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Comment on "Lanczos potential of Weyl field: interpretations and applications"

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Abstract We show that a proposal by Vishwakarma to realize conformal-covariance for the Weyl-Lanczos equation is nonviable.

In the recently published review article [1] Vishwakarma considers, among other topics, the question of how the Lanczos potential $L_{\mu\nu\sigma}$ in a four-dimensional Riemannian spacetime behaves under a conformal transformation. In attempting to formulate a consistent conformally-covariant treatment of the potential, the author is led to impose a complicated algebraic constraint among the components of $L_{\mu\nu\sigma}$. However, we show below that Vishwakarma's proposed constraint is incompatible with the existence of a Lanczos potential in a large class of spacetimes and therefore must be rejected. We briefly recap the author's argument (denoting equation numbers that appear in [1] with the prefix "V") and follow with our critique.[1]

Vishwakarma begins with the Weyl–Lanczos equation, expressing the Weyl conformal curvature tensor $C_{\mu\nu\sigma\rho}$ as a linear combination of first-derivatives of the potential $L_{\mu\nu\sigma}$:

$$\begin{aligned} C_{\mu\nu\sigma\rho} &= L_{\mu\nu\sigma;\rho} + L_{\sigma\rho\mu;\nu} - L_{\mu\nu\rho;\sigma} - L_{\sigma\rho\nu;\mu} \\ &\quad + g_{\nu\sigma}L_{\mu\rho} + g_{\mu\rho}L_{\nu\sigma} - g_{\nu\rho}L_{\mu\sigma} - g_{\mu\sigma}L_{\nu\rho} \\ &\quad + \frac{2}{3}L^{\lambda\kappa}{}_{\lambda;\kappa}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\nu\sigma}g_{\mu\rho}\right) \\ &\equiv W_{\mu\nu\sigma\rho} \end{aligned} \tag{V4}$$

where $L_{\mu\nu} = L_{\nu\mu} \equiv \frac{1}{2}\left(L_{\mu}{}^{\kappa}{}_{\nu;\kappa} + L_{\nu}{}^{\kappa}{}_{\mu;\kappa} - L_{\mu}{}^{\kappa}{}_{\kappa;\nu} - L_{\nu}{}^{\kappa}{}_{\kappa;\mu}\right)$. Next, the author considers how Eq. (V4) changes under the local conformal rescaling $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$, $L_{\mu\nu\sigma} \rightarrow \tilde{L}_{\mu\nu\sigma} = \Omega^s(x)L_{\mu\nu\sigma}$ (where s is to be determined) and finds that the right side transforms according to:

$$\begin{aligned} \tilde{W}^{\mu}{}_{\nu\sigma\rho} &= \Omega^{(s-2)}W^{\mu}{}_{\nu\sigma\rho} \\ &\quad + \frac{(s-3)}{2}\Omega^{(s-3)}g^{\mu\alpha}\left[2\left\{L_{\alpha\nu\sigma}\Omega_{,\rho} + L_{\nu\alpha\rho}\Omega_{,\sigma}\right.\right. \\ &\quad \left. + L_{\sigma\rho\alpha}\Omega_{,\nu} + L_{\rho\sigma\nu}\Omega_{,\alpha}\right\} \\ &\quad + \left\{g_{\alpha\rho}\left(L_{\nu}{}^{\kappa}{}_{\sigma} + L_{\sigma}{}^{\kappa}{}_{\nu}\right) + g_{\nu\sigma}\right. \\ &\quad \times\left(L_{\alpha}{}^{\kappa}{}_{\rho} + L_{\rho}{}^{\kappa}{}_{\alpha}\right) - g_{\alpha\sigma}\left(L_{\nu}{}^{\kappa}{}_{\rho} + L_{\rho}{}^{\kappa}{}_{\nu}\right) \\ &\quad \left. - g_{\nu\rho}\left(L_{\alpha}{}^{\kappa}{}_{\sigma} + L_{\sigma}{}^{\kappa}{}_{\alpha}\right)\right\}\Omega_{,\kappa} \\ &\quad + \left\{\left(g_{\nu\rho}L_{\sigma}{}^{\kappa}{}_{\kappa} - g_{\nu\sigma}L_{\rho}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\alpha}\right. \\ &\quad + \left(g_{\alpha\sigma}L_{\rho}{}^{\kappa}{}_{\kappa} - g_{\alpha\rho}L_{\sigma}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\nu} \\ &\quad + \left(g_{\rho\nu}L_{\alpha}{}^{\kappa}{}_{\kappa} - g_{\rho\alpha}L_{\nu}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\sigma} \\ &\quad \left. + \left(g_{\sigma\alpha}L_{\nu}{}^{\kappa}{}_{\kappa} - g_{\sigma\nu}L_{\alpha}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\rho}\right\} \\ &\quad \left. - \frac{4}{3}L^{\lambda\kappa}{}_{\kappa}\Omega_{,\lambda}\left(g_{\alpha\sigma}g_{\nu\rho} - g_{\nu\sigma}g_{\alpha\rho}\right)\right] \end{aligned} \tag{V31}$$

