Combinatorics of the Casselman-Shalika formula in type A

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The Casselman-Shalika formula

Theorem (Casselman-Shalika formula)

If $|\mathbf{z}^\alpha| < 1$ *for* $\alpha \in \Delta^+$ *and* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1})$ *is a dominant weight for* $\mathrm{GL}_{r+1}(\mathbb{C})$*, then*

$$W(t_\lambda) := \int_{N(F)} f_{\mathbf{z}}^\circ(w_0 n t_\lambda)\psi(n)\, dn = \delta^{1/2}(t_\lambda)\chi_\lambda(\mathbf{z}) \prod_{\alpha\in\Delta^+} (1 - q^{-1}\mathbf{z}^\alpha),$$

where $t_\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_{r+1}})$*,* ϖ *is a uniformizer in* $\mathfrak{o}$*, and* Δ *is the root system of* $\mathrm{GL}_{r+1}(\mathbb{C})$*.*

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- The term $\prod_{\alpha\in\Delta^+}(1 - q^{-1}\mathbf{z}^\alpha)\chi_\lambda(\mathbf{z})$ is a q-deformation of a Weyl character for the irreducible highest weight representation $V(\lambda+\rho)$.
- Expresses the value of the spherical Whittaker function in terms of a Weyl character.

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- Expresses the value of the spherical Whittaker function in terms of a Weyl character.

Goal

Express the product as a sum over the crystal $\mathcal{B}(\lambda+\rho)$ realized as the set of semistandard Young tableaux.

Definition

For a given reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$ for the longest element w_0 of the Weyl group, define the *BZL path* of $b \in \mathcal{B}(\lambda+\rho)$ as follows. Inductively, let

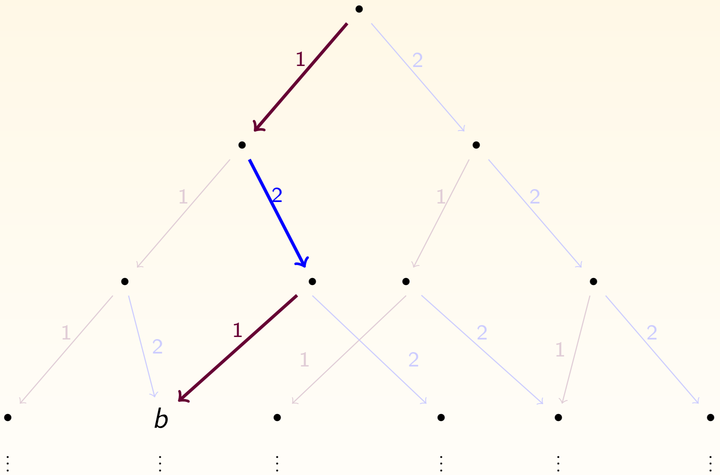
$$a_1 = \max\{k : \widetilde{e}_{i_1}^k b \neq 0\}, \qquad a_j = \max\{k : \widetilde{e}_{i_j}^k \widetilde{e}_{i_{j-1}}^{a_{j-1}} \cdots \widetilde{e}_{i_2}^{a_2} \widetilde{e}_{i_1}^{a_1} b \neq 0\}$$

for $j = 1, \dots, N$. Then we define $\psi_{\mathbf{i}}(b) = (a_1, \dots, a_N)$.

These are also known as *string parameterizations* or $\mathbf{i}$-*Kashiwara data*.

P. Littelmann proved that such a path terminates at the highest weight vector $b_{\lambda+\rho} \in \mathcal{B}(\lambda+\rho)$.

