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Combinatorics of Open Covers VI: Selectors for Sequences of Dense Sets

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Abstract

We consider the following two selection principles for topological spaces:

Principle 1: For each sequence of dense subsets, there is a sequence of points from the space, the n-th point coming from the n-th dense set, such that this set of points is dense in the space;

Principle 2: For each sequence of dense subsets, there is a sequence of finite sets, the n-th a subset of the n-th dense set, such that the union of these finite sets is dense in the space.

We show that for separable metric space X one of these principles holds for the space $\mathsf{C}_p(X)$ of realvalued continuous functions equipped with the pointwise convergence topology if, and only if, a corresponding principle holds for a special family of open covers of X. An example is given to show that these equivalences do not hold in general for Tychonoff spaces. It is further shown that these two principles give characterizations for two popular cardinal numbers, and that these two principles are intimately related to an infinite game that was studied by Berner and Juhász.

The following two selection hypotheses occur in many contexts in mathematics, especially in diagonalization arguments:[3] Let $\mathbb{N}$ denote the set of positive integers and let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set. The hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ states that for each sequence $(O_n : n \in \mathbb{N})$ with terms in $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n $T_n \in O_n$, and $\{T_n : n \in \mathbb{N}\} \in \mathcal{B}$. The hypothesis $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ states that for every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n T_n is a finite subset of O_n, and $\cup_{n=1}^{\infty} T_n$ is an element of $\mathcal{B}$. A pair $(\mathcal{A},\mathcal{B})$ for which either of these hypotheses holds usually has a rich theory.

Consider the following game which is inspired by $\mathsf{S}_1(\mathcal{A},\mathcal{B})$: Players ONE and TWO play an inning per $n \in \mathbb{N}$. In the n-th inning ONE selects a set $O_n \in \mathcal{A}$, after which TWO selects an element $T_n \in O_n$. A play $(O_1, T_1, O_2, T_2, \ldots)$ is won by TWO if $\{T_n : n \in \mathbb{N}\}$ is in $\mathcal{B}$; otherwise ONE wins. Let $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ denote this game. The hypothesis $\mathsf{H}_1(\mathcal{A},\mathcal{B})$ states that ONE has no winning strategy in $\mathsf{G}_1(\mathcal{A},\mathcal{B})$. We have the implication

$$\mathsf{H}_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathsf{S}_1(\mathcal{A},\mathcal{B}).$$

For several important examples of $\mathcal{A}$ and $\mathcal{B}$ it happens that the converse implication is also true. When this happens the game is a powerful tool to extract mathematical information about $\mathcal{A}$ and $\mathcal{B}$.

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For $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ the corresponding game is $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ and is played as follows: ONE and TWO play an inning per $n\in\mathbb{N}$. In the n-th inning ONE selects a set $O_n\in\mathcal{A}$, after which TWO selects a finite subset T_n of O_n. A play $(O_1,T_1,O_2,T_2,\ldots)$ is won by TWO if $\cup_{n=1}^{\infty}T_n$ is in $\mathcal{B}$; otherwise ONE wins. The hypothesis $\mathsf{H}_{fin}(\mathcal{A},\mathcal{B})$ states that ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$.

These selection hypotheses and games were studied in previous papers for a variety of topologically significant families $\mathcal{A}$ and $\mathcal{B}$. We continue this investigation for the case when $\mathcal{A}$ and $\mathcal{B}$ both are $\mathfrak{D}$, the collection of dense subsets of a T_3–space. Happily there is a serious connection between this example and earlier studies. The remainder of this introduction is used to describe a part of this connection, and to give a brief overview of the paper.

For a given space X let $\mathcal{O}$ denote the collection of all its open covers and let Ω denote the collection of all its ω–covers ($\mathcal{U}$ is an *ω–cover* if it is an open cover, X is not a member of it, and every finite subset of X is contained in an element of $\mathcal{U}$). The symbol $\mathsf{C}_p(X)$ denotes the set of continuous functions from X to the real line $\mathbb{R}$, endowed with the topology of pointwise convergence. We show in Theorem 13 that if X is an infinite separable metric space, then the following are equivalent:

1. $\mathsf{S}_1(\Omega,\Omega)$ holds for X;
2. $\mathsf{H}_1(\Omega,\Omega)$ holds for X;
3. $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$ holds for $\mathsf{C}_p(X)$;
4. $\mathsf{H}_1(\mathfrak{D},\mathfrak{D})$ holds for $\mathsf{C}_p(X)$.

In [4] Berner and Juhász introduced for a space Y the point-picking game $\mathsf{G}^D_\omega(Y)$ which is played as follows: ONE and TWO play an inning per $n\in\mathbb{N}$. In the n–th inning ONE first chooses a nonempty open subset O_n of Y; TWO responds by choosing a point $T_n\in O_n$. ONE wins a play $(O_1,T_1,O_2,T_2,\ldots)$ if $\{T_n:n\in\mathbb{N}\}\subseteq Y$ is dense; otherwise, TWO wins. In Theorems 7 and 8 we show:

1. ONE has a winning strategy in $\mathsf{G}^D_\omega(Y)$ if, and only if, TWO has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Y;
2. TWO has a winning strategy in $\mathsf{G}^D_\omega(Y)$ if, amd only if, ONE has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$.

For a separable metric space X these results allow us to treat the point-picking game on $\mathsf{C}_p(X)$ as a selection hypothesis. On account of results regarding $\mathsf{S}_1(\Omega,\Omega)$ and $\mathsf{G}_1(\Omega,\Omega)$ in [13] and [20] this connection plus the postulate that the real line is not the union of fewer than $2^{\aleph_0}$ first category sets (known also as Martin's Axiom for countable partially ordered sets) leads to new examples of spaces where neither player has a winning strategy in the point-picking game. Previous examples of Berner and Juhász in [4], and of Dow and Gruenhage in [6] used much stronger postulates to give such examples.

Another spinoff of these two theorems is that we get new characterizations of the countable strong fan tightness of $\mathsf{C}_p(X)$ for X separable and metrizable. For a nonisolated point y of a space Y define $\Omega_y = \{A \subseteq Y : y \in \overline{A} \setminus A\}$. According to Sakai Y is said to have countable strong fan tightness at y if $\mathsf{S}_1(\Omega_y, \Omega_y)$ holds; Y is said to have countable strong fan tightness if it has this property at each point. Since $\mathsf{C}_p(X)$ is homogeneous, countable strong fan tightness of $\mathsf{C}_p(X)$ is equivalent to countable strong fan tightness at some $f \in \mathsf{C}_p(X)$. In [18] Sakai proved for $\mathsf{T}_{3\frac{1}{2}}$–spaces X that $\mathsf{C}_p(X)$ has countable strong fan tightness if, and only if, X has property $\mathsf{S}_1(\Omega, \Omega)$; in [20] I gave more characterizations for this, among others each of $\mathsf{H}_1(\Omega, \Omega)$ for X and $\mathsf{H}_1(\Omega_f, \Omega_f)$ for $\mathsf{C}_p(X)$ at some f is equivalent to the countable strong fan tightness of $\mathsf{C}_p(X)$. Thus, we find from the results here that for X separable and metrizable, the countable strong fan tightness of $\mathsf{C}_p(X)$ is equivalent to TWO not having a winning strategy in $\mathsf{G}_\omega^D(\mathsf{C}_p(X))$.

Our methods also give the equivalence of the following statements when X is an infinite separable metric space (Theorem 35):

1. X satisfies $\mathsf{S}_{fin}(\Omega, \Omega)$;
2. X satisfies $\mathsf{H}_{fin}(\Omega, \Omega)$;
3. $\mathsf{C}_p(X)$ satisfies $\mathsf{S}_{fin}(\mathfrak{D}, \mathfrak{D})$;
4. $\mathsf{C}_p(X)$ satisifes $\mathsf{G}_{fin}(\mathfrak{D}, \mathfrak{D})$.

According to Arkhangel'skiĭ a space Y has countable fan tightness at $y \in Y$ if $\mathsf{S}_{fin}(\Omega_y, \Omega_y)$ holds. In [1] it is shown for X a $\mathsf{T}_{3\frac{1}{2}}$–space that $\mathsf{C}_p(X)$ has countable fan tightness if, and only if, each finite power of X satisfies $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$; in [13] it was shown that this condition on the finite powers of X is equivalent to X satisfying $\mathsf{S}_{fin}(\Omega, \Omega)$. In [20] I showed that this is equivalent to $\mathsf{C}_p(X)$ satisfying $\mathsf{H}_{fin}(\Omega_f, \Omega_f)$ at some (each) $f \in \mathsf{C}_p(X)$. Theorem 35 now gives equivalent conditions for the countable fan tightness of $\mathsf{C}_p(X)$ when X is separable and metrizable in terms of $\mathsf{G}_{fin}(\mathfrak{D}, \mathfrak{D})$ and $\mathsf{S}_{fin}(\mathfrak{D}, \mathfrak{D})$.

We give an example that shows that the hypothesis in Theorems 13 and 35 that X be separable and metrizable cannot be weakened to $\mathsf{T}_{3\frac{1}{2}}$–ness. Other examples illustrate that in general spaces $\mathsf{S}_1(\mathfrak{D}, \mathfrak{D})$ and $\mathsf{H}_1(\mathfrak{D}, \mathfrak{D})$, as well as $\mathsf{S}_{fin}(\mathfrak{D}, \mathfrak{D})$ and $\mathsf{H}_{fin}(\mathfrak{D}, \mathfrak{D})$ are not equivalent, and that $\mathsf{H}_1(\mathfrak{D}, \mathfrak{D})$ is not preserved by finite powers.

