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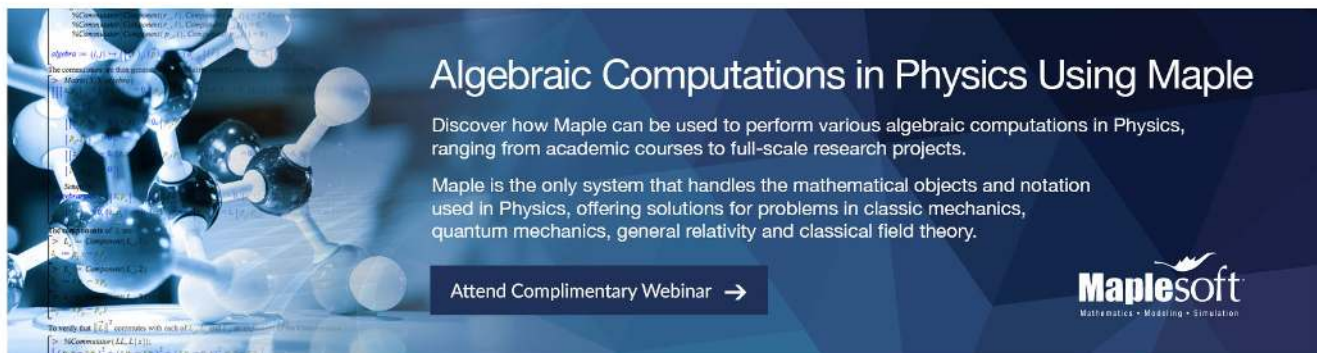
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Combined state-adding and state-deleting approaches to type III multi-step rationally extended potentials: Applications to ladder operators and superintegrability

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Type III multi-step rationally extended harmonic oscillator and radial harmonic oscillator potentials, characterized by a set of k integers m_1, m_2, \dots, m_k , such that $m_1 < m_2 < \dots < m_k$ with m_i even (resp. odd) for i odd (resp. even), are considered. The state-adding and state-deleting approaches to these potentials in a supersymmetric quantum mechanical framework are combined to construct new ladder operators. The eigenstates of the Hamiltonians are shown to separate into $m_k + 1$ infinite-dimensional unitary irreducible representations of the corresponding polynomial Heisenberg algebras. These ladder operators are then used to build a higher-order integral of motion for seven new infinite families of superintegrable two-dimensional systems separable in cartesian coordinates. The finite-dimensional unitary irreducible representations of the polynomial algebras of such systems are directly determined from the ladder operator action on the constituent one-dimensional Hamiltonian eigenstates and provide an algebraic derivation of the superintegrable systems whole spectrum including the level total degeneracies. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4901006>]

I. INTRODUCTION

In one-dimensional quantum mechanics, there has been a continuing interest in constructing exactly solvable potentials. For such a purpose, supersymmetric quantum mechanics (SUSYQM)¹ (combined with the concept of (translationally) shape invariant potentials (SIP)²) has provided a very powerful tool consistent with the Schrödinger factorization method^{3,4} and the Darboux intertwining one.⁵ On using first-order intertwining operators expressed in terms of some nodeless seed solution of a SIP Schrödinger equation, one can indeed build a partner potential, whose spectrum may differ by at most one level from that of the initial one. The ground-state energy is removed if the seed solution is the ground-state wavefunction of the latter (state-deleting case). If the seed solution has an energy below the ground state, it gives rise to an extra bound state below the initial spectrum (state-adding case) or it leads to the same bound-state spectrum (isospectral case) according to whether its inverse is normalizable or not.

More flexibility may be achieved by resorting to n th-order intertwining operators with $n > 1$, constructed from n seed solutions of the initial Hamiltonian Schrödinger equation.^{6–14} On the mathematical side, this corresponds to the Crum extension¹⁵ of the Darboux method. Crum¹⁵ treated more specifically the example of seed solutions that are consecutive wavefunctions starting from the ground-state one, which is the simplest case considered in higher-order SUSYQM. Later on, Krein¹⁶

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and Adler¹⁷ independently generalized the Crum¹⁵ results to chains of bound-state wavefunctions that may be lacunary with even gaps. This modification enriched the possibilities offered by higher-order SUSYQM at its very beginning,^{9,10} but along the years the latter has continued developing itself as a very efficient tool for spectral design, i.e., for building quantum systems with desired features (for a recent review see Ref. 18).

On the other hand, during the last few years, a lot of research activity has been devoted to the construction of exceptional orthogonal polynomials (EOP),^{19–48} which are new complete and orthogonal polynomial systems X_m extending the classical families of Hermite, Laguerre, and Jacobi.⁴⁹ In contrast with the latter, the former admit some gaps in the sequence of their degrees, the total number of them being referred to as the codimension m .

The first examples of EOP, the so-called Laguerre and Jacobi X_1 families, were proposed in the context of Sturm-Liouville theory,^{19,20} but it soon appeared that, apart from some gauge factor, they constituted the main part of the bound-state wavefunctions of some rationally extended SIP, which were themselves shape invariant.²⁸ From that time onwards, it has proved useful to adopt a SUSYQM or Darboux approach to the problem.^{29,30} EOP of arbitrarily large codimension m have then been constructed.^{36,37} A further extension has been the appearance of multi-indexed families X_{m_1, m_2, \dots, m_k} ^{22,39} in the context of higher-order SUSYQM or its variants. All of these EOP turn out to be associated with some solvable rational extensions of well-known SIP.

From the very beginning,³⁰ three different classes of X_m EOP have been identified. Later on, they were shown to be connected with discrete symmetries of the starting SIP.⁴² The rationally extended SIP corresponding to type I or type II EOP turn out to be strictly isospectral to the starting potential and to be shape invariant as the latter. In contrast, those associated with type III EOP (which are the only ones in the case of Hermite EOP) have an extra bound state below the starting potential spectrum and are not shape invariant. In the multi-indexed case, combining these three types may therefore in general lead to a rich variety of solvable rationally extended potentials. In some recent works,^{26,40} special attention has been paid to higher-order (pure) type III extensions. It has been shown, in particular, that, up to some energy shift, adding k new states below a SIP energy spectrum is equivalent to deleting some (appropriately chosen) excited states, thereby generalizing previous observations.^{50,51}

Going from one to two dimensions, some superintegrable systems related to EOP have been recently considered.^{52–54} Let us recall that a two-dimensional superintegrable quantum Hamiltonian system H is characterized by the existence of two integrals of motion X and Y , which are well-defined operators and form with H an algebraically independent set (for a recent review see Ref. 55). The best known of them are the quadratically superintegrable systems, i.e., those allowing two (at most) second-order integrals of motion. Their study began in the mid 1960s⁵⁶ and by now they have been completely classified in conformally flat spaces.^{57–66} In the higher-order case, the direct approach for determining integrals of motion becomes more and more difficult as their order increases, as it has recently been shown for third^{67–70} and quartic order.⁷¹ For such a reason, some other approaches, based on ladder operators,⁷² recurrence relations,^{73,74} or SUSYQM,^{75,76} have been proposed. As superintegrable systems related to EOP belong to such a higher-order case, the recurrence relation and the ladder operator methods have been applied in Ref. 52 and in Refs. 53 and 54, respectively.

In Ref. 53, some two-dimensional systems connected with type III Hermite EOP, as well as to type I, II, or III Laguerre EOP, were analyzed. The results proved entirely satisfactory for type I or II, but for type III it was not possible to derive the whole energy spectrum from the representations of the polynomial algebra generated by the integrals of motion. For the constituent one-dimensional Hamiltonians related to EOP, this study used ladder operators obtained by combining those for the partner SIP with the supercharges of the SUSYQM approach. Such a type of ladder operators is a well-known one and the polynomial Heisenberg algebras (PHA) they close with the Hamiltonian have been the topic of several studies.^{14,77,78}

With the aim of providing an adequate approach to superintegrable systems connected with type III EOP, some novel ladder operators were introduced in Ref. 54 for the case of a rationally extended harmonic oscillator derived in first-order SUSYQM and related to type III Hermite EOP.

The purpose of the present paper is twofold: first to propose a generalization of the latter ladder operators for type III multi-step rationally extended harmonic oscillator or radial harmonic

oscillator potentials, based on the (essential) equivalence of the state-adding and state-deleting approaches referred to above,^{26,40} and second to use them for constructing new superintegrable systems, extending some of those considered in Ref. 53, as well as for providing an algebraic derivation of their spectrum.

In Sec. II, the state-adding and state-deleting approaches to type III multi-step rationally extended harmonic oscillator and radial harmonic oscillator are briefly reviewed. In Sec. III, they are combined to construct new ladder operators for such extensions. The PHA generated by these ladder operators and their unitary irreducible representations (unirreps) are also determined. In Sec. IV, from the one-dimensional systems considered in Secs. II and III, some new two-dimensional superintegrable systems are built and the polynomial algebras generated by their integrals of motion are obtained. The finite-dimensional unirreps of the latter are also determined from the known action of the ladder operators on the eigenstates of the constituent Hamiltonians, thereby providing an algebraic derivation of the superintegrable system spectrum. Finally, Sec. V contains the conclusion.

II. STATE ADDING VERSUS STATE DELETING FOR TYPE III MULTI-STEP RATIONALLY EXTENDED POTENTIALS

In n th-order SUSYQM,⁶⁻¹⁴ the two partner Hamiltonians

$$H^{(i)} = -\frac{d^2}{dx^2} + V^{(i)}(x), \quad i = 1, 2, \quad (2.1)$$

intertwine with two n th-order differential operators \mathcal{A} and \mathcal{A}^\dagger as

$$\mathcal{A}H^{(1)} = H^{(2)}\mathcal{A}, \quad \mathcal{A}^\dagger H^{(2)} = H^{(1)}\mathcal{A}^\dagger. \quad (2.2)$$

We consider here the case where the latter can be factorized as

$$\mathcal{A} = A^{(n)} \dots A^{(2)}A^{(1)}, \quad \mathcal{A}^\dagger = A^{(1)\dagger}A^{(2)\dagger} \dots A^{(n)\dagger}, \quad (2.3)$$

into products of n first-order differential operators

$$A^{(i)} = \frac{d}{dx} + W^{(i)}(x), \quad A^{(i)\dagger} = -\frac{d}{dx} + W^{(i)}(x), \quad i = 1, 2, \dots, n, \quad (2.4)$$

with real superpotentials $W^{(i)}(x)$, $i = 1, 2, \dots, n$. Such superpotentials, which are assumed to be obtained from n different seed solutions $\varphi_i(x)$, $i = 1, 2, \dots, n$, of the Schrödinger equation associated with $H^{(1)}$, can be written as^{14,15}

$$W^{(i)}(x) = -\frac{d}{dx} \log \varphi^{(i)}(x), \quad i = 1, 2, \dots, n, \quad (2.5)$$

with

$$\varphi^{(1)}(x) = \varphi_1(x), \quad \varphi^{(i)}(x) = \frac{\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_i)}{\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_{i-1})}, \quad i = 2, 3, \dots, n. \quad (2.6)$$

Here, $\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_i)$ denotes the Wronskian of $\varphi_1(x), \varphi_2(x), \dots, \varphi_i(x)$.⁷⁹ Since the potentials of the two partner Hamiltonians are linked by the relationship^{14,15}

$$V^{(2)}(x) = V^{(1)}(x) - 2 \frac{d^2}{dx^2} \log \mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_n), \quad (2.7)$$

a regular potential $V^{(2)}(x)$ is obtained provided the Wronskian $\mathcal{W}(\varphi_1, \varphi_2, \dots, \varphi_n)$ does not vanish on the defining interval of $V^{(1)}(x)$.

