

plastic by the stresses induced during sectioning. The surface contour is clearly seen in juxtaposition with the surface slip markings. Uncut material extending into the sectioned surface corresponds to depressions in the surface. Caution should be exercised to distinguish between actual slip contours and artifacts introduced by irregularities in the cutting edge of the knife. At least one slip band can be seen extending a considerable distance into the sectioned surface, corresponding to a depth of 0.17 $\mu$ . The average pitch of the "stairstep" contour is  $\sim 0.02\mu$ .

Experiment has shown that fatigue cracks always originate in the deeper, "persistent" slip bands,<sup>5</sup> which may be considered as notches created by the cyclic slip characteristic of fatigue. A convenient and rapid method of viewing these notches simultaneously in depth and on the surface is afforded by the microtomed-taper sections reported here.

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<sup>1</sup>W. A. Wood and R. L. Segall, Bull. Inst. Metals 3, 160 (1957).  
<sup>2</sup>W. A. Wood, Phil. Mag. 3, 692 (1958).  
<sup>3</sup>P. B. Hirsch, P. G. Partridge, and R. L. Segall, Phil. Mag. 4, 721 (1959).  
<sup>4</sup>W. A. Backofen (private communication).  
<sup>5</sup>N. Thompson, N. J. Wadsworth, and N. Louat, Phil. Mag. 1, 113 (1956).

### Comments on a Ruptured Soap Film

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SUBSEQUENT to puncturing at a point, a horizontal soap film develops a hole whose edge, owing to surface tension, propagates outward from the point of puncture at apparently constant velocity. Measurements by Ranz<sup>1</sup> yielded results roughly 10% lower than those calculated on the basis of a simple energy conservation suggested by Rayleigh.<sup>2</sup> The discrepancy was attributed to an additional retarding viscous stress not included in the analysis. It appears, however, that the energy balance quoted<sup>1</sup> neglects an important contribution, indeed related to the viscous effect noted by Ranz, but which reduces the calculated values to 20% below those measured. A more detailed analysis of the motion of the edge gives this result; the neglected contribution arises from inelastic acceleration of the undisturbed fluid up to the velocity of the edge. The concomitant loss in mechanical energy may be identified with viscous dissipation which is estimated to be confined to a relatively thin region. Lack of agreement between calculated and measured values of the edge velocity seems to be caused by a second-order effect in the method used<sup>1</sup> to determine the thickness of the film.

The film is assumed to have uniform thickness  $h$  and uniform properties (in particular constant surface tension  $\sigma$  everywhere), so the flow is ideally symmetric about an axis through  $A$ , the point of puncturing. While the fluid from the destroyed portion of the film collects around the edge, the film outside the edge is, according to observation, undisturbed. Consider a sector in the plane of the film of small included angle  $\alpha$  and vertex at  $A$ ; the fraction of rolled edge within this sector and located at distance  $r$  from  $A$  may be treated as a system of variable mass  $m$  as it progresses outward at velocity  $v = dr/dt$ . Then relative to the point  $A$ , one may write for the sum of forces acting on the rolling edge, ignoring the effects of gravity and aerodynamic drag

$$F = \frac{d}{dt}(mv) = \frac{d}{dr}\left(\frac{mv^2}{2}\right) + \frac{v^2}{2} \frac{dm}{dr}$$

$$Fdr = d\left(\frac{mv^2}{2}\right) + \frac{v^2}{2} dm.$$

By conservation of mass,  $m = \frac{1}{2}\rho r^2 h \alpha$  and integrating over  $r$ ,

$$\int_0^r Fdr = \frac{1}{4}\rho r^2 h v^2 \alpha + \frac{\rho h \alpha}{2} \int_0^r v^2 r dr.$$