Finally, our results are used to give new characterizations of two well–studied cardinal numbers, $\mathsf{cov}(\mathcal{M})$ and $\mathfrak{d}$ (both defined later), associated with structures on the real line.

1 $\mathsf{S}_1(\mathfrak{D}, \mathfrak{D})$ and $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$.

Reduction to countable spaces.

The purpose of this section is mainly to show that the selection hypothesis and associated game studied here are closely tied up with countable spaces. Besides some lemmas that will be of further use, we are not going to use this result about countability in this paper.

A subset $\mathcal{P}$ of X is a π–base if it consists of nonempty open sets such that every nonempty open subset of X contains a set from $\mathcal{P}$. The π–*weight* of X is the minimal cardinality of a π–base; $\pi(X)$ denotes this cardinal number.

If X has an uncountable dense subset no countable subset of which is dense, then ONE has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$: Confront TWO with that dense set in each inning. Thus assume that every dense subset of X has a countable dense subset. If we let $\delta(X)$ denote the least κ such that every dense subset of X has a subset of cardinality at most κ which is dense in X, this assumption can be abbreviated by:

Assumption 1 $\delta(X) = \aleph_0$.

If $\delta(X) = \aleph_0$ then for every infinite open subset U of X, $\delta(U) = \aleph_0$.

Since all isolated points of X belong to every dense set, ONE must present TWO with these each inning. Since $\delta(X) = \aleph_0$, the set I of isolated points is countable; if it is a dense subset of X, then TWO has an easy winning strategy. Thus, assume that I is not dense in X. Then the open set $X \setminus \overline{I}$ is nonempty. Using standard ideas one can prove:

Lemma 1 *A player has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X if, and only if, that player has a winning strategy in the game $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on $(X \setminus \overline{I})$.*

Thus, when studying the game $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ we may assume:

Assumption 2 *X has no isolated points.*

If X has a countable π-base, then TWO has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$: TWO enumerates such a countable π–base using the positive integers, and then in the n–th inning chooses a point from ONE's dense set O_n which is also a member of the n–th element of the π–base.

Lemma 2 *Let σ be a strategy for TWO in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X, and let $(D_1,\dots,D_n)$ be a sequence of dense subsets of X (this may be the empty sequence). Then there is a nonempty open subset U of X such that for each $x \in U$ there is a dense set D of X such that $x = \sigma(D_1,\dots,D_n,D)$.*

Proof : Let E be the set of points *not* of the form $\sigma(D_1,\dots,D_n,D)$ for dense subsets D of X. Then E itself is not a dense subset of X: Otherwise we have the contradiction that the point $\sigma(D_1,\dots,D_n,E)$ is in E by the rules of the game, and not in E by the definition of members of E. Let U be $X \setminus \overline{E}$. $\square$

Theorem 3 *The following are equivalent:*

1. *TWO has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.*

2. $\pi(X) = \aleph_0$.

Proof : Assume the negation of 2 and let σ be a strategy for player TWO. By Lemma 2 choose a nonempty open set $U_{\emptyset}$ such that there is for each $x \in U$ a dense subset D of X with $x = \sigma(D)$. Let $(x_{(n)} : n < \omega)$ enumerate a dense subset of $U_{\emptyset}$; for each $n < \omega$ choose a dense subset $D_{(n)}$ of X such that $x_{(n)} = \sigma(D_{(n)})$.

Applying Lemma 2 again, choose for each n a nonempty open subset $U_{(n)}$ of X such for that each $x \in U_{(n)}$ there is a dense set D with $x = \sigma(D_{(n)}, D)$. Then let $(x_{(n,m)} : m < \omega)$ enumerate a dense subset of $U_{(n)}$, and for each m let $D_{(n,m)}$ be a dense subset of X such that $x_{(n,m)} = \sigma(D_{(n)}, D_{(n,m)})$. Continuing in this manner recursively choose families $(U_\nu : \nu \in {}^{<\omega}\omega)$; $(D_\nu : \nu \in {}^{<\omega}\omega \setminus \{\emptyset\})$; $(x_\nu : \nu \in {}^{<\omega}\omega \setminus \{\emptyset\})$ such that each U_ν is a nonempty open subset of X, each D_ν is a dense subset of X and each x_ν is an element of X, satisfying:

1. Each element of $U_{(n_1,\dots,n_k)}$ is of the form $\sigma(D_{(n_1)}, \dots, D_{(n_1,\dots,n_k)}, D)$ for some dense subset D of X;

2. $(x_{(n_1,\dots,n_k,m)} : m < \omega)$ enumerates a dense subset of $U_{(n_1,\dots,n_k)}$, and

3. $x_{(n_1,\dots,n_k)} = \sigma(D_{(n_1)}, \dots, D_{(n_1,\dots,n_k)})$.

Since $\pi(X) > \aleph_0$ fix a nonempty open subset V of X such that no U_ν is a subset of V. Since X is T_3, choose a nonempty open set W such that $\overline{W} \subseteq V$. Then we have for each $(n_1, \dots, n_k)$ that $U_{(n_1,\dots,n_k)} \setminus \overline{W} \neq \emptyset$.

Recursively choose elements of X as follows: Choose n_1 with $x_{(n_1)} \in U_{\emptyset} \setminus \overline{W}$, then choose n_2 with $x_{(n_1,n_2)} \in U_{(n_1)} \setminus \overline{W}$, then choose n_3 with $x_{(n_1,n_2,n_3)} \in U_{(n_1,n_2,n_3)} \setminus \overline{W}$, and so on. The sequence $D_{(n_1)}, x_{(n_1)}, D_{(n_1,n_2)}, x_{(n_1,n_2)}, \dots$ is a play during which TWO used the strategy σ and lost because the points chosen by σ were all outside the nonempty open subset W of X. $\Box$

Assumption 3 $\aleph_0 < \pi(X)$.

Lemma 4 *Every dense subspace of X has the same π-weight as X.*

Proof : Let κ be the π-weight of X and let $\mathcal{B}$ be a π-base of cardinality κ. Let Y be a dense subset of X. Then $\{B \cap Y : B \in \mathcal{B}\}$ is a π-base for Y. This shows that the π-weight of Y is at most κ. To see that it is also at least κ, let $\mathcal{C}$ be a π–base for Y. Then define $\mathcal{A}$ to be the collection consisting of sets of the form $\mathsf{Int}(\overline{U})$ where U is in $\mathcal{C}$, and the interior- and closure operations are those of X. To see that $\mathcal{A}$ is a π–base for X, let V be a nonempty open subset of X. Since X is T_3, we find a nonempty open set W of X such that $\overline{W} \subseteq V$. Since Y is a dense subset of X, $W \cap Y$ is a nonempty open subset of Y. Choose $C \in \mathcal{C}$ such that $C \subseteq W \cap Y$. Then $\overline{C} \subseteq \overline{W}$, and so $\mathsf{Int}(\overline{C}) \subseteq V$. $\Box$

Theorem 5 *The following are equivalent for X:*

1. *TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.*
2. *For each dense $Y \subseteq X$, TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Y.*
3. *For some dense subset Y of X, TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Y.*

Proof : $1 \Rightarrow 2$: By Theorem 3 X has uncountable π-weight. Lemma 4 implies that every dense subset of X has uncountable π–weight. Since $\delta(X) = \aleph_0$, Theorem 3 implies that for each dense $Y \subseteq X$, TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Y.
$2 \Rightarrow 3$: This needs no explanation.
$3 \Rightarrow 1$: Let Y be a dense subset of X such that TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Y. Then ONE confines plays of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X to dense subsets of Y – since these are also dense in X they are legitimate moves of ONE. Now apply 3. $\Box$

Theorem 6 *The following are equivalent for a space X:*

1. *ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.*
2. *For every dense subset Z of X, ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Z.*
3. *For every countable dense subset Z of X ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on Z.*

Proof : $1 \Rightarrow 2$: A strategy of ONE which confines ONE's moves to subsets of a dense subset Z of X is a strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ for both spaces.
$2 \Rightarrow 3$: This requires no proof.
$3 \Rightarrow 1$: Let σ be a strategy for ONE. Since $\delta(X) = \aleph_0$, we may assume that σ in each inning calls on ONE to play a countable dense subset of X.

Define an array $(x_\tau : \tau \in {}^{<\omega}\omega)$ of elements of X as follows:
$(x_{(n)} : n < \omega)$ is a bijective enumeration of $\sigma(\emptyset)$, ONE's first move; $(x_{(n,m)} : m < \omega)$ is a bijective enumeration of $\sigma(x_{(n)})$, ONE's response to TWO's move $x_{(n)}$, and so on. In general, $(x_{(n_1,\dots,n_k,m)} : m < \omega)$ is a bijective enumeration of $\sigma(x_{(n_1)}, x_{(n_1,n_2)}, \dots, x_{(n_1,\dots,n_k)})$.

