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Combustion instability due to the nonlinear interaction between sound and flame

By Xuesong Wu†, Meng Wang AND Parviz Moin

1. Introduction

Combustion instability generally refers to the sustained pressure fluctuations of acoustic nature in a chamber where unsteady combustion takes place. It is essentially a self-excited oscillation, involving a complex interplay between unsteady heat release, the acoustic fluctuation and the vorticity field, which according to experimental observations (e.g. Poinsot *et al.* 1987, Yu, Trouve & Daily 1991, Schadow & Gutmark 1992), may be described as follows. Unsteady heat release produces sound, which then generates (Kelvin-Helmholtz) instability waves at the inlet (via a receptivity mechanism as it is referred to in laminar-turbulent transition). These waves amplify and roll up on the shear layer and finally break down into small-scale motions, thereby affecting the heat release. The whole process forms a closed loop.

An important insight into the effect of unsteady heat release on sound amplification is provided by the Rayleigh criterion, which states that an acoustic wave will amplify if its pressure and the heat release are 'in phase', i.e. the integral of the product of the pressure and the unsteady heat release over a cycle is positive. The difficulty in applying this criterion is that unsteady heat release is often part of the solution and thus not known *a priori.* A usual remedy is to extrapolate, by using available experimental data, some empirical relations between the heat release and sound fluctuation. This then leads to a thermo-acoustic problem. Such an approach has been employed by Bloxsidge, Dowling & Langhorne (1988) to describe 'reheat buzz' (Langhorne 1988). Dowling (1995) formulated this approach in a more general setting, and discussed, *inter alia*, the effects of the mean Mach number and heat distribution.

In the above semi-empirical approach, the hydrodynamic (and chemical) processes of combustion are completely by-passed. To understand the acoustic-flame coupling on a first-principles basis, one has to look into the structure of the flame as well as its associated hydrodynamic field. Fortunately, for premixed flames much knowledge about the last two aspects above has been obtained by using the powerful asymptotic approach based on the large-activation-energy assumption (Williams 1985). The reader is referred to Clavin (1985, 1994) for detailed reviews of the subject. This framework as well as relevant previous results will be used in our work. Detailed discussions will be presented in Section 2.

A thorough theoretical treatment of sound-flame coupling is unrealistic at the present for a practical combustor, where the flow is strongly vortical and turbulent. As a first step, it is necessary to restrict to the simple case where the hydrodynamic motion is primarily due to unsteady heat release and remains laminar.

A formal formulation of acoustic-flame coupling has been given by Harten, Kapila & Matkowsky (1984) for what may be called the 'high-frequency' regime, where the acoustic time scale is comparable to the transit time of the flame, $O(d/U_L)$, where d and U_L stand

† Permanent address: Department of Mathematics, Imperial College, UK

for the flame thickness and speed respectively. The resulting system is nonlinear and requires a major numerical attack. Harten *et al.* considered the flat-flame case in the limits of low frequency and small heat release, and obtained in each limit the solution which describes the effect of acoustic pressure on the flame. However, they did not consider how the flame influences the sound. This inverse process was investigated by Clavin, Pelce & He (1990), who also removed the assumption of small heat release. By closing the loop, they were able to show that the mutual interaction leads to amplification of sound, i.e. to acoustic instability. For a *flat flame*, the hydrodynamics is completely absent, with the sole coupling being through the acoustic pressure affecting the temperature.

For a *curved flame*, there exists an additional coupling mechanism. As was pointed out by Markstein (1970), the sound pressure modulates the flame and hence alters its surface area. This in turn leads to modulation of heat release, thereby affecting the sound itself. The mechanism was further analyzed by Pelce & Rochwerger (1992) in connection with the experiments of Searby (1992), who observed that sound was generated when a curved flame was propagating downwards in a tube. The curved flame is due to the well known Darrieus-Landau (D-L) instability. In developing a mathematical mode, Pelce & Rochwerger represented the curved flame by the neutrally stable D-L instability mode (which exists due to the stabilizing effect of gravity). A constant amplitude is prescribed in calculating the growth rate of sound. They showed that this coupling mechanism could be stronger by an order of magnitude than that considered in Clavin *et al.* (1990).

The present work is aimed at improving the model of Pelce & Rochwerger (1992) in two somewhat related respects. First, we note that, like any marginally-stable mode, a neutral D-L mode must modulate in a weakly-nonlinear fashion rather than stay completely neutral. According to classical weakly-nonlinear theory (Stuart 1960), if the typical magnitude of the mode is ϵ, the time scale of modulation is $O(\epsilon^{-2})$, comparable with the time scale over which the sound amplifies. Second, Searby's (1992) experiments showed that the flame was evolving, and that the sound amplified mainly as the flame was evolving from a curved pattern to a flat one. Therefore for both mathematical and physical reasons, it is necessary to take into account the evolving nature of the flame as well as the back reaction of sound on the flame. For this purpose, we give a general formulation for the sound-flame interaction in what may be regarded as the 'low-frequency' regime in the sense that the acoustic time is much longer than the transit time of the flame. By using this basic framework, the nonlinear evolution of the acoustic and flame instability modes is studied in a systematic manner.

2. Formulation

Consider the combustion of a homogeneous premixed combustible mixture in a duct of width h^*; see Fig. 1. For simplicity, a one-step irreversible exothermic chemical reaction is assumed. The gaseous mixture consists of a single deficient reactant and an abundant component, and is assumed to obey the state equation for a perfect gas.

The fresh mixture has a density $\rho_{-\infty}$ and temperature $\Theta_{-\infty}$. Due to steady heat release, the mean temperature (density) behind the flame increases (decreases) to Θ_{∞} (ρ_{∞}). The flame propagates into the fresh mixture at a mean speed U_L, and it has an intrinsic thickness d. Let (x, y, z) and t denote the coordinates and time variables, normalized by h^* and h^*/U_L respectively. The velocity $\mathbf{u} \equiv (u, v, w)$, density ρ, temperature θ, and pressure p are non-dimensionalized by U_L, $\rho_{-\infty}$, $\Theta_{-\infty}$, and $\rho_{-\infty}U_L^2$ respectively.

