets of its dual) are in (1-1)-correspondence with the unordered pairs $(+A_k,-A_l)$ where $k\neq l$, and $\pm A_0,\ldots,\pm A_d$ are vertices of the dual of $e\alpha_d$ (up to homothety).*

(ii) *The facet of the dual of $e\alpha_d$ corresponding to the vertex $(-1,1,0,\ldots,0)$ is a parallelepiped spanned by the 2^{d-1} vertices $-A_0, +A_1$ and any $A_1+\sum A_{k_i}$ where the sum ranges over any subset $\{k_1,\ldots,k_i\}$ of $\{2,3,\ldots,d\}$. (Note that $\sum_{k=1}^{d} A_k=-A_0$.)*

Proof. (i) The sum $A_k+(-A_l)$ is clearly a multiple of one of the vertices of $e\alpha_d$, and the pair (k,l) is uniquely determined by that property. The scalar product of A_k with the vertices of $e\alpha_d$ obtains its maximum at the $d+1$ vertices

$$(\underbrace{-1,0,\ldots,0}_{k}, 1, \underbrace{0,\ldots,0}_{d-k}),\ldots,(\underbrace{0,\ldots,0}_{k}, 1, \underbrace{0,\ldots,0,-1}_{d-k}).$$

These are exactly all the simplicial facets of $e\alpha_d$ and therefore special vertices of its dual (cf. [2]).

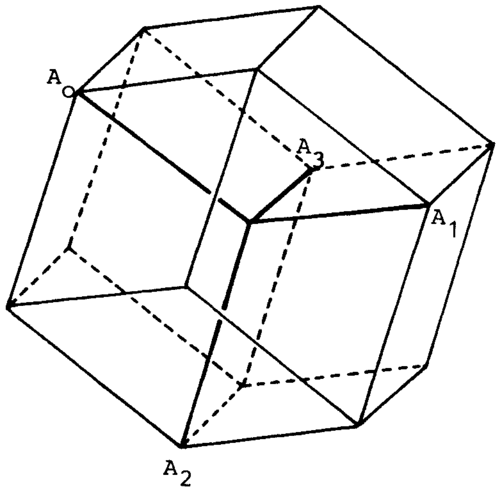

Fig. 1

(ii) Clearly, the scalar product of the vector $(-1, 1, 0, \ldots, 0)$ with $-A_0$ and $+A_1$ equals $d+1$, and its scalar product with $+A_2, \ldots, +A_d$ is zero and, therefore, all these 2^{d-1} vertices above lie in a supporting hyperplane. Note that $\sum_{k=0}^{d} A_k = 0$ and that, therefore, the $-A_k$'s occur as sums of certain $+A_k$'s. Consequently there are no more vertices in the same hypersurface. The facet spanned by these 2^{d-1} vertices is, combinatorially, a $(d-1)$-cube (see [13, 7C2]). See Fig. 1 for the case $d=3$. □

Lemma 2. *There is a simplicial decomposition of the boundary complex of the dual of $e\alpha_d$ into $(d+1)!$ simplices which is invariant under the simplex-transitive action of the full symmetric group S_{d+1}.*

Proof. Clearly S_{d+1} acts on $e\alpha_d$ (and its dual) by permuting the $d+1$ coordinates, i.e., permuting $+A_0, +A_1, \ldots, +A_d$. In addition it is centrally symmetric ($+A_i \leftrightarrow -A_i$). Each facet of the dual of $e\alpha_d$ is, combinatorially, a cube with the main diagonal $\langle +A_k, -A_l \rangle$ ($k \neq l$) by the preceding lemma. There is a standard triangulation of the $(d-1)$-dimensional cube which has $(d-1)!$ simplices and which is invariant under an S_{d-1}-action (see [11] and [12]). In the case $k=0$, $l=1$ we introduce the simplex $\langle -A_0, A_1, A_1+A_2, A_1+A_2+A_3, \ldots, A_1+\cdots+A_{d-1} \rangle$. The permutations of $\{A_2, \ldots, A_d\}$ will lead to a collection of $(d-1)!$ simplices triangulating the facet whose main diagonal is $\langle -A_0, A_1 \rangle$. Then all permutations of $\{A_0, \ldots, A_d\}$ will lead to a triangulation of the dual of $e\alpha_d$. □