$r = 2$, $\mathbf{i} = (1, 2, 1)$, $\lambda + \rho \gg 0$



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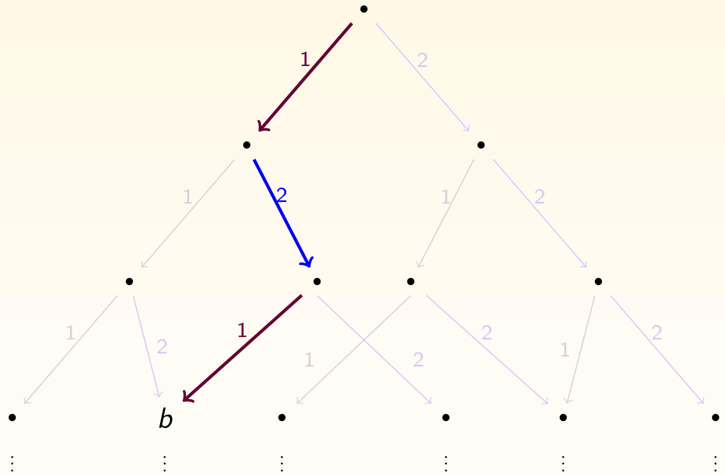
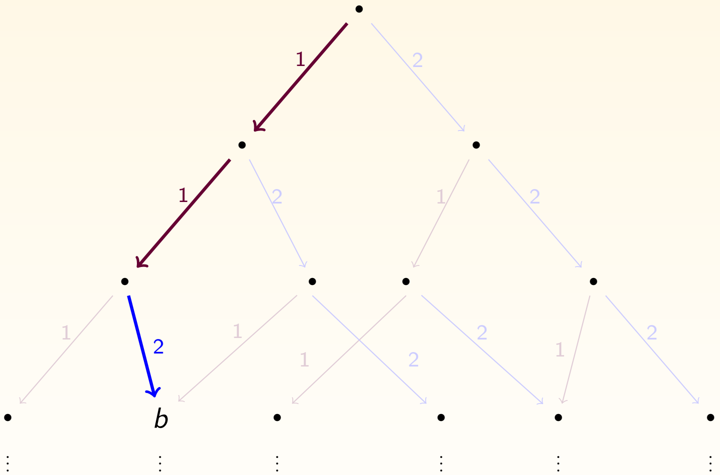


$$\psi_{\mathbf{i}}(b) = (1; 1, 1)$$

$r = 2$, $\mathbf{i} = (2,1,2)$, $\lambda + \rho \gg 0$



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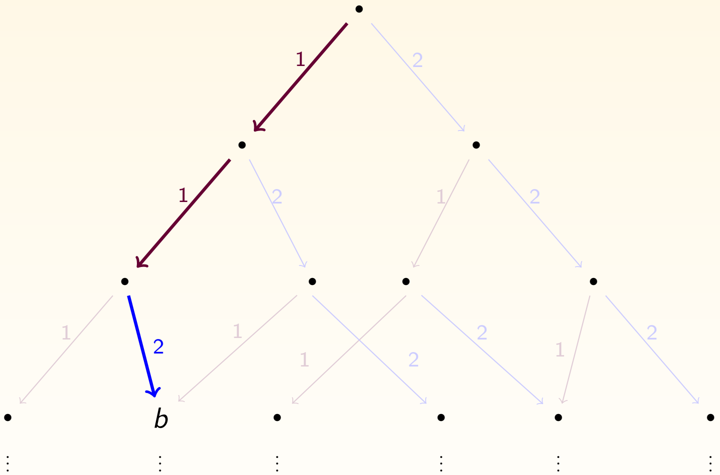

$$\psi_{\mathbf{i}}(b) = (1; 2, 0)$$

The circling and boxing rules

Write the BZL paths in triangles of the following form:

$$\psi_{\mathbf{i}}(b) = \begin{matrix} & & & a_1 & & & \\ & & a_2 & & a_3 & & \\ & a_4 & & a_5 & & a_6 & \\ \iddots & & \vdots & & \vdots & & \ddots \end{matrix} = \begin{matrix} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & a_{3,1} & & a_{3,2} & & a_{3,3} & \\ \iddots & & \vdots & & \vdots & & \ddots \end{matrix}$$

This triangular array looks more natural if we use Littelmann's result that

$$a_{1,1} \geq 0; \quad a_{2,1} \geq a_{2,2} \geq 0; \quad a_{3,1} \geq a_{3,2} \geq a_{3,3} \geq 0; \quad \ldots.$$

Entries outside the triangle are understood to be 0.

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Definition (Brubaker-Bump-Friedberg, 2011; Bump-Nakasuji, 2010)

- If the entry $a_{j,\ell-1} = a_{j,\ell}$, then we *circle* $a_{j,\ell-1}$.
- If $\widetilde{f}_{i_j}\widetilde{e}_{i_{j-1}}^{a_{j-1}} \cdots \widetilde{e}_{i_1}^{a_1} b = 0$, then *box* a_j.