We define the Mach number $M = U_L/a^*$, where the speed of sound $a^* = (\gamma p_{-\infty}/\rho_{-\infty})^{\frac{1}{2}}$, with γ being the ratio of specific heats.

A key simplifying assumption is that of large activation energy, corresponding to

$$\beta \equiv E(\Theta_\infty - \Theta_{-\infty})/\mathcal{R}\Theta_\infty^2 \gg 1 \ , \tag{2.1}$$

where E is the activation energy and $\mathcal{R}$ is the universal gas constant. Under this assumption the reaction occurs in a thin region of width $O(d/\beta)$ centered at the flame front. Assuming that the front is given by $x = f(y, z, t)$, it is convenient to introduce a coordinate system attached to the front,

$$\xi = x - f(y, z, t) \ , \quad \eta = y \quad \zeta = z \ ,$$

and to split the velocity $\mathbf{u}$ as $\mathbf{u} = u\mathbf{i} + \mathbf{v}$, where $\mathbf{i}$ is the unit vector along the duct. Then the governing equations can be written as (Matalon & Matkowsky 1982)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho s}{\partial \xi} + \triangledown \cdot (\rho \mathbf{v}) = 0 \ , \tag{2.2}$$

$$\rho\frac{\partial u}{\partial t} + \rho s\frac{\partial u}{\partial \xi} + \rho\,\mathbf{v}\cdot\triangledown\, u = -\frac{\partial p}{\partial \xi} + \delta Pr\Big\{\triangle u + \frac{1}{3}\frac{\partial}{\partial \xi}\Big(\frac{\partial s}{\partial \xi} + \triangledown\cdot\mathbf{v}\Big)\Big\} - \rho G \ , \tag{2.3}$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho s\frac{\partial \mathbf{v}}{\partial \xi} + \rho\,\mathbf{v}\cdot\triangledown\,\mathbf{v} = -\triangledown\, p + \triangledown f\frac{\partial p}{\partial \xi}$$
$$+\delta Pr\Big\{\triangle\,\mathbf{v} + \frac{1}{3}\Big(\triangledown - \triangledown f\frac{\partial}{\partial \xi}\Big)\Big(\frac{\partial s}{\partial \xi} + \triangledown\cdot\mathbf{v}\Big)\Big\} \ , \tag{2.4}$$

$$\rho\frac{\partial Y}{\partial t} + \rho s\frac{\partial Y}{\partial \xi} + \rho\,\mathbf{v}\cdot\triangledown Y = \delta Le^{-1}\triangle Y - \delta\Omega \ , \tag{2.5}$$

$$\rho\frac{\partial \theta}{\partial t} + \rho s\frac{\partial \theta}{\partial \xi} + \rho\,\mathbf{v}\cdot\triangledown\theta = \delta\triangle\theta + \delta q\Omega \ , \tag{2.6}$$

supplemented by the state equation $\gamma M^2 p = \rho\theta$, where $\delta = d/h^*$, $s = u - f_t - \mathbf{v}\cdot\triangledown f$,

$$\triangle = \Big[1 + (\triangledown f)^2\Big]\frac{\partial^2}{\partial \xi^2} + \triangledown^2 - \triangledown^2 f\frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \xi}(\triangledown f\cdot\triangledown) \ ;$$

here the operators $\triangledown$ and $\triangledown^2$ are defined with respect to η and ζ. Pr and Le denote the Prandtl and Lewis numbers respectively, and $G = gh^*/U_L^2$ is the normalized gravity force. The reaction rate Ω is taken to be described by the Arrhenius law:

$$\Omega \sim \delta^{-2}\rho Y \exp\Big\{\beta\Big(\frac{1}{\Theta_+} - \frac{1}{\theta}\Big)\Big\} \ , \tag{2.7}$$

where $\Theta_+ = 1+q$ is the adiabatic flame temperature. The large-activation-energy asymptotic approach requires the Lewis number Le to be close to unity, or more precisely

$$Le = 1 + \beta^{-1}l \quad \text{with} \quad l = O(1) \ . \tag{2.8}$$

To make analytical progress, we assume, in addition to $\beta \gg 1$, that

$$\delta \ll 1 \ , \quad M \ll 1 \ . \tag{2.9}$$

The whole flow field then is described by four distinct asymptotic regions as illustrated in Fig. 1. In addition to the thin reaction and preheated zones, there are also hydrodynamic and acoustic regions, which scale on h^* and h^*/M respectively. In the reaction zone, the

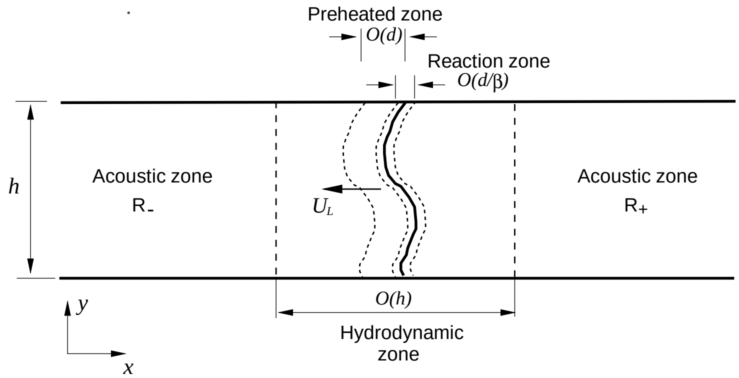

FIGURE 1. Sketch of the problem and the asymptotic structure

heat release due to the reaction balances the thermal diffusion, and the species variation balances the mass diffusion (Matkowsky & Sivashinsky 1979). In the preheated zone, the dominant balance is between the advection and diffusion. All the four regions are interactive, in that the complete solution relies on the investigation of all these regions.