Demanding scale invariance, i.e., $\tilde{W}^{\mu}{}_{\nu\sigma\rho} = W^{\mu}{}_{\nu\sigma\rho}$, Vishwakarma then sets $s = 2$ and discards the additive term proportional to $(s-3)$ by proposing, without proof, that the potential $L_{\mu\nu\sigma}$ can always be chosen to satisfy the intricate "symmetry" condition:

$$\begin{aligned} &\left[2\left\{L_{\alpha\nu\sigma}\Omega_{,\rho} + L_{\nu\alpha\rho}\Omega_{,\sigma} + L_{\sigma\rho\alpha}\Omega_{,\nu} + L_{\rho\sigma\nu}\Omega_{,\alpha}\right\}\right. \\ &\quad + \left\{g_{\alpha\rho}\left(L_{\nu}{}^{\kappa}{}_{\sigma} + L_{\sigma}{}^{\kappa}{}_{\nu}\right) + g_{\nu\sigma}\left(L_{\alpha}{}^{\kappa}{}_{\rho} + L_{\rho}{}^{\kappa}{}_{\alpha}\right)\right. \\ &\quad \left. - g_{\alpha\sigma}\left(L_{\nu}{}^{\kappa}{}_{\rho} + L_{\rho}{}^{\kappa}{}_{\nu}\right) - g_{\nu\rho}\left(L_{\alpha}{}^{\kappa}{}_{\sigma} + L_{\sigma}{}^{\kappa}{}_{\alpha}\right)\right\}\Omega_{,\kappa} \\ &\quad + \left\{\left(g_{\nu\rho}L_{\sigma}{}^{\kappa}{}_{\kappa} - g_{\nu\sigma}L_{\rho}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\alpha}\right. \\ &\quad + \left(g_{\alpha\sigma}L_{\rho}{}^{\kappa}{}_{\kappa} - g_{\alpha\rho}L_{\sigma}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\nu} \\ &\quad + \left(g_{\rho\nu}L_{\alpha}{}^{\kappa}{}_{\kappa} - g_{\rho\alpha}L_{\nu}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\sigma} \\ &\quad \left. + \left(g_{\sigma\alpha}L_{\nu}{}^{\kappa}{}_{\kappa} - g_{\sigma\nu}L_{\alpha}{}^{\kappa}{}_{\kappa}\right)\Omega_{,\rho}\right\} \\ &\quad \left. - \frac{4}{3}L^{\lambda\kappa}{}_{\kappa}\Omega_{,\lambda}\left(g_{\alpha\sigma}g_{\nu\rho} - g_{\nu\sigma}g_{\alpha\rho}\right)\right] = 0 \end{aligned} \tag{V32}$$

for arbitrary Ω.

But this is easily falsified without any need to compute Lanczos potentials and test them in the cumbersome rank-4

[1] Tensor calculations were performed in Wolfram *Mathematica* using the *xAct* suite of packages for computer algebra [2].