The set $D := \{x_\tau : \tau \in {}^{<\omega}\omega\}$ is a countable dense subset of X and the strategy σ of ONE is a strategy of ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on D. Thus σ is not a winning strategy for ONE on D. A σ–play of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on D which is lost by ONE is also a σ–play on X which is lost by ONE. Thus σ is not a winning strategy for ONE on X. $\Box$

The point-picking game.

Theorem 7 *For a space X the following are equivalent:*

1. *ONE has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.*

2. *TWO has a winning strategy in* $\mathsf{G}_\omega^D(X)$.

Proof : $1 \Rightarrow 2$: Let σ be a winning strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. Define a strategy τ for TWO in $\mathsf{G}_\omega^D(X)$ as follows: Let $\sigma(X)$ denote the first move of ONE of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. For nonempty open set U define $\tau(U)$ so that $\tau(U) \in U \cap \sigma(X)$. Let a finite sequence $(U_1,\dots,U_{n+1})$ of nonempty open sets be given, and assume that τ has already been defined for sequences of length less than $n+1$ of open sets. Define $\tau(U_1,\dots,U_{n+1})$ to be an element of $U_{n+1} \cap \sigma(\tau(U_1),\dots,\tau(U_1,\dots,U_n))$. To see that τ is a winning strategy for TWO in $\mathsf{G}_\omega^D(X)$, consider a τ–play $O_1, \tau(O_1), O_2, \tau(O_1,O_2), \dots$. A recursive computation shows that $\tau(O_1) \in O_1 \cap \sigma(X)$, and for each n, $\tau(O_1,\dots,O_{n+1}) \in O_{n+1} \cap \sigma(\tau(O_1),\dots,\tau(O_1,\dots,O_n))$. This implies that

$$\sigma(X), \tau(O_1), \sigma(\tau(O_1)), \tau(O_1,O_2), \sigma(\tau(O_1),\tau(O_1,O_2)), \dots$$

is a σ–play of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$, and thus won by ONE. But then the set $\{\tau(O_1,\dots,O_n) : n = 1,2,3,\dots\}$ is not a dense subset of X, showing that TWO won the play of $\mathsf{G}_\omega^D(X)$.
$2 \Rightarrow 1$: Let τ be a winning strategy for TWO in $\mathsf{G}_\omega^D(X)$. Define a strategy σ for ONE of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ as follows:
$\sigma(X) = \{\tau(U) : U \text{ nonempty and open}\}$. For any $\tau(U_1) \in \sigma(X)$, $\sigma(\tau(U_1)) = \{\tau(U_1,U) : U \text{ nonempty and open}\}$. For any $\tau(U_1,U_2) \in \sigma(\tau(U_1))$ define: $\sigma(\tau(U_1),\tau(U_1,U_2)) = \{\tau(U_1,U_2,U) : U \text{ nonempty and open}\}$, and so on.

To see that σ is a winning strategy for ONE of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$, consider a play $\sigma(X), T_1, \sigma(T_1), T_2, \sigma(T_1,T_2), \dots$ A recursive computation shows that there is a sequence $U_1,\dots,U_n,\dots$ of nonempty open subsets of X such that for each n, $T_n = \tau(U_1,\dots,U_n)$. Since $U_1, \tau(U_1), U_2, \tau(U_1,U_2), \dots$ is a play of $\mathsf{G}_\omega^D(X)$ won by TWO of that game, $\{T_1,\dots,T_n,\dots\}$ is not dense in X, and so ONE won the σ–play of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. $\square$

Theorem 8 *For a space X the following are equivalent:*

1. *TWO has a winning strategy in* $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$.

2. *ONE has a winning strategy in* $\mathsf{G}_\omega^D(X)$.

Proof : $1 \Rightarrow 2$: Let σ be a winning strategy for TWO in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. We are now going to use Lemma 2 to define a strategy τ for ONE of $\mathsf{G}_\omega^D(X)$.

To begin, choose by Lemma 2 a nonempty open subset O_1 of X such that each element of O_1 is of the form $\sigma(D)$ for D some dense subset of X. Then define $\tau(X) = O_1$. For x_1 any point from O_1, choose a dense subset D_1 of X such that $x_1 = \sigma(D_1)$. Then by Lemma 2 let $O_2 = \tau(x_1)$ be a nonempty open set such that each element of O_2 is of the form $\sigma(D_1,D)$ where D is a dense subset of X. For x_2 an element of $\sigma(x_1)$, choose a dense subset D_2 of X such that $x_2 = \sigma(D_1,D_2)$ Then, let $O_3 = \tau(x_1,x_2)$ be a nonempty open subset of X such that each element of O_3 is of the form $\sigma(D_1,D_2,D)$ where D is some dense subset of X, and so on.

This procedure defines a strategy τ for ONE of $\mathsf{G}^D_\omega(X)$. To see that τ is a winning strategy, consider a τ–play $O_1, T_1, O_2, T_2, \ldots$. A recursive computation shows that there are dense subsets $D_1, D_2, \ldots$ of X such that for each n, $T_n = \sigma(D_1, \ldots, D_n) \in O_n$. SInce $D_1, T_1, \ldots, D_n, T_n, \ldots$ is a σ–play of $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ won by TWO of that game, $\{T_1, T_2, \ldots\}$ is a dense subset of X, and so the τ–play of $\mathsf{G}^D_\omega(X)$ is won by ONE.
$2 \Rightarrow 1$: Let τ be a winning strategy for ONE of $\mathsf{G}^D_\omega(X)$. Define a strategy σ for TWO of $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ as follows: For dense set D_1 define $\sigma(D_1)$ to be a point of the set $D_1 \cap \tau(X)$. For dense sets D_1 and D_2 define $\sigma(D_1, D_2)$ to be an element of the set $D_2 \cap \tau(\sigma(D_1))$, and so on. Then σ is a winning strategy for TWO in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$. $\square$

The function space $\mathsf{C}_p(X)$.

$\mathbb{R}^X$ denotes the Cartesian product of X copies of the real line $\mathbb{R}$, endowed with the Tychonoff product topology. The subset of *continuous* functions from X to $\mathbb{R}$ with the topology it inherits from $\mathbb{R}^X$ is denoted $\mathsf{C}_p(X)$; this is the *topology of pointwise convergence*. The real-valued function on X with only value 0 is denoted $\underline{\mathbf{o}}$.

Theorem 9 *Let X be a Tychonoff space such that $\mathsf{C}_p(X)$ satisfies $\mathsf{S}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$. Then there is for each sequence $(A_n : n < \infty)$ of elements of $\Omega_{\underline{\mathbf{o}}}$ a pairwise disjoint sequence $(B_n : n < \infty)$ of elements of $\Omega_{\underline{\mathbf{o}}}$ such that for each n, $B_n \subseteq A_n$.*

Proof : Let $(A_n : n < \infty)$ be a sequence of elements of $\Omega_{\underline{\mathbf{o}}}$ for $\mathsf{C}_p(X)$. Since $\mathsf{C}_p(X)$ has property $\mathsf{S}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$ we may assume that each A_n is countable. Enumerate A_n bijectively as $(f^n_m : m < \infty)$.

Define a strategy σ for ONE in $\mathsf{G}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$ as follows: ONE's first move is $\sigma(\mathsf{C}_p(X)) = (|f^1_m| : m < \infty)$. If TWO chooses $|f^1_{m_1}|$, then ONE's response is $\sigma(|f^1_{m_1}|)$, defined as $\{|f^1_m| + |f^2_k| : |\{f^1_{m_1}, f^1_m, f^2_k\}| = 3\}$. If TWO now chooses $|f^1_{m^2_1}| + |f^2_{m^2_2}|$, then ONE responds with $\sigma(|f^1_{m_1}|, |f^1_{m^2_1}| + |f^2_{m^2_2}|)$, which is the set $\{|f^1_m| + |f^2_n| + |f^3_o| : |\{f^1_{m_1}, f^1_{m^2_1}, f^2_{m^2_2}, f^1_m, f^2_n, f^3_o\}| = 6\}$, and so on.

Since ONE has no winning strategy in $\mathsf{G}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$, σ is not winning for ONE. Consider a sequence of moves of TWO which defeats σ. It is of the form $|f^1_{m_1}|, |f^1_{m^2_1}| + |f^2_{m^2_2}|, |f^1_{m^3_1}| + |f^2_{m^3_2}| + |f^3_{m^3_3}|, \ldots$ Since this sequence constitutes a set in $\Omega_{\underline{\mathbf{o}}}$, and since for each i and $j \leq i$ $|f^j_{m^i_j}|$ is pointwise no larger than the j-th move of TWO, for each j the set $C_j = \{f^j_{m^i_j} : i \geq j\}$ is in $\Omega_{\underline{\mathbf{o}}}$, and for $i \neq j$, $C_i \cap C_j$ is finite. For each j define $B_j = C_j \setminus (\cup_{i<j} C_i)$. The B_j's are as required. $\square$

By Urysohn's metrization theorem second countable T_3–spaces are metrizable. Thus we lose no generality in stating some results (like the next one) for separable metric spaces instead of second countable T_3–spaces. The following standard result as well as many others we use can for example be deduced from general results in the text [2].