The direct interaction between the sound and the flame is through the hydrodynamic region, which we now consider. In this region, the solution expands as

$$\left.\begin{array}{rcl}(\rho,\theta) & = & (R_0,\Theta)+\delta(\rho_1,\theta_1)+\dots \\ (u,\,\mathbf{v},\,f) & = & (u_0,\,\mathbf{v}_0,\,f_0)+\delta(u_1,\,\mathbf{v}_1,\,f_1)+\dots \\ p & = & (R_0G\xi)+p_0+\delta p_1+\dots\end{array}\right\} . \tag{2.10}$$

The solution for the density (Pelce & Clavin 1982, Matalon & Matkowsky 1982),

$$R_0=\begin{cases}1\equiv R_- & \xi<0\\ (1+q)^{-1}\equiv R_+ & \xi>0\ ,\end{cases}$$

is accurate to all orders in δ. In the following, the subscript '0' will be omitted. Substitution of Eq. (2.10) into Eqs. (2.2)–(2.4) leads to the equations governing $(u_0,\,\mathbf{v}_0,\,p_0)$:

$$\frac{\partial s_0}{\partial\xi}+\bigtriangledown\cdot\mathbf{v}_0=0\ , \tag{2.11}$$

$$R\left\{\frac{\partial u_0}{\partial t}+s_0\frac{\partial u_0}{\partial\xi}+\mathbf{v}_0\cdot\bigtriangledown\ u_0\right\}=-\frac{\partial p_0}{\partial\xi}\ , \tag{2.12}$$

$$R\left\{\frac{\partial\,\mathbf{v}_0}{\partial t}+s_0\frac{\partial\,\mathbf{v}_0}{\partial\xi}+\mathbf{v}_0\cdot\bigtriangledown\mathbf{v}_0\right\}=-\bigtriangledown p_0+\bigtriangledown f_0\frac{\partial p_0}{\partial\xi}-RG\bigtriangledown f_0\ , \tag{2.13}$$

where $s_0=u_0-f_{0,t}-\mathbf{v}_0\cdot\bigtriangledown f_0$.

Embedded in the hydrodynamic zone are the preheated zone and the much thinner reaction zone. The jump conditions across the preheated zone were first derived by Pelce & Clavin (1982) for $\mathbf{v},\,f\ll O(1)$, and by Matalon & Matkowsky (1982) in the general case $\mathbf{v},\,f\sim O(1)$. These are

$$[u_0]=q[1+(\bigtriangledown f_0)^2]^{-\frac{1}{2}}\ ,\quad[\mathbf{v}_0]=-q\bigtriangledown f_0/(1+(\bigtriangledown f_0)^2)^{1/2}\ ,\quad[p_0]=-q\ . \tag{2.14}$$

The front evolution is governed by the equation

$$f_{0,t}=u_0(0^-,\eta,\zeta,t)-\mathbf{v}_0(0^-,\eta,\zeta,t)\cdot\bigtriangledown f_0-[1+(\bigtriangledown f_0)^2]^{\frac{1}{2}}\ . \tag{2.15}$$

The results Eqs. (2.14)–(2.15) were originally derived by assuming that the flow is incompressible. Fortunately, they are valid for low-Mach-number flows because the acoustic pressure does not directly affect the preheated zone or the reaction zone to leading order. It only contributes a small correction of a higher order (see Clavin *et al.* 1990).

The leading-order system suffices for most part of our work. However, a more general result may be derived if we include the jumps at $O(\delta)$, which were derived by Pelce & Clavin (1982) and Matalon & Matkowsky (1982). The full version is rather complex, but our subsequent analysis requires only the linearized results:

$$[u_1] = -\tfrac{1}{2} l D(q)(\bigtriangledown^2 f_0 + \bigtriangledown \cdot \mathbf{v}_0) \ , \quad [p_1] = -2\,[u_1] + q\,\bigtriangledown^2 f_0 + \ln(1+q)\frac{\partial u_0}{\partial t} \ , \qquad (2.16)$$

$$[\mathbf{v}_1] = Pr\left[\frac{\partial\,\mathbf{v}_0}{\partial \xi}\right] + \ln(1+q)\left\{\frac{\partial}{\partial t}(\mathbf{v}_0 + \bigtriangledown f_0) + G \bigtriangledown f_0\right\} - q \bigtriangledown f_1 \ , \qquad (2.17)$$

where $D(q) = \displaystyle\int_0^\infty \ln(1 + q\,\mathrm{e}^{-x})\,\mathrm{d}\,x$, u_0 and $\mathbf{v}_0$ as well as their derivatives are evaluated at the front $\xi = 0^-$. The function f_1 satisfies the equation

$$f_{1,t} = u_1(0, \eta, \zeta, t) + \left\{\frac{1+q}{q}\ln(1+q) + \frac{1}{2q} l D(q)\right\}\left\{\bigtriangledown^2 f_0 + \bigtriangledown \cdot \mathbf{v}_0\right\} . \qquad (2.18)$$

3. Strongly-nonlinear sound-flame interaction: a general formulation

3.1. *Acoustic zone*

The appropriate variable describing the acoustic motion in this region is

$$\tilde{\xi} = M\xi \ . \qquad (3.1)$$

Because the transverse length is much smaller than the longitudinal length, the motion is a longitudinal oscillation about the uniform mean background, and the solution, for the velocity and pressure say, can be written as

$$u = U_\pm + u_a(\tilde{\xi}, t) + \dots \ , \quad p = \frac{1}{\gamma M^2} + M^{-1} p_a(\tilde{\xi}, t) + \dots \ , \qquad (3.2)$$

where $U_\pm$ are the mean velocities behind and in front of the flame respectively, with $U_+ - U_- = q$. The pressure p_a and velocity u_a satisfy the linearized equations

$$R\frac{\partial^2 p_a}{\partial t^2} - \frac{\partial^2 p_a}{\partial \tilde{\xi}^2} = 0 \ , \quad \text{and} \quad R\frac{\partial u_a}{\partial t} = \frac{\partial p_a}{\partial \tilde{\xi}} \ . \qquad (3.3)$$

As $\tilde{\xi} \to \pm 0$,

$$u_a \to u_a(0^\pm, t) + \dots \ , \quad p_a \to p_a(0, t) + p_a'(0^\pm, t)\tilde{\xi} + \dots \ .$$

As will be shown in Section 3.2, the acoustic pressure is continuous across the flame, but the flame induces a jump in u_a i.e.

$$[p_a] = 0 \ , \quad [u_a] = q\left\{\overline{(1 + (\bigtriangledown F_0)^2)^{\frac{1}{2}}} - 1\right\} , \qquad (3.4)$$

where $\overline{\phi}$ stands for the space average of ϕ in the (η, ζ) plane, and F_0 is defined in Eq. (3.5).

