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Cite as: J. Math. Phys. **52**, 084101 (2011); https://doi.org/10.1063/1.3626942
Submitted: 18 July 2010 • Accepted: 02 August 2011 • Published Online: 29 August 2011

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(Received 18 July 2010; accepted 2 August 2011; published online 29 August 2011)

Hall and Mitchell described a family of heat kernels (or equivalently coherent states) and an associated resolution of the identity for a quantum particle whose classical configuration space is the d-dimensional sphere S^d J. Math. Phys. **43**(3), 1211 (2002). These heat kernels were chosen intelligently but in the case of d = 2, "one" of the formulas for the heat kernel must be corrected. © *2011 American Institute of Physics*. [doi:10.1063/1.3626942]

Hall and Mitchell[1,2] (hereafter referenced as HM) have presented the heat kernels for a movement quantum particle in which its phase space is cotangent bundle $T^*(S^d)$. These heat kernels are connected to the coherent states that satisfy the resolution of the identity. All of these heat kernels depend on two primary heat kernels ($d = 1, 2$). It should be pointed out that the first formula of HM (p. 1223) is correct, but formula (42) on p. 1224 is incorrect. To obtain the correct form, we use the heat kernel on the two-sphere "S^2" that was presented by Camporesi (see Eq. (4.39) of Ref. 3), i.e.,

$$K_{S^2}(\theta', \tau) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)e^{-\frac{\tau}{2}l(l+1)}P_l(\cos\theta'), \tag{1}$$

where $0 < \theta' < \pi$. By analytic continuation, we can write the heat kernel on the complex sphere $S^2_{\mathbb{C}}$ as follows:

$$K_{S^2_{\mathbb{C}}}(\theta, \tau) = \rho^2_\tau(\mathbf{a}, \mathbf{x}) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)e^{-\frac{\tau}{2}l(l+1)}P_l(\cos\theta), \tag{2}$$

where θ is a complex with $0 < Re(\theta) < \pi$. In the other words, in Eq. (42) of HM paper, the factor of $\sqrt{2l+1}$ should actually be $(2l+1)$ and there should be a normalization factor of $\frac{1}{4\pi}$. The reader can directly obtain this relation by the Poisson summation formula (see Appendix).

APPENDIX: THE CALCULATION OF HEAT KERNEL ON THE COMPLEX SPHERE

To obtain the correct form we start with the given equation on p. 1223 of Ref. 1:

$$\rho^2_\tau(\overrightarrow{a}, \overrightarrow{x}) = e^{\frac{\tau}{8}}\frac{(2\pi\tau)^{-1}}{\sqrt{\pi\tau}}\int_\theta^\pi \frac{1}{\sqrt{\cos\theta - \cos\phi}}\sum_{n=-\infty}^{\infty}(-1)^n(\phi - 2\pi n)e^{-\frac{(\phi-2\pi n)^2}{2\tau}}d\phi,$$

where $0 < Re(\theta) < \pi$. We can rewrite this equation by

$$\rho^2_\tau(\overrightarrow{a}, \overrightarrow{x}) = -e^{\frac{\tau}{8}}\frac{(2\pi)^{-1}}{\sqrt{\pi\tau}}\int_\theta^\pi \frac{1}{\sqrt{\cos\theta - \cos\phi}}\frac{d}{d\phi}\Big(\sum_{n=-\infty}^{\infty}e^{-in\pi}e^{-\frac{(\phi-2\pi n)^2}{2\tau}}\Big)d\phi.$$

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By using the Poisson summation formula,

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \widehat{f}(n),$$

where

$$\widehat{f}(n) = \int_{-\infty}^{\infty} f(y) e^{2\pi i n y} dy = e^{-in\pi} e^{-\dfrac{(\phi - 2\pi n)^2}{2\tau}},$$

we obtain

$$\rho_\tau^2(\overrightarrow{a}, \overrightarrow{x}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) e^{-\frac{\tau}{2} n(n+1)} \Big(\frac{\sqrt{2}}{\pi} \int_\theta^\pi \frac{\sin \phi (n + \frac{1}{2})}{\sqrt{\cos\theta - \cos\phi}} d\phi \Big).$$

The Legendre integral representation (see Refs. 4 or 5) leads us to rewrite this heat kernel as follows:

$$\rho_\tau^2(\mathbf{a}, \mathbf{x}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) e^{-\frac{\tau}{2} n(n+1)} P_n(\cos\theta).$$

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