Lemma 10 *If X is a separable metric space then $\delta(\mathsf{C}_p(X)) = \aleph_0$.*

Let Y be an infinite separable metric space with a countable dense subset D, enumerated bijectively as $\{d_n : n < \infty\}$. Let $\mathcal{B}$ be the family of sets of the form: each element of $\mathcal{B}$ is different from Y and a union of finitely many open spheres with centers in D and rational radius, whose closures are pairwise disjoint and all sets of this form which are not equal to Y are elements of $\mathcal{B}$. Then $\mathcal{B}$ is a countable ω–cover of Y and it has the property that: For every finite set $F \subseteq Y$ and for every open set V containing F, there is an $O \in \mathcal{B}$ such that $F \subseteq O \subseteq V$.

Next, associate with $\mathcal{B}$ the following subset $\mathsf{S}(\mathcal{B})$ of $\mathsf{C}_p(Y)$: For the element $B_1 \cup \ldots \cup B_k$ of $\mathcal{B}$ where $B_1, \ldots, B_k$ are open spheres with disjoint closures and rational radii and for $(q_1, \ldots, q_k)$ a k–tuple of nonzero rational numbers let $F(B_1, \ldots, B_k, q_1, \ldots, q_k)$ be a continuous function f on Y which has the following properties:

1. $f\lceil_{Y \setminus (B_1 \cup \ldots \cup B_k)}$ is identically zero;
2. f on any B_j has only nonzero values between 0 and q_j;
3. On the open sphere concentric with B_j but of one third the radius, f is equal to q_j.

$\mathsf{S}(\mathcal{B})$ is the set of functions of the form $F(B_1, \ldots, B_k, q_1, \ldots, q_k)$, $k \in \mathbb{N}$.

Lemma 11 *$\mathsf{S}(\mathcal{B})$ is a dense subset of $\mathsf{C}_p(Y)$.*

Proof : Consider a typical basic open subset of $\mathsf{C}_p(Y)$: It is of the form

$$[F, g, \epsilon] = \{f \in \mathsf{C}_p(Y) : (\forall x \in F)(|f(x) - g(x)| < \epsilon)\}$$

where F is a finite subset of Y, g is an element of $\mathsf{C}_p(Y)$ and ϵ is a positive real number. Enumerate F bijectively as $\{x_1, \ldots, x_m\}$. For $1 \leq j \leq m$ choose open spheres B_j centered at elements of D and with rational radii such that x_j is a member of the open the sphere concentric with B_j but with a third of the radius, and such that the closures of the B_j's are pairwise disjoint. For each j choose a rational number q_j in the interval $(g(x_j) - \epsilon, g(x_j) + \epsilon)$. Then the member $F(B_1, \ldots, B_m, q_1, \ldots, q_m)$ of $\mathsf{S}(\mathcal{B})$ is in $[F, g, \epsilon]$. $\square$

For E a subset of $S(\mathcal{B})$ let $\Omega(E)$ be the set of those proper subsets A of Y for which there is an element g of E such that $A = \{x \in Y : g(x) \neq 0\}$. Thus, $\Omega(E)$ consists of unions of finite sets of open spheres centered at elements of D and with rational radii, whose closures are pairwise disjoint.

Lemma 12 *If E is a dense subset of $S(\mathcal{B})$ then $\Omega(E)$ is an ω–cover of Y.*

Proof : Consider a finite (nonempty) subset F of Y. By our definition Y is not a member of $\Omega(E)$. Pick a point $y \in Y \setminus F$ (which is possible since Y is infinite). Then let g be a continuous function such that for each $x \in F$ $g(x) = 1$, and $g(y) = 0$. Also let ϵ be $\frac{1}{10}$.

Since E is dense, it has an element in $[F, g, \epsilon]$, say f. Since $\epsilon = \frac{1}{10}$ we see that for each $x \in F$, $f(x) > 0$. But this implies that $F \subseteq \{x \in Y : f(x) \neq 0\}$, and this is an element of $\Omega(E)$. $\square$

Theorem 13 *For X a separable metric space, the following are equivalent:*

1. *X has property $\mathsf{S}_1(\Omega, \Omega)$.*
2. *ONE has no winning strategy in $\mathsf{G}_1(\Omega, \Omega)$ on X.*
3. *$\mathsf{C}_p(X)$ has property $\mathsf{S}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$.*
4. *ONE has no winning strategy in $\mathsf{G}_1(\Omega_0, \Omega_0)$ on $\mathsf{C}_p(X)$.*
5. *ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on $\mathsf{C}_p(X)$.*
6. *$\mathsf{C}_p(X)$ has property $\mathsf{S}_1(\mathfrak{D}, \mathfrak{D})$.*

Proof : The equivalence of 1, 2, 3 and 4 has been established in [20]. We show that $4 \Rightarrow 5$ and $6 \Rightarrow 1$.
<u>$4 \Rightarrow 5$:</u> Let σ be a strategy for ONE in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on $\mathsf{C}_p(X)$. By Lemma 10 we may assume that in each inning σ calls on ONE to choose a countable dense set. Define the following array of elements of $\mathsf{C}_p(X)$: $\sigma(\mathsf{C}_p(X)) = (f_n : n < \infty)$, $\sigma(f_{n_1}) = (f_{n_1,n} : n < \infty)$, $\sigma(f_{n_1}, f_{n_1,n_2}) = (f_{n_1,n_2,n} : n < \infty)$, and so on. For each $n_1, \dots, n_k$, $\sigma(f_{n_1}, \dots, f_{n_1,\dots,n_k})$ is a countable dense subset of $\mathsf{C}_p(X)$.

Also let $(g_m : m < \infty)$ be a countable dense subset of $\mathsf{C}_p(X)$, and define the following strategy, τ, for ONE in the game $\mathsf{G}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$ of $\mathsf{C}_p(X)$:

1. $\tau(\mathsf{C}_p(X)) = (|f_n - g_1| : n < \infty)$;
2. $\tau(|f_{n_1} - g_1|) = (|f_{n_1,i} - g_1| + |f_{n,j} - g_2| : i, j < \infty)$;
3. $\tau(|f_{n_1} - g_1|, |f_{n_1,i_1} - g_1| + |f_{n_1,i_2} - g_2|)$ is the set $(|f_{n_1,i_1,j_1} - g_1| + |f_{n_1,i_1,j_2} - g_2| + |f_{n_1,i_1,j_3} - g_3| + |f_{n_1,i_2,k_1} - g_1| + |f_{n_1,i_2,k_2} - g_2| + |f_{n_1,i_2,k_3} - g_3| : j_1, j_2, j_3, k_1, k_2, k_3 < \infty)$, and so on.

Since this strategy is not winning for ONE in $\mathsf{G}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$, we find a play which is won by TWO, say the list of consecutive moves by TWO are: $|f_{n_1} - g_1|$, $|f_{n_1,n_1^2} - g_1| + |f_{n_1,n_2^2} - g_2|$, $|f_{n_1,n_1^2,n_1^3} - g_1| + |f_{n_1,n_1^2,n_2^3} - g_2| + |f_{n_1,n_1^2,n_3^3} - g_3| + |f_{n_1,n_2^2,n_4^3} - g_1| + |f_{n_1,n_2^2,n_5^3} - g_2| + |f_{n_1,n_2^2,n_6^3} - g_3|$, and so on. Since this sequence of moves is an element of $\Omega_{\underline{\mathbf{o}}}$ and $\mathsf{C}_p(X)$ has property $\mathsf{S}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$, apply Theorem 9 to partition this set into countably many disjoint sets $S_k, k < \infty$, such that each S_k is in $\Omega_{\underline{\mathbf{o}}}$. For each k let I_k be the set of n for which an element of S_k was selected by TWO in the n–th inning. By making appropriate finite modifications to the S_k's we may assume that for each k, $\min(I_k) \geq k$.

We shall now select $m_1, m_2, \dots, m_k, \dots$ such that the sequence of moves of TWO against σ given by f_{m_1}, f_{m_1,m_2}, $\dots$, $f_{m_1,\dots,m_k}, \dots$ is a dense subset of $\mathsf{C}_p(X)$. Since $1 \in I_1$, the only choice we have for m_1 is n_1. Consider 2; We have $2 \in I_1$ or $2 \in I_2$. Choose m_2 such that $|f_{m_1,m_2} - g_j|$ is a term of TWO's second move in $\mathsf{G}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$ where $2 \in I_j$. In general, when choosing m_{i+1}, identify the $j \leq i+1$ with $i+1 \in I_j$ and then choose m_{i+1} so that $|f_{m_1,\dots,m_i,m_{i+1}} - g_j|$ is a term of TWO's move in inning $i+1$ of $\mathsf{G}_1(\Omega_{\underline{\mathbf{o}}}, \Omega_{\underline{\mathbf{o}}})$.

Since for each k we have $S_k \in \Omega_{\underline{\mathbf{o}}}$, and thus $\{|f_{m_1,\dots,m_j} - g_k| : j \in I_k\}$ in $\Omega_{\underline{\mathbf{o}}}$, it follows from the density of the set of g_m's that $\{f_{m_1,\dots,m_j} : j < \infty\}$ is dense in $\mathsf{C}_p(X)$.
<u>$6 \Rightarrow 1$:</u> Let X be a separable metric space satisfying the hypotheses. Let D be a countable dense subset of X. Every ω–cover of X has a refinement which is a subset of $\mathcal{U}$, the ω–cover whose elements are unions of finitely many open spheres with rational radii, centered at elements of D, and with pairwise disjoint closures.

