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Comments on bases in dependence structures

Richard A. Brualdi

Dependence structures (in the finite case, matroids) arise when one tries to abstract the properties of linear dependence of vectors in a vector space. With the help of a theorem due to P. Hall and M. Hall, Jr concerning systems of distinct representatives of families of finite sets, it is proved that if B_1 and B_2 are bases of a dependence structure, then there is an injection $\sigma : B_1 \to B_2$ such that $(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$ is a basis for all e in B_1. A corollary is the theorem of R. Rado that all bases have the same cardinal number. In particular, it applies to bases of a vector space. Also proved is the fact that if B_1 and B_2 are bases of a dependence structure then given e in B_1 there is an f in B_2 such that both $(B_1 \setminus \{e\}) \cup \{f\}$ and $(B_2 \setminus \{f\}) \cup \{e\}$ are bases. This is a symmetrical kind of replacement theorem.

One of the most fundamental theorems in combinatorial theory is a theorem about systems of distinct representatives. If $(A_i : i \in I)$ is a family, indexed by I, of (not necessarily distinct) subsets of a set E, then the family $(e_i : i \in I)$ of *distinct* elements of E is a *system of distinct representatives* for $(A_i : i \in I)$ provided $e_i \in A_i$ for each $i \in I$. The theorem referred to above is the following.

If $(A_i : i \in I)$ is a family of finite subsets of E, then $(A_i : i \in I)$ has a system of distinct representatives if and only if for all finite subsets J of I

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$$|\bigcup_{i \in J} A_i| \geq |J| .$$

We are using the symbol $|X|$ to denote the cardinal number of the set X. For a finite index set I, this theorem is due to P. Hall [1]; for an infinite index set I it is due to M. Hall, Jr [2]. In [2] M. Hall, Jr used his theorem to give an interesting proof that in an infinite dimensional vector space any two bases have the same cardinal number. Our purpose here is to use the idea in Hall's proof to obtain some interesting properties of bases. Moreover we shall do this in a more abstract setting.

The study of the theory of abstract linear dependence and independence has received considerable attention recently. One reason for this is its wide applicability to many kinds of combinatorial problems. The initiative for this study seems to have been taken by H. Whitney [3]. In his pioneering paper Whitney set forth various sets of axioms for an abstract dependence structure on a finite set and proved their equivalence. We shall use here a set of axioms in terms of the minimal dependent sets. The axioms are actually suggested by graph theory. They have been used by Whitney and W. Tutte [4] in the finite situation and also by D. Asche [5] in the general situation.

Let E be a nonempty set. Let a collection of nonempty finite subsets of E, called *circuits*, be specified which satisfy the following two conditions.

(1) *No circuit is a proper subset of another circuit.*

(2) *If* C_1 *and* C_2 *are distinct circuits and* $a \in C_1 \cap C_2$, *then there is a circuit* C_3 *with*

$$C_3 \subseteq (C_1 \cup C_2) \setminus \{a\} .$$

It is known (see e.g. [5]) that (2) in the presence of (1) is equivalent to

(2') *If* C_1 *and* C_2 *are circuits with* $a \in C_1 \cap C_2$ *and* $b \in C_1 \setminus C_2$, *then there exists a circuit* C_3 *with*

$$b \in C_3 \subseteq (C_1 \cup C_2) \setminus \{a\} .$$

We shall call the set E along with a collection of circuits satisfying (1) and (2) a *dependence structure*. If E is finite, then we refer to a *finite dependence structure*, often called a *matroid*.

Conditions (1) and (2) are easily verified if E is a set of vectors

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from a vector space and the circuits are the nonempty linearly dependent subsets of E for which every proper subset is linearly independent.

In a dependence structure an *independent set* is a subset of E which contains no circuit. Since circuits are finite sets, a subset A of E is independent if and only if every finite subset of A is independent. Thus independence is a property of finite character. Using Zorn's Lemma in conjunction with the finite character of independence, we immediately deduce that maximal independent sets exist, in fact that every independent set is contained in a maximal independent set. A maximal independent subset of E is called a *basis*.

We require two lemmas.