The integral on the left-hand side is the total work done by the film on the edge in the motion from  $A$  to  $r$ , and is, therefore, equal to the energy originally associated with the surface which has disappeared  $\sigma r^2 \alpha$ ; alternatively, the force acting on the edge is  $F = 2\sigma r \alpha$ . On assuming  $v$  to be independent of  $r$ ,

$$\sigma r^2 \alpha = \frac{1}{2}\rho \left(\frac{r^2 h \alpha}{2}\right) v^2 + \frac{1}{2}\rho \left(\frac{r^2 h \alpha}{2}\right) v^2.$$

The left-hand side is the energy liberated by the surface of the destroyed film; the first term on the right-hand side is the kinetic energy of the rolled edge, while the remaining term, neglected in the previously cited energy balance, is associated with the inelastic process of accelerating the fluid in the undisturbed film up to the velocity of the edge. This energy, which always appears in a problem involving "plastic pick-up" of mass, is evidently lost as regards the mechanical energy of the system; it is dissipated by internal fluid friction. The expression for the velocity of the edge is now

$$v = \left(\frac{2\sigma}{h\rho}\right)^{\frac{1}{2}},$$

which is  $1/\sqrt{2}$  smaller than that computed by Rayleigh and is equal to the propagation speed of elastic waves in the sheet.

That the calculated values of velocity differ so greatly from Ranz' measurements is probably more correctly ascribed to errors in the values of the ratio  $\sigma/h$ .  $\sigma$  has been assumed known (with questionable accuracy) and  $\sigma/h$  was measured in a manner which might lead to quite large errors. For example, as the film sags, since it does not rupture, ensuing gradients in concentration lead to increasing values of  $\sigma$  away from the lowest point. Assume variations in  $h$  to be moderate, but approximate  $\sigma$  by  $\sigma \approx \sigma_0 + \sigma'x$  in the coordinate system adopted by Ranz (Fig. 4)<sup>1</sup>; the force balance for a small circular section of film centered at the lowest point is

$$\rho g \pi x^2 h \approx 4\pi x(\sigma_0 + \sigma'x) dy/dx,$$

which can be integrated to

$$y = \frac{\rho g h}{4\sigma_0} \left[ \frac{x}{\sigma_1} - \frac{1}{\sigma_1^2} \log(1 + \sigma_1 x) \right]; \quad \sigma_1 = \frac{\sigma'}{\sigma_0}.$$

For small  $\sigma_1 x$ ,

$$y \approx \frac{\rho g h}{4\sigma_0} \left[ \frac{x^2}{2} - \frac{\sigma_1 x^3}{3} - \dots \right].$$

The second term in brackets constitutes a correction to Ranz' Eq. (5), which assumed  $\sigma' = 0$ . The sign of this correction implies that the shape of the film is something between a paraboloid and a plane; that is, for a given  $h/\sigma_0$  the focal length  $f$  is larger,

$$f = \frac{x^2}{4y} = \frac{2\sigma_0}{\rho g h}; \quad \sigma \text{ const}$$

$$f \approx \frac{x^2}{4y} \approx \frac{2\sigma_0}{\rho g h} \left[ 1 + \frac{2}{3}\sigma_1 x + \dots \right]; \quad \sigma = \sigma_0 + \sigma'x.$$

Conversely, a given measured  $f$  implies a smaller actual  $h/\sigma_0$  than that deduced from Ranz' equation. Consequently, the calculated values of  $v$  should be higher, reducing the difference between the experimental results and those computed from the expression derived in the foregoing.

Since the measured focal length is proportional to the radius of curvature of the film, it is likely to be quite sensitive to small variations in the film shape. On the other hand, the expression for  $v$  probably predicts the edge velocity quite accurately providing local values of  $h/\sigma$  are used.