Thus, when checking whether X has property $\mathsf{S}_1(\Omega, \Omega)$, we may confine our attention to sequences $(\mathcal{U}_n : n \in \mathbb{N})$ where for each n $\mathcal{U}_n \subseteq \mathcal{U}$ is an ω–cover of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be such a sequence.

For each n define $D_n \subseteq \mathsf{C}_p(X)$ to be the set consisting of all functions of the form

$$F(B_1, \dots, B_m, q_1, \dots, q_m) \in D_n.$$

where $B_1, \dots, B_m$ are open spheres centered at elements of D, have rational radii and pairwise disjoint closures, $B_1 \cup \dots \cup B_m \in \mathcal{U}_n$, and $(q_1, \dots, q_m)$ is an m–tuple of non–zero rational numbers. As in the proof of Lemma 11 one can show that each D_n is a dense subset of $\mathsf{C}_p(X)$.

Now apply 6 to the sequence $(D_n : n \in \mathbb{N})$ of dense subsets of $\mathsf{C}_p(X)$ and choose for each n a $d_n \in D_n$ such that $\{d_n : n \in \mathbb{N}\}$ is dense. Then $\Omega(\{d_n : n \in \mathbb{N}\})$ is an ω–cover of X, and for each n the set $\{x \in X : d_n(x) \neq 0\}$, a member of $\Omega(\{d_n : n \in \mathbb{N}\})$, is in $\mathcal{U}_n$. $\square$

Examples

Example 1 (MA(countable)) *A Tychonoff space X where neither player has a winning strategy in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$.*

In Theorem 4.1 of [4] Berner and Juhász show that the postulate $\diamondsuit$ implies the existence of a space for which neither player has a winning strategy in the game $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$. In their Question 4.2 they ask if such examples can be found in classical mathematics. In [6] Dow and Gruenhage give another example of this phenomenon, this time using the much weaker postulate $\mathsf{MA}(\sigma - \mathit{centered})$ (Martin's Axiom for σ–centered partially ordered sets).

Theorem 13 gives another example under an even weaker postulate, $\mathsf{MA}(\mathit{countable})$ (Martin's Axiom for countable partially ordered sets). If X is an uncountable set of real numbers then $\mathsf{C}_p(X)$ has uncountable π–weight, and so TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on $\mathsf{C}_p(X)$. If moreover ONE has no winning strategy in $\mathsf{G}_1(\Omega, \Omega)$ on X, then neither player has a winning strategy in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on $\mathsf{C}_p(X)$.

Now Martin's Axiom for countable partially ordered sets can be used to construct an uncountable set X of real numbers such that X has property $\mathsf{S}_1(\Omega,\Omega)$ - a proof of this can be decoded from the proof of Theorem 2.11 of [13].

The additional postulate is to some extent needed to obtain an example of the form $\mathsf{C}_p(X)$ where X is metrizable. On account of a result of Laver in [14], it is consistent that the only sets of real numbers which have property $\mathsf{S}_1(\Omega,\Omega)$ are the countable ones. For X countable $\pi(\mathsf{C}_p(X)) = \aleph_0$, and TWO has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. In Laver's model, if X is an uncountable set of real numbers, then ONE has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. One might think that the real line is just the wrong metric space for the task. But by Carlson's theorem in [5], the only metric spaces in Laver's model with property $\mathsf{S}_1(\Omega,\Omega)$ are the countable ones.

Example 2 *A Tychonoff space X which satisfies $\mathsf{S}_1(\Omega,\Omega)$, but for which $\mathsf{C}_p(X)$ does not satisfy $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$.*

In [20] we showed that 1 through 4 of Theorem 13 are equivalent for Tychonoff spaces. Additional properties are needed to obtain their equivalence to 5 and 6. A space X which illustrates this was defined in [16] and described in convenient form in [22]. Here it is: Let Λ denote the set of limit ordinals in ω_1. For each $\alpha \in \Lambda$ choose a strictly increasing function $s_\alpha : \mathbb{N} \to \alpha$ which converges to α. Define X to be

$$\{f \in {}^{\omega_1}2 : \textit{support}(f) \text{ is finite, or } \{s_\alpha(n) : n \in \mathbb{N}\} \text{ for some } \alpha \in \Lambda\}.$$

For $f \in X$ and $\alpha < \omega_1$ define $B(f,\alpha) = \{g \in X : g\lceil_\alpha = f\lceil_\alpha\}$. The topology τ of X is such that for each $f \in X$ the set $\{B(f,\alpha) : \alpha < \omega_1\}$ is a neighborhood basis for f.

Then (X,τ) is a zero-dimensional Tychonoff space. Put $X_\beta = \{f \in X : \textit{support}(f) \subseteq \beta\}$. Each X_β is countable. For $F \subseteq X$ finite and for $\alpha < \omega_1$ the symbol $B(F,\alpha)$ denotes the set $\cup_{f\in F} B(f,\alpha)$. The set $\mathcal{U} = \{B(F,\alpha) : F \subseteq X \text{ finite}, \alpha < \omega_1 \text{ and } X \neq B(F,\alpha)\}$ is an ω–cover of X. Each ω–cover of X has a refinement which is a subset of $\mathcal{U}$. Thus, when we study $\mathsf{S}_1(\Omega,\Omega)$ for X we may assume that each ω–cover in question is a subset of $\mathcal{U}$.

For $\alpha \in \Lambda$ let x_α denote the unique element of X which has support $s_\alpha[\mathbb{N}]$. Before we prove that X has property $\mathsf{S}_1(\Omega,\Omega)$, we first show that it has the following weakened form of this property:

Proposition 14 *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω–covers of X and for each ordinal $\alpha < \omega_1$, there is a sequence $(U_n : n \in \mathbb{N})$ and a countable limit ordinal $\beta > \alpha$ such that*

1. *For each n, $U_n \in \mathcal{U}_n$;*
2. *$\{U_n : n \in \mathbb{N}\}$ is an ω–cover of $X \setminus \{x_\beta\}$.*

Proof : Let $\alpha < \omega_1$ as well as a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω–covers of X be given. We may assume that for each n $\mathcal{U}_n$ is a subset of $\mathcal{U}$.

Put $\beta_0 = \alpha+\omega$. For each k choose $U_{2^k} \in \mathcal{U}_{2^k}$ such that each element of X_{β_0} is in all but finitely many of the U_{2^k}. Each U_{2^k} is of the form $B(F_{2^k}, \alpha_{2^k})$ where $F_{2^k} \subseteq X$ is finite and $\alpha_{2^k} < \omega_1$. Put $\beta_1 = \sup\{\alpha_{2^k} : k \in \mathbb{N}\}+\beta_0+\omega$. For each k choose $U_{3^k} \in \mathcal{U}_{3^k}$ such that each element of X_{β_1} is in all but finitely many of the U_{3^k}. Each U_{3^k} is of the form $B(F_{3^k}, \alpha_{3^k})$ where each $F_{3^k} \subseteq X$ is finite and $\alpha_{3^k} < \omega_1$. Put $\beta_2 = \sup\{\alpha_{3^k} : k \in \mathbb{N}\} + \beta_1 + \omega$, and repeat the process for X_{β_2}.

In this manner we obtain an increasing sequence $\beta_1 < \beta_2 < \ldots < \beta_n < \ldots$ of countable limit ordinals and for p_n the n–th prime number a sequence $(U_{p_n^k} : k \in \mathbb{N})$ such that for each k $U_{p_n^k} \in \mathcal{U}_{p_n^k}$, and each element of X_{β_n} is in all but finitely many of the $U_{p_n^k}$. Let β be the limit of the β_n's. Then the sequence $(U_{p^k} : p$ prime and $k \in \mathbb{N})$ forms an ω–cover of $X \setminus \{x_\beta\}$. $\Box$

Proposition 15 *X has property $\mathsf{S}_1(\Omega, \Omega)$.*

Proof : Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω–covers of X. Write $\mathbb{N} = \cup_{n<\infty} Y_n$, where the Y_n's are pairwise disjoint and infinite.

From the sequence $(\mathcal{U}_m : m \in Y_1)$ of ω–covers of X find $(U_m : m \in Y_1)$ and an infinite ordinal $\beta_1 < \omega_1$ such that for each m $U_m \in \mathcal{U}_m$, and $\{U_m : m \in Y_n\}$ is an ω–cover of $X \setminus \{x_{\beta_1}\}$. Then from $(\mathcal{U}_m : m \in Y_2)$ find a sequence $(U_m : m \in Y_2)$ and a limit ordinal $\beta_2 > \beta_1$ such that for each m $U_m \in \mathcal{U}_m$, and $\{U_m : m \in Y_2\}$ is an ω–cover of $X \setminus \{x_{\beta_2}\}$.