A. Harmonic oscillator case

If $V^{(1)}(x)$ is the harmonic oscillator potential

$$V^{(1)}(x) = x^2, \quad -\infty < x < \infty, \quad (2.8)$$

then the starting Hamiltonian $H^{(1)}$ has an infinite number of bound-state wavefunctions

$$\psi_\nu^{(1)}(x) \propto \psi_\nu(x) = H_\nu(x)e^{-\frac{1}{2}x^2}, \quad \nu = 0, 1, 2, \dots, \quad (2.9)$$

with corresponding energies

$$E_\nu^{(1)} = 2\nu + 1, \quad \nu = 0, 1, 2, \dots \quad (2.10)$$

Here, $H_\nu(x)$ denotes a ν th-degree Hermite polynomial.⁴⁹

This Hamiltonian also possesses raising and lowering operators a^\dagger and a , which are first-order differential operators given by

$$a = \frac{d}{dx} + x, \quad a^\dagger = -\frac{d}{dx} + x. \quad (2.11)$$

They satisfy the Heisenberg algebra of the form

$$\begin{aligned} [H^{(1)}, a^\dagger] &= 2a^\dagger, & [H^{(1)}, a] &= -2a, \\ [a, a^\dagger] &= Q(H^{(1)} + 2) - Q(H^{(1)}), \end{aligned} \quad (2.12)$$

with

$$Q(H^{(1)}) = H^{(1)} - 1. \quad (2.13)$$

The action of the lowering operator on the wavefunctions is given by

$$\begin{aligned} a\psi_\nu^{(1)} &= 0, & \nu &= 0, \\ a\psi_\nu^{(1)} &= [\nu]^{1/2}\psi_{\nu-1}^{(1)}, & \nu &= 1, 2, \dots \end{aligned} \quad (2.14)$$

In the state-adding case, let us choose $n = k$ seed functions among the polynomial-type eigenfunctions $\phi_m(x)$ of $H^{(1)}$ below the ground-state energy $E_0^{(1)}$. These eigenfunctions, associated with the eigenvalues $E_m = -2m - 1$, can be written as^{9,27,35}

$$\phi_m(x) = \mathcal{H}_m(x)e^{\frac{1}{2}x^2}, \quad m = 0, 1, 2, \dots, \quad (2.15)$$

where $\mathcal{H}_m(x) = (-i)^m H_m(ix)$ is a m th-degree pseudo-Hermite polynomial (called twisted Hermite polynomial in the mathematical literature⁸⁰). For even m values, the eigenfunctions $\phi_m(x)$ are nodeless on the whole real line, while for odd m ones, they have a single zero at $x = 0$.

With the choice $(\varphi_1, \varphi_2, \dots, \varphi_n) \rightarrow (\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k})$, the partner potential (2.7) of (2.8) turns out to be nonsingular if $m_1 < m_2 < \dots < m_k$ with m_i even (resp. odd) for i odd (resp. even).^{9,10,14} On using Eq. (2.15) and standard properties of Wronskians,⁷⁹ we can write $\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k}) = \exp(\frac{1}{2}kx^2)\mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \dots, \mathcal{H}_{m_k})$, so that $V^{(2)}(x)$ becomes

$$V^{(2)}(x) = x^2 - 2k - 2\frac{d^2}{dx^2} \log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \dots, \mathcal{H}_{m_k}). \quad (2.16)$$

Its spectrum is given by

$$E_\nu^{(2)} = 2\nu + 1, \quad \nu = -m_k - 1, \dots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots, \quad (2.17)$$

and the corresponding wavefunctions are¹⁴

$$\begin{aligned} \psi_\nu^{(2)}(x) &\propto \frac{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k}, \psi_\nu)}{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k})}, & \nu &= 0, 1, 2, \dots, \\ \psi_{-m_i-1}^{(2)}(x) &\propto \frac{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \check{\phi}_{m_i}, \dots, \phi_{m_k})}{\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \phi_{m_k})}, & i &= 1, 2, \dots, k. \end{aligned} \quad (2.18)$$

Here, $\mathcal{W}(\phi_{m_1}, \phi_{m_2}, \dots, \check{\phi}_{m_i}, \dots, \phi_{m_k})$ means that $\phi_{m_i}(x)$ is excluded from the Wronskian for $k > 1$, while the latter reduces to one for $i = k = 1$.

To get (essentially) equivalent results for the two partner potentials in the state-deleting case,^{26,40} we have to take (at least) $n = m_k + 1 - k$ bound-state wavefunctions of $H^{(1)}$ as seed functions: $(\varphi_1, \varphi_2, \dots, \varphi_n) \rightarrow (\psi_1, \psi_2, \dots, \check{\psi}_{m_k-m_{k-1}}, \dots, \check{\psi}_{m_k-m_2}, \dots, \check{\psi}_{m_k-m_1}, \dots, \psi_{m_k})$.⁸¹ The latter

will then be suppressed from the spectrum. To distinguish this case from the previous one, let us put a bar above all corresponding quantities. Hence, the two partner Hamiltonians are now denoted by $\bar{H}^{(i)}$, $i = 1, 2$, and the associated potentials are

$$\bar{V}^{(1)}(x) = V^{(1)}(x) = x^2 \tag{2.19}$$

and $\bar{V}^{(2)}(x)$ given by Eq. (2.7) with the appropriate choice for $(\varphi_1, \varphi_2, \dots, \varphi_n)$. Equation (2.9), combined with standard properties of Wronskians, leads to $\mathcal{W}(\check{\psi}_1, \check{\psi}_2, \dots, \check{\psi}_{m_k-m_{k-1}}, \dots, \check{\psi}_{m_k-m_1}, \dots, \check{\psi}_{m_k}) = \exp[-\frac{1}{2}(m_k + 1 - k)x^2] \mathcal{W}(H_1, H_2, \dots, \check{H}_{m_k-m_{k-1}}, \dots, \check{H}_{m_k-m_1}, \dots, H_{m_k})$, thence

$$\bar{V}^{(2)}(x) = x^2 + 2(m_k + 1 - k) - 2 \frac{d^2}{dx^2} \log \mathcal{W}(H_1, H_2, \dots, \check{H}_{m_k-m_{k-1}}, \dots, \check{H}_{m_k-m_1}, \dots, H_{m_k}). \tag{2.20}$$

This potential is nonsingular provided the gaps between the surviving levels with $\nu = 0, m_k - m_{k-1}, \dots, m_k - m_2, m_k - m_1, m_k + 1, m_k + 2, \dots$, correspond to even numbers of consecutive levels.^{16,17} This leads to the same conditions on m_1, m_2, \dots, m_k as in the state-adding case. The resulting spectra are now

$$\begin{aligned} \bar{E}_\nu^{(1)} &= 2\nu + 1, & \nu &= 0, 1, 2, \dots, \\ \bar{E}_\nu^{(2)} &= 2m_k + 2\nu + 3, & \nu &= -m_k - 1, \dots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots, \end{aligned} \tag{2.21}$$

with corresponding wavefunctions $\bar{\psi}_\nu^{(1)}(x) = \psi_\nu^{(1)}(x)$, $\nu = 0, 1, 2, \dots$, and

$$\begin{aligned} \bar{\psi}_\nu^{(2)}(x) &\propto \frac{\mathcal{W}(\psi_1, \psi_2, \dots, \check{\psi}_{m_k-m_{k-1}}, \dots, \check{\psi}_{m_k-m_1}, \dots, \psi_{m_k}, \psi_{m_k+1+\nu})}{\mathcal{W}(\psi_1, \psi_2, \dots, \check{\psi}_{m_k-m_{k-1}}, \dots, \check{\psi}_{m_k-m_1}, \dots, \psi_{m_k})}, \\ \nu &= 0, 1, 2, \dots, \\ \bar{\psi}_{-m_i-1}^{(2)}(x) &\propto \frac{\mathcal{W}(\psi_1, \psi_2, \dots, \check{\psi}_{m_k-m_{k-1}}, \dots, \psi_{m_k-m_i}, \dots, \check{\psi}_{m_k-m_1}, \dots, \psi_{m_k})}{\mathcal{W}(\psi_1, \psi_2, \dots, \check{\psi}_{m_k-m_{k-1}}, \dots, \check{\psi}_{m_k-m_1}, \dots, \psi_{m_k})}, \\ i &= 1, 2, \dots, k. \end{aligned} \tag{2.22}$$

Since it has been proved that the two Wronskians in Eqs. (2.16) and (2.20) only differ by some multiplicative constant,⁴⁰ the two obtained potentials $V^{(2)}(x)$ and $\bar{V}^{(2)}(x)$ are the same up to some additive constant

$$V^{(2)}(x) + 2m_k + 2 = \bar{V}^{(2)}(x). \tag{2.23}$$

This result agrees with the respective energies, given in Eqs. (2.17) and (2.21), and also implies that $\bar{\psi}_\nu^{(2)}(x) \propto \psi_\nu^{(2)}(x)$, $\nu = -m_k - 1, \dots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots$.

B. Radial harmonic oscillator case

The radial harmonic oscillator case is more complicated than that of the linear one because the potential $V_l(x) = \frac{1}{4}x^2 + \frac{l(l+1)}{x^2}$, $0 < x < \infty$, now depends on a parameter l (the angular momentum quantum number), which is changed in SUSYQM transformations.

So, in the state-adding case, to get a partner potential that is a rational extension of $V_l(x)$, we have to start from

$$\begin{aligned} V^{(1)}(x) = V_{l+k}(x) &= \frac{1}{4}x^2 + \frac{(l+k)(l+k+1)}{x^2}, & 0 < x < \infty, \\ &= \frac{1}{2}z + \frac{(2\alpha + 2k - 1)(2\alpha + 2k + 1)}{8z}, & 0 < z < \infty, \end{aligned} \tag{2.24}$$

with $z \equiv \frac{1}{2}x^2$, $\alpha \equiv l + \frac{1}{2}$. The bound-state energies and wavefunctions of the starting Hamiltonian $H^{(1)}$ are therefore

$$E_{l+k,\nu}^{(1)} = 2\nu + \alpha + k + 1, \quad \nu = 0, 1, 2, \dots, \tag{2.25}$$

and

$$\begin{aligned} \psi_{l+k,v}^{(1)}(x) &\propto \psi_v^{(l+k)}(x) = \eta_{l+k}(z) L_v^{(\alpha+k)}(z) \\ &\propto x^{l+k+1} e^{-\frac{1}{4}x^2} L_v^{(l+k+\frac{1}{2})}(\frac{1}{2}x^2), \quad v = 0, 1, 2, \dots, \end{aligned} \quad (2.26)$$

with

$$\eta_l(z) = z^{\frac{1}{4}(2\alpha+1)} e^{-\frac{1}{2}z}. \quad (2.27)$$

Here, $L_v^{(\alpha)}(z)$ denotes a v th-degree Laguerre polynomial.⁴⁹ The $n = k$ seed functions are chosen among the polynomial-type eigenfunctions of $H^{(1)}$,

$$\begin{aligned} \phi_m^{(l+k)}(x) &= (\eta_{l+k-1}(z))^{-1} L_m^{(-\alpha-k)}(-z) \propto x^{-l-k} e^{\frac{1}{4}x^2} L_m^{(-l-k-\frac{1}{2})}(-\frac{1}{2}x^2), \\ m &= 0, 1, 2, \dots, \end{aligned} \quad (2.28)$$

at an energy $E_m^{(l+k)} = \alpha + k - 2m - 1$ below the ground-state energy $E_{l+k,0}^{(1)}$.^{30,40,42}

This Hamiltonian also possesses raising and lowering operators a^\dagger and a , which are second-order differential operators given by

$$\begin{aligned} a &= \frac{1}{4} \left(2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \frac{1}{2}x^2 - \frac{2(l+k)(l+k+1)}{x^2} + 1 \right), \\ a^\dagger &= \frac{1}{4} \left(2 \frac{d^2}{dx^2} - 2x \frac{d}{dx} + \frac{1}{2}x^2 - \frac{2(l+k)(l+k+1)}{x^2} - 1 \right), \end{aligned} \quad (2.29)$$

and satisfy a polynomial Heisenberg algebra of the form

$$\begin{aligned} [H^{(1)}, a^\dagger] &= 2a^\dagger, \quad [H^{(1)}, a] = -2a, \\ [a, a^\dagger] &= Q(H^{(1)} + 2) - Q(H^{(1)}), \end{aligned} \quad (2.30)$$

with

$$Q(H^{(1)}) = \frac{1}{16} (2H^{(1)} - 3 - 2l - 2k) (2H^{(1)} - 1 + 2l + 2k). \quad (2.31)$$