LEMMA 1 *Let B be a basis and $e \notin B$. Then $B \cup \{e\}$ contains a unique circuit C. This circuit contains e, and moreover for $f \in B$, $(B \setminus \{f\}) \cup \{e\}$ is a basis if and only if $f \in C$.*

Proof Since B is a basis, $B \cup \{e\}$ contains a circuit C_1. This circuit must contain e, for B is independent. If $B \cup \{e\}$ contained two distinct circuits, then (2) would imply that there is a circuit contained in B, which is impossible. Thus C_1 is unique. Let $f \in B$. If $f \notin C_1$, then $(B \setminus \{f\}) \cup \{e\}$ contains C_1 and thus cannot be a basis. If $f \in C_1$, then surely $A = (B \setminus \{f\}) \cup \{e\}$ contains no circuit and hence is independent. If A were not a basis there would be an $x \notin A$ such that $A \cup \{x\}$ is independent. Now $x \neq f$, so that $x \notin B$. By what we have already proved $B \cup \{x\}$ contains a unique circuit C_2, which contains x. If $f \notin C_2$, then $C_2 \subseteq A$, a contradiction. Hence $f \in C_2$.

Also there is a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus \{f\} \subseteq A \cup \{x\}$ since $f \in C_1$, by (2), a contradiction. Therefore A is a basis and we are done.

LEMMA 2 *Let $C_1, C_2, \ldots, C_n$ be distinct circuits with $C_k \nsubseteq \bigcup_{i \neq k} C_i$, $k = 1, 2, \ldots, n$. If $D \subseteq E$ with $|D| < n$, then there exists a circuit C with*

$$C \subseteq \left(\bigcup_{i=1}^{n} C_i\right) \setminus D .$$

Lemma 2 is a special case of Theorem 3 of Asche [5].

We are now prepared to state and prove our main result.

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THEOREM 1 *If* B_1 *and* B_2 *are two bases of a dependence structure, then there exists a injection* $\sigma : B_1 \to B_2$ *such that*

$$(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$$

is a basis for all $e \in B_1$.

Proof It is enough to find an injection $\sigma : B_1 \setminus B_2 \to B_2 \setminus B_1$ having the above property, for we may then extend σ to B_1 by defining $\sigma(e) = e$ for all $e \in B_1 \cap B_2$.

For each $e \in B_1 \setminus B_2$ let C_e be the unique circuit contained in $B_2 \cup \{e\}$. Let $C'_e = C_e \cap (B_2 \setminus B_1)$. It follows that $C'_e \neq \phi$, for otherwise $C_e \subseteq B_1$, which is impossible since B_1 is independent. By Lemma 1 we also know that for each $f \in C'_e$, $(B_2 \setminus \{f\}) \cup \{e\}$ is a basis. Consider the family $(C'_e : e \in B_1 \setminus B_2)$ of finite sets. If this family has a system of distinct representatives then this means that there is an injection $\sigma : B_1 \setminus B_2 \to B_2 \setminus B_1$ such that $\sigma(e) \in C'_e$ for all $e \in B_1 \setminus B_2$ and thus $(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$ is a basis for all $e \in B_1 \setminus B_2$. Therefore to complete the proof we need only verify that Hall's condition for the existence of a system of distinct representatives for $(C'_e : e \in B_1 \setminus B_2)$ is fulfilled.

Suppose $\{e_1 , \ldots , e_n\}$ is a subset of finite cardinality n of $B_1 \setminus B_2$. If

$$|C'_{e_1} \cup \ldots \cup C'_{e_n}| < n ,$$

then by taking $D = C'_{e_1} \cup \ldots \cup C'_{e_n}$ in Lemma 2, there is a circuit C with

$$C \subseteq (\cup_{i=1}^{n} C_{e_i}) \setminus D \subseteq B_1 .$$

This contradicts the independence of B_1 and completes the proof of the theorem.

COROLLARY 1 (Rado [6]) *All bases have the same cardinal number.*

Proof If B_1 and B_2 are bases, then by Theorem 1 there are injections $\sigma : B_1 \to B_2$ and $\tau : B_2 \to B_1$.

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COROLLARY 2 *If one basis is finite, then all bases are finite, and given two bases* B_1 *and* B_2 *there is a bijection* $\sigma : B_1 \to B_2$ *such that* $(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$ *is a basis for all* $e \in B_1$.

COROLLARY 3 *If* B_1 *and* B_2 *are finite bases, then there exists a bijection* $\tau : B_1 \to B_2$ *such that* $(B_1 \setminus \{e\}) \cup \{\tau(e)\}$ *is a basis for all* $e \in B_1$.