3. Proof of the Theorem

Let us take the tiling of euclidean d-space by translated duals of the expanded simplex $e\alpha_d$ and assume that one of them is centered at the origin. Now introduce the origin as an additional vertex and triangulate the central copy by using the triangulation of its boundary complex described in Lemma 2. More precisely, let us triangulate the central dual of $e\alpha_d$ by $(d+1)!$ d-dimensional simplices, all containing the origin. Then translate this triangulation to all the other copies. This will lead to a triangulation of euclidean d-space. By construction this is invariant under the reflection at the origin and under arbitrary permutations of the $d+1$ coordinates of the ambient $(d+1)$-space.

In addition it is invariant under the $d+1$ translations by the vectors $\mathbf{A}_0, \ldots, \mathbf{A}_d$. To see this apply an arbitrary $\mathbf{A}_k$ to the starting simplex $\langle 0, -A_0. A_1, A_1+A_2, \ldots, A_1+\cdots+A_{d-1}\rangle$ according to Lemma 2. Observe that the $\mathbf{A}_k$-translates of the duals of $e\alpha_d$ will overlap each other. The simplest example is the case $d=2$ where the overlapping translates of the hexagons of the tesselation $\{6, 3\}$ will lead to the tesselation $\{3, 6\}$.

Now let us introduce the group H generated by the translations by the following vectors $\mathbf{B}_0, \ldots, \mathbf{B}_d$:

$$\mathbf{B}_k := 2\mathbf{A}_k - \mathbf{A}_{k+1} \qquad \text{where} \quad \mathbf{A}_{d+1} := \mathbf{A}_0.$$

Then the triangulation of the d-torus will be defined by the tesselation above modulo H. First, this is a combinatorial manifold because each vertex star remains unchanged. (Note that the diameter of the dual of $e\alpha_d$ is smaller than the length of any $\mathbf{B}_k$.)

Its topology is obviously that of the d-dimensional torus. The set of vertices is generated by the $2^{d+1}-1$ vertices $0, A_0, 2A_0, 3A_0, \ldots, (2^{d+1}-2)A_0$ according to the relation

$$\begin{aligned}(2^{d+1}-1)\mathbf{A}_k &= \sum_{i=0}^{d-1} (2^{d-i}-1)(2\mathbf{A}_{i+k} - \mathbf{A}_{i+k+1}) \\ &= \sum_{i=0}^{d-1} (2^{d-i}-1)\mathbf{B}_{i+k} \\ &\equiv 0 \quad \text{modulo } H \qquad \text{for} \quad k = 0, \ldots, d.\end{aligned}$$

From this it follows that the triangulation is 2-neighborly: each vertex star contains all vertices.

The normalizer G of H in the large crystallographic group above is generated by the translations by $\mathbf{A}_0, \ldots, \mathbf{A}_d$, by the reflection at the origin, and by the cyclic shift of the coordinates. Therefore G/H will have order $(2^{d+1}-1)\cdot 2\cdot(d+1)$. G/H clearly acts transitively on the set of vertices. It is easy to compute the number of d-dimensional simplices: every vertex star contains $(d+1)!$ simplices, each containing $d+1$ vertices, and there are $2^{d+1}-1$ vertices altogether.

This completes the proof of the theorem. □

4. An Alternative Description in Terms of a Finite Affine Geometry

The symmetry group G/H of the triangulated d-torus above admits a natural representation in the group of affine transformations

$$x \curvearrowright ax+b$$

over the ring $\mathbb{Z}_n$, $n := 2^{d+1}-1$. (Of course this makes sense only if a is a unit of $\mathbb{Z}_n$.)