The action of the lowering operator on the wavefunctions is given by

$$\begin{aligned} a \psi_{l+k,v}^{(1)} &= 0, \quad v = 0, \\ a \psi_{l+k,v}^{(1)} &= [v(v+l+k+\frac{1}{2})]^{1/2} \psi_{l+k,v-1}^{(1)}, \quad v = 1, 2, \dots \end{aligned} \quad (2.32)$$

With the choice $(\varphi_1, \varphi_2, \dots, \varphi_n) \rightarrow (\phi_{m_1}^{(l+k)}, \phi_{m_2}^{(l+k)}, \dots, \phi_{m_k}^{(l+k)})$, the partner potential (2.7) of (2.24) is nonsingular if $m_1 < m_2 < \dots < m_k$ with m_i even (resp. odd) for i odd (resp. even) and if in addition $\alpha + k > m_k$. By proceeding as in Subsection II A, it can be rewritten as

$$V^{(2)}(x) = V_l(x) - k - 2 \frac{d^2}{dx^2} \log \tilde{\mathcal{W}}(L_{m_1}^{(-\alpha-k)}(-z), L_{m_2}^{(-\alpha-k)}(-z), \dots, L_{m_k}^{(-\alpha-k)}(-z)), \quad (2.33)$$

where $\tilde{\mathcal{W}}(f_1(z), f_2(z), \dots, f_k(z))$ denotes the Wronskian of the functions $f_1(z), f_2(z), \dots, f_k(z)$ with respect to z . Its bound-state energies and corresponding wavefunctions are given by

$$E_{l,v}^{(2)} = 2v + \alpha + k + 1, \quad v = -m_k - 1, \dots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots, \quad (2.34)$$

and

$$\begin{aligned} \psi_{l,v}^{(2)}(x) &\propto \frac{\mathcal{W}(\phi_{m_1}^{(l+k)}, \phi_{m_2}^{(l+k)}, \dots, \phi_{m_k}^{(l+k)}, \psi_v^{(l+k)})}{\mathcal{W}(\phi_{m_1}^{(l+k)}, \phi_{m_2}^{(l+k)}, \dots, \phi_{m_k}^{(l+k)}), \quad v = 0, 1, 2, \dots, \\ \psi_{l,-m_i-1}^{(2)}(x) &\propto \frac{\mathcal{W}(\phi_{m_1}^{(l+k)}, \phi_{m_2}^{(l+k)}, \dots, \check{\phi}_{m_i}^{(l+k)}, \dots, \phi_{m_k}^{(l+k)})}{\mathcal{W}(\phi_{m_1}^{(l+k)}, \phi_{m_2}^{(l+k)}, \dots, \phi_{m_k}^{(l+k)})}, \quad i = 1, 2, \dots, k, \end{aligned} \quad (2.35)$$

respectively.

To obtain an (essentially) equivalent result for the partner potential derived in the state-deleting approach, we have to start this time from a potential $\bar{V}^{(1)}(x)$ that differs from $V^{(1)}(x)$, considered in (2.24), namely,

$$\bar{V}^{(1)}(x) = V_{l+k-m_k-1}(x), \tag{2.36}$$

where we assume $\alpha + k > m_k + 1$. Then, with the choice $n = m_k + 1 - k$ and $(\varphi_1, \varphi_2, \dots, \varphi_n) \rightarrow (\psi_1^{(l+k-m_k-1)}, \psi_2^{(l+k-m_k-1)}, \dots, \check{\psi}_{m_k-m_{k-1}}^{(l+k-m_k-1)}, \dots, \check{\psi}_{m_k-m_2}^{(l+k-m_k-1)}, \dots, \check{\psi}_{m_k-m_1}^{(l+k-m_k-1)}, \dots, \psi_{m_k}^{(l+k-m_k-1)})$ in (2.7), we get

$$\begin{aligned} \bar{V}^{(2)}(x) = V_l(x) + m_k + 1 - k - 2 \frac{d^2}{dx^2} \log \tilde{W} & (L_1^{(\alpha+k-m_k-1)}(z), L_2^{(\alpha+k-m_k-1)}(z), \dots, \\ & \check{L}_{m_k-m_{k-1}}^{(\alpha+k-m_k-1)}(z), \dots, \check{L}_{m_k-m_1}^{(\alpha+k-m_k-1)}(z), \dots, L_{m_k}^{(\alpha+k-m_k-1)}(z)), \end{aligned} \tag{2.37}$$

which can be shown⁴⁰ to satisfy the relation

$$V^{(2)}(x) + m_k + 1 = \bar{V}^{(2)}(x) \tag{2.38}$$

with the potential $V^{(2)}(x)$ given in (2.33). In (2.38), nonsingular potentials correspond to $\alpha + k > m_k + 1$ and $m_1 < m_2 < \dots < m_k$ with m_i even (resp. odd) for i odd (resp. even). In the case of (2.36) and (2.37), the bound-state energies and wavefunctions can be written as

$$\begin{aligned} \bar{E}_{l+k-m_k-1, \nu}^{(1)} &= 2\nu + \alpha + k - m_k, \quad \nu = 0, 1, 2, \dots, \\ \bar{E}_{l, \nu}^{(2)} &= 2\nu + \alpha + k + m_k + 2, \quad \nu = -m_k - 1, \dots, -m_2 - 1, -m_1 - 1, 0, 1, 2, \dots, \end{aligned} \tag{2.39}$$

and $\bar{\psi}_{l+k-m_k-1, \nu}^{(1)}(x) \propto \psi_{\nu}^{(l+k-m_k-1)}(x)$, as well as $\bar{\psi}_{l, \nu}^{(2)}(x)$ obtained from Eq. (2.22) by changing $\psi_j(x)$ into $\check{\psi}_j^{(l+k-m_k-1)}(x)$.

As above-mentioned, an important difference between the linear and radial oscillator cases is that in the latter $\bar{H}^{(1)}$ and $H^{(1)}$ do not coincide. As for the construction of ladder operators in Sec. III, it will be important to be able to relate these Hamiltonians, we show in Appendix A that this can be done, up to some energy shift, by a $(m_k + 1)$ th-order SUSYQM transformation using seed functions of type II.

III. LADDER OPERATORS FOR TYPE III MULTI-STEP RATIONALLY EXTENDED POTENTIALS

In the present section, we will avail ourselves of the existence of two different paths going from $H^{(1)}$ to $H^{(2)}$ (up to some addition constant) to build some ladder operators for $H^{(2)}$.

A. Harmonic oscillator case

From Eqs. (2.19) and (2.23), and the intertwining relations (2.2) (as well as their counterparts for $\bar{H}^{(1)}$, $\bar{H}^{(2)}$, $\bar{\mathcal{A}}$, and $\bar{\mathcal{A}}^\dagger$), it is clear that one can go from $H^{(2)}$ to $H^{(2)} + 2m_k + 2$ along the following path

$$H^{(2)} \xrightarrow{\mathcal{A}^\dagger} H^{(1)} = \bar{H}^{(1)} \xrightarrow{\bar{\mathcal{A}}} \bar{H}^{(2)} = H^{(2)} + 2m_k + 2 \tag{3.1}$$

\curvearrowright
 c

The $(m_k + 1)$ th-order differential operator

$$c = \bar{\mathcal{A}}\mathcal{A}^\dagger \tag{3.2}$$

that performs such a transformation is therefore a lowering operator for $H^{(2)}$. Together with its Hermitian conjugate, the creation operator

$$c^\dagger = \mathcal{A}\bar{\mathcal{A}}^\dagger, \tag{3.3}$$

and $H^{(2)}$, it satisfies the commutation relations

$$\begin{aligned} [H^{(2)}, c^\dagger] &= (2m_k + 2)c^\dagger, & [H^{(2)}, c] &= -(2m_k + 2)c, \\ [c, c^\dagger] &= Q(H^{(2)} + 2m_k + 2) - Q(H^{(2)}), \end{aligned} \quad (3.4)$$

defining a PHA of m_k th order.^{77,78} In Eq. (3.4), $Q(H^{(2)}) = c^\dagger c$ is indeed a $(m_k + 1)$ th-order polynomial in $H^{(2)}$, which we now plan to determine.

First, let us observe that since \mathcal{A}^\dagger annihilates the k wavefunctions $\psi_{-m_i-1}^{(2)}(x)$, $i = 1, 2, \dots, k$, whose energy is $E_{-m_i-1}^{(2)} = -2m_i - 1$, we can express the $2k$ th-order differential operator $\mathcal{A}\mathcal{A}^\dagger$ as

$$\mathcal{A}\mathcal{A}^\dagger = \prod_{i=1}^k (H^{(2)} + 2m_i + 1). \quad (3.5)$$

From the intertwining relations (2.2), we then get

$$\mathcal{A}^\dagger \mathcal{A} = \prod_{i=1}^k (H^{(1)} + 2m_i + 1). \quad (3.6)$$

Second, let us note that $\bar{\mathcal{A}}$ annihilates the $m_k + 1 - k$ excited states $\psi_j^{(1)}(x)$, $j = 1, 2, \dots, m_k - m_{k-1} - 1, m_k - m_{k-1} + 1, \dots, m_k - m_1 - 1, m_k - m_1 + 1, \dots, m_k$, of $\bar{H}^{(1)} = H^{(1)}$, corresponding to the energies $E_j^{(1)} = 2j + 1$; hence, the $2(m_k + 1 - k)$ th-order differential operator $\bar{\mathcal{A}}^\dagger \bar{\mathcal{A}}$ can be written as

$$\bar{\mathcal{A}}^\dagger \bar{\mathcal{A}} = \prod_{\substack{j=1 \\ j \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (\bar{H}^{(1)} - 2j - 1), \quad (3.7)$$

with a similar expression for $\bar{\mathcal{A}}\bar{\mathcal{A}}^\dagger$ in terms of $\bar{H}^{(2)}$. In terms of $H^{(1)}$ and $H^{(2)}$, this leads to the relations

$$\bar{\mathcal{A}}^\dagger \bar{\mathcal{A}} = \prod_{\substack{j=1 \\ j \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (H^{(1)} - 2j - 1), \quad (3.8)$$

$$\bar{\mathcal{A}}\bar{\mathcal{A}}^\dagger = \prod_{\substack{j=1 \\ j \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (H^{(2)} + 2m_k - 2j + 1). \quad (3.9)$$

It only remains to combine Eqs. (3.5)–(3.9) with the intertwining relations to obtain

$$Q(H^{(2)}) = \left(\prod_{i=1}^k (H^{(2)} + 2m_i + 1) \right) \left(\prod_{\substack{j=1 \\ j \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (H^{(2)} - 2j - 1) \right), \quad (3.10)$$

which completes the definition of the PHA (3.4).

The action of the operator $Q(H^{(2)})$ on the wavefunctions $\psi_v^{(2)}(x)$, $v = -m_k - 1, \dots, -m_1 - 1, 0, 1, 2, \dots$, is readily obtained by replacing $H^{(2)}$ on the right-hand side of (3.10) by the corresponding eigenvalues $E_v^{(2)}$, given in Eq. (2.17). If we choose the normalization constants of the $\psi_v^{(2)}$'s in such a way that all matrix elements of the ladder operators c^\dagger, c are nonnegative, then such

an action yields the following results:

$$c\psi_v^{(2)} = 0, \quad v = -m_k - 1, \dots, -m_1 - 1, 1, 2, \dots, m_k - m_{k-1} - 1, \\ m_k - m_{k-1} + 1, \dots, m_k - m_1 - 1, m_k - m_1 + 1, \dots, m_k, \quad (3.11)$$

$$c\psi_0^{(2)} = \left[2^{m_k+1}(m_k + 1)! \left(\prod_{i=1}^{k-1} \frac{m_i + 1}{m_k - m_i} \right) \right]^{1/2} \psi_{-m_k-1}^{(2)}, \quad (3.12)$$

$$c\psi_{m_k-m_i}^{(2)} = \left[2^{m_k+1}(m_k + 1)(2m_k - m_i + 1)(m_k - m_i - 1)!m_i! \right. \\ \left. \times \left(\prod_{j=1}^{i-1} \frac{m_k + m_j - m_i + 1}{m_i - m_j} \right) \left(\prod_{l=i+1}^{k-1} \frac{m_k + m_l - m_i + 1}{m_l - m_i} \right) \right]^{1/2} \psi_{-m_i-1}^{(2)}, \quad (3.13) \\ i = 1, 2, \dots, k - 1,$$

$$c\psi_v^{(2)} = \left[2^{m_k+1}(v + m_k + 1) \frac{(v - 1)!}{(v - m_k - 1)!} \left(\prod_{i=1}^{k-1} \frac{v + m_i + 1}{v + m_i - m_k} \right) \right]^{1/2} \psi_{v-m_k-1}^{(2)}, \quad (3.14) \\ v = m_k + 1, m_k + 2, \dots$$

The action of c^\dagger on $\psi_v^{(2)}(x)$ is finally deduced from Eqs. (3.12)–(3.14) by using Hermitian conjugation.