Finally, it is a simple matter to estimate the extent of the region in which the acceleration of the fluid, and hence the dissipation of energy, occur. Suppose the acceleration of the undisturbed fluid to take place in a distance  $\delta$  at the outer boundary of the rolled edge where the thickness is still approximately  $h$ . The

volume in which dissipation occurs is then approximately  $\alpha r h \delta$ ; the rate at which energy is dissipated is

$$\mu \int \left( \frac{dv}{dr} \right)^2 d\tau,$$

the integral being over volume. Since  $dv/dr \sim v/\delta$ , mechanical energy is lost to the system at the rate

$$\mu \frac{v^2}{\delta^2} (\alpha r h \delta).$$

From the foregoing results, this must equal half the rate at which surface energy in the film is released,

$$\begin{aligned} \mu \frac{v^2}{\delta^2} (\alpha r h) &= \frac{1}{2} \frac{d}{dt} (\alpha r^2 \sigma), \\ \frac{\delta}{h} &\sim \frac{\mu v}{\sigma}. \end{aligned}$$

With  $v \approx 500$  cm/sec,  $\mu \approx 10^{-2}$  poises and  $\sigma \approx 26$  d/cm,  $\delta/h \sim 0.2$ , indicating that the viewpoint adopted here seems to be consistent. The temperature rise of the rolled edge can easily be computed to be negligibly small, of the order of  $0.006^\circ\text{C}$ , assuming all the heat to be retained in the edge.

Associated with the surface of the rolling edge itself is an amount of surface energy, which is contantly increasing as the edge collects fluid. This energy need not be included in previous considerations since it does not derive from the destroyed film but rather from the work done by the pressure difference across the surface of the edge. For, consider a unit length of edge, very nearly a cylinder, of radius  $a$ . The surface energy is then  $2\pi a\sigma$  (neglecting the small area intersected by the undisturbed film); the pressure difference across the surface is  $\Delta p = \sigma/a$  and the net normal force on the surface is  $\Delta p \cdot 2\pi a$ . Hence, in a small radial expansion  $\delta a$ , the work done by the normal force is  $2\pi\sigma\delta a$ , which exactly equals the change in surface energy  $\delta(2\pi\sigma a) = 2\pi\sigma\delta a$ . This work done by the fluid internal to the edge of course results in a slight lowering of the temperature which, however, is much less than the increase computed in the previous paragraph.

It is rather an amusing result that the viscous stresses can have such a significant influence on the energy distribution. Another example of this is Rayleigh's problem<sup>3</sup> of the infinite plate bounded on one side by an incompressible fluid and started impulsively in motion with constant velocity at  $t=0$ . One can easily compute that at any later time, of the work done in overcoming surface friction, the  $\sqrt{2}/2$  part has been dissipated by internal friction, the remainder appearing as kinetic energy.

<sup>1</sup> W. E. Ranz, *J. Appl. Phys.* **30**, 1950 (1959).

<sup>2</sup> Lord Rayleigh, *Sci. Papers* **3**, 441 (1902).

<sup>3</sup> Lord Rayleigh, *Sci. Papers* **1**, 474 (1899).

### More Experiments on Liquid Films

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WHEN two opposed vertical jets collide, a stable liquid horizontal sheet of radius  $r_0$  is formed. Some experiments on such liquid films were reported recently by Ranz<sup>1</sup> and Taylor.<sup>2</sup> The former rediscovered a simple relation between  $r_0$  and the surface tension of the liquid. This relation first derived and checked experimentally with water, mercury, and aqueous solutions of soaps and alcohols by Bond and Puls<sup>3-5</sup> is based on a straight momentum balance, assuming no losses either in the region of impact or in the film itself.