BZL paths and the Casselman-Shalika formula

Theorem (Bump-Nakasuji; Brubaker-Bump-Friedberg; Tokuyama)

If $\mathbf{i} = (1, 2, 1, 3, 2, 1, \dots, r, r-1, \dots, 2, 1)$, *then*

$$\chi_\lambda(\mathbf{z}) \prod_{\alpha\in\Delta^+} (1 - q^{-1}\mathbf{z}^\alpha) = \sum_{b\in\mathcal{B}(\lambda+\rho)} G_{\mathbf{i}}(b) q^{-\langle w_0(\mathrm{wt}(b)-\lambda-\rho),\rho\rangle} \mathbf{z}^{w_0(\mathrm{wt}(b)-\rho)}$$

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Applying the longest element w_0 to both sides gives

$$\mathbf{z}^\rho \chi_\lambda(\mathbf{z}) \prod_{\alpha\in\Delta^+} (1 - q^{-1}\mathbf{z}^{-\alpha}) = \sum_{b\in\mathcal{B}(\lambda+\rho)} G_{\mathbf{i}}(b) q^{\langle \mathrm{wt}(b)-\lambda-\rho,\rho\rangle} \mathbf{z}^{\mathrm{wt}(b)}$$

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Essentially, the right-hand side has the form

$$\sum_{b\in\mathcal{B}(\lambda+\rho)} (-q^{-1})^{\#\text{boxes}} (1-q^{-1})^{\#\text{neither circled nor boxed}} \mathbf{z}^{\mathrm{wt}(b)}.$$

However, b with an entry in $\psi_{\mathbf{i}}(b)$ which is both circled and boxed yields a coefficient of 0.

Crystals of tableaux

Theorem (M. Kashiwara and T. Nakashima, 1994)

The vertices of the highest weight $\mathfrak{sl}_{r+1}$-crystal $\mathcal{B}(\lambda + \rho)$ are in bijection with the semistandard Young tableaux of shape $\lambda + \rho$ over the alphabet $\{1, \ldots, r+1\}$.

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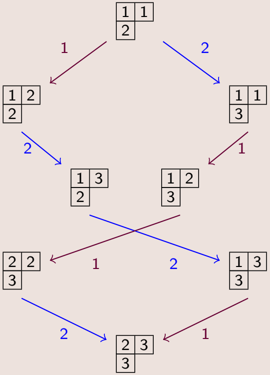
Example

$r = 2 \implies \mathcal{B}(\rho) =$



$\mathbf{a}(T)$ and $\mathbf{b}(T)$

Definition

Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableau. Define $\mathbf{a}_{i,j}$ to be the number of $(j+1)$-colored boxes in rows 1 through i for $1 \le i \le j \le r$, and define

$$\mathbf{a}(T) = \begin{matrix} \mathbf{a}_{1,1} & \mathbf{a}_{1,2} & \cdots & \mathbf{a}_{1,r} \\ & \mathbf{a}_{2,2} & \cdots & \mathbf{a}_{2,r} \\ & & \ddots & \vdots \\ & & & \mathbf{a}_{r,r} \end{matrix}$$

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Let $T \in \mathcal{B}(\lambda+\rho)$ be a tableau. The number $\mathbf{b}_{i,j}$ is defined to be the number of boxes in the ith row which have color greater or equal to $j+1$ for $1 \le i \le j \le r$. Set

$$\mathbf{b}(T) = \begin{matrix} \mathbf{b}_{1,1} & \mathbf{b}_{1,2} & \cdots & \mathbf{b}_{1,r} \\ & \mathbf{b}_{2,2} & \cdots & \mathbf{b}_{2,r} \\ & & \ddots & \vdots \\ & & & \mathbf{b}_{r,r} \end{matrix}$$

Boxing and circling from tableaux

For $\lambda \in P^+$, write $\lambda + \rho$ as

$$\lambda + \rho = (\ell_1 > \ell_2 > \cdots > \ell_r > \ell_{r+1} = 0),$$

and define $\theta_i = \ell_i - \ell_{i+1}$ for $i = 1, \ldots, r$. Let $\theta = (\theta_1, \ldots, \theta_r)$.

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Attach θ to the array $\mathbf{b}(T)$:

$$(\mathbf{b}(T), \theta) = \begin{array}{ccccc} \mathbf{b}_{1,1} & \mathbf{b}_{1,2} & \cdots & \mathbf{b}_{1,r} & (\theta_1) \\ & \mathbf{b}_{2,2} & \cdots & \mathbf{b}_{2,r} & (\theta_2) \\ & & \ddots & \vdots & \\ & & & \mathbf{b}_{r,r} & (\theta_r) \end{array}$$

New circling and boxing rules

Definition

Box $\mathbf{a}_{i,j}$ if $\mathbf{b}_{i,j} = \theta_i + \mathbf{b}_{i+1,j+1}$. Circle $\mathbf{a}_{i,j}$ if $\mathbf{a}_{i,j} = \mathbf{a}_{i-1,j}$.

