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COMBINATORICS OF CURVATURE, AND THE BIANCHI IDENTITY

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ABSTRACT. We analyze the Bianchi Identity as an instance of a basic fact of combinatorial groupoid theory, related to the Homotopy Addition Lemma. Here it becomes formulated in terms of 2-forms with values in the gauge group bundle of a groupoid, and leads in particular to the (Chern-Weil) construction of characteristic classes. The method is that of synthetic differential geometry, using "the first neighbourhood of the diagonal" of a manifold as its basic combinatorial structure. We introduce as a tool a new and simple description of wedge (= exterior) products of differential forms in this context.

Introduction

We shall give a proof of the Bianchi Identity in differential geometry. This is an old identity; the novelty of the proof we present is that it derives the identity from a basic identity in combinatorial groupoid theory, and permits rigorous pictures to be drawn of the mathematical objects (connections, curvature,...) that enter.

The method making this possible here, is that of synthetic differential geometry ("S-DG"; see e.g. [Koc81]), which has been around for more than 20 years; but in contrast to most of the published work which utilizes this method, the basic notion in the present paper is the first order notion of neighbour elements $x \sim y$ in a manifold M, rather than the notion of tangent vector $D \to M$ (which from the viewpoint of logic is a second order notion). This latter notion really belongs to the richer world of kinematics, rather than to geometry: tangent vectors are really *motions*.

In so far as the the specific theory of connections is concerned, it has been dealt with synthetically both from the viewpoint of neighbours, and from the viewpoint of tangents; for the latter, cf. [KR79], [MR91], and more recently and completely, [Lav96], (which also contains a proof of the Bianchi identity, but in a completely different spirit).

On the other hand, the theory of connections, and the closely related theory of differential forms, has only been expounded from the combinatorial/geometric (neighbour-) viewpoint in early unpublished work by Joyal, and by myself, [Koc80], [Koc83], [Koc82], [Koc81] (section I.18), [Koc85]; but it has only been mentioned in passing in the monographs [MR91] (p. 384), [Lav96] (p. 139 and 180).

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My contention is that the combinatorial/geometric viewpoint comes closer to the geometry of some situations, and permits more pictures to be drawn. One may compare the theory of differential forms, as expounded here, (and in [Koc82] and [Koc81] Section I.18.)), with the theory of differential forms as seen from the viewpoint of tangents, and expounded synthetically in [KRV80], [Koc81] Section I.14, [MR91], [Min88], [Lav96].

Thus, the fact that differential k-forms ($k \geq 2$) should be *alternating* is something that is *deduced* in the combinatorial/geometric approach, but is *postulated* in the more standard synthetic approach (as in [Koc81], Def. I.14.2). Likewise, the definition of wedge product of forms as presented in the present paper from the combinatorial/geometric viewpoint (for the first time, it seems), is more self-explanatory than the theory of [Min88] and [Lav96] p. 123 (their theory is here anyway quite close to the "classical" formalism).

The piece of drawable geometry, which, in the approach here is the main fact behind the Bianchi identity, is the following: consider a tetrahedron. *Then the four triangular boundary loops compose (taken in a suitable order) to a loop which is null-homotopic inside the 1-skeleton of the tetrahedron* (the loop opposite the first vertex should be "conjugated" back to the first vertex by an edge, in order to be composable with the other three loops). - This fact is stated and proved in Theorem 9.1 (in the context of combinatorial groupoid theory). Ronald Brown has pointed out to me that this is the Homotopy Addition Lemma, in one of its forms, cf. e.g. [BH82] (notably their Proposition 2).

In so far as the differential-geometric substance of the paper is is concerned (i.e. leaving the question of method and pictures aside), the content of the paper is part of the classical theory of connections in vector bundles, leading to the Chern-Weil construction of characteristic classes for vector bundles. I learned this material from [Mad88] Section 11, (except for the groupoids, which in this context are from Ehresmann); cf. also [MS74]. One cannot say that I *chose* to couch this theory into synthetic terms; this is not a matter of choice, but rather a *necessity* to see what connection theory means in terms of the neighbour relation, now that this relation has forced itself onto our minds.

I would like to acknowledge some correspondence with Professor van Est in 1991, where he suggested a relationship between combinatorial groupoid theory and the neighbourhood combinatorics of synthetic differential geometry. He also sent me his unpublished [vE76], where in particular some differential-geometric terminology (curvature, Bianchi identity,..) is used for notions in combinatorial groupoid theory.

1. Preliminaries

We shall need to recall a few notions from synthetic differential geometry. The main thing is that for any manifold M, there is, for each integer k, a notion of when two elements $x, y \in M$ are k-neighbours, denoted $x \sim_k y$. In particular, $x \sim_1 y$ is denoted $x \sim y$. (The "set" of pairs $(x, y) \in M \times M$ with $x \sim y$ is what in algebraic geometry is called the *first neighbourhood of the diagonal*, or $M_{(1)}$; if M is an affine scheme $Spec(A)$, then $M \times M$ is $Spec(A \otimes A)$ and $M_{(1)} = Spec((A \otimes A)/I^2)$ where I is the kernel of the multiplication map $A \otimes A \to A$.) There exist categories containing the category of smooth manifolds as a full subcategory, but in which any manifold M acquires such relations like $\sim$ and $\sim_k$ as new subobjects of $M \times M$; one talks about these new (sub-)objects in terms of their "elements", although $M_{(1)}$, say, does not have any more *points* than M itself does.

The set of k-neighbours of an element $x \in M$ is denoted $\mathcal{M}_k(x)$ and called the *k-monad* around x. The union of all the k-monads around x is denoted $\mathcal{M}_\infty(x)$.

In particular, the k-monad around $0 \in \mathbf{R}^n$, i.e. the "set" (object) of k-neighbours of $0 \in \mathbf{R}^n$ is denoted $D_k(n)$; if $k = 1$, we just write $D(n)$. So

$$\mathcal{M}_1(0) = D(n).$$

One may also describe $D_k(n) \subseteq \mathbf{R}^n$ as the set

$$\{(d_1, \ldots, d_n) \mid \text{ the product of any } k+1 \text{ of the } d_i\text{'s is } 0\}.$$

Let us explicitly note the following property of $D(n)$: If any multilinear function has an element from $D(n)$ as argument in two different places, then the value is zero.

The sub"set" $D_k(n) \subseteq \mathbf{R}^n$ is one of the important kinds of "infinitesimal" objects in SDG. Another one is $\Lambda^k D(n) \subseteq \mathbf{R}^n \times \ldots \times \mathbf{R}^n$ (k times) which we shall describe now.

First: an *infinitesimal k-simplex* in a manifold M is a k+1-tuple of elements $(x_0, \ldots, x_k)$, such that for any i, j, $x_i \sim x_j$. Such a simplex is called *degenerate* if two of the vertices x_i and x_j are equal ($i \neq j$).

A k-tuple of elements $(d_1, \ldots, d_k)$ in $\mathbf{R}^n$ is said to belong to $\Lambda^k D(n)$ if the $k+1$-tuple $(0, d_1, \ldots, d_k)$ is an infinitesimal k-simplex; so not only are all d_i in $D(n)$, but also each $d_i - d_j$. This infinitesimal subobject of $\mathbf{R}^n \times \ldots \times \mathbf{R}^n$ (k times) was introduced in [Koc81]; there it was denoted $\tilde{D}(k, n)$.

From [Koc81] Section I.18, we quote the following result, which seems not to have been considered in any of the subsequent treatises on SDG, but which is crucial to the following.