Continuing like this we find for each k a sequence $(U_m : m \in Y_k)$ and a limit ordinal β_k such that for each $m \in Y_k$, $U_m \in \mathcal{U}_m$, $\beta_k > \beta_i$ whenever $i < k$, and $\{U_m : m \in Y_k\}$ is an ω-cover of $X \setminus \{x_{\beta_k}\}$. But then $\{U_n : n \in \mathbb{N}\}$ is an ω–cover of X. $\Box$

Proposition 16 *Each element of $\mathsf{C}_p(X)$ has countable range.*

Proof : Let $f \in \mathsf{C}_p(X)$ be given. For each $x \in X$ and for each $n \in \mathbb{N}$, choose an $\alpha_n(x) < \omega_1$ such that $f[B(x, \alpha_n(x))] \subseteq (f(x) - \frac{1}{n}, f(x) + \frac{1}{n})$. Let $\alpha(x)$ be the supremum of the $\alpha_n(x)$'s. Then for each x f is constant on $B(x, \alpha(x))$. Since X is a Lindelöf space, countably many of the $B(x, \alpha(x))$'s cover X. But then the values of f at that countable set of x's constitute the range of f. $\Box$

Proposition 17 *There is a zero-dimensional Tychonoff space X with property $\mathsf{S}_1(\Omega, \Omega)$ and $d(\mathsf{C}_p(X)) \geq \aleph_1$.*

Proof : Let $\{f_n : n \in \mathbb{N}\}$ be a countable subspace of $\mathsf{C}_p(X)$. By Proposition 16 each f_n has countable range. Fix an ordinal $\alpha < \omega_1$ such that for each n, $range(f_n) = f_n[X_\alpha]$, and f_n is constant on $B(x, \alpha)$ for each $x \in X_\alpha$.

Let $\alpha_2 > \alpha_1 > \alpha$ be limit ordinals with $x_{\alpha_1}\lceil_\alpha = x_{\alpha_2}\lceil_\alpha$. Then choose $x \in X_\alpha$ such that both x_{α_1} and x_{α_2} are in the neighborhood $B(x, \alpha)$. Let $f : X \to \mathbb{R}$ be a continuous function such that for $i = 1, 2$, $f(x_{\alpha_i}) = i$. Then no f_n is in the neighborhood

$[\{x_{\alpha_1}, x_{\alpha_2}\}, f, \frac{1}{10}]$ of f. $\square$

Example 3 (CH) *A* T_3*-space* X *which satisfies* $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$*, but for which ONE has a winning strategy in* $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$*.*

Another important aspect of Theorem 13 is that though for the space $\mathsf{C}_p(X)$ the selection hypothesis $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$ and the nonexistence of a winning strategy for ONE in the game $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ are equivalent when X is a separable metric, this is not the case for spaces in general. In [4] Berner and Juhász construct with the aid of the Continuum Hypothesis a subspace of the Tychonoff power of $\aleph_1$ copies of the discrete space $2 = \{0, 1\}$ which has the following properties:

1. X is an HFD and
2. TWO has a winning strategy in the game $\mathsf{G}^D_\omega(X)$.

By Theorem 7 2 means that ONE has a winning strategy in the game $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.

We show that 1 implies that X has property $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$. For a set A let $\mathsf{Fin}(A, 2)$ be the set of finite binary sequences with domains finite subsets of A. A subset X of ${}^{\omega_1}2$ is an HFD if it is infinite and there is for each infinite subset S of X an ordinal $\alpha < \omega_1$ such that if σ is any element of $\mathsf{Fin}(\omega_1 \setminus \alpha, 2)$, then the basic open set $[\sigma] = \{f \in X : \sigma \subseteq f\}$ has nonempty intersection with S. One may assume that an HFD has no isolated points.

Let X be an HFD. For S an infinite subset of X define D_S to be the set of $\alpha < \omega_1$ such that for any $\sigma \in \mathsf{Fin}(\alpha, 2)$, if $[\sigma] \cap S$ is infinite, then for each $\tau \in \mathsf{Fin}(\omega_1 \setminus \alpha, 2)$, $[\sigma] \cap [\tau] \cap S$ is nonempty. One can show:

Proposition 18 *For every infinite subset* S *of* X *the set* D_S *is closed and unbounded.*

Proposition 19 *Every* HFD *satisfies* $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$*.*

Proof : Let $(D_n : n \in \mathbb{N})$ be a sequence of dense subsets of X.

Choose $\beta_1 \in D_X$ with $\beta_1 \geq \omega$. Let $(\sigma^1_n : n \in \mathbb{N})$ enumerate $\mathsf{Fin}(\beta_1, 2)$ in such a way that each element is listed infinitely many times. For each $k \in \mathbb{N}$ choose an $f_{2^k} \in D_{2^k} \cap [\sigma^1_k] \setminus \{f_{2^j} : j < k\}$, and put $A_1 = \{f_{2^k} : k \in \mathbb{N}\}$.

Choose $\alpha_1 \in D_{A_1}$ with $\alpha_1 > \beta_1$ and let $(\sigma^2_n : n \in \mathbb{N})$ enumerate $\mathsf{Fin}(\alpha_1, 2)$ such that each element is listed infinitely often. For each $k \in \mathbb{N}$ choose $f_{3^k} \in D_{3^k} \cap [\sigma^2_k] \setminus \{f_{3^j} : j < k\}$, and put $A_2 = A_1 \cup \{f_{3^k} : k \in \mathbb{N}\}$.

Choose $\alpha_2 \in D_{A_1} \cap D_{A_2}$ with $\alpha_1 < \alpha_2$ and let $(\sigma^3_n : n \in \mathbb{N})$ enumerate $\mathsf{Fin}(\alpha_2, 2)$ such that each element is listed infinitely often. For each $k \in \mathbb{N}$ choose $f_{5^k} \in D_{5^k} \cap [\sigma^3_k] \setminus \{f_{5^j} : j < k\}$. Put $A_3 = A_2 \cup \{f_{5^k} : k \in \mathbb{N}\}$, then choose $\alpha_3 \in D_{A_1} \cap D_{A_2} \cap D_{A_3}$ with $\alpha_3 > \alpha_2$, and so on.

Finally, put $A = \cup_{k \in \mathbb{N}} A_k$. Then A is dense in X and its construction requires choosing at most one point per D_k. $\square$

Example 4 (MA(σ-centered)) *A countable Tychonoff space X such that ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X, but ONE has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X^2.*

The example we discuss here was given in [6]. According to E. Hewitt - [9] - a topological space is *irresolvable* if no two dense subsets of it are disjoint. At the other end of the spectrum from irresolvability is *splittability*. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a set X. We say that X satisfies $\mathsf{Split}(\mathcal{A},\mathcal{B})$ if there is for each element A of $\mathcal{A}$, pairwise disjoint elements B_1 and B_2 of $\mathcal{B}$ such that $B_1 \cup B_2 \subset A$.

Theorem 20 *If TWO has a winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X then X satisfies $\mathsf{Split}(\mathfrak{D},\mathfrak{D})$.*

Proof : Let D_1 and D_2 be dense sets, and let F be a winning strategy for TWO in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. Consider the following two runs of the game. We have two boards, I and II, each with its player ONE competing against the strategy F of TWO. The symbol O_n^I denotes the move in inning n for player ONE of board I, while O_n^{II} denotes that of player ONE of board II.

Define $O_1^I := D_1$ and $O_1^{II} := D_2 \setminus \{F(O_1^I)\}$. Recursively define for each n:

$$O_{n+1}^I := O_n^I \setminus \{F(O_1^I,\dots,O_n^I), F(O_1^{II},\dots,O_n^{II})\}$$

and

$$O_{n+1}^{II} := O_n^{II} \setminus \{F(O_1^I,\dots,O_n^I,O_{n+1}^I), F(O_1^{II},\dots,O_n^{II})\}$$

Since TWO wins the games on each of the boards, the sets $E_I := \{F(O_1^I,\dots,O_n^I) : n \in \mathbb{N}\}$ and $E_{II} := \{F(O_1^{II},\dots,O_n^{II}) : n \in \mathbb{N}\}$ are both dense, and they are disjoint. Moreover, $E_I \subseteq D_1$ and $E_{II} \subseteq D_2$. $\Box$

Corollary 21 *For X irresolvable TWO has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.*

Proposition 22 *If ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X^2, then for every pair D_1 and D_2 of dense subsets of X, there are dense subsets $E_1 \subset D_1$ and $E_2 \subset D_2$ such that $E_1 \cap E_2 = \emptyset$. (In particular, X satisfies $\mathsf{Split}(\mathfrak{D},\mathfrak{D})$.)*

Proof : Let D_1 and D_2 be dense subsets of X. Then $D_1 \times D_2$ is a dense subset of X^2. Consider the strategy σ of ONE which is defined as follows:

$\sigma(\emptyset)$ is the set $\{(x,y) \in D_1 \times D_2 : x \neq y\}$. For (x_1,y_1) an element of this set,

$$\sigma((x_1,y_1)) = \{(x,y) \in D_1 \times D_2 : \{x,y\} \cap \{x_1,y_1\} = \emptyset \text{ and } x \neq y\}.$$

For (x_2,y_2) from this set,

$$\sigma((x_1,y_1),(x_2,y_2)) = \{(x,y) \in (D_1 \setminus \{x_1,x_2,y_1,y_2\}) \times (D_2 \setminus \{x_1,x_2,y_1,y_2\}) : x \neq y\}$$

and so on.

Since σ is not a winning strategy for ONE, consider a σ–play lost by ONE, say $\sigma(\emptyset), (x_1, y_1), \sigma((x_1, y_1)), (x_2, y_2), \sigma((x_1, y_1), (x_2, y_2)), \ldots$.