Let us first describe G/H in terms of three generators R, S, and T where R is the reflection at the origin, S is the cyclic shift of the $d+1$ coordinates, and T is the translation by $\mathbf{A}_0$. These satisfy the following relations:

$$R^2 = S^{d+1} = T^n = E,$$
$$RS = SR,$$
$$RT = T^{-1}R,$$
$$ST = T^2 S.$$

We easily observe the same relations for the following transformations over $\mathbb{Z}_n$:

$$R \triangleq (x \curvearrowright -x),$$
$$S \triangleq (x \curvearrowright 2x),$$
$$T \triangleq (x \curvearrowright x+1).$$

Consequently, there is a natural labeling of the n vertices by elements of $\mathbb{Z}_n$ where 0 is the label of the origin. Then the translations by $\mathbf{A}_k$ correspond to translations in $\mathbb{Z}_n$ and the cyclic shift corresponds to the multiplication by 2. In particular, the vertices $A_0, A_1, A_2, \dots, A_d$ correspond to the numbers $1, 2, 4, 8, \dots, 2^d$. The starting simplex is nothing but $\langle -1\ \ 0\ \ 2\ \ 6\ \ 14\ \ \cdots\ \ \sum_{i=1}^{d-1} 2^i \rangle$ or after applying T $\langle 0\ \ 1\ \ 3\ \ 7\ \ 15\ \ \cdots\ \ 2^d-1 \rangle$.

The complete list of the $n \cdot d!$ d-simplices follows by application of the two procedures:

(1) Permute the d distances in

$$\langle 0\quad 1\quad 3\quad 7\quad 15\quad \cdots \rangle.$$

(2) Take $\mathbb{Z}_n$-translations of these $d!$ simplices.

There are other combinatorial manifolds with this kind of generation (cf. [8]). However, it is easy to see that our examples above have the maximal numbers of vertices among those which are 2-neighborly and are generated by (1) and (2).

Particular Cases

$d=1$: start with $\langle 0\ \ 1 \rangle$ and take the translates $\langle 1\ \ 2 \rangle$, $\langle 2\ \ 0 \rangle$. This leads to the boundary of a triangle.

$d=2$: start with $\langle 0\ 1\ 3\rangle$, interchange the distances $\langle 0\ 2\ 3\rangle$ and take the $\mathbb{Z}_7$-translates

$$\begin{array}{ll}
\langle 0\ \ 1\ \ 3\rangle & \langle 0\ \ 2\ \ 3\rangle \\
\langle 1\ \ 2\ \ 4\rangle & \langle 1\ \ 3\ \ 4\rangle \\
\langle 2\ \ 3\ \ 5\rangle & \langle 2\ \ 4\ \ 5\rangle \\
\langle 3\ \ 4\ \ 6\rangle & \langle 3\ \ 5\ \ 6\rangle \\
\langle 4\ \ 5\ \ 0\rangle & \langle 4\ \ 6\ \ 0\rangle \\
\langle 5\ \ 6\ \ 1\rangle & \langle 5\ \ 0\ \ 1\rangle \\
\langle 6\ \ 0\ \ 2\rangle & \langle 6\ \ 1\ \ 2\rangle .
\end{array}$$

This is the well-known 7-vertex torus.

$d=3$: start with $\langle 0\ 2\ 3\ 7\rangle$ and find the $6=3!$ many interchangings:

$$\begin{array}{lll}
\langle 0\ \ 1\ \ 3\ \ 7\rangle & \langle 0\ \ 1\ \ 5\ \ 7\rangle & \langle 0\ \ 2\ \ 3\ \ 7\rangle \\
\langle 0\ \ 2\ \ 6\ \ 7\rangle & \langle 0\ \ 4\ \ 5\ \ 7\rangle & \langle 0\ \ 4\ \ 6\ \ 7\rangle .
\end{array}$$

Then apply the $\mathbb{Z}_{15}$-translation. The complete scheme of the 90 tetrahedra can be found in [9].

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