We conclude that the PHA generated by $H^{(2)}$, c^\dagger , and c has $m_k + 1$ infinite-dimensional unirreps spanned by the states $\{\psi_{i+(m_k+1)j}^{(2)} \mid j = 0, 1, 2, \dots\}$ with $i = -m_k - 1, \dots, -m_1 - 1, 1, 2, \dots, m_k - m_{k-1} - 1, m_k - m_{k-1} + 1, \dots, m_k - m_1 - 1, m_k - m_1 + 1, \dots, m_k$, respectively.

It is worth observing that the ladder operators c^\dagger , c , constructed in this subsection, extend to arbitrary k values those proposed in Ref. 54 for $k = 1$.

B. Radial harmonic oscillator case

The construction of ladder operators for the Hamiltonian $H^{(2)}$, obtained from Eq. (2.33), is similar to that performed in Subsection III A, except for the need of using an intermediate transformation from $H^{(1)}$ to $\tilde{H}^{(1)}$ by means of the additional intertwining operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^\dagger$, derived in Appendix A. Instead of (3.1), let us therefore consider the following path:

$$H^{(2)} \xrightarrow{\tilde{\mathcal{A}}^\dagger} H^{(1)} \xrightarrow{\tilde{\mathcal{A}}} \tilde{H}^{(2)} = \tilde{H}^{(1)} + m_k + 1 \xrightarrow{\tilde{\mathcal{A}}} \tilde{H}^{(2)} + m_k + 1 = H^{(2)} + 2m_k + 2 \quad (3.15)$$

$\underbrace{\hspace{15em}}_c$

The corresponding lowering operator c and its Hermitian conjugate c^\dagger are now defined by

$$c = \tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger, \quad c^\dagger = \tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger, \quad (3.16)$$

respectively. They are $(2m_k + 2)$ th-order differential operators, closing a PHA of $(2m_k + 1)$ th order with $H^{(2)}$. For such a PHA, Eq. (3.4) remains valid with an expression of $Q(H^{(2)}) = c^\dagger c$ different from (3.10).

To derive the latter, let us observe that an argument similar to that used in Subsection III A leads to the equations

$$\begin{aligned}
 \mathcal{A}\mathcal{A}^\dagger &= \prod_{i=1}^k (H^{(2)} - \alpha - k + 2m_i + 1), \\
 \mathcal{A}^\dagger\mathcal{A} &= \prod_{i=1}^k (H^{(1)} - \alpha - k + 2m_i + 1) = \prod_{i=1}^k (\tilde{H}^{(1)} - \alpha - k + 2m_i + 1), \\
 \tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}} &= \prod_{\substack{n=1 \\ n \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (\tilde{H}^{(1)} - \alpha - 2n + m_k - k) \\
 &= \prod_{\substack{n=1 \\ n \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (\tilde{H}^{(2)} - \alpha - 2n - k - 1), \tag{3.17} \\
 \tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger &= \prod_{\substack{n=1 \\ n \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (\tilde{H}^{(2)} - \alpha - 2n + m_k - k) \\
 &= \prod_{\substack{n=1 \\ n \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (H^{(2)} - \alpha - 2n + 2m_k - k + 1).
 \end{aligned}$$

In the same way, since the operator $\tilde{\mathcal{A}}$ annihilates the nonnormalizable states $\tilde{\phi}_j^{(l+k)}(x), j = 0, 1, \dots, m_k$, of energy $E_{l+k,j}^{\text{II}} = -\alpha - k + 2j + 1$, considered in Appendix A, we can express the $(2m_k + 2)$ th-order differential operator $\tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}}$ as

$$\tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}} = \prod_{j=0}^{m_k} (\tilde{H}^{(1)} + \alpha + k - 2j - 1) = \prod_{j=0}^{m_k} (H^{(1)} + \alpha + k - 2j - 1). \tag{3.18}$$

From the intertwining relations, we then also get

$$\tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger = \prod_{j=0}^{m_k} (\tilde{H}^{(2)} + \alpha + k - 2j - 1) = \prod_{j=0}^{m_k} (\tilde{H}^{(1)} + \alpha + k + m_k - 2j). \tag{3.19}$$

Equations (3.17)–(3.19) finally yield

$$\begin{aligned}
 Q(H^{(2)}) &= \left(\prod_{i=1}^k (H^{(2)} - \alpha + 2m_i - k + 1) \right) \left(\prod_{j=0}^{m_k} (H^{(2)} + \alpha - 2j + k - 1) \right) \\
 &\quad \times \left(\prod_{\substack{n=1 \\ n \neq m_k - m_{k-1}, \dots, m_k - m_1}}^{m_k} (H^{(2)} - \alpha - 2n - k - 1) \right). \tag{3.20}
 \end{aligned}$$

Let us stress that this relation and Eq. (3.4) have to be completed with the condition $\alpha + k > m_k + 1$.

On proceeding as in Subsection III A, from Eqs. (2.34) and (3.20), we can easily derive the following results for the action of c on the wavefunctions $\psi_{l,\nu}^{(2)}(x), \nu = -m_k - 1, \dots, -m_1 - 1$,

0, 1, 2, . . . , of $H^{(2)}$:

$$c\psi_{l,v}^{(2)} = 0, \quad v = -m_k - 1, \dots, -m_1 - 1, 1, 2, \dots, m_k - m_{k-1} - 1, \tag{3.21}$$

$$m_k - m_{k-1} + 1, \dots, m_k - m_1 - 1, m_k - m_1 + 1, \dots, m_k,$$

$$c\psi_{l,0}^{(2)} = 2^{m_k+1} \left[(m_k + 1)! \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + k - m_k)} \left(\prod_{i=1}^{k-1} \frac{m_i + 1}{m_k - m_i} \right) \right]^{1/2} \psi_{l,-m_k-1}^{(2)}, \tag{3.22}$$

$$c\psi_{l,m_k-m_i}^{(2)} = 2^{m_k+1} \left[(m_k + 1)(2m_k - m_i + 1)(m_k - m_i - 1)! m_i! \right. \\ \times \frac{\Gamma(\alpha + m_k - m_i + k + 1)}{\Gamma(\alpha - m_i + k)} \left(\prod_{j=1}^{i-1} \frac{m_k + m_j - m_i + 1}{m_i - m_j} \right) \tag{3.23}$$

$$\left. \times \left(\prod_{n=i+1}^{k-1} \frac{m_k + m_n - m_i + 1}{m_n - m_i} \right) \right]^{1/2} \psi_{l,-m_i-1}^{(2)}, \quad i = 1, 2, \dots, k - 1,$$

$$c\psi_{l,v}^{(2)} = 2^{m_k+1} \left[(v + m_k + 1) \frac{(v - 1)!}{(v - m_k - 1)!} \frac{\Gamma(v + \alpha + k + 1)}{\Gamma(v + \alpha + k - m_k)} \right. \tag{3.24}$$

$$\left. \times \left(\prod_{i=1}^{k-1} \frac{v + m_i + 1}{v + m_i - m_k} \right) \right]^{1/2} \psi_{l,v-m_k-1}^{(2)}, \quad v = m_k + 1, m_k + 2, \dots,$$

and corresponding results for the action of c^\dagger .

When we compare Eqs. (3.21)–(3.24) with the corresponding results for the extended harmonic oscillator, contained in Eqs. (3.11)–(3.14), we note that there are very few changes, namely, all matrix elements of c given in (3.12)–(3.14) are multiplied by a generic factor $[2^{m_k+1}\Gamma(v + \alpha + k + 1)/\Gamma(v + \alpha + k - m_k)]^{1/2}$, which is nonvanishing for any $v \in \{0, m_k - m_{k-1}, \dots, m_k - m_1, m_k + 1, m_k + 2, \dots\}$. Hence, as before, the present PHA has $m_k + 1$ infinite-dimensional unirreps, spanned this time by the states $\{\psi_{l,i+(m_k+1)j}^{(2)} \mid j = 0, 1, 2, \dots\}$ with $i = -m_k - 1, \dots, -m_1 - 1, 1, 2, \dots, m_k - m_{k-1} - 1, m_k - m_{k-1} + 1, \dots, m_k - m_1 - 1, m_k - m_1 + 1, \dots, m_k$, respectively.

The energy levels of the three pairs of partner Hamiltonians $(H^{(1)}, H^{(2)})$, $(\tilde{H}^{(1)}, \tilde{H}^{(2)})$, and $(\bar{H}^{(1)}, \bar{H}^{(2)})$, as well as the action of the corresponding intertwining operators, are displayed in Fig. 1

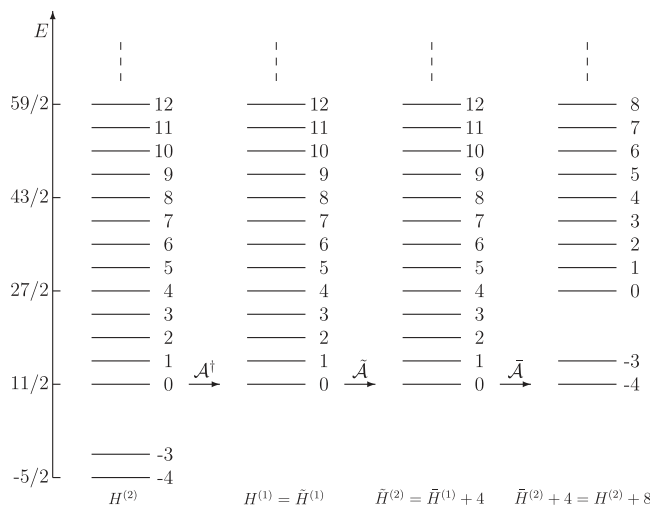


FIG. 1. Energy spectra of $H^{(2)}$, $H^{(1)}$, $\tilde{H}^{(2)}$, $H^{(2)} + 8$, and action of \mathcal{A}^\dagger , $\tilde{\mathcal{A}}$, $\bar{\mathcal{A}}$ for $m_1 = 2, m_2 = 3, \alpha = 5/2$. The v values are indicated on the right.

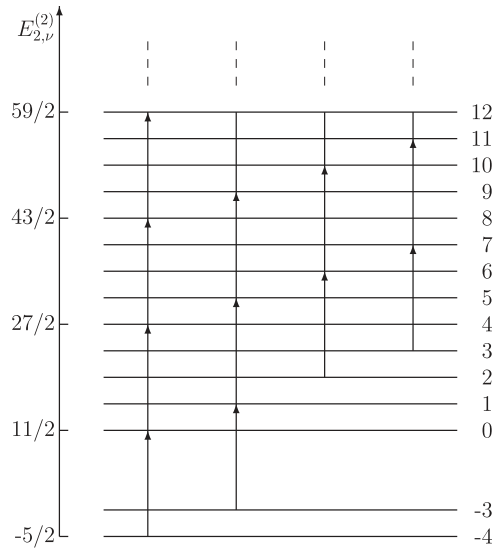


FIG. 2. Action of c^\dagger on the $H^{(2)}$ eigenstates for $m_1 = 2, m_2 = 3, \alpha = 5/2$. The ν values are indicated on the right.

for the case where $k = 2, m_1 = 2, m_2 = 3, l = 2$, and $\alpha = 5/2$. The resulting four PHA unirreps are shown in Fig. 2.