Then $\{(x_n, y_n) : n = 1, 2, 3, 4, \ldots\}$ is dense in X^2, whence each of $\{x_n : n = 1, 2, 3, \ldots\} \subset D_1$ and $\{y_n : n = 1, 2, 3, \ldots\} \subset D_2$ is dense in X. But these two dense sets are disjoint. $\Box$

Theorem 23 *If X is irresolvable then ONE has a winning strategy in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on X^2.*

Proof : Let D be a countable dense subset of X. Since X is T_3 and has no isolated points, for every finite subset F of X the set $D \setminus F$ is dense in X. We now define a strategy σ for ONE in the game $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on X^2. To begin:

$$\sigma(\emptyset) = \{(x, y) : x, y \in D \text{ and } x \neq y\}.$$

For a selection $(x_1, y_1) \in \sigma(\emptyset)$ by TWO, define

$$\sigma((x_1, y_1)) = \{(x, y) : \{x, y\} \subset D \setminus \{x_1, y_1\} \text{ and } x \neq y\}.$$

For a selection $(x_2, y_2) \in \sigma((x_1, y_1))$ by TWO, define

$$\sigma((x_1, y_1), (x_2, y_2)) = \{(x, y) : \{x, y\} \subset D \setminus \{x_1, y_1, x_2, y_2\} \text{ and } x \neq y\},$$

and so on. Then σ is a winning strategy for ONE. To see this we must check:

1. Each move prescribed by σ is a legitimate move for ONE;
2. Each play according to σ is lost by TWO.

Regarding 1: Let A be a nonempty open subset of X^2. Choose nonempty open subsets U and V of X such that $U \times V \subseteq A$. Let F be any finite subset of D and choose $x \in (U \cap D) \setminus F$ and $y \in (V \cap D) \setminus (F \cup \{x\})$. Then $(x, y) \in U \times V$ and $(x, y) \in (D \setminus F) \times (D \setminus F)$, and $x \neq y$.
Regarding 2: Consider a σ–play $\sigma(\emptyset), (x_1, y_1), \sigma((x_1, y_1)), (x_2, y_2), \ldots$ of $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$ on X^2. By the definition of σ the two subsets $\{x_1, x_2, \ldots, x_n, \ldots\}$ and $\{y_1, y_2, \ldots, y_n, \ldots\}$ of D are disjoint. By irresolvability they cannot both be dense in (X, τ). But if $\{(x_n, y_n) : n = 1, 2, 3, \ldots\}$ were a dense subset of X^2, then as the projections onto each coordinate are open mappings, each of the sets $\{x_1, x_2, \ldots, x_n, \ldots\}$ and $\{y_1, y_2, \ldots, y_n, \ldots\}$ would be dense – known not to be the case. $\Box$

To finish the example let X be the space obtained by Dow and Gruenhage in [6]: The underlying point set for this space is ω, and the topology τ is such that (ω, τ) is an irresolvable T_3–space for which ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D}, \mathfrak{D})$. $\mathsf{MA}(\sigma$–centered) is used to obtain this example. Some such hypothesis is needed, because the existence of such a space implies the existence on $\mathbb{N}$ of a semiselective filter which is the intersection of countably many ultrafilters; Dow and Gruenhage prove in [6] that it is consistent that no such filters exist.

I have not been able to answer the following two questions:

Problem 1 *If X is irresolvable, does $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$ imply $\mathsf{H}_1(\mathfrak{D},\mathfrak{D})$?*

Problem 2 *If X is irresolvable does $\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$ for X imply the same for X^2?*

$\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ and the Baire category theorem.

In a 1938 paper [17] Rothberger introduced property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$; Galvin introduced the game $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ in [8] and Pawlikowski proved in [15]:

Theorem 24 (Pawlikowski) *For any space X, ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ if, and only if, X has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

Let $\mathcal{M}$ denote the ideal of first category subsets of the real line. According to the Baire category theorem the real line is not a union of countably many first category sets. Let $\mathsf{cov}(\mathcal{M})$ denote the least cardinal number for which the Baire category theorem fails – *i.e.*, the least κ such that the real line is a union of κ first category sets.

In Theorem 29 we connect show that $\mathsf{cov}(\mathcal{M})$ is intimately connected with $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. In this proof we need to refer the reader to some facts from the literature – in particular [7], [13], [17], [18], [19] and [20]. The following theorems summarize some of the results we use.

Theorem 25 (Fremlin-Miller) $\mathsf{cov}(\mathcal{M})$ *is the minimal possible cardinality for a separable metric space which does not have property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

Theorem 26 *For a set X of real numbers the following are equivalent:*

1. *X has property $\mathsf{S}_1(\Omega,\Omega)$.*
2. *ONE does not have a winning strategy in the game $\mathsf{G}_1(\Omega,\Omega)$ played on X.*
3. *Every finite power of X has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

Theorem 27 ([13], Theorem 4.8) *The minimal cardinality of a set of real numbers which does not have property $\mathsf{S}_1(\Omega,\Omega)$ is* $\mathsf{cov}(\mathcal{M})$.

For a collection $\mathcal{B}$ of subsets of ω let $X_{\mathcal{B}}$, the set of characteristic functions of elements of $\mathcal{B}$, represents $\mathcal{B}$ as a subspace of 2^{ω}.

Theorem 28 *If (ω,τ) is a T_3–space with no isolated points and $\mathcal{B}$ is a π-base such that $X_{\mathcal{B}}$ is a $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ space, then ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$.*

Proof : Let σ be a strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on (ω,τ). Define a strategy ρ for ONE in $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ on $X_{\mathcal{B}}$ as follows: $\rho(X_{\mathcal{B}}) = \{[\{(x,1)\}] : x \in \sigma(\emptyset)\}$. Since $\sigma(\emptyset)$ is dense it meets each element of $\mathcal{B}$, and thus $\rho(X_{\mathcal{B}})$ is an open cover of $X_{\mathcal{B}}$. TWO responds by selecting $T_1 \in \rho(X_{\mathcal{B}})$.

The set T_1 is of the form $[\{(t_1,1)\}]$ where $t_1 \in \sigma(\emptyset)$. In order to respond to T_1, ONE first computes $\sigma(t_1)$ and then defines $\rho(T_1) = \{[\{(x,1)\}] : x \in \sigma(t_1)\}$, to which TWO responds with $T_2 \in \rho(T_1)$.

Now T_2 is of the form $[\{(t_2,1)\}]$ where $t_2 \in \sigma(t_1)$. ONE computes $\sigma(t_1,t_2)$, and then defines $\rho(T_1,T_2) = \{[\{(x,1)\}] : x \in \sigma(t_1,t_2)\}$ and so on. By Theorem 24 ρ is not a winning strategy for ONE. Find a play against it won by TWO, say $\rho(X_{\mathcal{B}}), T_1, \rho(T_1), T_2, \rho(T_1,T_2), \ldots$. From the definition of ρ compute a corresponding sequence $t_1, t_2, \ldots$ of elements of ω such that $\sigma(\emptyset), t_1, \sigma(t_1), t_2, \sigma(t_1,t_2), t_3, \ldots$ is a play of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ where ONE used σ, and for each n we have $T_n = [\{(t_n,1)\}]$. Since $\{T_n : n = 1,2,3,\ldots\}$ is a cover of $X_{\mathcal{B}}$, each element of $\mathcal{B}$ has nonempty intersection with $\{t_n : n = 1,2,3,\ldots\}$, and so TWO won the σ–play of $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$. $\square$

The following theorem was given in [12] with a different proof:

Theorem 29 *For an infinite cardinal number κ, the following are equivalent:*

1. *For each T_3–space X with $\delta(X) = \aleph_0$ and $\pi(X) \leq \kappa$, ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X.*
2. *$\kappa < \mathsf{cov}(\mathcal{M})$.*

Proof : $1 \Rightarrow 2$: Let X be a set of real numbers of cardinality κ. Then $\delta(\mathsf{C}_p(X)) = \aleph_0$, and $\pi(\mathsf{C}_p(X)) = |X| = \kappa$. By hypothesis ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on $\mathsf{C}_p(X)$. Theorems 13 and 26 imply that X has property $\mathsf{S}_1(\Omega,\Omega)$. Thus, every set of real numbers of cardinality κ has property $\mathsf{S}_1(\Omega,\Omega)$. Theorem 27 implies that $\kappa < \mathsf{cov}(\mathcal{M})$.
$2 \Rightarrow 1$: Let σ be a strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X. Since $\delta(X) = \aleph_0$, we use the method of Theorem 6 to find a countable dense subset D of X such that σ is a strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on D. Then $\pi(D) = \pi(X) < \mathsf{cov}(\mathcal{M})$. Letting $\mathcal{B}$ be a π-base of D of minimal cardinality, and considering D as ω, we see that $X_{\mathcal{B}}$ has cardinality less than $\mathsf{cov}(\mathcal{M})$ – consequently $X_{\mathcal{B}}$ has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. By Theorem 28, σ is not a winning strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on D, and thus also not in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on X. $\square$

Corollary 30 *If X is an irresolvable T_3–space with no isolated points such that $\delta(X) = \aleph_0$, then $\pi(X) \geq \mathsf{cov}(\mathcal{M})$.*

Proof : Since $\pi(X^2) = \pi(X)$, Theorems 23 and 29 imply the result. $\square$

We can also use these results to gain more information regarding the phenomenon discussed in Example 4.