3.2. *Hydrodynamic zone*

In the hydrodynamic zone, u_a and $p_{a,\tilde{\xi}}$ appear spatially uniform on either side of the flame, and can be approximated by their values at $\tilde{\xi} = 0^\pm$. To facilitate the matching with the solution in the acoustic region, we subtract from the total field the acoustic components as well as the mean background flow, by writing

$$u_0 = U_\pm + u_a(0^\pm, t) + U_0 \,, \quad p_0 = \frac{1}{\gamma M^2} + P_\pm + p_a'(0^\pm, t)\xi + P_0 \,, \quad f_0 = F_a + F_0 \,, \tag{3.5}$$

where $P_\pm$ is the mean pressure (with $P_+ - P_- = q$), and $F_a' = U_- - 1 + u_a(0^-, t)$. Let $\mathbf{v}_0 = \mathbf{V}_0$. Then the leading-order hydrodynamic field satisfies the following equations

$$\frac{\partial U_0}{\partial \xi} + \bigtriangledown \cdot \mathbf{V}_0 = \frac{\partial \mathbf{V}_0}{\partial \xi} \cdot \bigtriangledown F_0 \,, \tag{3.6}$$

$$\frac{\partial U_0}{\partial \xi} + R\left\{\frac{\partial \tilde{U}_0}{\partial t} + S_0 \frac{\partial U_0}{\partial \xi} + \mathbf{V}_0 \cdot \bigtriangledown U_0\right\} = -\frac{\partial P_0}{\partial \xi} - R\mathcal{J}h(\xi)\frac{\partial U_0}{\partial \xi} \,, \tag{3.7}$$

$$\frac{\partial \mathbf{V}_0}{\partial \xi} + R\left\{\frac{\partial \mathbf{V}_0}{\partial t} + S_0 \frac{\partial \mathbf{V}_0}{\partial \xi} + \mathbf{V}_0 \cdot \bigtriangledown \mathbf{V}_0\right\} = -\bigtriangledown P_0 + \bigtriangledown F_0 \frac{\partial P_0}{\partial \xi} - R\mathcal{J}h(\xi)\frac{\partial \mathbf{V}_0}{\partial \xi}$$
$$-RG \bigtriangledown F_0 + p_a'(0^\pm, t) \bigtriangledown F_0 \,, \tag{3.8}$$

while the flame front is governed by

$$F_{0,t} = U_0 - \mathbf{V}_0 \cdot \bigtriangledown F_0 - \left\{(1 + (\bigtriangledown F_0)^2)^{\frac{1}{2}} - 1\right\} \,, \tag{3.9}$$

where $h(\xi)$ is the Heaviside step function, $\mathcal{J} = [u_a]$, and $S_0 = U_0 - F_{0,t} - \mathbf{V}_0 \cdot \bigtriangledown F_0$. Matching with the outer acoustic solution requires that

$$U_0 \to 0 \,, \quad \mathbf{V}_0 \to 0 \,, \quad P_{0,\xi} \to 0 \quad \text{as} \quad \xi \to \pm\infty \,. \tag{3.10}$$

The unsteady pressure and transverse velocity jumps are

$$[P_0] = 0 \,, \quad [\mathbf{V}_0] = -q \bigtriangledown F_0/(1 + (\bigtriangledown F_0)^2)^{1/2} \,. \tag{3.11}$$

The hydrodynamic motion affects the ambient acoustic regions by inducing a longitudinal velocity jump. To derive this key result, we take the spatial average of Eq. (3.6) in the (η, ζ) plane, and integrate with respect to ξ to obtain $\overline{U_0} = \overline{\mathbf{V}_0 \cdot \bigtriangledown F_0}$, where the overbar denotes the mentioned spatial average. Inserting the first relation in Eq. (3.5) into Eq. (2.14), and taking the spatial average and using the second relation in Eq. (3.11), we find

$$\mathcal{J} = [u_a] = q\left\{\overline{(1 + (\bigtriangledown F_0)^2)^{\frac{1}{2}} - 1}\right\} \,. \tag{3.12}$$

On the scale of acoustic wavelength, the right-hand side is equivalent to the rate of a concentrated unsteady heat release, which is shown to be proportional to the change of the surface area of the flame.

The jump condition for U_0 becomes

$$[U_0] = q\left\{(1 + (\bigtriangledown F_0)^2)^{-\frac{1}{2}} - \overline{(1 + (\bigtriangledown F_0)^2)^{\frac{1}{2}}}\right\} \,. \tag{3.13}$$

The hydrodynamic equations, Eqs. (3.6)–(3.9), and the acoustic equations Eq. (3.3) form an overall interactive system via Eq. (3.4): the acoustic pressure modulates the flame, which in turn drives sound by producing unsteady heat release. This system uses two distinct spatial variables to describe two distinct motions so that, in terms of $\tilde{\xi}$, the acoustic motion has an $O(1)$ characteristic speed (see Eq. (3.3)), comparable with the hydrodynamic velocity. This has a significant advantage from the numerical point of view, because the acoustic speed does not impose a severe restriction on the time step.

4. A weakly nonlinear case

A flat flame may become unstable owing to differential diffusivity of mass and heat, or to the hydrodynamic effect associated with gas expansion. The latter is the D-L instability mentioned in Section 1. An interesting question is: how large-scale combustion instability is related to flame instabilities, which occur over small scales over which the unsteady flow can be treated as incompressible. A natural proposal is that combustion instability arises when acoustic modes of the chamber are excited and amplified by the flame instabilities through mutual resonance. D-L instability perhaps is the most important candidate for driving combustion instability since, for most mixtures, the Lewis number is close to unity so that the instability due to differential diffusivity is ruled out.