IV. APPLICATION TO SOME SUPERINTEGRABLE TWO-DIMENSIONAL SYSTEMS

A. Superintegrability and polynomial algebras

It is known from previous papers that integrals of motion of two-dimensional Hamiltonians in cartesian⁷² or polar⁷³ coordinates can be constructed using ladder operators or recurrence relations for one-dimensional Hamiltonians. Here, we restrict the treatment to the case of two-dimensional Hamiltonians allowing separation of variables in cartesian coordinates with scalar potentials of the form

$$H = H_x + H_y = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + V_x(x) + V_y(y), \tag{4.1}$$

and we assume that there exist ladder operators (a_x^\dagger, a_x) and (a_y^\dagger, a_y) in both axes that are differential operators of order k_1 and k_2 , respectively, and satisfy the defining relations of two PHAs,

$$\begin{aligned} [H_x, a_x^\dagger] &= \lambda_x a_x^\dagger, & [H_x, a_x] &= -\lambda_x a_x, & [a_x, a_x^\dagger] &= Q(H_x + \lambda_x) - Q(H_x), \\ [H_y, a_y^\dagger] &= \lambda_y a_y^\dagger, & [H_y, a_y] &= -\lambda_y a_y, & [a_y, a_y^\dagger] &= S(H_y + \lambda_y) - S(H_y). \end{aligned} \tag{4.2}$$

In Eq. (4.2), the constants λ_x and λ_y correspond to the shift in the energy of the wavefunctions by application of the ladder operators. By construction, such ladder operators only admit combinations of finite or infinite sequences of equidistant levels. However, interesting patterns can be created by these combinations. In Eq. (4.2), $Q(H_x)$ and $S(H_y)$ are polynomials that can be expressed further as

$$Q(H_x) = \prod_{i=1}^{k_1} (H_x - \epsilon_i^{(x)}), \tag{4.3}$$

$$S(H_y) = \prod_{j=1}^{k_2} (H_y - \epsilon_j^{(y)}). \tag{4.4}$$

The constants $\epsilon_i^{(x)}$ and $\epsilon_j^{(y)}$ will be related with zero modes of the lowering operators. As seen in Secs. II and III, and also previous papers,^{53,54} ladder operators can themselves have an underlying

structure. This can take the form of products of supercharges of various types. The intertwining and factorization relations satisfied by these supercharges can thus simplify the calculation of the polynomial Heisenberg algebra in both axes.

The separation of variables in cartesian coordinates implies the existence of a second-order integral of motion $H_x - H_y$, showing that the two-dimensional system (4.1) is integrable. From the ladder operators, one can construct additional polynomial operators commuting with H , $a_x^{\dagger n_1} a_y^{n_2}$ and $a_x^{n_1} a_y^{\dagger n_2}$, where $n_1, n_2 \in \mathbb{Z}^+$ are chosen such that $n_1 \lambda_x = n_2 \lambda_y = \bar{\lambda}$.⁷² Hence, system (4.1) possesses three algebraically independent integrals of motion and thus has the superintegrability property.

The integrals of motion, written in the following factorized form:

$$K = \frac{1}{2\bar{\lambda}}(H_x - H_y), \quad I_+ = a_x^{\dagger n_1} a_y^{n_2}, \quad I_- = a_x^{n_1} a_y^{\dagger n_2}, \quad (4.5)$$

generate the polynomial algebra of the system

$$\begin{aligned} [K, I_{\pm}] &= \pm I_{\pm}, & [I_-, I_+] &= F_{n_1, n_2}(K + 1, H) - F_{n_1, n_2}(K, H), \\ F_{n_1, n_2}(K, H) &= \prod_{i=1}^{n_1} Q\left(\frac{H}{2} + \bar{\lambda}K - (n_1 - i)\lambda_x\right) \prod_{j=1}^{n_2} S\left(\frac{H}{2} - \bar{\lambda}K + j\lambda_y\right), \end{aligned} \quad (4.6)$$

which is of order $k_1 n_1 + k_2 n_2 - 1$. This is the symmetry algebra of the two-dimensional superintegrable systems (4.1).

B. Representations of the polynomial algebras

In previous papers,^{53,54,69,70,72} the Daskaloyannis approach⁶² was used in order to obtain finite-dimensional unirreps of polynomial algebras for various superintegrable Hamiltonians and furthermore to calculate algebraically the energy spectrum and the level total degeneracies. This approach is based on calculating the Casimir operator, establishing realizations of the polynomial algebras as deformed oscillator ones, generated by $\{N, b, b^{\dagger}, 1\}$ with commutation relations

$$[N, b^{\dagger}] = b^{\dagger}, \quad [N, b] = -b, \quad b^{\dagger} b = \Phi(N, E, u), \quad b b^{\dagger} = \Phi(N + 1, E, u), \quad (4.7)$$

and imposing three constraints on the structure function $\Phi(N, E, u)$ (where N is the number operator). These constraints are

$$\Phi(0, E, u) = 0, \quad \Phi(p + 1, E, u) = 0, \quad \Phi(x, E, u) > 0, \quad x = 1, \dots, p. \quad (4.8)$$

The resulting system of equations formed by these constraints involves the representation-dependent constant u , the energy E , the integer p , and can be solved, but we need to remove some non-physical solutions. In addition, it is not guaranteed the whole spectrum can be recovered, as in various cases the method does not provide the entire spectrum and degeneracies.^{53,69,70,72}

In a recent paper,⁵⁴ we demonstrated that this method can be applied to families of systems related with one-step extensions of the harmonic oscillator, containing as special cases Gravel systems for which no algebraic derivation of the complete energy spectrum and degeneracies was previously available.⁶⁸⁻⁷⁰ We showed that such an algebraic calculation can be carried out, provided a new set of integrals based on ladder operators with a different structure in terms of zero modes was constructed. We also obtained that unions of unirreps needed to be considered to obtain the total degeneracies, a phenomenon not observed in quadratically superintegrable systems. These results also showed how the Daskaloyannis method can have wider applications to systems with higher-order integrals. A negative point of this approach, however, remained the existence of nonphysical solutions that had to be eliminated.

The Daskaloyannis technique is particularly convenient in the case no underlying structure of integrals of motion is known. However, as described in Subsection IV A, we consider here some systems (4.1) whose individual constituents H_x and H_y have known ladder operators. The integrals of motion are thus products of the latter, as shown in Eq. (4.5). In addition, the ladder operators used are themselves composed of supercharges, so that their action on physical states is known (see

Eqs. (3.11)–(3.14) and (3.21)–(3.24)). Taking advantage of this knowledge will therefore provide us with an alternative method to study the polynomial algebra representations.

The finite-dimensional unirreps of the polynomial algebra given by (4.6) (which is a deformed $u(2)$) may be characterized by (N, s) and their basis states by

$$|N, \tau, s, \sigma\rangle, \quad (4.9)$$

where $\sigma = -s, -s + 1, \dots, s$ and τ distinguishes between repeated representations specified by the same s . Here, s may be any integer or half-integer, σ denotes the eigenvalue of $I_0 = K + c$, where c is some representation-dependent constant, and I_+, I_- are such that

$$I_+|N, \tau, s, \sigma\rangle = I_-|N, \tau, s, -\sigma\rangle = 0. \quad (4.10)$$

From the last properties, one can find the value of s realized for every N value.

In order to implement the method, we will denote the eigenstates of H as $|N, \nu_x\rangle$, where $N = \nu_x + \nu_y + 1$ and ν_x, ν_y are two quantum numbers associated with the eigenvalues of H_x and H_y , respectively, and whose allowed values will depend on the examples considered. Equation (4.10) will be solved by deriving the action of integrals from that of ladder operators on physical states in the $|N, \nu_x\rangle$ basis. Starting from states annihilated by I_- and acting iteratively with I_+ until we reach a state annihilated by the latter, we identify the corresponding value of s and the σ associated with each state forming this sequence. Moreover, on using the notation $N = \lambda n_1 n_2 + \mu$ with some appropriate values of λ and μ , the basis states $|N, \tau, s, \sigma\rangle$ are obtained (with values of τ related to values of λ and μ). In this way of deducing finite-dimensional unirreps, we do not need to remove nonphysical states as such a method only involves action on physical ones.

C. Combination of a k -step rationally extended potential with a standard one

Let us consider 2D Hamiltonians of the form given by Eq. (4.1). In the x -axis, we take a k -step extension of the harmonic or radial oscillator, given by Eq. (2.16) or Eq. (2.33), respectively. In the y -axis, we simply take the harmonic or radial oscillator. This provides us with four new infinite families of 2D superintegrable systems

$$H_a = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 + y^2 - 2k - 2\frac{d^2}{dx^2} \log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \dots, \mathcal{H}_{m_k}), \quad (4.11)$$

$$H_b = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + \frac{1}{4}x^2 + \frac{l(l+1)}{x^2} + y^2 - k - 2\frac{d^2}{dx^2} \log \tilde{\mathcal{W}}(L_{m_1}^{(-\alpha-k)}(-z), L_{m_2}^{(-\alpha-k)}(-z), \dots, L_{m_k}^{(-\alpha-k)}(-z)), \quad (4.12)$$

$$H_c = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 + \frac{1}{4}y^2 + \frac{l(l+1)}{y^2} - 2k - 2\frac{d^2}{dx^2} \log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \dots, \mathcal{H}_{m_k}), \quad (4.13)$$

$$H_d = -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + \frac{1}{4}x^2 + \frac{l(l+1)}{x^2} + \frac{1}{4}y^2 + \frac{l(l+1)}{y^2} - k - 2\frac{d^2}{dx^2} \log \tilde{\mathcal{W}}(L_{m_1}^{(-\alpha-k)}(-z), L_{m_2}^{(-\alpha-k)}(-z), \dots, L_{m_k}^{(-\alpha-k)}(-z)). \quad (4.14)$$

From the results of Subsections III A and III B, we know that for the four cases mentioned above, namely, $H_a, H_b, H_c,$ and H_d, H_x possess ladder operators associated with a polynomial Heisenberg algebra. On the other hand, in Sec. II, we presented the well-known ladder operators and polynomial Heisenberg algebras for the harmonic and radial oscillators. From the general construction of Subsection IV A, we obtain the integrals of motion as products of these ladder operators. We take a_x as given by Eq. (3.2) for Hamiltonians H_a and H_c , and as given by Eq. (3.16)

for H_b and H_d . Moreover, a_y is the usual ladder operator of the harmonic oscillator given by Eq. (2.11) for H_a and H_b , and the usual ladder operator of the radial oscillator given by Eq. (2.29) (with $k = 0$) for H_c and H_d . In all cases, $\lambda_x = 2m_k + 2$ and $\lambda_y = 2$ and correspondingly $n_1 = 1$ and $n_2 = m_k + 1$, so that the order of the resulting integrals I_+ and I_- is, in the four cases, $2m_k + 2$, $3m_k + 3$, $3m_k + 3$, and $4m_k + 4$, respectively.

The polynomial algebra can be determined explicitly using Eq. (4.6). The polynomial Q is given by Eq. (3.10) for H_a and H_c and by Eq. (3.20) for H_b and H_d . On the other hand, for the polynomial S we use Eq. (2.13) for H_a and H_b and Eq. (2.31) (with $k = 0$) for H_c and H_d .

The form of the potentials, the ladder operators and consequently the integrals of motion differs in the four cases. However, the structure of the energy spectrum is similar and also the action of the ladder operators a_x and a_y on the physical states $|v_x\rangle_1$ and $|v_y\rangle_2$, respectively. From unirreps of the polynomial algebras we can therefore perform an algebraic derivation of the energy spectrum and of the total degeneracies in an uniform manner for the four cases.