1.1. Theorem. *Any map $\omega : \Lambda^k D(n) \to \mathbf{R}$ which for each $i = 1, \ldots, k$ has the property that $\omega(d_1, \ldots, d_k) = 0$ if d_i is 0, is the restriction of a unique k-linear alternating map $\overline{\omega} : (\mathbf{R}^n)^k \to \mathbf{R}$.*

From the Theorem, we may immediately deduce a more general one, where the codomain $\mathbf{R}$ is replaced by any other finite dimensional vector space V.

2. Logarithms on General Linear Groups

Let V be a finite dimensional vector space (of dimension q, say). Then we have the ring $End(V)$, whose additive group is a vector space of dimension q^2, and whose group of units is a Lie group, denoted $GL(V) \subseteq End(V)$. The identity element $\in GL(V)$ is denoted e.

If $a \sim_k 0$ in $End(V)$, it follows by matrix calculations (identifying V with $\mathbf{R}^q$, and $End(V)$ with the vector space of $q \times q$ matrices) that a^{k+1} is the zero endomorphism, so the exponential series $\sum_0^\infty \frac{a^p}{p!}$ has only finitely many non-zero terms, and it is actually an automorphism $V \to V$ (due to the first term which is e); it is thus an element of $GL(V)$, and it is denoted $\exp(a)$. Similarly, if $b \sim_k e$ in $GL(V)$, $\log(b) \in End(V)$ is defined by the logarithmic series $\log(b) = (b-e) - \frac{(b-e)^2}{2} + \frac{(b-e)^3}{3} - \ldots$, which likewise terminates. The fact that exp and log establish mutually inverse bijections $\mathcal{M}_k(0) \cong \mathcal{M}_k(e)$ for all k, and hence $\mathcal{M}_\infty(0) \cong \mathcal{M}_\infty(e)$, is the standard power series calculation. Likewise, if a and b are *commuting* endomorphisms in $\mathcal{M}_\infty(0) \subseteq End(V)$, the formula

$$\exp(a+b) = \exp(a) \cdot \exp(b) \tag{1}$$

comes by a standard calculation with series. Similarly for log on *commuting* elements in $End(V)$.

In particular, if V is 1-dimensional, we get the standard homomorphisms $exp : (\mathbf{R}, +) \to (\mathbf{R}^*, \cdot)$ and log (natural logarithm), which here happen to be extendible beyond the respective ∞-monads, and also are group homomorphisms.

The following is a basic fact in Lie theory:

2.1. Proposition. *Let α and β be $\sim e \in GL(V)$. Then the logarithm of the group theoretic commutator is the ring theoretic commutator of their logarithms.*

Proof. Let a and b be the logarithms of α and β, respectively, so $\alpha = e + a$. From $\alpha \sim e$ follows $a \sim 0 \in End(V)$. Similarly for b. Using coordinate calculations in the algebra of $q \times q$ matrices, it is immediate from $a \sim 0$ that $a \cdot a = 0$, and hence that the multiplicative inverse of $\alpha = e + a$ is $e - a$; similarly for β. Thus

$$\alpha\beta\alpha^{-1}\beta^{-1} = (e+a)(e+b)(e-a)(e-b)$$

which multiplies out in $End(V)$ by distributivity to give 16 terms. Some of these, like aba contain a repeated a or b factor, and are therefore 0, again by matrix calculations, using $a \sim 0$, resp. $b \sim 0$. The "first order" tems are $a, b, -a$, and $-b$, so they cancel. Also aa and bb vanish, and we are left with $e + ab - ba$; since $ab - ba \sim 0$ (being 0 if $a = 0$), we get that $\log(\alpha\beta\alpha^{-1}\beta^{-1}) = ab - ba = [a, b]$, proving the Proposition. ∎

In particular, 1-neighbours of e in $GL(V)$ commute if and only if their logarithms do in $End(V)$ (this actually also holds for k-neighbours of e). Among pairs which always commute in $End(V)$ are pairs of form $\phi(d), \psi(d)$, where ϕ and ψ are linear maps $\mathbf{R}^m \to End(V)$ and $d \in D(m)$. This again follows from matrix calculations.

We shall give a sample of other commutation laws that can be derived from this principle, and which are to be used later. Assume f and g are bilinear maps $\mathbf{R}^m \times \mathbf{R}^m \to End(V)$, and assume $0, d_1, d_2, d_3$ form an infinitesimal 3-simplex in $\mathbf{R}^m$. Then $f(d_1, d_2)$ commutes with $g(d_1, d_3)$ as well as with $g(d_2 - d_1, d_3 - d_1)$. For the former, this follows by keeping d_2 and d_3 fixed; then each of the two elements depend linearly on d_1, and we are back at the previous situation; similarly, for the second assertion, we rewrite $g(d_2 - d_1, d_3 - d_1)$ as a sum of four terms, using bilinearity of g; then each of the four terms has either a d_1 or a d_2 factor, and hence commutes with $f(d_1, d_2)$.

The results of this section generalize to "abstract" Lie groups G, without representing these as subgroups of some $GL(V)$; this depends on consideration of fragments of the

Campbell-Baker-Hausdorff series. In particular, if $\exp : D(p) \to \mathcal{M}_1(e) \subseteq G$ is any bijection taking 0 to e, additive inversion $v \mapsto -v$ in $D(p)$ corresponds to multiplicative inversion in G.

3. Differential forms with values in Lie groups

Let M be a manifold, and let G be a Lie group. For $k = 0, 1, \ldots$, we define a *differential k-form with values in G* to be a law ω which to each infinitesimal k-simplex σ in M associates an element $\omega(\sigma) \in G$; the only axiom is that if the simplex σ is degenerate, then $\omega(\sigma) = e$, the neutral element of G. (Remark: it even suffices to assume that $\omega(\sigma) = e$ for any σ which is degenerate in the special way that it contains the vertex x_0 repeated.)

3.1. Proposition. *Let ω be a differential k-form on a manifold M with values in a Lie group G. Then ω is alternating, in the sense that swapping two of the vertices in the simplex implies a "sign change":*

$$\omega(x_0, \ldots, x_i, \ldots, x_j, \ldots, x_k) = \omega(x_0, \ldots, x_j, \ldots, x_i, \ldots, x_k)^{-1}, \tag{2}$$

for $i \neq j$.

Proof. Since the question is local, we may assume that $M = \mathbf{R}^m$. Also, since values of differential forms are always 1-neighbours of the neutral element in the value group G, only $\mathcal{M}_1(e) \subseteq G$ is involved; and $\mathcal{M}_1(G) \cong \mathcal{M}_1(0) \subseteq \mathbf{R}^n$, i.e. $\mathcal{M}_1(e) \cong D(n)$, where n is the dimension of G. Now, (as mentioned above), under this isomorphism, the inversion $g \mapsto g^{-1}$ in $\mathcal{M}_1(e)$ corresponds to the inversion $v \mapsto -v$ in $D(n)$. Therefore, the problem reduces to the case where $M = \mathbf{R}^m$ and $G = \mathbf{R}^n$ (under addition). Now any infinitesimal k-simplex $(x_0, \ldots, x_k)$ in $\mathbf{R}^m$ may be written $(x_0, x_0 + d_1, \ldots, x_0 + d_k)$, where $(d_1, \ldots, d_k) \in \Lambda^k D(m)$. Write

$$\omega(x_0, x_0 + d_1, \ldots, x_0 + d_k) = f(x_0, d_1, \ldots, d_k). \tag{3}$$