In general, D-L instability occurs at all wavenumbers. However it can be stabilised by gravity effect, which introduces a small-wavenumber cut-off (Pelce & Clavin 1982). The mode with this cut-off wavenumber is nearly neutral. On the other hand, an acoustic mode is neutral on a linear basis. A mutual interaction can take place between the two when their magnitudes are still small. Such a weakly-nonlinear coupling will be analysed by using the general formulation in Section 3. The present analysis is motivated by the experiments of Searby (1992), where such an effect apparently operates.

4.1. *Analysis of the hydrodynamics of the flame*

For simplicity, we assume that the flame is two dimensional. The flame is stable when the flame speed U_L is less than the critical value $U_L = (gh^*/(\pi(1+q)))^{\frac{1}{2}}$, as was shown by Pelce & Clavin (1982); see also below. Suppose that the magnitude of the nearly neutral D-L mode is of $O(\epsilon)$. Then the weakly nonlinear interaction takes place over the time scale of $O(\epsilon^{-2})$ (Stuart 1960), and thus we introduce the slow variable

$$\tau = \epsilon^2 t \ . \tag{4.1}$$

In keeping with this, U_L is allowed to deviate from its critical value by $O(\epsilon^2)$, and thus we write

$$\frac{gh^*}{U_L^2} = \pi(1+q) + \epsilon^2 g_d \equiv G_c + \epsilon^2 g_d \quad \text{with} \quad g_d = O(1) \ . \tag{4.2}$$

To take account of the effect of Markstein length, we assume that $\delta = O(\epsilon^2)$, and without losing generality we take $\epsilon^2 = \delta$.

The velocity and pressure in the hydrodynamic region expand as

$$(U_0, V_0, P_0) = \epsilon(\hat{U}_1, \hat{V}_1, \hat{P}_1) + \epsilon^2(\hat{U}_2, \hat{V}_2, \hat{P}_2) + \epsilon^3(\hat{U}_3, \hat{V}_3, \hat{P}_3) + \dots \ . \tag{4.3}$$

The expansion of F_0 is somewhat unusual and has the form

$$F_0 = \hat{F}_0(\tau) + \epsilon\hat{F}_1 + \epsilon^2\hat{F}_2 + \epsilon^3\hat{F}_3 + \dots \ , \tag{4.4}$$

where the $O(1)$ term is due to the advection of the front by the accumulated streaming effect. By substituting the expansion into Eqs. (3.6)–(3.9) and expanding to $O(\epsilon^3)$, we obtain a sequence of equations at $O(\epsilon^n)$ $(n = 1, 2, 3)$.

The leading-order solution is given by (cf. Pelce & Clavin 1982)

$$\left.\begin{array}{rcl} (\hat{U}_1,\, \hat{P}_1,\, \hat{F}_1) &=& A(\tau)\left\{(-P^{\pm}\,\mathrm{e}^{-k\xi} + C^{\pm}),\, (P^{\pm}\,\mathrm{e}^{-k\xi} - R_{\pm}G_cF_1),\, F_1\right\}(\mathrm{e}^{\mathrm{i}\,k\eta} + c.c.) \\ \hat{V}_1 &=& A(\tau)P^{\pm}\,\mathrm{e}^{-k\xi}(\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,k\eta} + c.c.) \end{array}\right\} \tag{4.5}$$

where A is the amplitude function of the D-L mode, and $C^- = 0$ to satisfy the upstream matching condition. The wavenumber $k = \pi$ so that $\hat{V}_1 = 0$ at $\eta = 0, 1$. The front equation implies that $P^- = 0$, while the jump conditions are given by the linearized version of Eq. (3.11) and Eq. (3.13), i.e.

$$P^+ - R_+ G_c F_1 = -R_- G_c F_1, \quad -P^+ + C^+ = 0, \quad P^+ = -qkF_1 \ .$$

The requirement of a non-zero solution gives the eigen-relation: $gh^*/U_L^2 = (1+q)\pi$. The eigenfunction is normalized by setting $F_1 = 1$, and then $P^+ = C^+ = -q\pi \equiv P$.

The $O(\epsilon^2)$ terms in Eq. (4.3) and Eq. (4.4) are governed by the following equations

$$\frac{\partial \hat{U}_2}{\partial \xi} + \frac{\partial \hat{V}_2}{\partial \eta} = \frac{\partial \hat{V}_1}{\partial \xi}\frac{\partial \hat{F}_1}{\partial \eta} \ , \tag{4.6}$$

$$R\frac{\partial \hat{U}_2}{\partial t} + \frac{\partial \hat{U}_2}{\partial \xi} = -\frac{\partial \hat{P}_2}{\partial \xi} - R\Big\{\hat{U}_1\frac{\partial \hat{U}_1}{\partial \xi} + \hat{V}_1\frac{\partial \hat{U}_1}{\partial \eta}\Big\} \ , \tag{4.7}$$

$$R\frac{\partial \hat{V}_2}{\partial t} + \frac{\partial \hat{V}_2}{\partial \xi} = -\frac{\partial \hat{P}_2}{\partial \eta} - R\Big\{\hat{U}_1\frac{\partial \hat{V}_1}{\partial \xi} + \hat{V}_1\frac{\partial \hat{V}_1}{\partial \eta}\Big\} \tag{4.8}$$

$$+\frac{\partial \hat{P}_1}{\partial \xi}\frac{\partial \hat{F}_1}{\partial \eta} - RG_c\frac{\partial \hat{F}_2}{\partial \eta} + p'_{a,1}(0^\pm, t)\frac{\partial \hat{F}_1}{\partial \eta} \ , \tag{4.9}$$

$$\hat{F}_{2,t} = \hat{U}_2(0^-, \eta, t) - \hat{V}_1(0, \eta, t)\frac{\partial \hat{F}_1}{\partial \eta} - \frac{1}{2}\Big(\frac{\partial \hat{F}_1}{\partial \eta}\Big)^2 \ , \tag{4.10}$$

subject to the jump conditions

$$\Big[\hat{U}_2\Big] = -\frac{1}{2}q\Big[(\bigtriangledown \hat{F}_1)^2 + \overline{(\bigtriangledown \hat{F}_1)^2}\Big] \ , \quad \Big[\hat{V}_2\Big] = -q \bigtriangledown \hat{F}_2 \ , \quad \Big[\hat{P}_2\Big] = 0 \ . \tag{4.11}$$