Proof Reversing the rôles of B_1 and B_2 in Corollary 2, we know there exists a bijection $\rho : B_2 \to B_1$ such that $(B_1 \setminus \{\rho(e)\}) \cup \{e\}$ is a basis for all $e \in B_2$. The corollary follows by taking τ to be ρ^{-1} .

Corollary 3 remains valid if B_1 and B_2 are allowed to be infinite but $B_1 \setminus B_2$ (and thus $B_2 \setminus B_1$) is finite.

Suppose we have a dependence structure with finite bases B_1 and B_2 . We now know there exist two bijections $\sigma : B_1 \to B_2$ and $\tau : B_1 \to B_2$ such that $(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$ and $(B_1 \setminus \{e\}) \cup \{\tau(e)\}$ are bases for all $e \in B_1$. The question arises as to whether we can choose $\sigma = \tau$. That is, does there exist a bijection $\pi : B_1 \to B_2$ such that both

$$(B_2 \setminus \{\pi(e)\}) \cup \{e\}$$

and

$$(B_1 \setminus \{e\}) \cup \{\pi(e)\}$$

are bases for all $e \in B_1$? Before this question should be answered the following simpler question should be answered. Given $e \in B_1$, does there exist a $f \in B_2$ such that both $(B_1 \setminus \{e\}) \cup \{f\}$ and $(B_2 \setminus \{f\}) \cup \{e\}$ are bases? The following theorem resolves this affirmatively with no finiteness restrictions.

THEOREM 2 *Let* B_1 *and* B_2 *be two bases of a dependence structure. Given* $e \in B_1$ *there exists a* $f \in B_2$ *such that*

$$(B_1 \setminus \{e\}) \cup \{f\} , (B_2 \setminus \{f\}) \cup \{e\}$$

are both bases.

Proof If $e \in B_1 \cap B_2$, we may choose $f = e$. Let $e \in B_1 \setminus B_2$ and let C be the unique circuit contained in $B_2 \cup \{e\}$. Surely $C \cap (B_2 \setminus B_1) \neq \phi$. For each $x \in C \cap (B_2 \setminus B_1)$, $B_1 \cup \{x\}$ contains a

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unique circuit. If none of these circuits contain e, then after a number of applications of property (2') we obtain a circuit containing e but contained within B_1. This is a contradiction. Hence there exists a $f \in C \cap (B_2 \setminus B_1)$ such that the unique circuit contained in $B_1 \cup \{f\}$ contains e. Hence by Lemma 1, both $(B_2 \setminus \{f\}) \cup \{e\}$ and $(B_1 \setminus \{e\}) \cup \{f\}$ are bases.

In spite of Theorem 2 the answer to the first question we raised is no. To obtain a counterexample, let E be the set of vectors of the vector space of 3-tuples over a finite field of two elements. Let 'dependence' and 'independence' mean 'linear dependence' and 'linear independence' respectively. Consider the two bases

$$B_1 = \{(1, 0, 0), (1, 0 . 1), (0, 1, 1)\}$$

$$B_2 = \{(0, 1, 0), (0, 0, 1), (1, 1, 0)\} .$$

Then $(B_1 \setminus \{(1,0,0)\}) \cup \{x\}$ is a basis for $x = (0,1,0)$ or $(0,0,1)$ but not for $x = (1,1,0)$. Also $(B_2 \setminus \{x\}) \cup \{(1,0,0)\}$ is a basis for $x = (0,1,0)$ or $(1,1,0)$ but not for $x = (0,0,1)$. Therefore

$$(B_1 \setminus \{(1, 0, 0)\}) \cup \{x\}, \quad (B_2 \setminus \{x\}) \cup \{(1, 0, 0)\}$$

are both bases only for $x = (0, 1, 0)$ of B_2. In a similar way, one concludes that

$$(B_1 \setminus \{(0, 1, 1)\}) \cup \{x\}, \quad (B_2 \setminus \{x\}) \cup \{(0, 1, 1)\}$$

are both bases only for $x = (0, 1, 0)$ of B_2. Thus the bijection whose existence was questioned cannot exist in this case.

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Department of Mathematics,
University of Wisconsin,
Madison, Wisconsin, USA.