Then for each x_0, $f(x_0, -, \ldots, -)$ is a function $\Lambda^k(D(m)) \to \mathbf{R}^n$, (with value zero if one of the blanks is filled with a zero), and hence, by Theorem 1.1, it extends to a k-linear alternating function $(\mathbf{R}^m)^k \to \mathbf{R}^n$, which we also denote $f(x_0, -, \ldots, -)$. From this, it is clear that we get a minus on the value of $\omega(x_0, x_1, \ldots, x_k)$ if x_i is swapped with x_j for $i, j \geq 1$, since this amounts to swapping d_i and d_j. (For the same reason, it is also clear that we get 0 if $x_i = x_j$ for some $i \neq j$; from this, the parenthetical remark prior to the statement of the Proposition follows.) The argument for the case of swapping x_0 with one of the other x_i's is a little different. Without loss of generality, we may consider swapping x_0 and $x_1 = x_0 + d_1$. This we do by a Taylor expansion in the first variable for the function $f : \mathbf{R}^m \times (\mathbf{R}^m)^k \to \mathbf{R}^n$, which is multilinear in the last k arguments. The values to be compared are $f(x_0, d_1, d_2, \ldots, d_k)$ and $f(x_0 + d_1, -d_1, d_2 - d_1, \ldots, d_k - d_1)$, and the Taylor development of this latter expression gives

$$f(x_0, -d_1, d_2 - d_1, \ldots, d_k - d_1) + D_1 f(x_0, -d_1, d_2 - d_1, \ldots, d_k - d_1)(d_1),$$

where $D_1 f$ denotes the differential of f in the first variable; it is linear in the original k variables as well as in the extra variable. Because d_1 appears in two places in the $k+1$ linear expression $D_1 f$, this term vanishes, so we are left with $f(x_0, -d_1, d_2-d_1, \ldots, d_k-d_1)$ which we expand into a sum, using its multilinearity. In this sum, all but one term contain a $-d_1$ as argument in two different places, so vanishes. The remaining term is $f(x_0, -d_1, \ldots, d_k) = -f(x_0, d_1, \ldots, d_k) = -\omega(x_0, x_1, \ldots, x_k)$, as desired. This proves the Proposition. ∎

REMARK. It is tempting, (but unjustified without reference to Theorem 1.1), to attempt to start the argumentation: "keep $x_2, \ldots, x_k$ fixed, and consider $\omega(-,-,x_2,\ldots,x_k)$ as a function of the two variables x_0 and $x_1 = x_0 + d_1$ only; as a function f of d_1, it extends by the basic axiom of SDG to a linear map on the whole vector space". This is unjustified; for, it would require that f is already defined on the whole of $D(m)$, but d_1 is not free to range over the whole of $D(m)$, since it is still tied by the condition that $x_0 + d_1$ should be neighbour to all the $x_2, \ldots$ that we are keeping fixed.

The fact that the "alternating" property comes so easily, namely just by proving that the value on degenerate simplices is trivial, leads to a very simple description of wedge product of forms; this we deal with in Section 5.

There is an evident way to multiply together two k-forms on M with values in the group G:

$$(\omega \cdot \theta)(x_0, \ldots, x_k) := \omega(x_0, \ldots, x_k) \cdot \theta(x_0, \ldots, x_k);$$

The set of k-forms in fact becomes a group $\Omega^k(M, G)$ under this multiplication; the neutral element is the "zero form" z given by $z(x_0, \ldots, x_k) = e$.

4. Coboundary (commutative case)

For an infinitesimal $k+1$-simplex $\sigma = (x_0, \ldots, x_{k+1})$ in the manifold M, its i'th face $\partial_i(\sigma)$ ($i = 0, \ldots, k+1$) is the infinitesimal k-simplex obtained by deleting the vertex x_i. If now ω is a k-form with values in the vector space V, we define its coboundary $d\omega$ to be the $k+1$ form given by the usual simplicial formula

$$d\omega(\sigma) = \sum_{i=0}^{k+1} (-1)^i \omega(\partial_i(\sigma)). \tag{4}$$

The fact that this expression vanishes when two vertices in σ are equal is clear: all but two of the terms in the sum contain two equal vertices, so vanish since ω is a differential form, and the remaining two terms cancel out because of the sign $(-1)^i$ and because ω is alternating, by Proposition 3.1.

Clearly, d is a linear map $\Omega^k(M, V) \to \Omega^{k+1}(M, V)$.

The following fact is the standard calculation from simplicial theory, so the proof will not be given.

4.1. Proposition. *For any k-form ω, $d(d(\omega)) = 0$, the zero $k+2$ form.*

So we have the deRham complex $\Omega^*(M, V)$ for M (with coefficients in the vector space V). But formally, it is more like an Alexander-Spanier cochain complex from topology.

5. Wedge products

Let ω be a k-form on a manifold M with values in (the additive group of) a vector space U, and let θ be an l-form on M with values in a vector space V. If now $U \times V \dot{\rightarrow} W$ is a bilinear map into a third vector space W, we can manifacture a $k+l$-form $\omega \wedge \theta$ on M with values in W as follows. We put

$$(\omega \wedge \theta)(x_0, \ldots, x_{k+l}) := \omega(x_0, \ldots, x_k) \cdot \theta(x_0, x_{k+1}, \ldots, x_{k+l}). \tag{5}$$

To see that it is a form, we have to see that the value is 0 if two of the x's are equal. Since the question is local, it suffices to consider the case where $M = \mathbf{R}^m$ and $x_0 = 0$. When ω is considered as a function of the remaining $(x_1, \ldots, x_k) \in \Lambda^k D(m)$, it extends by Theorem 1.1 to a multilinear map $\overline{\omega} : (\mathbf{R}^m)^k \rightarrow V$, and similarly θ extends to a multilinear $\overline{\theta}$, and then since $\cdot$ is bilinear, the whole expression in (5) is multilinear in the the arguments $x_1, \ldots, x_{k+l}$, thus vanishes if two of them are equal (the x_i's being 1-neighbours of 0).

Remark. The reason we have chosen to let the θ-factor in (5) have an x_0 in its first position, rather than the aesthetically more pleasing x_k, is that then the formula also makes sense for forms with values in vector *bundles* U, V and W over M, with $\cdot$ now being a fibrewise bilinear map. Then both factors in (5) are in the same fibre, namely the one over x_0. It can be proved that for forms with values in constant bundles, we get the same value in either case. For ease of future references, we display also the aesthetically pleasing definition (equivalent to the above one, for constant bundles)

$$(\omega \wedge \theta)(x_0, \ldots, x_{k+l}) := \omega(x_0, \ldots, x_k) \cdot \theta(x_k, x_{k+1}, \ldots, x_{k+l}). \tag{6}$$

It should also be remarked that our wedge product agrees with the "small" classical one: there are two conventions on how many $k!$, $l!$ and $(k+l)!$ to apply. The "big" one gives the determinant as $dx_1 \wedge \ldots \wedge dx_p$, the small one gives $\frac{1}{p!}$ times the determinant as $dx_1 \wedge \ldots \wedge dx_p$, in other words, the volume of a simplex, rather than the volume of the parallellepiped it spans. With the emphasis on simplices in our treatment, this is anyway quite natural.

Since our notions (form, coboundary, wedge) agree with the classical ones in contexts where the comparison can be made (models of SDG), the calculus of differential forms (d being an antiderivation w.r.to $\wedge$ etc.) holds, so there is no need to prove them in the context here, except for simplification; this task I therefore postpone - I believe it will be calculations that one could copy from those of Alexander-Spanier cohomology. (Also, our notions agree with the "linearized" ones employed in other treatises of forms in SDG, say [Koc81], [MR91], [Min88], or [Lav96]; this is essentially argued in [Koc81], Section I.18,, except for the wedge.)