More experiments on liquid films are summarized here to show that the second assumption is correct but that the first one applies only to liquids of low viscosity. The electrical con-

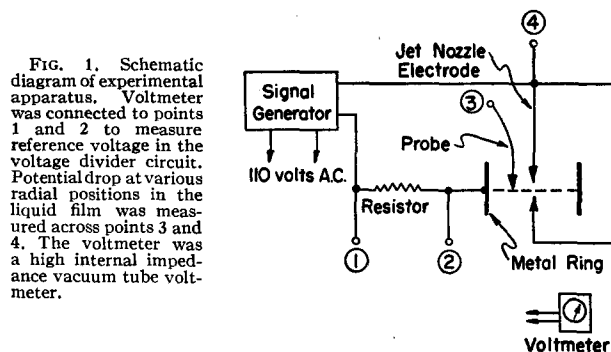


FIG. 1. Schematic diagram of experimental apparatus. Voltmeter was connected to points 1 and 2 to measure reference voltage in the voltage divider circuit. Potential drop at various radial positions in the liquid film was measured across points 3 and 4. The voltmeter was a high internal impedance vacuum tube voltmeter.

ductivity of the films was measured in the apparatus shown in Fig. 1. Clearly, as the distance  $r$  between the probe and the external metal ring is increased, the measured voltage drop should increase linearly if the cross-sectional area  $A$  of the film perpendicular to the direction of current and fluid flow stays constant. This is indeed the case as shown by the data of Fig. 2. But if  $A$  does not change with  $r$ , no detectable momentum loss takes place in the film itself.

On the other hand, comparison between the slopes of the straight lines for water (viscosity: 1 cps) and a glycerol water solution (viscosity: 91 cps) reveals that, in the latter case, the calculated area  $A$  is now larger. This implies a loss of momentum in the region of impact for the more viscous fluid.

A similar situation obtains for mixing between the liquid jets. While some mixing occurs at the region of impact, no detectable mixing takes place in the sheet itself. This again was shown by

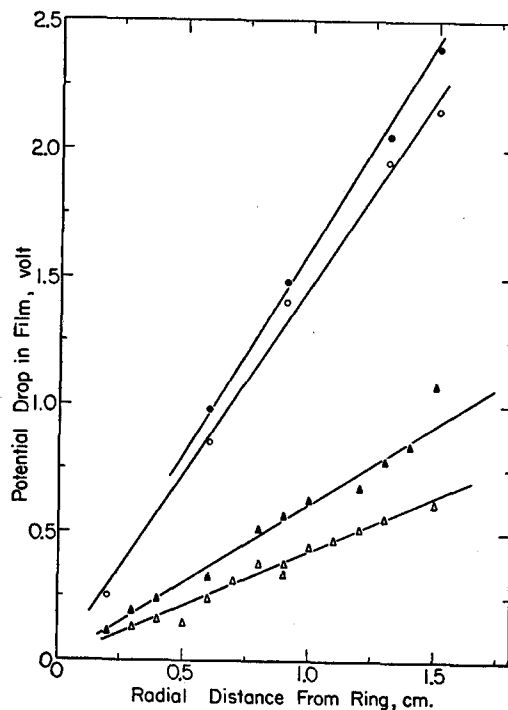


FIG. 2. Potential drop in film (volt) vs radial distance from ring (cm). Upper nozzle diameter: 0.191 cm; lower jet diameter: 0.193 cm.  $\blacktriangle$  Upper jet: dilute water solution of NaCl, viscosity 1 cps, velocity at the nozzle: 392 cm/sec; lower jet: same liquid, velocity at the nozzle: 384 cm/sec.  $\triangle$  Upper jet: a mixture of water, glycerol, and NaCl, viscosity 91 cps, velocity at the nozzle: 236 cm/sec; lower jet: same liquid, velocity at the nozzle: 231 cm/sec.  $\bullet$  Upper jet: dilute water solution of  $\text{Ba}(\text{OH})_2$ , velocity at the nozzle: 381 cm/sec; lower jet: dilute water solution of  $\text{H}_2\text{SO}_4$ , velocity at the nozzle: 392 cm/sec.  $\circ$  Upper jet: dilute water solution of  $\text{Ba}(\text{OH})_2$ , velocity at the nozzle: 213 cm/sec; lower jet: dilute water solution of  $\text{H}_2\text{SO}_4$ , velocity at the nozzle: 228 cm/sec.