On using Eqs. (2.17) and (2.34), we can calculate the energy spectrum directly. This yields

$$E_{i,N} = 2N + \gamma_i, \quad i = a, b, c, d, \tag{4.15}$$

where $\gamma_a = 0$, $\gamma_b = \alpha + k$, $\gamma_c = \alpha$, $\gamma_d = 2\alpha + k$, and

$$N = v_x + v_y + 1, \quad v_x = -m_k - 1, \dots, -m_1 - 1, 0, 1, 2, \dots, \quad v_y = 0, 1, 2, \dots \tag{4.16}$$

The additive constant γ_i in the energy eigenvalues does not affect the structure of the levels nor the degeneracies. In all cases, a direct calculation leads to the following results:

$$\text{deg}(E_N) = \begin{cases} k - j + 1 & \text{if } N = -m_j, -m_j + 1, \dots, -m_{j-1} - 1 \text{ for } j = 2, 3, \dots, k, \\ k & \text{if } N = -m_1, -m_1 + 1, \dots, 0, \\ N + k & \text{if } N = 1, 2, 3, \dots \end{cases} \tag{4.17}$$

Let us remark how the degeneracies differ from the well-known isotropic harmonic oscillator ones (i.e., for two harmonic oscillators in both axes), which are simply given by $N + 1$. For the four families of systems considered here, the pattern is more complicated and there are bands of levels with degeneracies 1 to k before the degeneracies increase as $N + k$. The width of these bands is related to parameters m_j (with $j = 1, \dots, k$). We will show how even such a more complex structure can be deduced using the polynomial algebras and their finite-dimensional unirreps.

The eigenfunctions of the Hamiltonians H_a , H_b , H_c , and H_d are given in terms of Hermite polynomials, Laguerre ones, type III Hermite EOP, or type III Laguerre EOP. In spite of this, they can be expressed in a uniform way. We can observe that on using $N = v_x + v_y + 1$, the states $|v_x\rangle_1|v_y\rangle_2$ of the four systems can be written as

$$|N, v_x\rangle = |v_x\rangle_1|N - v_x - 1\rangle_2, \tag{4.18}$$

where the values of N and v_x are presented in Table I. We also list in this table the corresponding values of $v_y = N - v_x - 1$ as they are used to calculate the action.

TABLE I. States of superintegrable systems H_a, H_b, H_c , and H_d in the $|N, v_x\rangle$ basis with corresponding values of $v_y = N - v_x - 1$. In this table, j runs over $2, 3, \dots, k$.

N	v_x	$N - v_x - 1$
$-m_j, -m_j + 1, \dots,$	$-m_k - 1, -m_{k-1} - 1, \dots,$	$N + m_k, N + m_{k-1}, \dots,$
$-m_{j-1} - 1$	$-m_j - 1$	$N + m_j$
$-m_1, -m_1 + 1, \dots,$	$-m_k - 1, -m_{k-1} - 1, \dots,$	$N + m_k, N + m_{k-1}, \dots,$
0	$-m_1 - 1$	$N + m_1$
1, 2, ...	$-m_k - 1, -m_{k-1} - 1, \dots,$	$N + m_k, N + m_{k-1}, \dots,$
	$-m_1 - 1, 0, 1, \dots, N - 1$	$N + m_1, N - 1, N - 2, \dots, 0$

The action of the integral K associated with separation of variables is given by

$$K|N, \nu_x\rangle = \frac{1}{2(m_k + 1)}(2\nu_x + 1 - N)|N, \nu_x\rangle. \quad (4.19)$$

That of the integrals I_+ and I_- is calculated from the action of the ladder operators. We use iteratively Eqs. (2.14), (2.32), (3.11)–(3.14) and (3.21)–(3.24) according to the values of n_1 and n_2 . The states annihilated by I_+ are identified directly from $a_y^{n_2}|v_y\rangle_2$ and those by I_- from $a_x^{n_1}|v_x\rangle_1$. The states annihilated by I_+ and I_- are presented in Tables VI and VII of Appendix B. They are given in terms of the $|N, \nu_x\rangle$ basis.

For all other states, the action of the integrals I_+ and I_- is given by

$$\begin{aligned} I_+|N, \nu_x\rangle &= \alpha(N, \nu_x, m_k)|N, \nu_x + m_k + 1\rangle, \\ I_-|N, \nu_x\rangle &= \beta(N, \nu_x, m_k)|N, \nu_x - m_k - 1\rangle. \end{aligned} \quad (4.20)$$

The coefficients $\alpha(N, \nu_x, m_k)$ and $\beta(N, \nu_x, m_k)$ can be determined from the explicit action of the ladder operators. They differ in the four cases considered in this paper, but these explicit expressions do not play any role in the calculation of degeneracies using the finite-dimensional unirreps and we thus omit them.

Furthermore, as mentioned in Subsection IV B, we also write all the states using the notation $N = \lambda(m_k + 1) + \mu$, $\mu = 0, 1, 2, \dots, m_k$. The negative values of N can be obtained by taking the following values for λ and μ :

$$\begin{aligned} N &= -m_j, -m_j + 1, \dots, -m_{j-1} - 1, \quad j = 2, 3, \dots, k : \\ \lambda &= -1, \quad \mu = m_k - m_j + 1, m_k - m_j + 2, \dots, m_k - m_{j-1}, \\ N &= -m_1, -m_1 + 1, \dots, -1 : \\ \lambda &= -1, \quad \mu = m_k - m_1 + 1, m_k - m_1 + 2, \dots, m_k. \end{aligned} \quad (4.21)$$

We consider the states formed by the repeated action of the integral I_+ on the states ξ annihilated by I_- , presented in Table VII. From the smallest integer $n + 1$ such that

$$(I_+)^{n+1}\xi = 0, \quad (4.22)$$

we deduce that $(I_+)^n\xi$ is one of the zero modes of I_+ of Table VI, denoted by χ . This process allows us to obtain the integer or half-integer s (given by $\frac{n}{2}$) such that Eq. (4.10) is satisfied. A value of σ is assigned to each member of this finite sequence. In this way, we calculate the values of s realized for every N . In Table II, we list the values of λ , μ , and s , together with the associated number \mathcal{N} of unirreps per level, and the total level degeneracy.

The results agree with those previously obtained for $k = 1$ using the Daskaloyannis approach and presented in Table I of Ref. 54. For this special value of k , lines 1, 2, 5, 6, 9, and 10 of Table II would be missing in the present approach. In order to show the results coincide for both methods, we also need to take into account the following relations between the parameters and integrals:

$$s = \frac{p}{2}, \quad I_0 = N - \frac{p}{2}, \quad c = -u - \frac{p}{2}. \quad (4.23)$$

Note in addition that for the next value of k , namely, $k = 2$, lines 2, 6, and 10 would be missing in Table II.

D. Combination of two k -step rationally extended potentials

Let us consider 2D Hamiltonians of the type (4.1) with a k -step extension of the harmonic oscillator, given by Eq. (2.16), and/or a k -step extension of the radial oscillator, given by Eq. (2.33), in both axes. We can form three new infinite families of superintegrable systems, which include

TABLE II. Set of s values with their number of occurrences, number \mathcal{N} of unirreps per level, and total level degeneracy for the polynomial algebra (4.6) corresponding to Hamiltonian H_a, H_b, H_c , or H_d . In this table, j runs over $2, 3, \dots, k - 1$.

λ	μ	s	\mathcal{N}	deg (E_N)
-1	$1, \dots, m_k - m_{k-1}$	0	1	1
-1	$m_k - m_j + 1, \dots, m_k - m_{j-1}$	0^{k-j+1}	$k - j + 1$	$k - j + 1$
-1	$m_k - m_1 + 1, \dots, m_k$	0^k	k	k
0	0	0^k	k	k
0	$1, \dots, m_k - m_{k-1}$	$\frac{1}{2}$ $0^{\mu+k-2}$	$\mu + k - 1$	$N + k$
0	$m_k - m_j + 1, \dots, m_k - m_{j-1}$	$(\frac{1}{2})^{k-j+1}$ $0^{\mu-k+2j-2}$	$\mu + j - 1$	$N + k$
0	$m_k - m_1 + 1, \dots, m_k$	$(\frac{1}{2})^k$ $0^{\mu-k}$	μ	$N + k$
1, 2, ...	0	$(\frac{\lambda}{2})^k$ $(\frac{\lambda-1}{2})^{m_k-k+1}$	$m_k + 1$	$N + k$
1, 2, ...	$1, \dots, m_k - m_{k-1}$	$\frac{\lambda+1}{2}$ $(\frac{\lambda}{2})^{\mu+k-2}$	$m_k + 1$	$N + k$
1, 2, ...	$m_k - m_j + 1, \dots, m_k - m_{j-1}$	$(\frac{\lambda-1}{2})^{m_k-\mu-k+2}$ $(\frac{\lambda+1}{2})^{k-j+1}$ $(\frac{\lambda}{2})^{\mu-k+2j-2}$ $(\frac{\lambda-1}{2})^{m_k-\mu-j+2}$	$m_k + 1$	$N + k$
1, 2, ...	$m_k - m_1 + 1, \dots, m_k$	$(\frac{\lambda+1}{2})^k$ $(\frac{\lambda}{2})^{\mu-k}$ $(\frac{\lambda-1}{2})^{m_k-\mu+1}$	$m_k + 1$	$N + k$

some of those introduced in previous papers^{53,54}

$$\begin{aligned}
 H_e = & -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 + y^2 - 2k_1 - 2k_2 \\
 & - 2\frac{d^2}{dx^2} \log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \dots, \mathcal{H}_{m_{k_1}}) - 2\frac{d^2}{dy^2} \log \mathcal{W}(\mathcal{H}_{n_1}, \mathcal{H}_{n_2}, \dots, \mathcal{H}_{n_{k_2}}),
 \end{aligned} \tag{4.24}$$

$$\begin{aligned}
 H_f = & -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + \frac{1}{4}x^2 + \frac{l(l+1)}{x^2} + \frac{1}{4}y^2 + \frac{l(l+1)}{y^2} - k_1 - k_2 \\
 & - 2\frac{d^2}{dx^2} \log \tilde{\mathcal{W}}(L_{m_1}^{(-\alpha-k_1)}(-z), L_{m_2}^{(-\alpha-k_1)}(-z), \dots, L_{m_{k_1}}^{(-\alpha-k_1)}(-z)) \\
 & - 2\frac{d^2}{dy^2} \log \tilde{\mathcal{W}}(L_{n_1}^{(-\alpha-k_2)}(-z), L_{n_2}^{(-\alpha-k_2)}(-z), \dots, L_{n_{k_2}}^{(-\alpha-k_2)}(-z)),
 \end{aligned} \tag{4.25}$$

$$\begin{aligned}
 H_g = & -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 + \frac{1}{4}y^2 + \frac{l(l+1)}{y^2} - 2k_1 - k_2 \\
 & - 2\frac{d^2}{dx^2} \log \mathcal{W}(\mathcal{H}_{m_1}, \mathcal{H}_{m_2}, \dots, \mathcal{H}_{m_{k_1}}) \\
 & - 2\frac{d^2}{dy^2} \log \tilde{\mathcal{W}}(L_{n_1}^{(-\alpha-k_2)}(-z), L_{n_2}^{(-\alpha-k_2)}(-z), \dots, L_{n_{k_2}}^{(-\alpha-k_2)}(-z)).
 \end{aligned} \tag{4.26}$$

By construction, these systems are superintegrable for any k_1 and k_2 . We take a_x as given by Eq. (3.2) for H_e and H_g , and as given by Eq. (3.16) for H_f , while for a_y we assume Eq. (3.2) for H_e and Eq. (3.16) for H_f and H_g . In all cases, we have $\lambda_x = 2m_{k_1} + 2$ and $\lambda_y = 2n_{k_2} + 2$, so that $n_1 = n_{k_2} + 1$ and $n_2 = m_{k_1} + 1$. The order of the integrals I_+ and I_- is $2(m_{k_1} + 1)(n_{k_2} + 1)$, $4(m_{k_1} + 1)(n_{k_2} + 1)$, and $3(m_{k_1} + 1)(n_{k_2} + 1)$ for these three cases, respectively. The polynomial

algebra can be determined explicitly using Eq. (4.6). In this equation, the polynomial Q is given by Eq. (3.10) for H_e and H_g and by Eq. (3.20) for H_f . On the other hand, for the polynomial S we use Eq. (3.10) for H_e and Eq. (3.20) for H_f and H_g .