Lemma 31 *Let τ be a T_3–topology with no isolated points on ω and let $\mathcal{B}$ be a π-base for τ. If the subspace $X_{\mathcal{B}}$ of 2^ω has property $\mathsf{S}_1(\Omega,\Omega)$, then for each n ONE has no winning strategy in the game $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on $(\omega,\tau)^n$.*

Proof : Fix a natural number n and consider $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on $(\omega,\tau)^n$. Since $X_{\mathcal{B}}$ has property $\mathsf{S}_1(\Omega,\Omega)$ Theorem 26 implies that each finite power of $X_{\mathcal{B}}$ has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.

With σ be a strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on $(\omega,\tau)^n$, define a strategy ρ for ONE in $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ on $X^n_{\mathcal{B}}$ along the lines of the proof of Theorem 28 and then argue as there. $\Box$

Corollary 32 *Let (ω,τ) be a T_3-space with no isolated points such that $\pi(\omega,\tau)<\mathsf{cov}(\mathcal{M})$. For each n ONE has no winning strategy in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$ on $(\omega,\tau)^n$.*

Corollary 33 *If (ω,τ) is an irresolvable T_3–space then for each π–base $\mathcal{B}$, $X_{\mathcal{B}}$ is not an $\mathsf{S}_1(\Omega,\Omega)$–subspace of 2^ω.*

But we may say more about the π–weight of irresolvable T_3–spaces: A subset S of $\mathbb{N}$ is said to *split* the infinite subset T of $\mathbb{N}$ if both $T\setminus S$ and $T\cap S$ are infinite. If a family of infinite subsets of $\mathbb{N}$ is small enough (cardinality-wise), then a single subset of $\mathbb{N}$ can be found which splits each element of that family. No subset of $\mathbb{N}$ splits all infinite subsets of $\mathbb{N}$. Thus, the cardinal number denoted $\mathfrak{r}$, and defined to be the least κ such that there is a family of κ infinite subsets of $\mathbb{N}$ for which no single subset of $\mathbb{N}$ splits all κ members of the family, exists. It is known that $\mathsf{cov}(\mathcal{M})\leq\mathfrak{r}\leq 2^{\aleph_0}$, and no equality is provable.

Proposition 34 *If X is an irresolvable T_3–space with no isolated points such that $\delta(X)=\aleph_0$, then $\pi(X)\geq\mathfrak{r}$.*

Proof : Let Y be a countable dense subset of X and let $\kappa<\mathfrak{r}$ be a cardinal number. For each $\alpha<\kappa$ let U_α be a nonempty open set, and define $D_\alpha=Y\cap U_\alpha$. Then $\{D_\alpha:\alpha<\kappa\}$ is a family of κ infinite subsets of the countable set Y. Choose a set $Z\subseteq Y$ which splits each D_α. Then both Z and $Y\setminus Z$ meets each U_α. Since X is irresolvable, so is Y. This means that $\{U_\alpha:\alpha<\kappa\}$ is not a π–base of X. $\Box$

2 $\mathsf{S}_{fin}(\mathfrak{D},\mathfrak{D})$ and $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$.

This section is mainly a summary of results which can be proved using minor variations of the methods used so far.

If TWO has a winning strategy in $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X, then X satisfies $\mathsf{Split}(\mathfrak{D},\mathfrak{D})$. Thus, for X irresolvable, TWO has no winning strategy in $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X. Since the existence of a winning strategy for ONE in $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ implies the existence of a winning strategy for ONE in $\mathsf{G}_1(\mathfrak{D},\mathfrak{D})$, Example 4 gives (under MA(σ-centered)) a countable T_3–space X such that neither player has a winning strategy in $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X.

For function space $\mathsf{C}_p(X)$ one can prove:

Theorem 35 *For X a separable metric space the following are equivalent:*

1. *X has property $\mathsf{S}_{fin}(\Omega,\Omega)$.*

2. *ONE has no winning strategy in* $\mathsf{G}_{fin}(\Omega,\Omega)$ *on* X.

3. $\mathsf{C}_p(X)$ *has property* $\mathsf{S}_{fin}(\Omega_{\underline{\mathbf{o}}},\Omega_{\underline{\mathbf{o}}})$.

4. *ONE has no winning strategy in* $\mathsf{G}_{fin}(\Omega_{\underline{\mathbf{o}}},\Omega_{\underline{\mathbf{o}}})$ *on* $\mathsf{C}_p(X)$.

5. *ONE does not have a winning strategy in* $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ *on* $\mathsf{C}_p(X)$.

6. $\mathsf{C}_p(X)$ *has property* $\mathsf{S}_{fin}(\mathfrak{D},\mathfrak{D})$.

The Tychonoff space X of Example 2 is a space which satisfies $\mathsf{S}_1(\Omega,\Omega)$ and thus $\mathsf{S}_{fin}(\Omega,\Omega)$, but $\mathsf{C}_p(X)$ has no countable dense subset. This shows that in Theorem 35 we need to assume more than simply that X is a Tychonoff space.

Let ${}^{\omega}\omega$ denote the set of functions from ω to ω. For f and g in ${}^{\omega}\omega$ we say that g *eventually dominates* f, and we write $f \prec g$, if $\{n : g(n) \leq f(n)\}$ is finite. Let $\mathfrak{d}$ be the least cardinal number such that there is a subset $\mathcal{F}$ of ${}^{\omega}\omega$ of that cardinality with each element of ${}^{\omega}\omega$ eventually dominated by some element of $\mathcal{F}$. The cardinal number $\mathfrak{d}$ is also well-studied; it is for example well–known that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$, and that equality is not provable.

Hurewicz proved in [10] that a property introduced by Menger in 1925 is equivalent to the covering property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. In Theorem 10 of [10] Hurewicz proved:

Theorem 36 (Hurewicz) *A space* X *has property* $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ *if, and only if, ONE has no winning strategy in the game* $\mathsf{G}_{fin}(\mathcal{O},\mathcal{O})$.

By another theorem of Hurewicz proved in [11]:

Theorem 37 (Hurewicz) *The minimal cardinality of a separable metric space which does not have property* $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ *is* $\mathfrak{d}$.

Theorem 38 ([13], Theorem 4.6) $\mathfrak{d}$ *is the least cardinality of a set of real numbers which does not have property* $\mathsf{S}_{fin}(\Omega,\Omega)$.

Let σ be a strategy for player ONE in $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X. Since $\delta(X) = \aleph_0$ we may assume that σ calls on ONE to play countable sets. Define an array of points of X as follows:

Enumerate $\sigma(\emptyset)$ bijectively as $(d_{(n)} : n < \omega)$. For each finite subset F of ω, enumerate $\sigma(\{d_{(j)} : j \in F\})$ bijectively as $(d_{(F,m)} : m < \omega)$. For each pair F_1, F_2 of finite subsets of ω, enumerate $\sigma(\{d_{(j)} : j \in F_1\}, \{d_{(F_1,i)} : i \in F_2\})$ bijectively as $(d_{(F_1,F_2,m)} : m < \omega)$, and so on. In general, with $(F_1,\dots,F_k)$ a specified sequence of finite subsets of ω, enumerate $\sigma(\{d_{(j)} : j \in F_1\}, \{d_{(F_1,j)} : j \in F_2\}, \dots, \{d_{(F_1,\dots,F_{k-1},j)} : j \in F_k\})$ bijectively as $(d_{(F_1,\dots,F_k,m)} : m < \omega)$.

Let $\mathcal{B}$ be a π-base of X of cardinality $\pi(X)$. For $B \in \mathcal{B}$ define

$$\Psi(B) = \{(F_1,\dots,F_k) : \sigma(\{d_j : j \in F_1\}, \dots, \{d_{(F_1,\dots,F_{k-1},j)} : j \in F_k\}) \cap B \neq \emptyset\}.$$

Then $\Psi(B)$ is a countably infinite subset of ${}^{<\omega}([\omega]^{<\aleph_0})$, a countable set. If we endow the set $2 := \{0,1\}$ with the discrete topology, then ${}^{{}^{<\omega}([\omega]^{<\aleph_0})}2$ is homeomorphic to the Cantor space. Let $X_{\mathcal{B}}$ be the set of characteristic functions of the family $\{\Psi(B) : B \in \mathcal{B}\}$. Then $X_{\mathcal{B}}$ is a subspace of our version of the Cantor space.

Theorem 39 *If $X_{\mathcal{B}}$ has property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, then ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X.*

Proof : The proof is similar to that of Theorem 28, except that in place of Pawlikowski's theorem we use Hurewicz's theorem. $\square$

Theorem 40 *For an infinite cardinal number κ the following are equivalent:*

1. $\kappa < \mathfrak{d}$.
2. *For each T_3–space X such that $\delta(X) = \aleph_0$ and $\pi(X) = \kappa$, ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X.*

Proof : $1 \Rightarrow 2$: Let X be a T_3–space such that $\delta(X) = \aleph_0$ and $\pi(X) = \kappa$. Let $\mathcal{B}$ be a π–base of X of cardinality κ. Then $X_{\mathcal{B}}$ from our previous construction has cardinality at most κ. Since $\kappa < \mathfrak{d}$ Theorem 37 implies that $X_{\mathcal{B}}$ satisfies $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. Then Theorem 39 implies that ONE has no winning strategy in the game $\mathsf{G}_{fin}(\mathfrak{D},\mathfrak{D})$ on X.
$2 \Rightarrow 1$: Put Theorems 38 and 35 together as in the proof of Theorem 29. $\square$

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