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Consider the tableaux

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} \end{array} .$$

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Then

$$\mathbf{a}(T) = \begin{array}{ccc} 2 & 1 & 0 \\ & 3 & 0 \\ & & 1 \end{array} , \quad (\mathbf{b}(T), \theta) = \begin{array}{cccc} 3 & 1 & 0 & (2) \\ & 2 & 0 & (1) \\ & & 1 & (2) \end{array} .$$

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Comparison of circling and boxing rules

Lemma

Let $T \in \mathcal{B}(\lambda + \rho)$*. Then the sequences* $\psi_{\mathbf{i}}(T) = (a_{i,j})$ *and* $\mathbf{a}(T) = (\mathbf{a}_{i,j})$ *are related via the formula* $a_{i,j} = \mathbf{a}_{i-j+1,i}$*.*

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Proposition

An entry $a_{i,j}$ *in* $\psi_{\mathbf{i}}(T)$ *is circled (by the original rule) if and only if the corresponding entry in* $\mathbf{a}(T)$ *is circled (by the new rule).*

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An entry $a_{i,j}$ in $\psi_{\mathbf{i}}(T)$ is boxed (by the original rule) if and only if the corresponding entry in $\mathbf{a}(T)$ is boxed (by the new rule).

Definition

Say $T \in \mathcal{B}(\lambda+\rho)$ is *strict* if no entry of $\mathbf{a}(T)$ is both circled and boxed.

The CS formula using tableaux

Let $T \in \mathcal{B}(\lambda + \rho)$.

- $\mathrm{non}(T)$ = number of entries in $\mathbf{a}(T)$ which are neither circled nor boxed
- $\mathrm{box}(T)$ = number of entries in $\mathbf{a}(T)$ which are boxed

Define

$$C_\lambda(T; q^{-1}) = \begin{cases} (-q^{-1})^{\mathrm{box}(T)}(1 - q^{-1})^{\mathrm{non}(T)} & \text{if } T \text{ is strict,} \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem (K.-H. Lee, P. Lombardo, and S)

$$\mathbf{z}^\rho \chi_\lambda(\mathbf{z}) \prod_{\alpha \in \Delta^+} (1 - q^{-1}\mathbf{z}^{-\alpha}) = \sum_{T \in \mathcal{B}(\lambda+\rho)} C_\lambda(T; q^{-1}) \mathbf{z}^{\mathrm{wt}(T)}.$$

Segments and the Gindikin-Karpelevich formula

Example (J. Hong and H. Lee, 2008)

$$r = 3 \Longrightarrow \mathcal{B}(\infty) = \left\{ \text{tableaux of the form below} \right\}$$

1	1 · · · 1	1	1 · · · 1	1 · · · 1	1	2 · · · 2	3 · · · 3	4 · · · 4
2	2 · · · 2	2	3 · · · 3	4 · · · 4				
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1	$1\cdots1$	1	$1\cdots1$	$1\cdots1$	1	$2\cdots2$	$3\cdots3$	$4\cdots4$
2	$2\cdots2$	2	$3\cdots3$	$4\cdots4$				
3	$4\cdots4$							

Theorem (Gindikin-Karpelevich formula)

If $|\mathbf{z}^\alpha| < 1$ *for all* $\alpha \in \Delta^+$, *then*

$$\int_{N(F)} f_{\mathbf{z}}^{\circ}(w_0 n t_\lambda)\, dn = \left(\prod_{\alpha\in\Delta^+} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} \right) (\delta^{1/2}\tau_{w_0\mathbf{z}})(t_\lambda).$$

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Theorem (Lee-S, 2012; Kim-Lee, 2011; Bump-Nakasuji, 2010)

$$\prod_{\alpha\in\Delta^+} \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha} = \sum_{T\in\mathcal{B}(\infty)} (1-q^{-1})^{\mathrm{seg}(T)} \mathbf{z}^{-\mathrm{wt}(T)}.$$

Segments and the Gindikin-Karpelevich formula

There exists an embedding

$$\Psi_{\lambda+\rho}\colon \mathcal{B}(\lambda+\rho) \hookrightarrow \mathcal{B}(\infty) \otimes \mathcal{T}_{\lambda+\rho}$$

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$$\Psi_{\lambda+\rho}\left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1&1&2&2&2&3&3&4 \\ \hline 2&2&3&3&3&4 \\ \cline{1-6} 4&4&4 \\ \cline{1-3} \end{array} \right) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1&1&1&1&1&1&1&1&1&1&2&2&2&3&3&4 \\ \hline 2&2&2&2&2&3&3&3&4 \\ \cline{1-9} 3&4&4&4 \\ \cline{1-4} \end{array}$$

Definition

Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableau.

1. Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableaux. We define a *k-segment* of T (in the ith row) to be a maximal consecutive sequence of k-boxes in the ith row for any $i + 1 \leq k \leq r + 1$. Denote the total number of k-segments in T by seg(T).