For the purpose of an algebraic derivation of the degeneracies, let us consider the particular case of two one-step extensions with m_1 and n_1 (without loss of generality we may assume $m_1 \geq n_1$). Only the case when $m_1 = n_1 = m$ has been published⁵⁴ and there are also some unpublished results for $m_1 = 4, m_2 = 2$, but the general problem is difficult to solve in the Daskaloyannis approach. Equations (2.17) and (2.34) lead to the following degenerate energy spectrum:

$$E_{i,N} = 2N + \delta_i, \quad i = e, f, g, \tag{4.27}$$

where $\delta_e = 0, \delta_f = 2\alpha + 2, \delta_g = \alpha + 1$, and

$$N = v_x + v_y + 1, \quad v_x = -m_1 - 1, 0, 1, 2, \dots, \quad v_y = -n_1 - 1, 0, 1, 2, \dots \tag{4.28}$$

The degeneracies can be written in the three cases as

$$\text{deg}(E_N) = \begin{cases} 1 & \text{if } N = -m_1 - n_1 - 1, -m_1, -m_1 + 1, \dots, -n_1 - 1, \\ 2 & \text{if } N = -n_1, -n_1 + 1, \dots, 0, \\ N + 2 & \text{if } N = 1, 2, \dots \end{cases} \tag{4.29}$$

In these cases, we note the presence of bands with the same total degeneracies as in the systems of Subsection IV C. The ladder operators of the k -step extensions of the harmonic and radial oscillators act on the physical states $|v_x\rangle_1 |v_y\rangle_2$ in a similar way and consequently we can use a uniform notation. Since $v_y = -n_1 - 1$ or $v_y \geq 0$, the relation $N = v_x + v_y + 1$ implies that $v_x = N + n_1$ or $v_x \leq N - 1$. The states can be rewritten as

$$|N, v_x\rangle = |v_x\rangle_1 |N - v_x - 1\rangle_2, \tag{4.30}$$

with the corresponding values of N and v_x presented in Table III. We also list there the corresponding values of $v_y = N - v_x - 1$ as they are useful to compute the action of the integrals of motion and to identify the states they annihilate.

The action of the integral K is given by the following equation:

$$K|N, v_x\rangle = \frac{1}{2(m_1 + 1)(n_1 + 1)}(2v_x + 1 - N)|N, v_x\rangle. \tag{4.31}$$

That of the integrals I_+ and I_- can be obtained directly from the action of the ladder operators given by Eqs. (3.11)–(3.14) and (3.21)–(3.24). It is important to distinguish the states annihilated by I_+ and I_- , which are presented in Tables VIII and IX of Appendix B. The action of the integrals on all other states is given by

$$\begin{aligned} I_+|N, v_x\rangle &= \tilde{\alpha}(N, v_x, m_1, n_1)|N, v_x + (m_1 + 1)(n_1 + 1)\rangle, \\ I_-|N, v_x\rangle &= \tilde{\beta}(N, v_x, m_1, n_1)|N, v_x - (m_1 + 1)(n_1 + 1)\rangle. \end{aligned} \tag{4.32}$$

The coefficients $\tilde{\alpha}(N, v_x, m_1, n_1)$ and $\tilde{\beta}(N, v_x, m_1, n_1)$ can be determined from the explicit action of the ladder operators. As they do not play any role in the derivation of the total degeneracies and would differ in the three cases, we omit them. The unirreps are calculated directly by acting

TABLE III. States of superintegrable systems H_e, H_f , and H_g (with $k_1 = k_2 = 1$ and $m_1 \geq n_1$) in the $|N, v_x\rangle$ basis and corresponding values of $v_y = N - v_x - 1$.

N	v_x	$N - v_x - 1$
$-m_1 - n_1 - 1$	$-m_1 - 1$	$-n_1 - 1$
$-m_1, -m_1 + 1, \dots, -n_1 - 1$	$-m_1 - 1$	$0, 1, \dots, m_1 - n_1 - 1$
$-n_1, -n_1 + 1, \dots, 0$	$-m_1 - 1, N + n_1$	$N + m_1, -n_1 - 1$
$1, 2, \dots$	$-m_1 - 1, 0, 1, \dots, N - 1, N + n_1$	$N + m_1, N - 1, N - 2, \dots, 0, -n_1 - 1$

TABLE IV. Set of s values with their number of occurrences, number \mathcal{N} of unirreps per level, and total level degeneracy for the polynomial algebra (4.6) corresponding to Hamiltonian H_e, H_f , or H_g with $k_1 = k_2 = 1$ and $m_1 \geq n_1$.

λ	μ	s	\mathcal{N}	deg (E_N)
-1	$m_1 n_1$	0	1	1
-1	$m_1 n_1 + n_1 + 1, \dots, m_1 n_1 + m_1$	0	1	1
-1	$m_1 n_1 + m_1 + 1, \dots, m_1 n_1 + m_1 + n_1$	0^2	2	2
0	$0, \dots, m_1 n_1 - 1$	$0^{\mu+2}$	$\mu + 2$	$N + 2$
0	$m_1 n_1$	$\frac{1}{2}$	$\mu + 1$	$N + 2$
		0^μ		
0	$m_1 n_1 + 1, \dots, m_1 n_1 + n_1$	$0^{\mu+2}$	$\mu + 2$	$N + 2$
0	$m_1 n_1 + n_1 + 1, \dots, m_1 n_1 + m_1$	$\frac{1}{2}$	$\mu + 1$	$N + 2$
		0^μ		
0	$m_1 n_1 + m_1 + 1, \dots, m_1 n_1 + m_1 + n_1$	$(\frac{1}{2})^2$	μ	$N + 2$
		$0^{\mu-2}$		
1, 2, ...	$0, \dots, m_1 n_1 - 1$	$(\frac{\lambda}{2})^{\mu+2}$	M	$N + 2$
		$(\frac{\lambda-1}{2})^{M-\mu-2}$		
1, 2, ...	$m_1 n_1$	$\frac{\lambda+1}{2}$	M	$N + 2$
		$(\frac{\lambda}{2})^\mu$		
		$(\frac{\lambda-1}{2})^{M-\mu-1}$		
1, 2, ...	$m_1 n_1 + 1, \dots, m_1 n_1 + n_1$	$(\frac{\lambda}{2})^{\mu+2}$	M	$N + 2$
		$(\frac{\lambda-1}{2})^{M-\mu-2}$		
1, 2, ...	$m_1 m_2 + n_1 + 1, \dots, m_1 n_1 + m_1$	$\frac{\lambda+1}{2}$	M	$N + 2$
		$(\frac{\lambda}{2})^\mu$		
		$(\frac{\lambda-1}{2})^{M-\mu-1}$		
1, 2, ...	$m_1 n_1 + m_1 + 1, \dots, m_1 n_1 + m_1 + n_1$	$(\frac{\lambda+1}{2})^2$	M	$N + 2$
		$(\frac{\lambda}{2})^{\mu-2}$		
		$(\frac{\lambda-1}{2})^{M-\mu}$		

repeatedly with I_+ on states annihilated by I_- until we reach a state that is annihilated. Here, we set $N = \lambda M + \mu, M = (m_1 + 1)(n_1 + 1), \mu = 0, 1, \dots, m_1 n_1 + m_1 + n_1$. The finite-dimensional unirreps are listed in Table IV.

The results agree with those previously obtained for $m_1 = n_1 = m$ and for $m_1 = 4, n_1 = 2$ by using Daskaloyannis method of calculating the finite-dimensional unirreps of superintegrable systems. In the case $m_1 = n_1 = m$, for instance, we have $N = \lambda(m + 1)^2 + \mu, \mu = 0, 1, \dots, m(m + 2)$ in the present approach. In Table II of Ref. 54, on the other hand, we used the notation $\mu = \rho(m + 1) + \sigma$ with ρ and σ running over $0, 1, \dots, m$ (note that this σ has nothing to do with σ introduced in Eq. (4.9)). The correspondence between the μ and (ρ, σ) values is given in Table V. On taking it into account, it can be easily checked that the results of Ref. 54 are recovered as a special case of Table IV.

TABLE V. Comparison with $N = \lambda(m + 1)^2 + \mu$ where $\mu = \rho(m + 1) + \sigma$.

μ	ρ	σ
$0, 1, \dots, m^2 - 1$	$0, 1, \dots, m - 2$	$0, 1, \dots, m$
$0, 1, \dots, m^2 - 1$	$m - 1$	0
m^2	$m - 1$	1
$m^2 + 1, m^2 + 2, \dots, m^2 + m$	$m - 1$	$2, 3, \dots, m$
$m^2 + 1, m^2 + 2, \dots, m^2 + m$	m	0
$m^2 + m + 1, m^2 + m + 2, \dots, m^2 + 2m$	m	$1, 2, \dots, m$

V. CONCLUSION

In this paper, for k -step extensions of the harmonic and radial oscillators related with type III EOP, we obtained new ladder operators that satisfy polynomial Heisenberg algebras with only infinite-dimensional unirreps. The construction is based on supersymmetric quantum mechanics and relies on a combination of the state-adding and state-deleting approaches. These results extend those obtained recently for one-step extensions of the harmonic oscillator.⁵⁴

Moreover, from these ladder operators we generated seven new infinite families of superintegrable systems with higher-order integrals of motion. These families generalize various superintegrable systems discovered by Gravel.⁶⁸ It is interesting to note that one- and two-step extensions of the harmonic oscillator (for specific values of m_1 and m_2) are also related with quantum systems involving the fourth Painlevé transcendent.^{68,70,72} The new ladder operators are not only significant in order to construct integrals of motion, but more importantly allow to obtain a higher-order polynomial algebra, from which an algebraic derivation of the whole energy spectrum and the total degeneracies can be done by using finite-dimensional unirreps. Such ladder operators are not of the lowest possible order in contrast with standard ladder operators obtained from supersymmetric quantum mechanics. The latter, however, admit k singlet states and, as a consequence, do not allow to create integrals of motion with a corresponding polynomial algebra whose finite-dimensional unirreps give rise to the whole energy spectrum and the total degeneracies.

Another novelty of the paper is the introduction of a different method than the Daskaloyannis approach used in previous works for the calculation of finite-dimensional unirreps. It takes advantage of the fact we know the underlying structure of the integrals. Tables II and IV list the obtained unirreps and degeneracies. We recover as particular cases results obtained from the Daskaloyannis approach.⁵⁴ A very interesting aspect of our results is the fact that by allowing k -step extensions of the harmonic or radial oscillator we can create more complex patterns for the degeneracies. The seven families discussed here have bands of levels with the same amount of total degeneracies. The length of these bands depends on the choice of the Hamiltonian parameters m_i and n_j . This may open the way of another type of spectral design, i.e., the possibility of creating systems with a given type of degeneracies. These systems and their new ladder operators themselves could have applications in other contexts. Let us mention that the anisotropic oscillator and its related degeneracies found applications in nuclear physics.⁸²

Let us make further remarks and describe how these results could also be generalized in various ways. One possible avenue is the extension to 3D systems. It would be of great interest to study their degeneracies through finite-dimensional unirreps. We also restricted our algebraic treatment to isotropic cases, but generalization to anisotropic ones could be investigated. Furthermore, from Ref. 35, we know that two-step extensions of the harmonic oscillator allow not only ladder operators with two singlet states, but also ladder operators with a doublet state. We showed in this paper there exist ladder operators with only infinite sequences of levels for k -step extensions of the harmonic oscillator that differ from the standard ones with k singlets. The study of other types of ladder operators and, in particular, those with multiplet states for k -step extensions needs to be done.

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APPENDIX A: GOING FROM $H^{(1)}$ TO $\tilde{H}^{(1)}$ IN THE RADIAL HARMONIC OSCILLATOR CASE

In addition to the two sets of partner Hamiltonians $(H^{(1)}, H^{(2)})$ and $(\tilde{H}^{(1)}, \tilde{H}^{(2)})$ considered in Subsection II B, we plan to introduce here a third one $(\tilde{H}^{(1)}, \tilde{\tilde{H}}^{(2)})$, such that $\tilde{H}^{(1)} = H^{(1)}$ while $\tilde{\tilde{H}}^{(2)}$ only differs from $\tilde{H}^{(2)}$ by some additive constant.