6. Fibre bundles

Let Φ be a groupoid over M (i.e. with M for its space of objects). To make a space over M, $E \to M$, into a fibre bundle for Φ, means to provide it with a left action by Φ. Similarly it makes sense to talk about a map $E \to E'$ being a fibre bundle homomorphism w.r. to fibre bundle structures for Φ. If $E \to M$ is a group bundle, there is an evident notion that the Φ-action consists of group homomorphisms, so is a "fibre bundle of groups", or a "group bundle" for Φ. Similarly for bundles of vector spaces, etc., so that we have a notion of "fibre bundle of vector spaces", or just *vector bundle*, for Φ.

To the groupoid Φ is associated the group bundle consisting of its vertex groups $\Phi(x, x)$ for $x \in M$; it is actually a fibre bundle of groups for Φ, because arrows of Φ acts from the left on the endo-arrows of Φ by conjugation: if $\phi \in \Phi(x, x)$ and $f : x \to y$ is an arrow in Φ, then ${}^f\phi := f \circ \phi \circ f^{-1} \in \Phi(y, y)$. This group bundle for Φ is what in the literature [Mac89] is called the *gauge group bundle* of Φ; we denote it *gauge*(Φ).

If $\Phi \to \Psi$ is a homomorphism of groupoids over M, any fibre bundle for Ψ canonically can be viewed as a fibre bundle for Φ also (in analogy with "restriction of scalars" for modules over rings). In particular, the gauge group bundle of Ψ is canonically a group fibre bundle for Φ.

7. Differential forms with bundle values

Let M be a manifold, and $p : E \to M$ a bundle of Lie groups over M. A *differential k-form* on M with values in $p : E \to M$ is a law ω which to any infinitesimal k simplex $(x_0, \ldots, x_k)$ in M associates an element $\omega(x_0, \ldots, x_k)$ in E_{x_0} (the fibre over x_0), in such a way that if the simplex is degenerate (two vertices equal), then the value of ω is e, the neutral element in E_{x_0}.

We have that k-forms are "alternating" in the restricted sense that swapping two of the vertices x_i and x_j for $i \neq j$ and $i, j \geq 1$ in the simplex implies a "sign change":

$$\omega(x_0, \ldots, x_i, \ldots, x_j, \ldots, x_k) = \omega(x_0, \ldots, x_j, \ldots, x_i, \ldots, x_k)^{-1}. \tag{7}$$

The result of swapping x_0 with x_i for $i \geq 1$ cannot be compared to the original value, since these lie in different fibres, in general. If the group bundle is "constant", i.e. of form $G \times M \to M$ for some Lie group G, forms are also alternating in the sense of swapping x_0 with x_i implies a sign change. This case is of course the same as the case of differential forms with values in the group G. The proof of the "restricted" case is identical to the one given above for Proposition 3.1. The sense in which ω is also changes sign when swapping x_0 and x_i is dealt with in Section 14.

There is an evident way to multiply together two k-forms with values in a group bundle:

$$(\omega \cdot \theta)(x_0, \ldots, x_k) := \omega(x_0, \ldots, x_k) \cdot \theta(x_0, \ldots, x_k);$$

The set of k-forms in fact becomes a group $\Omega^k(E)$ under this multiplication; the neutral element is the "zero form" z given by $z(x_0, \ldots, x_k) = e_{x_0}$.

Let G be a Lie group. For G-valued 1-forms ω, there is an evident notion of its coboundary $d\omega$, which is a G-valued 2-form:

$$d\omega(x_0, x_1, x_2) := \omega(x_0, x_1)\omega(x_1, x_2)\omega(x_2, x_0), \tag{8}$$

which we may think of as the "curve integral of ω around the boundary of the infinitesimal simplex (x_0, x_1, x_2)". (Warning: it differs, for non-commutative G, from the coboundary of the Lie-algebra-valued 1-form to which it gives rise, by a correction term $\omega \wedge \omega$; we return to this.)

Using that $\omega(x_2, x_0) = \omega(x_0, x_2)^{-1}$, and that an even permutation on the three arguments of $d\omega$ does not change the value (since $d\omega$ is alternating), we may rewrite the definition (8) into

$$d\omega(x_0, x_1, x_2) = \omega(x_1, x_2)\omega(x_0, x_2)^{-1}\omega(x_0, x_1). \tag{9}$$

For G-valued k-forms ω ($k \geq 2$), it is also possible, but less evident, to define its cobundary $k+1$-form $d\omega$; it is less evident in the sense that there is no geometrically compelling way for the order in which to multiply together those factors $\omega(x_0, \ldots, \hat{x_i}, \ldots, x_{k+1})$, whose product should constitute $d\omega(x_o, \ldots, x_{k+1})$, but it turns out (with proof in the spirit of Section 2) that these factors actually commute (for $k \geq 2$!), so that the question of order is irrelevant. We shall consider such "non-commutative" coboundary in Section 11.

For forms ω with values in a general group bundle $E \to M$, one needs one further piece of structure in order to define $d\omega$, namely a *connection* in $E \to M$. This is the topic of the next section.

8. Connections

There are two ways to define connections, both of which are in terms of the reflexive graph $M_{(1)} \rightrightarrows M$, the first neighbourhood of the diagonal; one is as an *action* of this graph on a bundle $E \to M$, the other is as a morphism of (reflexive) graphs $M_{(1)} \to \Phi$, where Φ is a groupoid with M for its object manifold. The former can be reduced to the latter, by taking Φ to be the groupoid of isomorphism from one fibre of $E \to M$ to another one. Here, we temporarily put the emphasis on the latter, groupoid theoretic, viewpoint (due to Ehresmann).

So, a *connection* in a groupoid Φ over M is a homorphism of reflexive graphs $\nabla : M_{(1)} \to \Phi$, where $M_{(1)}$ denotes the first neighbourhood of the diagonal. Equivalently, ∇ is a law which to any infinitesimal 1-simplex (x, y) associates an arrow $\nabla_{yx} : x \to y$ in Φ, in such a way that ∇_{xx} is the identity arrow at x. We shall assume throughout that Φ is a *Lie* groupoid, in the well known sense, cf [Mac87]. The additive group bundle of a vector bundle $E \to M$, as well as the groupoid $\mathbf{GL}(E)$ of a vector bundle considered in Section 10, are Lie groupoids. The gauge group bundle of a Lie groupoid is a bundle of Lie groups. Just as for forms, we may then prove that

$$\nabla_{xy} = \nabla_{yx}^{-1}. \tag{10}$$

One may also write $\nabla(y, x)$ instead of ∇_{yx}.

8.1. Proposition. *The set C of connections in Φ carries a canonical left action by the group $G = \Omega^1(gauge\Phi)$ of 1-forms with values in the gauge group bundle of Φ, and with this action, it is a translation space over G.*

(The sense of the term 'translation space' will be apparent from the proof.)