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2. Let $1 \leq i < k \leq r + 1$ and suppose ℓ is the smallest integer greater than k such that there exists an ℓ-segment in the $(i + 1)$st row of T. A k-segment in the ith row of T is called *flush* if the leftmost box in the k-segment and the leftmost box of the ℓ-segment are in the same column of T. If, however, no such ℓ exists, then this k-segment is said to be *flush* if the number of boxes in the k-segment is equal to θ_i. Denote the number of flush k-segments in T by flush(T).

Calculation of $C_\lambda(T, q^{-1})$

Corollary

Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableau.

1. *Let $1 \le i < k \le r$. Suppose the following two conditions hold.*
   - **(a)** *There is no k-segment in the ith row of T.*
   - **(b)** *Let ℓ be the smallest integer greater than k such that there exist an ℓ-segment in the ith row. There is no p-segment in the $(i+1)$st row, for $k+1 \le p \le \ell$, and the ℓ-segment is flush.*[a]

   Then $C_\lambda(T; q^{-1}) = 0$.

Calculation of $C_\lambda(T, q^{-1})$

Corollary

Let $T \in \mathcal{B}(\lambda + \rho)$ be a tableau.

1. *Let $1 \le i < k \le r$. Suppose the following two conditions hold.*
   - **(a)** *There is no k-segment in the ith row of T.*
   - **(b)** *Let ℓ be the smallest integer greater than k such that there exist an ℓ-segment in the ith row. There is no p-segment in the $(i+1)$st row, for $k+1 \le p \le \ell$, and the ℓ-segment is flush.*[a]

   Then $C_\lambda(T; q^{-1}) = 0$.

2. *If condition (1) is not satisfied, then*

$$C_\lambda(T; q^{-1}) = (-q^{-1})^{\mathrm{flush}(T)}(1 - q^{-1})^{\mathrm{seg}(T) - \mathrm{flush}(T)}.$$

[a] By convention, if no such ℓ exists, then condition (b) is not satisfied.

Example

Let $\lambda = \omega_2 + \omega_3$, $r = 3$, and

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 2 & 4 \\ \cline{1-4} 3 & 4 \\ \cline{1-2} \end{array} .$$

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As a check, we have

$$\mathbf{a}(T) = \begin{array}{ccc} \bigcirc\!\!\!\!\!\boxed{0} & \boxed{1} & \boxed{1} \\ & \textcircled{1} & 2 \\ & & \boxed{3} \end{array}, \quad (\mathbf{b}(T), \theta) = \begin{array}{cccc} 2 & 2 & 1 & (1) \\ & 1 & 1 & (2) \\ & & 1 & (1) \end{array} .$$

Example

Let $\lambda = \omega_1 + 2\omega_2 + 2\omega_3$, $r = 3$, and

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ \hline 2 & 2 & 3 & 3 & 3 & 4 \\ \cline{1-6} 4 & 4 & 4 \\ \cline{1-3} \end{array}\,.$$

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$$\mathbf{a}(T) = \begin{array}{ccc} \boxed{3} & \boxed{2} & 1 \\ & 5 & 2 \\ & & \boxed{5} \end{array}, \quad (\mathbf{b}(T), \theta) = \begin{array}{cccc} 6 & 3 & 1 & (2) \\ & 4 & 1 & (3) \\ & & 3 & (3) \end{array}.$$

Application of the $C_\lambda(-;q^{-1})$

For $\beta \in Q^+$, define a polynomial $H_\lambda(\beta;q^{-1}) \in \mathbb{Z}[q^{-1}]$ by

$$H_\lambda(\beta;q^{-1}) = \sum_{\substack{T\in\mathcal{B}(\lambda+\rho)\\ \mathrm{wt}(T)=\lambda+\rho-\beta}} C_\lambda(T,q^{-1}).$$

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Theorem (Brubaker, Bump, and Friedberg, 2011)

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Proposition (H. Kim and K.-H. Lee, 2012)

- $H_\lambda(\beta;0)$ *is the multiplicity of* $\lambda-\beta$ *in* $V(\lambda)$;
- $H_\lambda(\beta;-1)$ *is the multiplicity of* $\lambda+\rho-\beta$ *in* $V(\lambda)\otimes V(\mu)$;
- $H_\lambda(\beta;1) = \begin{cases} (-1)^{\ell(w)} & w(\lambda+\rho)-\rho = \lambda-\beta \text{ for some } w\in W,\\ 0 & \text{otherwise.}\end{cases}$

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- Are seg and flush useful elsewhere in combinatorics?

T	*H*	*A*	*N*	*K*
Y	*O*	*U*	!	