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condition (V32). We instead form a much simpler necessary condition of rank-0 by first contracting Eq. (V32) with $C^{\alpha\nu\sigma\rho}$ to get $8\,C^{\alpha\nu\sigma\rho}L_{\alpha\nu\sigma}\Omega_{,\rho} = 0$. Because the function $\Omega(x)$ is arbitrary, this requires that the coefficient vector $Q^\rho \equiv C^{\alpha\nu\sigma\rho}L_{\alpha\nu\sigma} = 0$ throughout all spacetime. And since Q^ρ vanishes everywhere, then so must its covariant derivative $Q^\rho{}_{;\tau}$ and, more specifically, its covariant divergence $Q^\rho{}_{;\rho}$. The scalar condition $Q^\rho{}_{;\rho} = 0$ is thus necessary (but of course, not sufficient) for the validity of Eq. (V32). To check whether this divergence does indeed vanish everywhere, we apply the product rule:

$$Q^\rho{}_{;\rho} = C^{\alpha\nu\sigma\rho}{}_{;\rho}L_{\alpha\nu\sigma} + C^{\alpha\nu\sigma\rho}L_{\alpha\nu\sigma;\rho}$$

and then use the once-contracted Bianchi identity:

$$C^{\alpha\nu\sigma\rho}{}_{;\rho} \equiv \tfrac{1}{2}\left(R^{\alpha\sigma;\nu} - R^{\nu\sigma;\alpha}\right) - \tfrac{1}{12}\left(g^{\alpha\sigma}R^{;\nu} - g^{\nu\sigma}R^{;\alpha}\right)$$

along with the "scalarized" Weyl–Lanczos equation obtained by the contraction of Eq. (V4) with the Weyl tensor:

$$C^{\alpha\nu\sigma\rho}C_{\alpha\nu\sigma\rho} = 4\,C^{\alpha\nu\sigma\rho}L_{\alpha\nu\sigma;\rho}$$

to rewrite it as:

$$Q^\rho{}_{;\rho} = \left(R^{\alpha\sigma;\nu} - \tfrac{1}{6}\,g^{\alpha\sigma}R^{;\nu}\right)L_{\alpha\nu\sigma} + \tfrac{1}{4}C^{\alpha\nu\sigma\rho}C_{\alpha\nu\sigma\rho}$$

Specializing now to Ricci-flat ("Rf") $R^{\alpha\sigma} = 0$ spacetimes, only the term quadratic in the Weyl-tensor survives on the right side:

$$Q^\rho_{(Rf);\rho} = \tfrac{1}{4}C^{\alpha\nu\sigma\rho}C_{\alpha\nu\sigma\rho} = \tfrac{1}{4}K$$

where K is the Kretschmann scalar. For example, in the Ricci-flat Schwarzschild ("Sc") spacetime with line element:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 \\ -r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

this divergence evaluates to $Q^\rho_{(Sc);\rho} = \frac{1}{4}K_{(Sc)} = \frac{12\,G^2M^2}{r^6}$, which is clearly *not* zero for any finite radius r or at any time t. In other words, the Lanczos potential $L_{(Sc)\alpha\nu\sigma}$ in a Schwarzschild spacetime with mass $M \neq 0$ violates Eq. (V32). Indeed, the potentials in *all* generic Ricci-flat spacetimes with $K \neq 0$ fail condition (V32), and we speculate that those in most Ricci-curved spacetimes likely fail as well.

So while there might exist particular spacetimes with Lanczos potentials that do happen to obey Eq. (V32), we conclude that Vishwakarma's scheme to implement conformal-covariance for the Weyl–Lanczos equation by setting $s = 2$ is not viable in the general case. Instead, we propose to accept that the Lanczos potential $L_{\mu\nu\sigma}$ most naturally scales with the weight $s = 3$ and investigate whether the Weyl–Lanczos equation itself can be altered to realize uniform overall scaling with this weighting. This possibility will be examined elsewhere.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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References

1. R.G. Vishwakarma, Lanczos potential of Weyl field: interpretations and applications. Eur. Phys. J. C **81**, 194 (2021). https://doi.org/10.1140/epjc/s10052-021-08981-5
2. J.M. Martín-García et al., xAct: efficient tensor computer algebra for the Wolfram language. http://www.xact.es (2002–2021). Accessed 1 June 2022