Proof. Given a connection ∇ in Φ, and a 1-form ω with values in the gauge group bundle, we get the connection $\omega \cdot \nabla$ by putting

$$(\omega \cdot \nabla)(x, y) := \omega(x, y) \circ \nabla(x, y). \tag{11}$$

(Note that $\omega(x, y)$ is an endo-arrow at x, so that the composition here does make sense.) Given two connections ∇ and Γ in Φ, their "difference" $\Gamma \circ \nabla^{-1}$ is the gauge-group-bundle valued 1-form given by

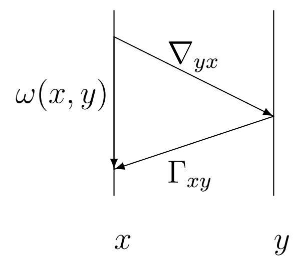
$$(\Gamma \circ \nabla^{-1})(x, y) := \Gamma(x, y) \circ \nabla(y, x) = \Gamma(x, y) \circ (\nabla(x, y))^{-1}. \tag{12}$$

Then clearly

$$(\Gamma \circ \nabla^{-1}) \cdot \nabla = \Gamma,$$

and it is unique with this property. The verifications are trivial. ■

Remark. For drawings, the viewpoint of a connection ∇ as an action by the graph $M_{(1)}$ on a bundle $E \to M$ is usually more appropriate. If ∇ and Γ are two such, we present here a drawing, which exhibits the action (via ∇ and Γ) of $x \sim y$ on suitable points in E, as well as the action of the "difference" 1-form ω, (which now takes values in the group bundle of automorphism groups of the fibres):



9. Combinatorial Bianchi identity

Given a connection ∇ in a Lie groupoid Φ, we define its *curvature* as a 2-form R with values in the gauge group bundle of Φ:

$$R(x, y, z) = \nabla(x, y) \circ \nabla(y, z) \circ \nabla(z, x). \tag{13}$$

for (x, y, z) an infinitesimal 2-simplex. Note that this is going around the boundary of the infinitesimal simplex (x, y, z) (in reverse order). We shall adopt a notational shortcut in the proof of the following formula, by writing yx for $\nabla(y, x) : x \to y$. Also, we omit commas; upper left indices denote conjugation. Then we have the combinatorial Bianchi identity (essentially a form of the Homotopy Addition Lemma, cf. the Introduction):

9.1. THEOREM. *Let ∇ be a connection in a groupoid Φ, and let R be its curvature. Then for any infinitesimal 3-simplex (x, y, z, u),*

$$id_x = {}^{xy}R(yzu) \circ R(xyu) \circ R(xuz) \circ R(xzy).$$

We shall below interpret the expression here as the *coboundary* or *covariant derivative* of R in a "complex" of group-bundle valued forms, with respect to ∇. This is still a purely "combinatorial" gadget, but we shall later specialize to the case of the general linear groupoid of a vector bundle with connection, and see that our formulation contains the classical one.

PROOF. With the streamlined notation mentioned, the identity to be proved is

$$xy \circ (yz \circ zu \circ uy) \circ yx \circ (xy \circ yu \circ ux) \circ (xu \circ uz \circ zx) \circ (xz \circ zy \circ yx) = id_x;$$

the proof is now simply repeated cancellation: first remove all parentheses, then keep cancelling anything that occurs, or is created, which is of the form $yx \circ xy$ etc., using (10); one ends up with nothing, i.e. the identity arrow at x. ∎

The reader may like to see the geometry of this proof by drawing a tetrahedron with vertices named x, y, z, u, and then trace that path (of length 14 units) which the left hand side of the above identity denotes; cf. the description in the Introduction.

10. Linear connections

We shall consider in particular linear connections in vector bundles $E \to M$; they are connections in the above groupoid theoretic sense, if we take the groupoid Φ to be the groupoid $\mathbf{GL}(E)$ of linear isomorphisms between the fibres of $E \to M$. So for $x \sim y$ in M, $\nabla_{xy} : E_y \to E_x$ is a linear isomorphism, and ∇_{xx} is the identity map of E_x. We then have the following comparison, which we shall use later. Note that the first term of the left hand side is just the curvature $R(x, y, z) \in GL(E_x)$, applied to $f(x)$. Since this is $\sim e \in GL(E_x)$, the left hand side itself is $(\log R)(x, y, z)$, applied to $f(x)$.

10.1. PROPOSITION. *Let f be a section of the vector bundle $E \to M$, and let ∇ be a linear connection on it. Then for any infinitesimal 2-simplex x, y, z*

$$\nabla_{xy}\nabla_{yz}\nabla_{zx}f(x) - f(x) = \nabla_{xy}\nabla_{yz}f(z) - \nabla_{xz}f(z). \tag{14}$$

PROOF. Since the question is local, we may assume that the bundle is $V \times M \to M$ for a vector space V, and we may assume that $M = \mathbf{R}^m$. Then there is a 1-form θ on M with values in $GL(V)$ such that

$$\nabla_{xy}(v, y) = (\theta(x, y)(v), x)$$

for all $x \sim y \in M$ and $v \in V$. Now given a 2-simplex x, y, z and a section f, there is a unique linear map $g : M = \mathbf{R}^m \to V$ such that $f(u) = f(x) + g(u - x)$ for all $u \in \mathcal{M}_1(x)$.

Rewriting ∇ in terms of θ, the left hand side of (14) is simply $d\theta(x, y, z)(f(x)) - f(x)$. Writing $f(z) = f(x) + g(z - x)$ and using linearity of $\theta(u, v) : V \to V$ for any $u \sim v$, we calulate the right hand side to be

$$\theta(x, y)\theta(y, z)f(x) - \theta(x, z)f(x) + \theta(x, y)\theta(y, z)g(z - x) - \theta(x, z)g(z - x). \tag{15}$$

Now since the values of θ are in $GL(V) \subseteq End(E)$ which is a vector space, there is a linear map $\overline{\theta}_x : \mathbf{R}^m \to End(V)$ such that $\theta(x, u) = id_V + \overline{\theta}_x(u - x)$. In particular, $\theta(x, z)g(z - x) = g(z - x) + \overline{\theta}_x(z - x)g(z - x)$, and the last term here vanishes, because it comes from a bilinear expression with $z - x \sim 0$ substituted in two places. So the last term in our expression (15) is just $-g(z - x)$. One may similarly see that the second but last term is $g(z-x)$ (rewrite $g(z-x)$ as $g(z-y)+g(y-x)$). So the two last terms cancel. Therefore (14) will follow if we prove in the ring $End(V)$ that

$$\theta(x, y)\theta(y, z)\theta(z, x) - id = \theta(x, y)\theta(y, z) - \theta(x, z). \tag{16}$$

Both sides clearly give 0 if the 2-simplex is degenerate, so both define 2-forms with values in (the additive group of) $End(V)$. In particular, let ρ denote the right hand side, so

$$\rho(x, y, z) = \theta(x, y)\theta(y, z) - \theta(x, z).$$

Also, there is an $End(V)$ valued 1-form $\overline{\theta}$ on M such that $\theta(u, v) = id + \overline{\theta}(u, v)$, for $u \sim v$. Then $\overline{\theta} \wedge \rho$ is an $End(V)$-valued 3-form, and therefore

$$0 = (\overline{\theta} \wedge \rho)(x, z, y, z) = \overline{\theta}(x, z)\rho(x, y, z),$$

and hence

$$\theta(z, x)\rho(x, y, z) = \rho(x, y, z).$$

So we may multiply the expression defining $\rho(x, y, z)$ on the left by $\theta(z, x)$ without changing the value. So the right hand side of (16) equals

$$\theta(z, x)\theta(x, y)\theta(y, z) - id. \tag{17}$$

But the expression $\theta(z, x)\theta(x, y)\theta(y, z)$ is a 2-form $d\theta(x, y, z)$, hence alternating, so that we may perform a cyclic permutation of the arguments x, y, z. Applying it to (17) gives the left hand side of (16), proving the Proposition. ∎

More generally, we may replace the section f in the above Proposition by an E-valued n-form, for any n:

10.2. Proposition. *Let $E \to M$ be a vector bundle equipped with a connection ∇. Let ω be an n-form on M with values in E. Then for any infinitesimal $n + 2$-simplex $x, y, z, z_1, \ldots, z_n$, we have*

$$\nabla_{xy}\nabla_{yz}\nabla_{zx}\omega(x, z_1, \ldots, z_n) - \omega(x, z_1, \ldots, z_n)$$
$$= \nabla_{xy}\nabla_{yz}\omega(z, z_1, \ldots, z_n) - \nabla_{xz}\omega(z, z_1, \ldots, z_n).$$