As the forcing terms on the right-hand side indicate, there exists a mutual interaction between the sound and flame as well as the self-interaction of the flame. The solution, for $\hat{U}_2$ and $\hat{F}_2$ say, takes the form

$$\left.\begin{array}{lcl} \hat{U}_2 & = & \hat{U}_{2,a}AB(\mathrm{e}^{\mathrm{i}\,k\eta} + c.c.)\,\mathrm{e}^{\mathrm{i}\,\omega t} + \hat{U}_{2,2}A^2(\mathrm{e}^{2\,\mathrm{i}\,k\eta} + c.c.) + \hat{U}_{2,0}A^2 \\ \hat{F}_2 & = & \hat{F}_{2,a}AB(\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,k\eta} + c.c.)\,\mathrm{e}^{\mathrm{i}\,\omega t} + \hat{F}_{2,2}A^2(\mathrm{e}^{2\,\mathrm{i}\,k\eta} + c.c.) \end{array}\right\} \ . \tag{4.12}$$

At cubic order, the governing equations are found to be

$$\frac{\partial \hat{U}_3}{\partial \xi} + \frac{\partial \hat{V}_3}{\partial \eta} = \frac{\partial \hat{V}_1}{\partial \xi}\frac{\partial \hat{F}_2}{\partial \eta} + \frac{\partial \hat{V}_2}{\partial \xi}\frac{\partial \hat{F}_1}{\partial \eta} \ , \tag{4.13}$$

$$R\frac{\partial \hat{U}_3}{\partial t} + \frac{\partial \hat{U}_3}{\partial \xi} = -\frac{\partial \hat{P}_3}{\partial \xi} - RA'\hat{U}_1 - R\Big\{\hat{U}_1\frac{\partial \hat{U}_2}{\partial \xi} + \hat{U}_2\frac{\partial \hat{U}_1}{\partial \xi} + \hat{V}_1\frac{\partial \hat{U}_2}{\partial \eta} + \hat{V}_2\frac{\partial \hat{U}_1}{\partial \eta}\Big\}$$

$$+R\Big\{\hat{F}_{0,\tau} + \hat{F}'_{2,a} + \hat{V}_1\frac{\partial \hat{F}_1}{\partial \eta}\Big\}\frac{\partial \hat{U}_1}{\partial \xi} - R\mathcal{J}h(\xi)\frac{\partial \hat{U}_1}{\partial \xi} + Pr \bigtriangledown^2 \hat{U}_1 \ , \tag{4.14}$$

$$R\frac{\partial \hat{V}_3}{\partial t} + \frac{\partial \hat{V}_3}{\partial \xi} = -\frac{\partial \hat{P}_3}{\partial \eta} - RA'\hat{V}_1 - R\Big\{\hat{U}_1\frac{\partial \hat{V}_2}{\partial \xi} + \hat{U}_2\frac{\partial \hat{V}_1}{\partial \xi} + \hat{V}_1\frac{\partial \hat{V}_2}{\partial \eta} + \hat{V}_2\frac{\partial \hat{V}_1}{\partial \eta}\Big\}$$

$$+p'_{a,1}(0,t)\frac{\partial \hat{F}_2}{\partial \eta} + \frac{\partial \hat{P}_1}{\partial \xi}\frac{\partial \hat{F}_2}{\partial \eta} + \frac{\partial \hat{P}_2}{\partial \xi}\frac{\partial \hat{F}_1}{\partial \eta} - R_\pm G_c\frac{\partial \hat{F}_3}{\partial \eta}$$

$$+R\Big\{\hat{F}_{0,\tau} + \hat{F}'_{2,a} + \hat{V}_1\frac{\partial \hat{F}_1}{\partial \eta}\Big\}\frac{\partial \hat{V}_1}{\partial \xi} - R\mathcal{J}h(\xi)\frac{\partial \hat{V}_1}{\partial \xi} + Pr \bigtriangledown^2 \hat{V}_1 \ , \tag{4.15}$$

where $\mathcal{J} = qk^2$. Expansion of the front equation gives

$$\hat{F}_{1,\tau} + \hat{F}_{3,t} = \hat{U}_3(0^-, t) - \hat{V}_1 \cdot \bigtriangledown \hat{F}_2 - \hat{V}_2 \cdot \bigtriangledown \hat{F}_1 - \bigtriangledown \hat{F}_1 \cdot \bigtriangledown \hat{F}_2 \ . \tag{4.16}$$

To this order, it is only necessary to consider the component that coincides with the fundamental of the D-L mode, and thus we write

$$(\hat{U}_3,\, \hat{P}_3,\, \hat{F}_3) = (\hat{U}_{3,1},\, \hat{P}_{3,1},\, \hat{F}_{3,1})(\mathrm{e}^{\mathrm{i}\,k\eta} + c.c.)\ , \quad \hat{V}_3 = \hat{V}_{3,1}(\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,k\eta} + c.c.)\ . \tag{4.17}$$

The jump conditions at this order need some attention. A direct expansion of Eq. (3.11) and Eq. (3.13) shows that at $O(\epsilon^3)$,

$$\left[\hat{U}_3\right] = -q \bigtriangledown \hat{F}_1 \cdot \bigtriangledown \hat{F}_2\ , \quad \left[\hat{P}_3\right] = 0\ , \quad \left[\hat{V}_3\right] = -q \bigtriangledown \hat{F}_3 + \frac{q}{2}(\bigtriangledown \hat{F}_1)^2 \bigtriangledown \hat{F}_1\ . \tag{4.18}$$