This can be achieved by considering n th-order differential operators $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^\dagger$ intertwining with $\tilde{H}^{(1)}$ and $\tilde{H}^{(2)}$ as in Eq. (2.2) and constructed from $n = m_k + 1$ polynomial-type seed solutions of $\tilde{H}^{(1)} = H^{(1)}$ of class II, i.e., $(\varphi_1, \varphi_2, \dots, \varphi_n) \rightarrow (\tilde{\phi}_0^{l+k}, \tilde{\phi}_1^{l+k}, \dots, \tilde{\phi}_{m_k}^{l+k})$, where³¹

$$\begin{aligned}\tilde{\phi}_j^{l+k}(x) &= \chi_{l+k}^{\text{II}}(z) L_j^{(-\alpha-k)}(z), \quad j = 0, 1, \dots, m_k, \\ \chi_{l+k}^{\text{II}}(z) &= z^{-\frac{1}{4}(2\alpha+2k-1)} e^{-\frac{1}{2}z} = z^{-\alpha-k} \eta_{l+k}(z).\end{aligned}\tag{A1}$$

The corresponding eigenvalue of $\tilde{\phi}_j^{l+k}(x)$ is given by $E_{l+k,j}^{\text{II}} = -\alpha - k + 2j + 1$.

The two partner potentials now read

$$\begin{aligned}\tilde{V}^{(1)}(x) &= V^{(1)}(x) = V_{l+k}(x), \\ \tilde{V}^{(2)}(x) &= \tilde{V}^{(1)}(x) - 2 \frac{d^2}{dx^2} \log \mathcal{W}(\tilde{\phi}_0^{(l+k)}(x), \tilde{\phi}_1^{(l+k)}(x), \dots, \tilde{\phi}_{m_k}^{(l+k)}(x)),\end{aligned}\tag{A2}$$

where the latter is nonsingular provided $\alpha + k > m_k$. As from standard properties of Wronskians,⁷⁹ it can be shown that $\mathcal{W}(\tilde{\phi}_0^{(l+k)}, \tilde{\phi}_1^{(l+k)}, \dots, \tilde{\phi}_{m_k}^{(l+k)}) \propto (\chi_{l+k}^{\text{II}}(z))^{m_k+1} z^{m_k(m_k+1)/4}$, it is straightforward to get the relation

$$\tilde{V}^{(2)}(x) = V_{l+k-m_k-1}(x) + m_k + 1 = \bar{V}^{(1)}(x) + m_k + 1,\tag{A3}$$

which establishes the announced property.

As a check, we may compare the bound-state energies and wavefunctions of $\tilde{V}^{(2)}(x)$, obtained from those of $\tilde{V}^{(1)}(x)$ in $(m_k + 1)$ th-order SUSYQM, with those of $\bar{V}^{(1)}(x)$ given in Subsection II B. Since type II seed functions lead to isospectral transformations, Eq. (2.25) yields

$$\tilde{E}_{l+k-m_k-1,v}^{(2)} = \tilde{E}_{l+k,v}^{(1)} = E_{l+k,v}^{(1)} = 2v + \alpha + k + 1, \quad v = 0, 1, 2, \dots,\tag{A4}$$

which is compatible with Eqs. (2.39) and (A3). Furthermore, in the wavefunctions

$$\tilde{\psi}_{l+k-m_k-1,v}^{(2)}(x) \propto \frac{\mathcal{W}(\tilde{\phi}_0^{(l+k)}, \tilde{\phi}_1^{(l+k)}, \dots, \tilde{\phi}_{m_k}^{(l+k)}, \psi_v^{(l+k)})}{\mathcal{W}(\tilde{\phi}_0^{(l+k)}, \tilde{\phi}_1^{(l+k)}, \dots, \tilde{\phi}_{m_k}^{(l+k)}),} \quad v = 0, 1, 2, \dots,\tag{A5}$$

the denominator is already known, while, on using Eq. (A1), the numerator can be readily shown to be given by

$$\begin{aligned}\mathcal{W}(\tilde{\phi}_0^{(l+k)}, \tilde{\phi}_1^{(l+k)}, \dots, \tilde{\phi}_{m_k}^{(l+k)}, \psi_v^{(l+k)}) \\ \propto (\chi_{l+k}^{\text{II}})^{m_k+2} z^{\alpha+k+\frac{1}{4}(m_k-2)(m_k+1)} L_v^{(\alpha+k-m_k-1)}(z).\end{aligned}\tag{A6}$$

Hence,

$$\begin{aligned}\tilde{\psi}_{l+k-m_k-1,v}^{(2)}(x) &\propto \chi_{l+k}^{\text{II}}(z) z^{\alpha+k-\frac{1}{2}(m_k+1)} L_v^{(\alpha+k-m_k-1)}(z) \\ &\propto \eta_{l+k-m_k-1}(z) L_v^{(\alpha+k-m_k-1)}(z) \propto \tilde{\psi}_{l+k-m_k-1,v}^{(1)}(x),\end{aligned}\tag{A7}$$

which completes the check.

APPENDIX B: STATES ANNIHILATED BY THE INTEGRALS I_+ AND I_- IN THE BASIS $|N, \nu_x\rangle$

We present in Tables VI and VII the states annihilated by the integrals I_+ and I_- in the case of the four Hamiltonians of Subsection IV C. They are given in the $|N, \nu_x\rangle$ basis.

We present in Tables VIII and IX the states annihilated by the integrals I_+ and I_- in the case of the three Hamiltonians of Subsection IV D. They are given in the $|N, \nu_x\rangle$ basis.

TABLE VI. The states χ annihilated by the integrals I_+ in the $|N, \nu_x\rangle$ basis for the superintegrable systems H_a, H_b, H_c , and H_d .

N	ν_x
$-m_j, -m_j + 1, \dots, -m_{j-1} - 1,$ $j = 2, 3, \dots, k$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_j - 1$
$-m_1, -m_1 + 1, \dots, 0$ $1, 2, \dots, m_k - m_{k-1}$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_1 - 1$ $-m_{k-1} - 1, -m_{k-2} - 1, \dots, -m_1 - 1,$ $0, 1, \dots, N - 1$
$m_k - m_j + 1, m_k - m_j + 2, \dots, m_k - m_j - 1,$ $j = 2, 3, \dots, k - 1$	$-m_{j-1} - 1, -m_{j-2} - 1, \dots, -m_1 - 1,$ $0, 1, \dots, N - 1$
$m_k - m_1 + 1, m_k - m_1 + 2, \dots, m_k + 1$	$0, 1, \dots, N - 1$
$m_k + 2, m_k + 3, \dots$	$N - m_k - 1, N - m_k, \dots, N - 1$

TABLE VII. The states ξ annihilated by the integrals I_- in the $|N, \nu_x\rangle$ basis for the superintegrable systems H_a, H_b, H_c , and H_d .

N	ν_x
$-m_j, -m_j + 1, \dots, -m_{j-1} - 1,$ $j = 2, 3, \dots, k$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_j - 1$
$-m_1, -m_1 + 1, \dots, 0$ $1, 2, \dots, m_k - m_{k-1}$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_1 - 1$ $-m_k - 1, -m_{k-1} - 1, \dots, -m_1 - 1,$ $1, 2, \dots, N - 1$
$m_k - m_j + 1, m_k - m_j + 2, \dots,$ $m_k - m_{j-1}, j = 2, 3, \dots, k - 1$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_1 - 1, 1, 2, \dots,$ $m_k - m_{k-1} - 1, m_k - m_{k-1} + 1, \dots,$ $m_k - m_j - 1, m_k - m_j + 1, \dots, N - 1$
$m_k - m_1 + 1, m_k - m_1 + 2, \dots, m_k + 1$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_1 - 1, 1, 2, \dots,$ $m_k - m_{k-1} - 1, m_k - m_{k-1} + 1, \dots,$ $m_k - m_1 - 1, m_k - m_1 + 1, \dots, N - 1$
$m_k + 2, m_k + 3, \dots$	$-m_k - 1, -m_{k-1} - 1, \dots, -m_1 - 1, 1, 2, \dots,$ $m_k - m_{k-1} - 1, m_k - m_{k-1} + 1, \dots,$ $m_k - m_1 - 1, m_k - m_1 + 1, \dots, m_k$

TABLE VIII. The states χ annihilated by the integrals I_+ in the $|N, \nu_x\rangle$ basis for the superintegrable systems H_e, H_f , and H_g with $k_1 = k_2 = 1$ and $m_1 \geq n_1$.

N	ν_x
$-m_1 - n_1 - 1, -m_1, -m_1 + 1, \dots,$ $-n_1 - 1$	$-m_1 - 1$
$-n_1, -n_1 + 1, \dots, 0$ $1, 2, \dots, m_1 n_1 - 1$	$-m_1 - 1, N + n_1$ $-m_1 - 1, 0, 1, \dots, N - 1, N + n_1$
$m_1 n_1$ $m_1 n_1 + 1, m_1 n_1 + 2, \dots,$ $(m_1 + 1)n_1$	$0, 1, \dots, m_1 n_1 - 1, (m_1 + 1)n_1$ $-m_1 - 1, 0, 1, \dots, N - 1, N + n_1$
$m_1 n_1 + n_1 + 1, m_1 n_1 + n_1 + 2, \dots,$ $m_1(n_1 + 1)$	$0, 1, \dots, N - 1, N + n_1$
$m_1(n_1 + 1) + 1, m_1(n_1 + 1) + 2, \dots,$ $(m_1 + 1)(n_1 + 1)$	$0, 1, \dots, N - m_1(n_1 + 1) - 2, N - m_1(n_1 + 1),$ $N - m_1(n_1 + 1) + 1, \dots, N - 1, N + n_1$
$(m_1 + 1)(n_1 + 1) + 1,$ $(m_1 + 1)(n_1 + 1) + 2, \dots$	$N - (m_1 + 1)(n_1 + 1), N - (m_1 + 1)(n_1 + 1) + 1,$ $\dots, N - m_1(n_1 + 1) - 2, N - m_1(n_1 + 1),$ $N - m_1(n_1 + 1) + 1, \dots, N - 1, N + n_1$

TABLE IX. The states ξ annihilated by the integrals I_- in the $|N, \nu_x\rangle$ basis for the superintegrable systems H_c , H_f , and H_g with $k_1 = k_2 = 1$ and $m_1 \geq n_1$.

N	ν_x
$-m_1 - n_1 - 1, -m_1, -m_1 + 1, \dots, -n_1 - 1$	$-m_1 - 1$
$-n_1, -n_1 + 1, \dots, 0$	$-m_1 - 1, N + n_1$
$1, 2, \dots, m_1 n_1 - 1$	$-m_1 - 1, 0, 1, \dots, N - 1, N + n_1$
$m_1 n_1$	$-m_1 - 1, 0, 1, \dots, m_1 n_1 - 1$
$m_1 n_1 + 1, m_1 n_1 + 2, \dots, (m_1 + 1)n_1$	$-m_1 - 1, 0, 1, \dots, N - 1, N + n_1$
$m_1 n_1 + n_1 + 1, m_1 n_1 + n_1 + 2, \dots, m_1 n_1 + m_1$	$-m_1 - 1, 0, 1, \dots, (m_1 + 1)n_1 - 1,$ $(m_1 + 1)n_1 + 1, \dots, N - 1, N + n_1$
$m_1 n_1 + m_1 + 1, m_1 n_1 + m_1 + 2, \dots,$	$-m_1 - 1, 0, 1, \dots, (m_1 + 1)n_1 - 1,$
$m_1 n_1 + m_1 + n_1 + 1$	$(m_1 + 1)n_1 + 1, \dots, N - 1$
$(m_1 + 1)(n_1 + 1) + 1, (m_1 + 1)(n_1 + 1) + 2, \dots$	$-m_1 - 1, 0, 1, \dots, (m_1 + 1)n_1 - 1,$ $(m_1 + 1)n_1 + 1, \dots, (m_1 + 1)n_1 + m_1$

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