PROOF. The proof is the same as that of the previous Proposition, except that the linear function g appearing there, and which is the degree 1 term of the Taylor series for the map $f : \mathbf{R}^m \to V$, is replaced by a linear map $\mathbf{R}^m \to Alt_n(\mathbf{R}^m, V)$, again the degree 1 term of the Taylor series at x for ω, viewed as a map from $M = \mathbf{R}^m$ to the vector space $Alt_n(\mathbf{R}^m, V)$ of n-linear alternating maps from $\mathbf{R}^m$ to V. ■

11. Covariant derivative

We consider here a group bundle $E \to M$ (typically the additive group bundle of a vector bundle). Also, let there be given a connection ∇ on it, i.e. a connection in the groupoid $\mathbf{GL}(E)$, or equivalently, an action by the neigbourhood graph $M_{(1)} \rightrightarrows M$ on $E \to M$, acting by group homomorphisms. As in Section 7, we consider the group $\Omega^k(E)$ of k-forms with values in E. We shall define a map (not a group homomorphism, unless the group bundle is commutative)

$$d^\nabla : \Omega^k(E) \to \Omega^{k+1}(E).$$

Let $\omega \in \Omega^k(E)$, and let $(x_0, \ldots, x_{k+1})$ be an infinitesimal $k+1$-simplex in M. As usual, ∂_i of such a simplex denotes the one obtained by omitting the i'th vertex. Then we put

$$(d^\nabla \omega)(x_0, \ldots, x_{k+1}) = \nabla(x_0, x_1)\omega(x_1, \ldots, x_{k+1}) \cdot \prod_1^{k+1} \omega(\partial_i(x_0, \ldots, x_{k+1})^{\pm 1},$$

where the sign in the exponent is $-$ if i is odd, $+$ if i is even. (It turns out that for $k \geq 2$, the factors commute (essentially by the arguments at the end of Section 2), so that their order is irrelevant.) For $k = 0$, and $f \in \Omega^0(E)$ (so f is a section of E), $d^\nabla(f)$ is the *covariant derivative* of f with respect to ∇,

$$d^\nabla(f)(x, y) = (\nabla(xy)f(y)) \cdot f(x)^{-1}. \tag{18}$$

If we consider a connection ∇ in a Lie groupoid $\Phi \rightrightarrows M$, we get a connection in the group bundle $gauge(\Phi)$, since Φ acts on $gauge(\Phi)$ by conjugation. We denote this connection $ad\nabla$.

We now have the following reformulation of the combinatorial Bianchi identity (Theorem 9.1):

11.1. THEOREM. *Let ∇ be a connection in a Lie groupoid $\Phi \rightrightarrows M$, and let R be its curvature, $R \in \Omega^2(gauge(\Phi))$. Then $d^{ad\nabla}(R)$ is the "zero" 3-form, i.e. takes only the neutral group elements in the fibres as values.*

PROOF. Let x, y, z, u form an infinitesimal 3-simplex. We have by definition of $d^{ad\nabla}$ that

$$(d^{ad\nabla}(R))(xyzu) = ad\nabla_{xy}R(yzu) \circ R(xzu)^{-1} \circ R(xyu) \circ R(xyz)^{-1}$$

(omitting commas for ease of reading). Now the two middle terms may be interchanged, by arguments as those of Section 2. We then get the expression in the combinatorial Bianchi identity in Theorem 9.1, and by the Theorem, it has value id_x. ■

It is not true that $d^\nabla \circ d^\nabla$ is the "zero" map, unless the curvature of the connection vanishes; see Section 13 below. To make the comparison with classical curvature, we calculate here $d^\nabla \circ d^\nabla : \Omega^0(E) \to \Omega^2(E)$ for the case of a commutative group bundle $E \to M$ (which we write additively):

11.2. Proposition. *Let $f \in \Omega^0(E)$, so f is a section of the bundle E, and let (x, y, z) be an infinitesimal 2-simplex. Then*

$$d^\nabla(d^\nabla(f))\ (x, y, z) = \nabla_{xy}\nabla_{yz} f(z) - \nabla_{xz} f(z).$$

Proof. For ω an E-valued 1-form, $d^\nabla\omega(x, y, z) = \omega(x, y) - \omega(x, z) + \nabla_{xy}\omega(y, z)$. Now let ω be given by the expression (18), but written additively, so $\omega(x, y) = d^\nabla f\ (x, y) = \nabla_{xy} f(y) - f(x)$; then we get, using additivity of ∇_{xy}, six terms, four of which cancel, and the two remaining ones give the expression claimed. ■

We recognize the right hand side here as the one we have in Proposition 10.1. So we get by combining Propositions 10.1 and 11.2:

11.3. Proposition. *For f a section of the vector bundle E with a linear connection ∇, we have*

$$(\log R)(x, y, z)(f(x)) = (d^\nabla d^\nabla f)(x, y, z),$$

where R is the combinatorial curvature of ∇, $R \in \Omega^2(GL(E))$.

The "classical" curvature of a connection ∇ in a vector bundle is usually defined in terms of the right hand side of this equation. So the Proposition establishes the comparison that *the logarithm of the combinatorial curvature equals the classical curvature.*

12. Classical Bianchi Identity

We consider a vector bundle $E \to M$ equipped with a linear connection ∇. Then we get induced connections $ad\nabla$ in the group-, resp. ring-bundle $GL(E)$, $End(E)$. We shall consider diagrams, for $k \geq 1$,

$$\begin{array}{ccc}
\Omega^k(GL(E)) & \xrightarrow{d^{ad\nabla}} & \Omega^{k+1}(GL(E)) \\
{\scriptstyle \Omega^k(\log)}\downarrow & & \downarrow{\scriptstyle \Omega^{k+1}(\log)} \\
\Omega^k(End(E)) & \xrightarrow[d^{ad\nabla}]{} & \Omega^{k+1}(End(E)).
\end{array}$$

Even though, for $x \in M$, $\log : GL(E_x) \xrightarrow{*} End(E_x)$ is only partially defined (namely on the respective monads $\mathcal{M}_\infty(e_x)$, at least), the vertical maps are globally defined, since for any form $\omega \in \Omega^k(GL(E))$, its values are $\sim e_x$. We shall address the question of commutativity of these diagrams. The (partial) map log is equivariant with respect to the connections $ad\nabla$ in $GL(E)$ and $End(E)$, but it is not a group homomorphism $GL(E_x) \to End(E_x)$, not even on $\mathcal{M}_\infty(e_x)$; due to this, there is no "cheap" reason why these diagrams should commute. However, log has the homomorphism property with respect to pairs of *commuting* elements in $GL(E_x)$; since the factors that define $d^{ad\nabla}\omega$ for $GL(E)$ do commute for $k \geq 2$, as stated also in Section 11, we conclude that the square above actually does commute for $k \geq 2$.