However, since $\delta = O(\epsilon^2)$, the $(\epsilon\delta)$ terms in Eq. (2.10) are of the same order as the $O(\epsilon^3)$ terms in Eq. (4.5). The jumps Eqs. (2.16)-(2.17) must be added to Eq. (4.18) to give

$$\left.\begin{aligned} \left[\hat{U}_{3,1}\right] &= \tfrac{1}{2} lD(k)k^2 A - 2qk^2 \hat{F}_{2,2} A^3 \\ \left[\hat{P}_{3,1}\right] &= -lD(q)k^2 A - qk^2 A \\ \left[\hat{V}_{3,1}\right] &= -kqF_{3,1} - Prqk^2 A + \ln(1+q)(kG_c A) + \tfrac{3}{2} qk^3 A^3 \end{aligned}\right\}\ . \tag{4.19}$$

The equation controlling the front motion is

$$A' = \hat{U}_{3,1}(0^-) - 2k^2 \hat{F}_{2,2} A^3 - k\hat{V}_{2,2}(0)A^3 - k^2 \Big\{ \frac{1+q}{q} \ln(1+q) + \frac{1}{2q} lD(q) \Big\} A\ . \tag{4.20}$$

After substituting the leading- and second-order solutions into the right-hand sides of Eqs. (4.13)-(4.15), the solution for $\hat{U}_{3,1}$, $\hat{V}_{3,1}$, etc. can be written down. Inserting it into Eq. (4.19) and Eq. (4.20), we obtain the amplitude equation

$$A' = \kappa A + \gamma_s A^3 - \gamma_b |B|^2 A\ , \tag{4.21}$$

$$\kappa = -\frac{q}{2(1+q)} g_d - \frac{1}{2} k^2 \left\{ q + \frac{1+q}{q} \Big((q+2)\ln(1+q) + lD(q) \Big) \right\}\ , \tag{4.22}$$

$$\gamma_s = \Big\{ -\frac{1}{2} q + \frac{3}{2} + \frac{2}{q} \Big\} k^3 = (4-q)(1+q)k^3/(2q)\ , \tag{4.23}$$

$$\gamma_b = \Big\{ 4(R_+ - R_-)^2 (1 + R_+/R_-) k\omega^2 \sin^2(R_-^{\frac{1}{2}} \sigma\omega L) \Big\} / \Big\{ (R_+ + R_-)^2 \omega^2 + 4k^2 \Big\}\ . \tag{4.24}$$

4.2. *Analysis of the acoustics*

The pressure and velocity of the acoustic fluctuation are expanded as

$$p_a = \epsilon B(\tau) p_{a,1} + \epsilon^3 p_{a,2} + \dots\ , \quad u_a = \epsilon B(\tau) u_{a,1} + \epsilon^3 u_{a,2} + \dots\ , \tag{4.25}$$

where B is the amplitude function.

To leading order, $p_{a,1}$ and $u_{a,1}$ satisfy Eq. (3.3), and they have the solution

$$\left.\begin{aligned} p_{a,1} &= \mathrm{e}^{\mathrm{i}\,\omega t} \Big[a_r^{\pm}\, \mathrm{e}^{-\mathrm{i}\, R_{\pm}^{\frac{1}{2}} \omega \tilde{\xi}} + a_l^{\pm}\, \mathrm{e}^{\mathrm{i}\, R_{\pm}^{\frac{1}{2}} \omega \tilde{\xi}} \Big] \\ u_{a,1} &= \mathrm{e}^{\mathrm{i}\,\omega t}\, R_{\pm}^{-\frac{1}{2}} \Big[a_r^{\pm}\, \mathrm{e}^{-\mathrm{i}\, R_{\pm}^{\frac{1}{2}} \omega \tilde{\xi}} - a_l^{\pm}\, \mathrm{e}^{\mathrm{i}\, R_{\pm}^{\frac{1}{2}} \omega \tilde{\xi}} \Big] \end{aligned}\right\} \tag{4.26}$$

where $a_r^{\pm}$ and $a_l^{\pm}$ are constants, and for convenience we take $a_l^- = \mathrm{e}^{\mathrm{i}\, R_-^{\frac{1}{2}} \sigma\omega L}$. The end conditions are: $u_{a,1} = 0$ at $\tilde{\xi} = -\sigma L$, and $p_{a,1} = 0$ at $\tilde{\xi} = (1-\sigma)L$, where L is related to the dimensional length of the duct l^* by $L = Ml^*/h^*$, and σ is a parameter characterizing the mean position of the flame front. Both $u_{a,1}$ and $p_{a,1}$ are continuous across the flame, i.e. $[u_{1,a}]=0$ and $[p_{1,a}]=0$, as the expansion of Eq. (3.4) shows. Application of these conditions leads to the dispersion relation of the acoustic mode (cf. Clavin *et al.* 1990),

$$(R_+/R_-)^{\frac{1}{2}}\tan(R_-^{\frac{1}{2}}\sigma\omega L)\tan(R_+^{\frac{1}{2}}(1-\sigma)\omega L)=1\ . \tag{4.27}$$

Inserting Eq. (4.25) into Eq. (3.3), and solving the resultant equations at $O(\epsilon^3)$, we find

$$p_{a,2}=\mathrm{e}^{\mathrm{i}\,\omega t}\Big\{\Big(b_r^{\pm}\,\mathrm{e}^{-\,\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}+b_l^{\pm}\,\mathrm{e}^{\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}\Big)-R_\pm^{\frac{1}{2}}B'\tilde{\xi}\,\Big(a_r^{\pm}\,\mathrm{e}^{-\,\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}-a_l^{\pm}\,\mathrm{e}^{\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}\Big)\Big\}\ ,$$

$$u_{a,2}=\mathrm{e}^{\mathrm{i}\,\omega t}\Big\{R_\pm^{-\frac{1}{2}}\Big(b_r^{\pm}\,\mathrm{e}^{-\,\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}-b_l^{\pm}\,\mathrm{e}^{\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}\Big)-B'\tilde{\xi}\Big(a_r^{\pm}\,\mathrm{e}^{-\,\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}+a_l^{\pm}\,\mathrm{e}^{\mathrm{i}\,R_\pm^{\frac{1}{2}}\omega\tilde{\xi}}\Big)\Big\}\ .$$

It follows from substituting F_0 into Eq. (3.4) and expanding to $O(\epsilon^3)$ that

$$[p_{2,a}]=0\ ,\qquad [u_{2,a}]=2qk^2\hat{F}_{2,a}A^2B\ ,$$

The above relations together with the end conditions, $u_{a,2}=0$ at $\tilde{\xi}=-\sigma L$ and $p_{a,2}=0$ at $\tilde{\xi}=(1-\sigma)L$, lead to the amplitude equation for the acoustic mode:

$$B'=\chi A^2B\ , \tag{4.28}$$

$$\chi=\frac{\mathrm{i}\,2qk^3R_-^{-\frac{1}{2}}(R_+-R_-)\Lambda}{L(\mathrm{i}(R_++R_-)\omega+2k)}\ , \tag{4.29}$$

$$\Lambda=\frac{\tan(R_-^{\frac{1}{2}}\sigma\omega L)}{\sigma\sec^2(R_-^{\frac{1}{2}}\sigma\omega L)+(1-\sigma)(R_+/R_-)\sec^2(R_+^{\frac{1}{2}}(1-\sigma)\omega L)\tan^2(R_-^{\frac{1}{2}}\sigma\omega L)}\ . \tag{4.30}$$

4.3. *Amplitude equations*

The sound-flame interaction is thus described by the coupled amplitude equations

$$A'(\tau)=\kappa A+\gamma_sA^3-\gamma_b|B|^2A\ ,\quad B'(\tau)=\chi A^2B\ . \tag{4.31}$$

Now if the flame amplitude A is taken to be a constant, then the equation for B reduces to the result of Pelce & Rochwerger (1992) with B growing exponentially. In their model, the coupling is one-way. The present work includes the back-effect of the sound on the flame, leading to a better description of the experiments of Searby (1992); see below.

The effects of the nonlinear interactions become clear if one inspects the signs of the coefficients. According to Eq. (4.29) and Eq. (4.24), $\Re(\chi)>0$ and $\gamma_b>0$, indicating that the flame always acts to amplify the acoustic field, while sound inhibits the flame. Note also that γ_s is positive (negative) for $q<4$ ($q>4$), and hence the self-nonlinearity of the flame is destabilizing for $q<4$ and stabilizing for $q>0$.

Assuming that the flame and sound are weak initially so that the nonlinear terms in the amplitude equations can be ignored, then the appropriate initial conditions are

$$A\sim\mathrm{e}^{\kappa\tau}\ ,\quad B\sim b_0\exp\{\chi\,\mathrm{e}^{2\kappa\tau}\}\quad \text{as}\quad \tau\to-\infty\ , \tag{4.32}$$

where $b_0\ll 1$. Figure 2 shows the evolution of A and B for $b_0=0.1$, 0.05, (with γ_s, γ_b and χ being arbitrarily taken to be unity). The background noise remains constant when the flame is of small amplitude, but starts to amplify when the latter has gained a certain strength. The amplification is extremely abrupt, taking place primarily when the curved flame evolves into a flat one. The flattening of the flame is caused by the back-reaction of the sound. Eventually the sound saturates at a constant level. For comparison purposes,

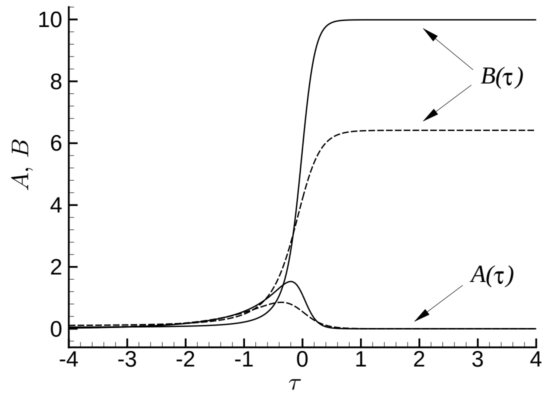

FIGURE 2. Nonlinear evolution of the acoustic amplitude B and flame amplitude A. ——— $b_0 = 0.05$; - - - - $b_0 = 0.1$.

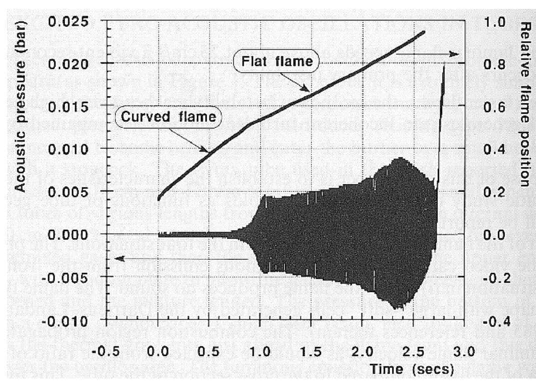

FIGURE 3. Time traces of the acoustic pressure and flame position from Searby's experiment (Fig. 3b of Searby (1992)).

Searby's experimental results are shown in Fig. 3. It is clear that the present theoretical predictions are entirely consistent with his observations in the qualitative sense.

5. Conclusions

In this paper, the acoustic-flame coupling, the key process underlying combustion instability, is studied by using matched-asymptotic-expansion techniques based on the assumptions of large activation energy and low Mach number. A general asymptotic formulation was given for the lower-frequency regime of practical relevance, for which the acoustic source is found to be directly linked to the shape of the flame. The basic framework was then used to study the weakly nonlinear interaction between an acoustic mode of the duct and a nearly-neutral D-L instability mode. A system of coupled amplitude equations was derived, and was found able to describe the experimental observations of Searby (1992) qualitatively.

We note that the present analysis can be extended to include the effect of 'weak turbulence' (i.e. convected gusts) in the oncoming fresh mixture. It would be interesting to solve the fully-nonlinear system in Section 3 numerically, with a view to addressing whether or not the coupling leads to self-sustained large-amplitude pressure oscillations.

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