Using the commutativity for $k = 2$, and the combinatorial Bianchi identity (in the form of Theorem 11.1), we shall prove

12.1. Theorem. *(Classical Bianchi Identity) Let $\overline{R}$ denote the curvature of a linear connection ∇ in a vector bundle $E \to M$. As an element $\overline{R} \in \Omega^2(End(E))$, $d^{ad\nabla}(\overline{R})$ is 0.*

Proof. Consider the combinatorial curvature $R \in \Omega^2(GL(E))$ of the connection ∇. By the combinatorial Bianchi Identity, in the guise of Theorem 11.1, it goes to the "zero" form by $d^{ad\nabla}$, hence also to the zero form by $\Omega^3(\log) \circ d^{ad\nabla}$. Chasing R the other way round in the square first gives $\log R$, which by Proposition 11.3 is the classical curvature $\overline{R}$ of ∇, so from the commutativity of the square follows $d^{ad\nabla}(\overline{R}) = 0$. ∎

For $k = 1$, the square does not commute, in general.This fact is related to the Maurer-Cartan formula. For the case of a constant Lie group bundle, this relationship was considered synthetically in [Koc82]. Here we do the more general case of a group *bundle* - but, on the other hand, it is more special in another direction, since it deals only with a group bundle of form $GL(E)$.

12.2. Proposition. *Let $\omega \in \Omega^1(GL(E))$, where $E \to M$ is a vector bundle with a connection ∇. Then*

$$\log(d^{ad\nabla}(\omega)) = d^{ad\nabla}(\log\omega) + \log\omega \wedge \log\omega.$$

(The wedge here is with respect to the multiplication $\circ$ in the ring $End(E)$, which is non-commutative; so there is no reason for the wedge of a 1-form with itself to vanish.)

Proof. Let us write $\omega(x, y) = e_x + \theta(x, y)$, so $\theta = \log\omega \in \Omega^1(End(E))$. Let us write conjugation by $\nabla(x, y)$ by an upper left index xy. Then for an infinitesimal 2-simplex x, y, z, we have

$$d^\nabla\omega(x, y, z) =^{xy} (e_y + \theta(yz)) \circ (e_x - \theta(x, z)) \circ (e_x + \theta(x, y)).$$

Multiplying out, we get

$$e_x +^{xy}\theta(yz) - \theta(x, z) + \theta(x, y) +$$

$$-^{xy}\theta(y, z) \circ \theta(x, z) +^{xy} \theta(y, z) \circ \theta(x, y) - \theta(x, z) \circ \theta(x, y)$$

plus a threefold product, which is easily seen to vanish (e.g. by coordinate calculations like the following). The first line in the formula is $(e_x + d^\nabla\theta)(x, y, z)$. We prove that the second line equals $\theta(x, y) \circ \theta(x, z) = (\theta \wedge \theta)(x, y, z)$. We claim that each of the three terms in the second line give plus or minus $(\theta \wedge \theta)(x, y, z)$ (twice plus, once minus). Let us consider the first only, the two others are similar. Since the question is local, we may assume that $E = M \times V$ for V a vector space, and that $M = \mathbf{R}^m$; and we may assume that $\theta(u, v) = f(u, v - u)$ for $u \sim v$, with $f : M \times M \to End(V)$ linear in the second variable. Let $y = x + d_1$ and $z = x + d_2$. Then

$$^{xy}\theta(y, z) \circ \theta(x, z) = {}^{xy} f(x + d_1, d_2 - d_1) \circ f(x, d_2);$$

expanding the first term out by linearity in the second variable of f, we get two terms, one of which vanishes because of "bilinear occurrence" of d_2, and we are left with $^{xy}f(x + d_1, -d_1) \circ f(x, d_2)$. Now we Taylor expand f in its first variable, and get $^{xy}f(x, -d_1) \circ f(x, d_2)$ plus a term which vanishes because of "bilinear occurrence" of d_1. Also, since we are conjugating with $\nabla(x, y) \in GL(V)$, whose logarithm depends linearly on $y - x = d_1$, $f(x, -d_1)$ is fixed under this conjugation. We are left with

$$f(x, -d_1) \circ f(x, d_2) = -f(x, d_1) \circ f(x, d_2) = -\theta(x, y) \circ \theta(x, z) = -(\theta \wedge \theta)(x, y, z).$$

All said, we conclude that

$$d^{ad\nabla}\omega(x, y, z) = e_x + d\theta(x, y, z) + (\theta \wedge \theta)(x, y, z),$$

where $\theta = \log(\omega)$. Now subtracting e_x from the right hand side of this expression gives something which is $\sim 0 \in End(E_x)$ (in coordinates, it depends linearly on $y - x$, say); and this something is therefore the logarithm. This proves the Proposition. ∎

13. Analyzing $d^\nabla \circ d^\nabla$

We sketch in this section how the curvature enters in describing how the composite $d^\nabla \circ d^\nabla$ fails to be the zero map in the complex of E-valued forms (where $E \to M$ is a vector bundle equipped with a linear connection ∇). This is classical, and of importance in constructing the characteristic classes for the bundle $E \to M$.

Since the ring bundle $End(E) \to M$ acts in a bilinear way on $E \to M$ simply by evaluation $End(E) \times_M E \stackrel{ev}{\to} E$, the wedge product $\overline{R} \wedge \omega$ is a well defined E-valued $k + 2$ form whenever ω is an E-valued k-form, and $\overline{R}$ an $End(E)$-valued 2-form, say the (classical) curvature of the connection ∇. This is what $\overline{R}$ denotes in the following Proposition.

13.1. Proposition. *Let ω be an E-valued k-form in the vector bundle E. Then*

$$d^\nabla(d^\nabla(\omega)) = \overline{R} \wedge \omega.$$

PROOF. We shall do the case $k = 1$ only. The calculation is much similar to the one in Proposition 11.2 (for any $k \geq 1$, in fact). For $k = 1$, we get 12 terms in $d^\nabla(d^\nabla(\omega))(x, y, z, u)$, where (x, y, z, u) is an infinitesimal 3-simplex. These terms are

$$\nabla_{xy}(\nabla_{yz}\omega(z,u) - \omega(y,u) + \omega(y,z))$$

$$-(\nabla_{xz}\omega(z,u) - \omega(x,u) + \omega(x,z))$$

$$+\nabla_{xy}\omega(y,u) - \omega(x,u) + \omega(x,y)$$

$$-(\nabla_{xy}\omega(y,z) - \omega(x,z) + \omega(x,y))$$

(in the first line, use linearity of ∇_{xy} to get three terms). Ten of these twelve terms cancel in pairs, and we are left with

$$\nabla_{xy}(\nabla_{yz}\omega(z,u) - \nabla_{xz}\omega(z,u).$$

This equals, by Proposition 10.2, (with $z_1 = u$)

$$\nabla_{xy}\nabla_{yz}\nabla_{zx}\omega(x,u) - \omega(x,u).$$

The first term here is the combinatorial curvature $R(x, y, z)$ of ∇, applied to $\omega(x, u)$, so when we subtract $\omega(x, u)$, we obtain its logarithm, i.e. the classical curvature $\overline{R}(x, y, z)$ (applied to $\omega(x, u)$). But by the definition (5) of wedge, (with evaluation $End(E) \times_M E \to E$ as the bilinear map),

$$\overline{R}(x,y,z)(\omega(x,u)) = (\overline{R} \wedge \omega)(x,y,z,u).$$

■

14. Curvature difference

We consider in this section a Lie groupoid $\Phi \rightrightarrows M$. When we have two connections ∇ and Γ in Φ, we may form their "difference", which is a *gauge*(Φ)-valued 1-form $\Gamma \circ \nabla^{-1}$, cf. Section 8. So $(\Gamma \circ \nabla^{-1})(x, y) = \Gamma_{xy} \circ \nabla_{yx}$. Also, we have two *gauge*(Φ)-valued 2-forms, namely their curvatures R^∇ and R^Γ.

14.1. PROPOSITION. *We have*

$$d^{ad\nabla}(\Gamma \circ \nabla^{-1}) = (R^\nabla)^{-1} \circ R^\Gamma. \tag{19}$$

PROOF. We first prove a Lemma which in some sense expresses how certain bundle valued forms also are alternating with respect to interchanging the first vertex with another one:

14.2. LEMMA. *Let θ be a 2-form with values in the gauge group bundle of a Lie groupoid Φ. For any connection ∇ in Φ, we have*

$$^{\nabla(z,x)}\theta(x,y,z) = \theta(z,x,y).$$

The similar result holds for any k-form θ with $k \geq 1$, using cyclic permutation of the vertices, and inserting the exponent $(-1)^k$.

PROOF. Since a Lie groupoid is locally trivial (cf. [Mac87]) and the question is local, we may assume that $\Phi = M \times G \times M$ for a Lie group G, and that θ is given by a G-valued 2-form ρ,

$$\theta(x,y,z) = (x, \rho(x,y,z), x) \in M \times G \times M,$$

and also that ∇ is given by a G-valued 1-form ω

$$\nabla_{xy} = (x, \omega(x,y), y).$$

Then the question reduces to whether $\rho(x,y,z)$ $(= \rho(z,x,y)$, since ρ is alternating by Proposition 3.1) is fixed by conjugation by $\omega(x,z)$, i.e. whether $\rho(x,y,z)$ and $\omega(x,z)$ commute. This is so, by arguments similar to those of Section 2 (bilinearity in $d_2 = z - x$). ∎

To prove (19), let z, x, y form an infinitesimal 2-simplex. We want to prove (omitting commas for ease of reading)

$$d^{ad\nabla}(\Gamma \circ \nabla^{-1})(zxy) = (R^{\nabla}(zxy))^{-1} \circ R^{\Gamma}(zxy).$$

We rewrite the left hand side, by using the Lemma twice (first for the connection Γ, then for the connection ∇) and get

$$^{\Gamma(zy)\nabla(yx)}d^{ad\nabla}(\Gamma \circ \nabla^{-1})(xyz),$$

where upper left index denotes "conjugation by". Now, expanding the left hand side into its constituents, using the definition of covariant derivative $d^{ad\nabla}$, we get

$$\Gamma_{zy}\nabla_{yx}((\nabla_{xy}(\Gamma_{yz}\nabla_{zy})\nabla_{yx})(\nabla_{xz}\Gamma_{zx})(\Gamma_{xy}\nabla_{yx}))\nabla_{xy}\Gamma_{yz},$$

where the parentheses are just meant as an aid for the checking of the correctness of the expansion. Removing all parentheses, and cancelling anything of the form $\nabla_{yx}\nabla_{xy}$ (and similarly for Γ) that occurs or is created, we end up with

$$\nabla_{zy}\nabla_{yx}\nabla_{xz}\Gamma_{zx}\Gamma_{xy}\Gamma_{yz} = R^{\nabla}(zyx)R^{\Gamma}(zxy) = (R^{\nabla}(zxy))^{-1}R^{\Gamma}(zxy).$$

This proves the Proposition. ∎

We shall use this Proposition in order to prove that the trace of $\overline{R}^{\nabla}$ (and its wedge powers) only depends on ∇ up to a coboundary, so its deRham cohomology class does not depend on the choice of connection.

15. Characteristic classes

We sketch how the theory of connections in vector bundles leads to characteristic classes. This is the classical Chern-Weil theory, and there is not much specifically synthetic about the way we present it here.

For a vector bundle $E \to M$, we have for each $x \in M$ the linear trace map

$$\text{Trace } : End(E_x) \to \mathbf{R},$$

and collectively, they define a linear map of vector bundles $End(E) \to M \times \mathbf{R}$ over M. Since for any linear isomorphism $\phi : E_x \to E_y$, and any $a \in End(E_x)$, Trace $(a) =$ Trace $(\phi \circ a \circ \phi^{-1})$, it follows that the vector bundle map

$$\text{Trace } : End(E) \to M \times \mathbf{R}$$

is equivariant for the groupoid $\Phi = \mathbf{GL}(E)$ (with the groupoid acting by conjugation on $End(E)$, and trivially on $M \times \mathbf{R}$). In particular, if ∇ is a linear connection in $E \to M$, we get homomorphisms

$$\text{Trace } : \Omega^k(End(E)) \to \Omega^k(M \times \mathbf{R}) = \Omega^k(M) \tag{20}$$

commuting with the d's (respectively $d^{ad\nabla}$ and the usual deRham d).

From the Bianchi identity, in the guise of Theorem 11.1, we therefore get that the trace of the (classical) curvature $\overline{R}$ of ∇ is a cocycle in $\Omega^2(M)$.

Since Trace $: End(E_x) \to \mathbf{R}$ does not preserve the multiplicative structure, the induced maps (20) will not preserve wedge products, in general. However, let $\omega \in \Omega^1(End(E))$. Then

$$\text{Trace } (\omega \wedge \omega) = 0.$$

For the left hand side, applied to an infinitesimal 2-simples x, y, z, yields

$$\text{Trace } (\omega \wedge \omega)(x, y, z) = \text{Trace } (\omega(x, y) \circ \omega(x, z)) = \text{Trace } (\omega(x, z) \circ \omega(x, y))$$

(the last equality by the fundamental property of trace that Trace $(a \circ b) =$ Trace $(b \circ a)$), and then we continue the equation

$$= \text{Trace } ((\omega \wedge \omega)(x, z, y)) = -\text{Trace } ((\omega \wedge \omega)(x, y, z)),$$

since forms are alternating. But then the total equation implies that Trace $(\omega \wedge \omega)(x, y, z) = 0$.

15.1. THEOREM. *Let ∇ and Γ be two linear connections on the vector bundle $E \to M$, with (classical) curvatures $\overline{R}^{\nabla}$ and $\overline{R}^{\Gamma}$, respectively. Then the deRham cocycles Trace $(\overline{R}^{\nabla})$ and Trace $(\overline{R}^{\Gamma}) \in \Omega^2(M)$ define the same cohomology class.*

Proof. Let $\omega = \Gamma \circ \nabla^{-1}$ be the "difference" 1-form of the two connections, viewed as connections with values in the groupoid $\mathbf{GL}(E)$. Thus, for the combinatorial curvatures R^{∇} and R^{Γ}, we have, by Proposition 14.1,

$$d^{ad\nabla}(\omega) = (R^{\nabla})^{-1} \circ R^{\Gamma};$$

now apply log; the values of R^{∇} and R^{Γ} commute, as observed in Section 12, so that log has the homomorphism property, and we get

$$\log d^{ad\nabla}\omega = \log R^{\Gamma} - \log R^{\nabla} = \overline{R}^{\Gamma} - \overline{R}^{\nabla}.$$

On the other hand, by Proposition 12.2,

$$\log d^{ad\nabla}\omega = d^{ad\nabla}\log\omega + \log\omega \wedge \log\omega.$$

Thus

$$\text{Trace } \overline{R}^{\Gamma} - \text{Trace } \overline{R}^{\nabla} = \text{Trace } (\overline{R}^{\Gamma} - \overline{R}^{\nabla}) = \text{Trace } (d^{ad\nabla}\log\omega + \log\omega \wedge \log\omega).$$

But this equals Trace $(d^{ad\nabla}\log\omega)$, since the trace of the wedge summand is 0, as we observed above. And this finally in turn equals dTrace $(\log\omega)$, since Trace commutes with differentials. This proves the Theorem. ∎

It is more generally true that the forms Trace $(R^{\nabla} \wedge \ldots \wedge R^{\nabla})$ are deRham cocycles, and that the cohomology classes they define do not depend on the choice of the connection ∇; this follows in essentially the same way, now using the result of Section 13, together with the relationship between $\wedge$ and d (not yet fully developed in the present context).

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